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STABILITY ANALYSIS OF FUZZY DIFFERENTIAL EQUATION FOR PREY–PREDATOR SYSTEM UNDER Z-NUMBER FUZZY AND GRANULAR DIFFERENTIABILITY CONCEPT

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Abstract. The stability analysis is one of the basic problems in the field of system, prey–predator, production inventory control, signal processing etc. The goal of stability analysis for prey–predator and production inventory control model is to determine the region of the different variables and parameters at which the system is stable or unstable. This paper explores an uncertain prey–predator model where both prey and predator populations are fuzzy variables. Also we have considered all the prey–predator parameters of the system are imprecise. This fuzzy prey–predator model is converted to an equivalent crisp model using granular differentiability approach. The condition of local stability is obtained mathematically by analysis the eigenvalues of characteristic equation. And also we introduced a production inventory model as a case study of fuzzy dynamical system to illustrate the efficiency of the result and practical aspects.

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1. Introduction

Mathematical modeling of ecosystems is a study of understanding the mechanisms that influence the growth of prey–predator, their existence and stability. Biological phenomena such as prey–predator system with infection, prey–predator fishery harvesting system, prey–predator interaction etc. can be exhibited by a non-autonomous or autonomous system of ordinary system.

1.1. Literature review

The theoretical biology in this area was first inaugurated by Thomas Malthus which is known as Malthusian growth model. Other famous examples of prey–predator equations are Lotka [1] and Volterra [2] equation. Then the researchers developed the significance of this area of population dynamics. The bio-economic exploitation of the renewable resource in fisheries management was presented by Clark [3, 4]. Zhang *et al.* [5], Fan and Wanng [6], Wang and Wang [7] inquire into optimal harvesting policies for a single population. Kar *et al.*

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[8] presented and resolved a prey–predator harvesting model incorporating a prey refuge. Stage structured prey–predator harvesting model solved by Chakraborty *et al.* [9,10], Qu and Wei [11] submitted a bifurcation analysis in a stage structure prey–predator growth model. From the above research works we can observe that the biological parameters which were used in the models are fixed but in reality depending on dynamical ecological conditions the biological parameters are not always fixed. Not only that due to lack of proper numerical information such as data collection, determining initial condition, measurement process some or all parameters become uncertain. Some researchers handled this problem in stochastic approach and also provide stochastic solution to the system. Some stochastic approach works are given by Beddington and May [12], Li *et al.* [13], Wu *et al.* [14], Liu *et al.* [15], Lande *et al.* [16], Ji *et al.* [17] and many others. In stochastic approach, the parameters are apprehending as random variables with some known probability distributions. To conquer these difficulties non-stochastic uncertain approaches are more suitable options. For non-stochastic uncertain approaches the parameters are replaced by fuzzy-random, rough, fuzzy etc. Ambiguity is inborn in most real world systems. One important ambiguity is fuzziness, which act as an indispensable part in the real actual situation. At first the scientist Zadeh [18] in 1965 presented the fuzzy set theory. The applications of fuzzy sets, including decision making [19], probability [20], medical studies [21], and the characterization of complex systems [22] are currently used in the majority of scientific disciplines. Two unique perspectives can be seen as fuzzy numbers, the first one is their membership function, and the second being their alpha-cut. Cheng and Zadeh [23] first introduced the fuzzy derivative. By following the fuzzy derivative Dubois and Parde [24] presented the extension principle in their article. Najariyan and Farahi [25] presented the strongly generalized Hukuhara (SGH) differentiability in fuzzy dynamical system and also design the fuzzy optimal control problem using generalized Hukuhara (gH) differentiability concept. Peixoto *et al.* [26] investigate a prey–predator model with fuzzy nature. Pal *et al.* [27], De *et al.* [28,29] presented optimal harvesting model with interval parameters for prey–predator system. Wang [30] presented stability analysis of prey–predator system with fuzzy impulsive control. The stability and bionomic analysis of prey–predator harvesting model using UFM for fuzzy parameters also presented by Pal *et al.* [31]. Khatua *et al.* [32] presented Stability of fuzzy dynamical system based on quasi-level-wise system. Hati [33] developed a sustainable pollution control fuzzy production inventory model. Mohammed Mustafa *et al.* [34] presented the solution of fuzzy optimal control problem with fuzzy variation under granular differentiability idea. De *et al.* [35] studied a fuzzy production model of assortment items under granular approach to control the preservation cost. Khatua *et al.* [36] presented a fuzzy production inventory control model under gr-differentiability approach. Hati *et al.* [37] presented a multi-objective pollution control innovation inventory model with stock dependent demand in finite time horizon. Hati *et al.* [38] developed a reliability dependent imperfect production inventory optimal control fractional order model for uncertain environment under granular differentiability. Hati *et al.* [39] presented a Product process innovation model of fuzzy optimal control of nonlinear system with finite time horizon under granular differentiability concept. Najariyan *et al.* [40] developed the stability of fuzzy linear dynamical systems. Najariyan *et al.* [41] presented a interval type-2 fuzzy differential equations and stability analysis.

1.2. Motivation of this study

Due to the globalization, green house effect and pollution the ecological system are continuously changeable and the world complexity is increased day to day that are the main cause of facing ambiguity in different configuration. Depending on dynamical ecological conditions the biological parameters are imprecise in nature. In recent years some researchers developed the fuzzy dynamical system and fuzzy production inventory optimal control model using strongly generalized hukuhara differentiability (SGH-differentiability), gH-differentiability, granular differentiability (gr-differentiability) approach. The drawbacks such as the unnatural behavior in modeling phenomenon, doubling feature and multiplicity solutions is created by using SGH-differentiability and gH-differentiability approach. But using gr-differentiability approach in fuzzy dynamical system and fuzzy production inventory control system, avoids all the above drawbacks created by using SGH-differentiability and gH-differentiability approach. The fuzzy prey–predator model or fuzzy production inventory models are extensively used for the analysis of the market stability condition of a production company, analysis of economic processes and phenomena due to competitive interaction of different production company. In real life situations, products, industries, technologies, brands, enterprises can compete with each other. Analogues of the population size indicator in models with economic applications can be the volume of production, capital, profits, profitability, the number and value of shares, the number of customers and so on. To analyze the development of internet resources, web services, social networks, commercial websites the modified prey–predator model are used successfully. When we try to control different type of fuzzy model such as fuzzy prey–predator model, fuzzy production inventory model with fuzzy dynamical system in different field, we have definitely needed to check the stability analysis of the said model.

The main achievements of the paper are as follows:

– In this paper we develop a fuzzy prey–predator model with all variables and parameters are fuzzy in nature.
– And also develop the stability of a autonomous fuzzy simultaneous differential equation with fuzzy variables under gr-differentiability approach.
– We developed two fuzzy differential equations for production inventory model and check the stability under gr-differentiability approach.
– Here we consider the fuzzy variables are triangular fuzzy number and defuzzified the triangular fuzzy number under granular differentiability approach.
– This paper explores that theoretically and practically the granular differentiability approach is more stable than the interval mathematics to convert the fuzzy dynamical system into crisp dynamical system.

Due to the lack of precise numerical information such as experimental part, the measurement process the data collection and determining the initial conditions, some parameters are fuzzy in nature. For the direct effect of those parameters on the variables, the variables become also fuzzy in nature. But when we try to control different type of fuzzy prey–predator or production inventory control models, then we have definitely essential to check the stability of the said model. There are so many methods have been used by researchers to convert an imprecise model in a deterministic form and to check the stability of the model. And they have been faced some difficulties when the said model is unstable. At first in this paper we explore the stability analysis theoretically using interval mathematics concept and granular differentiability concept. And also by developing a fuzzy prey–predator model and fuzzy production inventory model we check the stability analysis practically.

The rest of the paper is arranged accordingly: Section 2 briefly summarizes the essential preliminaries related to the definition and theory of fuzzy dynamical system. Section 3 provides the method of converting Z-number to regular fuzzy number. Section 4 presented the methods to check the stability analysis of fuzzy dynamical system. Section 5 presented the process of formulation of prey–predator model. Section 6 provides the process of local stability analysis of fuzzy dynamical model. Section 7 presented another case study to check stability analysis. Section 8 provides the discussion about the model. Section 9 presented the managerial implication of the model. Some conclusion of this model is presented in Section 10.

2. Fuzzy dynamical system

2.1. Stability of linear dynamical system

Consider a Linear dynamical system with constant co-efficient $\frac{\mathrm{d}x}{\mathrm{d}t} = Ax(t)$ where $x : [0, \infty) \to R^n$ and $A = [a_{ij}]_{n\times n}$, $a_{ij} \in R$. The eigenvalue of A, λ_i, $i = 1, 2, \ldots, n$ are the roots of the characteristic polynomial $\det(A - \lambda_i I)$, $i = 1, 2, \ldots, n$, where $\det(\cdot)$ means determinant and I is the $n \times n$ identity matrix.

Theorem 1 ([42]). *All solutions of the regular linear system $\dot{y}(t) = Ay(t) + f(t)$ have the same Liapunov stability property (unstable, stable, uniformly stable, uniformly and asymptotically stable, asymptotically stable). This is the same as that of the zero (or any other) solution of the homogeneous equation $\dot{x}(t) = Ax(t)$.*

Theorem 2 ([42]). *Let A be a constant matrix in the system $\dot{y}(t) = Ay(t)$ with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$.*

(i) *If the system is stable, then* $\mathrm{Re}(\lambda_i) \le 0$, $i = 1, 2, \ldots, n$.

(ii) *If either* $\mathrm{Re}(\lambda_i) < 0$, $i = 1, 2, \ldots, n$ *or if* $\mathrm{Re}(\lambda_i) \leq 0$, $i = 1, 2, \ldots, n$ *and there is no zero repeated eigenvalues, then the system is uniformly stable.*

(iii) *The system is asymptotically stable if and only if* $\mathrm{Re}(\lambda_i) < 0$, $i = 1, 2, \ldots, n$ *(and then it is also uniformly stable by (ii)).*

(iv) *If* $\mathrm{Re}(\lambda_i) > 0$, *for any* i, *the solution is unstable.*

In this section, we have discussed some necessary knowledge on fuzzy dynamical system. Here the set of all real numbers is denoted be R, the set of all fuzzy number on R by E_1.

Definition 1 ([43])**.** The fuzzy set $\widetilde{u} : R \to [0,1]$ is called a fuzzy number if it is normal, fuzzy convex, upper semi-continuous and compactly supported fuzzy subsets of real numbers. The fuzzy number $\widetilde{u}$ can be represented in a parametric form by the ordered pair of function $(\underline{u}^\mu, \overline{u}^\mu), 0 \leq \mu \leq 1$, satisfying the following properties:

(i) $\underline{u}^\mu$ is a bounded non-decreasing left continuous function in $(0, 1]$, and it is right continuous at $\mu = 0$.
(ii) $\overline{u}^\mu$ is a bounded non-decreasing left continuous function in $(0, 1]$, and it is right continuous at $\mu = 0$.
(iii) $\underline{u}^\mu \leq \overline{u}^\mu$.

Definition 2 ([44])**.** The fuzzy function $\widetilde{f} : (a, b) \subseteq R \to E_1$ is said to be generalized Hukuhara (gH) differentiable at $t \in (a, b)$, if there exists a fuzzy number $\frac{\mathrm{d}\widetilde{f}}{\mathrm{d}t} \in E_1$ such that the following limit exists $\lim_{h\to 0} \frac{\widetilde{f}(t+h)\Theta_{gh}\widetilde{f}(h)}{h} = \frac{\mathrm{d}\widetilde{f}(t)}{\mathrm{d}t}$ where "Θ_{gh}" stands for generalized Hukuhara difference and means $\widetilde{u}\Theta_{gh}\widetilde{v} = \widetilde{z} \Longleftrightarrow \widetilde{u} = \widetilde{v} + \widetilde{z}$ or $\widetilde{u} - \widetilde{z} = \widetilde{v}$.

Definition 3 ([45])**.** Let $\widetilde{u} : [a, b] \subseteq R \to [0, 1]$ be a fuzzy number. The horizontal membership function $u^{gr} : [0, 1] \times [0, 1] \to [a, b]$ is a representation of $\widetilde{u}(x)$ as $u^{gr}(\mu, \alpha_u) = x$ in which "gr" stands for the granule of information included in $x \in [a, b], \mu \in [0, 1]$ is the membership degree of x in $\widetilde{u}(x)$, $\alpha_u \in [0, 1]$ is called relative-distance-measure (RDM) variable, and $u^{gr}(\mu, \alpha_u) = \underline{u}^\mu + (\overline{u}^\mu - \underline{u}^\mu)\alpha_u$.

Note 1. The horizontal membership function of $\overline{u}(x) \in E_1$ is also denoted by $H(\widetilde{u}) = u^{gr}(\mu, \alpha_u)$. Furthermore, if the triangular fuzzy number $\widetilde{u} \in E_1$ is denoted by triple (a, b, c), $a \leq b \leq c$, then the horizontal membership function of $\widetilde{u} = (a, b, c)$ can be characterized as $H(\widetilde{u}) = [a + (b - a)\mu] + [(1 - \mu)(c - a)]\alpha_u$.

Note 2. The horizontal membership function of $\widetilde{u}(x) \in E_1$ is also denoted by $H(\widetilde{u}) = u^{gr}(\mu, \alpha_u)$.

Definition 4 ([46])**.** Let $\widetilde{a}$ and $\widetilde{b}$ be two fuzzy numbers whose horizontal membership functions are $a^{gr}(\mu, \alpha_a)$ and $b^{gr}(\mu, \alpha_b)$ respectively and "$\odot$" denote one of the four basic operations, *i.e.* addition, subtraction, multiplication, and division. Then $\widetilde{a} \odot \widetilde{b}$ is a fuzzy number $\widetilde{m}$ such that $H(\widetilde{a}) = a^{gr}(\mu, \alpha_a) \odot b^{gr}(\mu, \alpha_b)$. It should be noted that 0 is not belongs to $b^{gr}(\mu, \alpha_b)$ when "$\odot$" denotes the division operator.

Definition 5 ([46])**.** Let $\widetilde{f} : [a, b] \subseteq R \to E_1$ include $n \in N$ distinct fuzzy numbers $\widetilde{u}_1, \widetilde{u}_2, \ldots, \widetilde{u}_n$. The horizontal membership function of $\widetilde{f}(t)$ at the point $t \in [a, b]$ is denoted by $H(\widetilde{f}(t)) = f^{gr}(t, \mu, \alpha_f)$ and defined as $f^{gr} : [a, b] \times [0, 1] \times [0, 1] \times \ldots \times [0, 1] \to [a, b] \subseteq R$ in which $\alpha_f = (\alpha_{u_1}, \alpha_{u_2}, \ldots, \alpha_{u_n})$ are the RDM variables corresponding to the fuzzy numbers.

Definition 6 ([45])**.** The fuzzy function $\widetilde{f} : [a, b] \subseteq R \to E_1$ to be granular differentiable (gr-differentiable) at $t \in [a, b]$ if there exists a fuzzy number $\frac{\mathrm{d}\widetilde{f}(t)}{\mathrm{d}t} \in E_1$ such that the following limit exits:

$$\lim_{h\to 0} \frac{\widetilde{f}(t + h) - \widetilde{f}(h)}{h} = \frac{\mathrm{d}\widetilde{f}(t)}{\mathrm{d}t}.$$

Theorem 3. *The fuzzy function* $\widetilde{f} : [a, b] \subseteq R \to E_1$ *is granular differentiable at the point* $t \in [a, b]$ *if and only if its horizontal membership function is differentiable with respect to* t *at that point. However,* $H(\frac{\mathrm{d}\widetilde{f}(t)}{\mathrm{d}t}) = \frac{\partial f^{gr}(t,\mu,\alpha_f)}{\partial t}$.

Definition 7 ([47])**.** Let $\widetilde{A} \in E_1$ Describe a fuzzy restriction on values which a real valued uncertain variable X can take. The order pair $(\widetilde{A}, \widetilde{B})$ is called a Z-number if the fuzzy number $\widetilde{B}$ describe how sure we are that X is $\widetilde{A}$.

Note 3 ([47])**.** The fuzzy number $\widetilde{A}$ plays a role of possibility distribution of X as X is $\widetilde{A} \longrightarrow \text{Poss}(X = x) = \widetilde{A}(x)$, where x is a generic value of X, and $\widetilde{A}(x)$ outputs the membership degree of x.

In the Z-number $z = (\widetilde{A}, \widetilde{B})$, the component $\widetilde{B}$ is referred to as sureness or certainty. Certainty (or sureness) may be consider as probability when X is a random variable. This component plays a role of fuzzy constraint on the probability measure of $\widetilde{A}$.

Definition 8 ([48])**.** The function $f : [a, b] \subseteq R \to Z^+$ is called a Z-number valued function.

Note 4. The Z-number valued function $f(t)$ can also be written as $f(t) = (f_A(t), f_R(t))$ where $f_A(t)$ is a function mapping a real number to a fuzzy number; and $f_R(t)$ is a function mapping a real number to a random variable. The functions $f_A(t)$ and $f_R(t)$ may be considered as a fuzzy process and random process respectively.

Definition 9 ([48])**.** The Z-number valued function $f : [a, b] \subseteq R \to Z^+$ mapping $t \to f(t)$ is said to be Z-differentiable at $t \in [a, b]$ if there exists a Z^+-number ${}^zDf(t)$ such that the following limit exists: $\lim\limits_{h\to 0} \frac{f(t+h)-f(t)}{h} = {}^zDf(t)$.

3. The method of converting Z-number to regular fuzzy number

Assume a Z-number $Z = (\widetilde{A}, \widetilde{R})$. The left part of restriction and the right is the part of reliability. Let $\widetilde{A} = \{\langle x, u_{\widetilde{A}}(x)\rangle |\ x \in [0, 1]\}$ and $\widetilde{R}$=$\{\langle x, u_{\widetilde{R}}(x)\rangle |\ x \in [0, 1]\}$, $u_{\widetilde{A}}(x)$ is a trapezoidal membership function, $u_{\widetilde{R}}(x)$ is a triangular membership function [49].

(i) Convert the second part (reliability) into a crisp number

$$\alpha = \frac{\int x\mu_{\widetilde{R}}(x)\,\mathrm{d}x}{\int \mu_{\widetilde{R}}(x)\,\mathrm{d}x}$$

where $\int$ denotes an algebraic integration.

(ii) Add the weight of the second part (reliability) to the first part (restriction). The weighted Z-number can be denoted as $\widetilde{Z}^\alpha = \{\langle x, \mu_{\widetilde{A}^\alpha}(x)\rangle | \mu_{\widetilde{A}^\alpha}(x) = \alpha\mu_{\widetilde{A}}(x),\ x \in [0, 1]\}$.

(iii) Convert the irregular fuzzy number (weighted restriction) to regular fuzzy number. The regular fuzzy set can be denoted as $\widetilde{Z}' = \{\langle x, \mu_{\widetilde{Z}'}(x)\rangle | \mu_{\widetilde{Z}'}(x) = \mu_{\widetilde{A}}(\frac{x}{\sqrt{\alpha}}),\ x \in [0, 1]\}$.

For an example, we assume an expert gives his opinion as follows: $\widetilde{A} = (0.7, 0.8, 0.9, 1; 1)$ and his reliability is $\widetilde{R} = (0.8, 0.9, 1; 1)$.

The expert's knowledge can be expressed to Z-number as $\widetilde{Z} = (\widetilde{A}, \widetilde{R}) = [(0.7, 0.8, 0.9, 1; 1), (0.8, 0.9, 1; 1)]$.

At first, we should to convert expert's reliability into crisp number according to method we get, $\alpha = \frac{\int x\mu_{\widetilde{R}}(x)\,\mathrm{d}x}{\int \mu_{\widetilde{R}}(x)\,\mathrm{d}x} = 0.9$.

Second, add the weight of reliability to the constraint.

$$\widetilde{Z}^\alpha = (0.7, 0.8, 0.9, 1; 0.9).$$

Third, convert the weighted Z-number to regular fuzzy number according to the proposed approach.

$$\begin{aligned}\widetilde{Z}' &= \left(\sqrt{0.9} \times 0.7, \sqrt{0.9} \times 0.8, \sqrt{0.9} \times 0.9, \sqrt{0.9} \times 1; 1\right)\\ &= (0.6641, 0.7589, 0.8538, 0.9487; 1).\end{aligned}$$

4. Methods to check the stability of fuzzy dynamical system

In this section we developed and discussed some methods to check the stability of autonomous and non-autonomous system of fuzzy dynamical system.

4.1. To check the stability of autonomous non-homogeneous linear dynamical system with constant coefficient

Let us consider a autonomous non-homogeneous linear dynamical system with constant coefficient

$$\begin{cases} \dot{y}_1(t) = g(t) - y_2(t) \\ \dot{y}_2(t) = ay_1(t) - by_2(t) \end{cases} \tag{1}$$

where $\dot{y}_1(t) = \frac{\mathrm{d}y_1}{\mathrm{d}t}, \dot{y}_2(t) = \frac{\mathrm{d}y_2}{\mathrm{d}t}$ with $y_1(0) = c_1,\ y_2(0) = c_2\ a, b \geq 0$ and $g(t) \geq 0$ is bounded.

Now the equivalent homogeneous system of the system (1) (using Thm. 1) is

$$\begin{cases} \dot{y}_1(t) = -y_2(t) = F(y_1, y_2)\ \text{(say)} \\ \dot{y}_2(t) = ay_1(t) - by_2(t) = G(y_1, y_2)\ \text{(say)} \end{cases} \tag{2}$$

where $\dot{y}_1(t) = \frac{\mathrm{d}y_1}{\mathrm{d}t}, \dot{y}_2(t) = \frac{\mathrm{d}y_2}{\mathrm{d}t}$ with $y_1(0) = c_1,\ y_2(0) = c_2\ a, b \geq 0$ and $g(t) \geq 0$ is bounded.

Now we investigate about the stability of the linear dynamical system (2). The Jacobian matrix of the system (1) is

$$\begin{bmatrix} 0 & -1 \\ a & -b \end{bmatrix}.$$

The eigenvalues of the characteristic equation of the above matrix are $\frac{-b-\sqrt{(b^2-4a)}}{2}, \frac{-b+\sqrt{(b^2-4a)}}{2}$. If $b^2 - 4a < 0$, then $\sqrt{(b^2 - 4a)}$ is an imaginary part, this implies the real part of the eigenvalues are always negative that is the system is stable. If $b^2 - 4a \geq 0$, then $\sqrt{(b^2 - 4a)} < b$ holds always. This implies that the eigenvalues are always negative that is the system is always stable.

4.2. Methods for checking the stability of fuzzy dynamical system

Let us consider the fuzzy dynamical system is

$$\begin{cases} \dot{\widetilde{y}}_1(t) = g(t) - \widetilde{y}_2(t) \\ \dot{\widetilde{y}}_2(t) = \widetilde{a}\widetilde{y}_1(t) - b\widetilde{y}_2(t) \end{cases} \tag{3}$$

where $\dot{\widetilde{y}}_1(t) = \frac{\mathrm{d}\widetilde{y}_1}{\mathrm{d}t}, \dot{\widetilde{y}}_2(t) = \frac{\mathrm{d}\widetilde{y}_2}{\mathrm{d}t}$ with $\widetilde{y}_1(0) = \widetilde{c}_1,\ \widetilde{y}_2(0) = \widetilde{c}_2$.

4.2.1. Interval mathematics

To convert the fuzzy dynamical system (3) into the crisp system, we use the interval mathematics concept.

$$\begin{cases} \left[\dot{\underline{y}}_1^\alpha(t), \dot{\overline{y}}_1^\alpha(t)\right] = (g(t), g(t)) - \left[\underline{y}_2^\alpha(t), \overline{y}_2^\alpha(t)\right] \\ \left[\dot{\underline{y}}_2^\alpha(t), \dot{\overline{y}}_2^\alpha(t)\right] = \left[\underline{a}^\alpha \underline{y}_1^\alpha(t), \overline{a}^\alpha \overline{y}_1^\alpha(t)\right] + b\left[\underline{y}_2^\alpha(t), \overline{y}_2^\alpha(t)\right]. \end{cases} \tag{4}$$

The eigenvalues of the above homogeneous system are $\frac{\sqrt{\alpha_1}-\sqrt{\beta_1}}{2}, \frac{\sqrt{\alpha_1}+\sqrt{\beta_1}}{2}, \frac{-\sqrt{\alpha_2}-\sqrt{\beta_2}}{2}, \frac{-\sqrt{\alpha_2}+\sqrt{\beta_2}}{2}$, where $\alpha_1 = \frac{2b^2}{3} + \frac{2^{\frac{1}{3}}\beta}{3\alpha_3} + \frac{\alpha_3}{3.2^{\frac{1}{3}}}$, $\beta_1 = \frac{4b^2}{3} - \frac{2^{\frac{1}{3}}\beta}{3\alpha_3} - \frac{\alpha_3}{3.2^{\frac{1}{3}}} + \frac{2b(\overline{a}+\underline{a})}{\sqrt{\alpha_1}}$, $\alpha_3 = -2b^6 - 72b^2\underline{a}\overline{a} + 27b^2(\underline{a} + \overline{a})^2 + \sqrt{(-4(b^4 - 12\underline{a}\overline{a})^3 + (-2b^6 - 72b^2\underline{a}\overline{a} + 27b^2(\underline{a} + \overline{a})^2)^2)}$, $\beta = b^6 - 12\underline{a}\overline{a}$, $\alpha_2 = b^2 - (\underline{a}^\alpha + \overline{a}^\alpha)$, $\beta_2 = \frac{4b^2}{3} - \frac{2^{\frac{1}{3}}\beta_3}{3\alpha_3} - \frac{\alpha_3}{3.2^{\frac{1}{3}}} + \frac{2b(\overline{a}+\underline{a})}{\sqrt{\alpha_1}}$, $\beta_3 = -4\underline{a}^\alpha\overline{a}^\alpha + (-b^2 + \underline{a}^\alpha + \overline{a}^\alpha)^2$. From the above eigenvalues we see that the real part of at least one of two eigenvalues is positive. So the above system is unstable.

4.2.2. Granular differentiability concept

To convert the fuzzy dynamical system (3) into the crisp system, we use the granular differentiability concept.

$$\begin{cases} \frac{\mathrm{d}y_1^{gr}(\mu,\alpha_{y_1},t)}{\mathrm{d}t} = g(t) - y_2^{gr}(\mu,\alpha_{y_2},t) \\ \frac{\mathrm{d}y_2^{gr}(\mu,\alpha_{y_2},t)}{\mathrm{d}t} = a^{gr}(\mu,\alpha_a)y_1^{gr}(\mu,\alpha_{y_1},t) - by_2^{gr}(\mu,\alpha_{y_2},t) \end{cases} \tag{5}$$

$$\begin{bmatrix} 0 & -1 \\ a^{gr}(\mu,\alpha_a) & -b \end{bmatrix}.$$

The eigenvalues of the characteristic equation of the above matrix are $\frac{-b-\sqrt{(b^2-4a^{gr}(\mu,\alpha_a))}}{2}$, $\frac{-b+\sqrt{(b^2-4a^{gr}(\mu,\alpha_a))}}{2}$. If $b^2-4a^{gr}(\mu,\alpha_a) < 0$, then $\sqrt{(b^2-4a^{gr}(\mu,\alpha_a))}$ is an imaginary part, this implies the real part of the eigenvalues are always negative that is the system is stable. If $b^2-4a^{gr}(\mu,\alpha_a) \geq 0$, then $\sqrt{(b^2-4a^{gr}(\mu,\alpha_a))} < b$ holds always. This implies that the eigenvalues are always negative that is the system is always stable.

5. Formulation of prey–predator population model

In this paper under the following assumptions and notations we developed a prey–predator population model with fuzzy variables and fuzzy parameters and also Z-number variables and Z-number parameters.

5.1. Assumptions and notation

$X(t)$	The prey population density at any time t.
$Y(t)$	The predator population density at any time t.
g	The natural growth rate of prey population.
E	The environmental carrying capacity at any time t.
δ	Predation rate of prey population.
σ	Increasing rate of predator population due to successful predation of prey.
η	Decay rate of predator population due to natural death.
$\widetilde{X}(t)$	The prey population density at any time t, which is fuzzy in nature.
$\widetilde{Y}(t)$	The predator population density at any time t, which is fuzzy in nature.
$\widetilde{g}$	The natural fuzzy growth rate of prey population.
E	The environmental carrying capacity at any time t.
$\widetilde{\delta}$	Fuzzy Predation rate of prey population.
$\widetilde{\sigma}$	Increasing rate of fuzzy predator population due to successful predation of prey.
$\widetilde{\eta}$	Decay rate of predator population due to natural death which is also fuzzy in nature.
$X^z(t)$	The prey population density at any time t which is a Z-number.
$Y^z(t)$	The predator population density at any time t which is a Z-number.
g^z	The natural growth rate is Z-number of prey population.
E	The environmental carrying capacity at any time t.
δ^z	Predation rate of prey population is a Z-number.
σ^z	Increasing rate of predator population due to successful predation of prey is a Z-number.
η^z	Decay rate of predator population due to natural death is also a Z-number.

5.2. Mathematical model description in crisp environment

To check the theoretical implication of stability analysis in different environment, first we developed a prey–predator model in crisp environment. The prey population followings a logistic growth model with g as intrinsic growth rate and E to be environmental carrying capacity. The prey population decays due to predation by the predator at a rate α and the predation function is XY. Again the predator population increase by consuming the preys at a rate μ and decays due to natural death at a rate η. The rate of change of prey population and rate of change of predator population can be presented as follows

$$\frac{\mathrm{d}X(t)}{\mathrm{d}t} = gX(t)\left(1 - \frac{X(t)}{E}\right) - \delta X(t)Y(t) \tag{6}$$

$$\frac{\mathrm{d}Y(t)}{\mathrm{d}t} = \sigma X(t)Y(t) - \eta Y(t). \tag{7}$$

5.3. Mathematical model description in imprecise environment

Due to globalization and different ecological conditions the prey–predator population parameters are not always fixed but rather it may vary. So this prey–predator model will be more realistic by considering the prey–predator parameters are fuzzy nature in the field of mathematical biology. Along with the prey X and the predator Y population, it has been considered that the predation rate δ, growth rate g, and increase rate σ to be fuzzy in nature. Then the system become

$$\frac{\mathrm{d}\widetilde{X}(t)}{\mathrm{d}t} = \widetilde{g}\widetilde{X}(t)\left(1 - \frac{\widetilde{X}(t)}{E}\right) - \widetilde{\delta}\widetilde{X}(t)\widetilde{Y}(t) \tag{8}$$

$$\frac{\mathrm{d}\widetilde{Y}(t)}{\mathrm{d}t} = \widetilde{\sigma}\widetilde{X}(t)\widetilde{Y}(t) - \widetilde{\eta}\widetilde{Y}(t). \tag{9}$$

Now by using granular computing method (Defs. 4, 6 and Thm. 3) the system reduced to the following

$$\frac{\mathrm{d}X^{gr}(\mu,\alpha_x,t)}{\mathrm{d}t} = g^{gr}(\mu,\alpha_g)X^{gr}(\mu,\alpha_x,t)\left(1 - \frac{X^{gr}(\mu,\alpha_x,t)}{E}\right) - \delta^{gr}(\mu,\delta_\alpha)X^{gr}(\mu,\alpha_x,t)Y^{gr}(\mu,\alpha_y,t) \tag{10}$$

$$\frac{\mathrm{d}Y^{gr}(\mu,\alpha_y,t)}{\mathrm{d}t} = \sigma^{gr}(\mu,\alpha_\sigma)X^{gr}(\mu,\alpha_x,t)Y^{gr}(\mu,\alpha_y,t) - \eta^{gr}(\mu,\alpha_\eta)Y^{gr}(\mu,\alpha_y,t). \tag{11}$$

5.4. Mathematical model description in Z-number fuzzy environment

In the environment we consider the prey X and the predator Y population are situated in Z-number, it has been considered that the predation rate δ, growth rate g, and increase rate σ all are Z-numbers in nature. Then the system become

$$\frac{\mathrm{d}X^z(t)}{\mathrm{d}t} = g^zX^z(t)\left(1 - \frac{X^z(t)}{E}\right) - \delta^zX^z(t)Y^z(t) \tag{12}$$

$$\frac{\mathrm{d}Y^z(t)}{\mathrm{d}t} = \sigma^zX^z(t)Y^z(t) - \eta^zY^z(t). \tag{13}$$

Using the method of converting Z-number to regular fuzzy number, we convert all Z-numbers g^z, $X^z(t)$, $Y^z(t)$, σ^z, η^z, δ^z to fuzzy numbers we get, $\widetilde{g}^z$, $\widetilde{X}^z(t)$, $\widetilde{Y}^z(t)$, $\widetilde{\sigma}^z$, $\widetilde{\eta}^z$, $\widetilde{\delta}^z$.

Then the system become

$$\frac{\mathrm{d}\widetilde{X}^z(t)}{\mathrm{d}t} = \widetilde{g}^z\widetilde{X}^z(t)\left(1 - \frac{\widetilde{X}^z(t)}{E}\right) - \widetilde{\delta}^z\widetilde{X}^z(t)\widetilde{Y}^z(t) \tag{14}$$

$$\frac{\mathrm{d}\widetilde{Y}^z(t)}{\mathrm{d}t} = \widetilde{\sigma}^z\widetilde{X}^z(t)\widetilde{Y}^z(t) - \widetilde{\eta}^z\widetilde{Y}^z(t). \tag{15}$$

Now by using granular computing method the system reduced to the following

$$\frac{\mathrm{d}X^{z^{gr}}(\mu,\alpha_x,t)}{\mathrm{d}t} = g^{z^{gr}}(\mu,\alpha_g)X^{z^{gr}}(\mu,\alpha_x,t)\left(1 - \frac{X^{z^{gr}}(\mu,\alpha_x,t)}{E}\right) - \delta^{z^{gr}}(\mu,\delta_\alpha)X^{z^{gr}}(\mu,\alpha_x,t)Y^{z^{gr}}(\mu,\alpha_y,t) \tag{16}$$

TABLE 1. Input data.

Parameters	Fuzzy values	Horizontal membership function
$\widetilde{g}$	$(1.08, 1.12, 1.15)$	$g^{gr} = 1.08 + 0.04\mu + 0.07(1-\mu)\alpha_g$
$\widetilde{\delta}$	$(0.009, 0.011, 0.015)$	$\delta^{gr} = 0.009 + 0.002\mu + 0.006(1-\mu)\alpha_\delta$
$\widetilde{\eta}$	$(0.084, 0.087, 0.092)$	$\eta^{gr} = 0.084 + 0.003\mu + 0.008(1-\mu)\alpha_\eta$
$\widetilde{\sigma}$	$(0.001, 0.0012, 0.0016)$	$\sigma^{gr} = 0.001 + 0.0002\mu + 0.0006(1-\mu)\alpha_\sigma$

$$\frac{\mathrm{d}Y^{z^{gr}}(\mu,\alpha_y,t)}{\mathrm{d}t} = \sigma^{z^{gr}}(\mu,\alpha_\sigma)X^{z^{gr}}(\mu,\alpha_x,t)Y^{z^{gr}}(\mu,\alpha_y,t) - \eta^{z^{gr}}(\mu,\alpha_\eta)Y^{z^{gr}}(\mu,\alpha_y,t). \tag{17}$$

6. Local stability analysis

6.1. Stability analysis in imprecise environment

The steady state solutions are given by

$$X^{*gr}(\mu,\alpha_x,t) = \frac{\eta^{gr}(\mu,\alpha_\eta)}{\sigma^{gr}(\mu,\alpha_\sigma)}$$
$$Y^{*gr}(\mu,\alpha_y,t) = \frac{g^{gr}(\mu,\alpha_g)(E\sigma^{gr}(\mu,\alpha_\sigma)-\eta^{gr}(\mu,\alpha_\eta))}{E\sigma^{gr}(\mu,\alpha_\sigma)\alpha^{gr}(\mu,\alpha_\alpha)}.$$

The Jacobian matrix for the system in steady state $(X^{*gr}(\mu,\alpha_x,t), Y^{*gr}(\mu,\alpha_y,t))$ is given by

$$\begin{bmatrix} g^{gr}(\mu,\alpha_g) - \frac{2\eta^{gr}(\mu,\alpha_\eta)g^{gr}(\mu,\alpha_g)}{E\sigma^{gr}(\mu,\alpha_\sigma)} - \frac{g^{gr}(\mu,\alpha_g)(E\sigma^{gr}(\mu,\alpha_\sigma)-\eta)}{E\sigma^{gr}(\mu,\alpha_\sigma)} & -\frac{\delta^{gr}(\mu,\delta_\alpha)\eta^{gr}(\mu,\alpha_\eta)}{\sigma^{gr}(\mu,\alpha_\sigma)} \\ \frac{g^{gr}(\mu,\alpha_g)(E\sigma^{gr}(\mu,\alpha_\sigma)-\eta^{gr}(\mu,\alpha_\eta))}{E\delta^{gr}(\mu,\delta_\alpha)} & -\eta^{gr}(\mu,\alpha_\eta) \end{bmatrix}$$

and the corresponding eigen values of the above matrix are given by $\frac{1}{2E\sigma^{gr}(\mu,\alpha_\sigma)}(-c+\sqrt{c^2-4a}$ and $\frac{1}{2E\sigma^{gr}(\mu,\alpha_\sigma)}(-c-\sqrt{c^2-4a}$, where $c = (\eta^{gr}(\mu,\alpha_\eta)g^{gr}(\mu,\alpha_g) + \eta^{gr}(\mu,\alpha_\eta)E\sigma^{gr}(\mu,\alpha_\sigma))$ and $a = E^2\eta^{gr}(\mu,\alpha_\eta)g^{gr}(\mu,\alpha_g)\sigma^{gr}(\mu,\alpha_\sigma)$. Then if $\sqrt{c^2-4a} < 0$ then $\sqrt{c^2-4a}$ is an imaginary part and real part of the eigenvalues are negative. And if $\sqrt{c^2-4a} > 0$ then also all the eigenvalues of the given system are real and negative because $\sqrt{c^2-4a} < c$. Therefore the given system is stable. and the above theory is illustrated in the following numerical experiment.

By using the numerical data from Table 1 and $E = 150$ the eigenvalues are $-0.30020+0.09415i$ and $-0.30020-0.09415i$. Since the real part of the eigenvalues are negative. Therefore the steady state solution are stable (using Thm. 2).

6.2. Stability analysis in Z-number fuzzy environment

The steady state solutions are given by

$$X^{z*gr}(\mu,\alpha_x,t) = \frac{\eta^{z^{gr}}(\mu,\alpha_\eta)}{\sigma^{z^{gr}}(\mu,\alpha_\sigma)}$$
$$Y^{z*gr}(\mu,\alpha_y,t) = \frac{g^{z^{gr}}(\mu,\alpha_g)\big(E\sigma^{z^{gr}}(\mu,\alpha_\sigma)-\eta^{z^{gr}}(\mu,\alpha_\eta)\big)}{E\sigma^{z^{gr}}(\mu,\alpha_\sigma)\alpha^{z^{gr}}(\mu,\alpha_\alpha)}.$$

TABLE 2. Input data.

Parameters	Z-number
$\widetilde{g}$	$(1.08, 1.12, 1.15)(0.8, 0.9, 1)$
$\widetilde{\delta}$	$(0.009, 0.011, 0.015)(0.8, 0.85, 0.9)$
$\widetilde{\eta}$	$(0.084, 0.087, 0.092)(0.85, 0.9, 0.95)$
$\widetilde{\sigma}$	$(0.001, 0.0012, 0.0016)(0.75, 0.85, 0.95)$

TABLE 3. Values of all variables and parameters after using the method of converting Z-number to fuzzy number.

Parameters	Z-number	Fuzzy values	Horizontal membership function
$\widetilde{g}$	$(1.08, 1.12, 1.15)(0.8, 0.9, 1)$	$(1.02, 1.06, 1.09)$	$g^{gr} = 1.02 + 0.04\mu + 0.07(1-\mu)\alpha_g$
$\widetilde{\delta}$	$(0.009, 0.011, 0.015)(0.8, 0.85, 0.9)$	$(0.008, 0.010, 0.014)$	$\delta^{gr} = 0.008 + 0.002\mu + 0.006(1-\mu)\alpha_\delta$
$\widetilde{\eta}$	$(0.084, 0.087, 0.092)(0.85, 0.9, 0.95)$	$(0.080, 0.0825, 0.087)$	$\eta^{gr} = 0.080 + 0.0025\mu + 0.007(1-\mu)\alpha_\eta$
$\widetilde{\sigma}$	$(0.001, 0.0012, 0.0016)(0.75, 0.85, 0.95)$	$(0.0009, 0.0011, 0.0015)$	$\sigma^{gr} = 0.0009 + 0.0002\mu + 0.0006(1-\mu)\alpha_\sigma$

The Jacobian matrix for the system in steady state $(X^{z^{*}gr}(\mu,\alpha_x,t), Y^{*gr}(\mu,\alpha_y,t))$ is given by

$$\begin{bmatrix} g^{z^{gr}}(\mu,\alpha_g) - \frac{2\eta^{z^{gr}}(\mu,\alpha_\eta)g^{z^{gr}}(\mu,\alpha_g)}{E\sigma^{z^{gr}}(\mu,\alpha_\sigma)} - \frac{g^{z^{gr}}(\mu,\alpha_g)(E\sigma^{z^{gr}}(\mu,\alpha_\sigma)-\eta)}{E\sigma^{z^{gr}}(\mu,\alpha_\sigma)} & -\frac{\delta^{gr}(\mu,\delta_\alpha)\eta^{z^{gr}}(\mu,\alpha_\eta)}{\sigma^{z^{gr}}(\mu,\alpha_\sigma)} \\ \frac{g^{z^{gr}}(\mu,\alpha_g)(E\sigma^{z^{gr}}(\mu,\alpha_\sigma)-\eta^{z^{gr}}(\mu,\alpha_\eta))}{E\delta^{z^{gr}}(\mu,\delta_\alpha)} & -\eta^{z^{gr}}(\mu,\alpha_\eta) \end{bmatrix}$$

and the corresponding eigen values of the above matrix are given by $\frac{1}{2E\sigma^{z^{gr}}(\mu,\alpha_\sigma)}(-c + \sqrt{c^2-4a}$ and $\frac{1}{2E\sigma^{z^{gr}}(\mu,\alpha_\sigma)}(-c - \sqrt{c^2-4a}$, where $c = (\eta^{z^{gr}}(\mu,\alpha_\eta)g^{z^{gr}}(\mu,\alpha_g) + \eta^{z^{gr}}(\mu,\alpha_\eta)E\sigma^{z^{gr}}(\mu,\alpha_\sigma))$ and $a = E^2\eta^{z^{gr}}(\mu,\alpha_\eta)g^{z^{gr}}(\mu,\alpha_g)\sigma^{z^{gr}}(\mu,\alpha_\sigma)$. Then if $\sqrt{c^2-4a} < 0$ then $\sqrt{c^2-4a}$ is an imaginary part and real part of the eigenvalues are negative. And if $\sqrt{c^2-4a} > 0$ then also all the eigenvalues of the given system are real and negative because $\sqrt{c^2-4a} < c$. Therefore the given system is stable (using Thm. 2). And the above theory is illustrated in the following numerical experiment (Tab. 2).

By using the numerical data from Table 3 and $E = 150$ the eigenvalues are $-0.30492 + 0.010867i$ and $-0.30492 - 0.010867i$. Since the real part of the eigenvalues are negative. Therefore the steady state solution are stable.

7. Another case study

In this section we have developed two fuzzy differential equations for production inventory model. The following assumption and notations are used to develop the model.

7.1. Assumption

(i) Demand of the product depends on quality of product, stock of the product and advertisement rate of the product.
(ii) Unit production is considered as fuzzy variable due to creating the different problem in real life situation.

7.2. Notations

$\widetilde{X}(t)$ The fuzzy stock level at time t.

$\widetilde{D}(t)$ The fuzzy demand rate at time t.
$\widetilde{U}(t)$ The fuzzy unit production rate at time t.
$\widetilde{v}$ The fuzzy advertisement rate.
$\widetilde{\delta}$ The fuzzy defective parameter.
$\widetilde{\beta}$ The fuzzy depreciation rate of demand.
$\widetilde{\rho}$ A fuzzy constant.
$\widetilde{q}$ Quality of the product which is fuzzy parameter.

7.3. Model formulation

To check the stability of fuzzy dynamical system under granular differentiability concept we develop a production inventory model with fuzzy variable and fuzzy parameters. In traditional model formulation, all the data or variables or parameters in the form of input or output are deterministic. Normally production inventory optimal control models are presented with crisp demand. In real life demand for an item in a competitive market varies and fluctuates. Due to fluctuating demand, we cannot fit exact demand, for this, the demand can be expressed as a fuzzy number. Then it is called fuzzy demand. The traditional models are presented with deterministic variables, parameters and constraints whereas fuzzy models take on imprecise/vague variables, parameters and constraints which are more realistic and practical. The advertisement policy through electronic media and print media has a positive influence to increase the customers. So the advertisement policy is to create the demand rate. In this production inventory model we consider the demand of any product depends upon the stock level of product, quality of product and advertisement rate of the product. Due to competitive market and long run business the advertisement cost is not fixed and also to control the demand we consider that the advertisement rate, stock level of product and quality of product all are fuzzy in nature to make this model more realistic and implicate. Therefore the rate of change of demand is also fuzzy in nature. The fuzzy differential equation of the rate of change of demand is represented by

$$\dot{\widetilde{D}}(t) = \frac{\widetilde{q}\widetilde{v}\widetilde{\rho}\widetilde{X}(t)}{1+\widetilde{v}} - \widetilde{\beta}\widetilde{D}(t). \tag{18}$$

Here we consider a single item defective production inventory model with production rate $\widetilde{U}$ of which $\widetilde{\delta}\frac{\widetilde{U}(t)}{1+\widetilde{U}(t)}$ is defective rate. Due to defectiveness the stock level decreases at time t. The demands rate $\widetilde{D}(t)$ of the customers is from the inventory. The fuzzy differential equation for the rate of change of stock level is

$$\dot{\widetilde{X}}(t) = \left(1 - \widetilde{\delta}\frac{\widetilde{U}(t)}{1+\widetilde{U}(t)}\right)\widetilde{U}(t) - \widetilde{D}(t). \tag{19}$$

Now by using granular computing method the system reduced to the following

$$\dot{X^{gr}}(\mu,\alpha_x,t) = \left(1 - \delta^{gr}(\mu,\alpha_\delta)\frac{U^{gr}(\mu,\alpha_u,t)}{1+U^{gr}(\mu,\alpha_u,t)}\right)U^{gr}(\mu,\alpha_u,t) - D^{gr}(\mu,\alpha_d,t) \tag{20}$$

$$\dot{D^{gr}}(\mu,\alpha_d,t) = \frac{q^{gr}(\mu,\alpha_q)v^{gr}(\mu,\alpha_v)\rho^{gr}(\mu,\alpha_\rho)X^{gr}(\mu,\alpha_x,t)}{1+v^{gr}(\mu,\alpha_v)} - \beta^{gr}(\mu,\alpha_\beta)D^{gr}(\mu,\alpha_d,t). \tag{21}$$

7.4. Stability analysis

The Jacobian matrix for the system in steady state is given by

$$\begin{bmatrix} 0 & -1 \\ \frac{q^{gr}(\mu,\alpha_q)v^{gr}(\mu,\alpha_v)\rho^{gr}(\mu,\alpha_\rho)X^{gr}(\mu,\alpha_x,t)}{1+v^{gr}(\mu,\alpha_v)} & \beta^{gr}(\mu,\alpha_\beta) \end{bmatrix}.$$

TABLE 4. Input data.

Parameters	Fuzzy values	Horizontal membership function
$\widetilde{q}$	$(0.53, 0.55, 0.58)$	$q^{gr} = 0.53 + 0.02\mu + 0.05(1-\mu)\alpha_q$
$\widetilde{\rho}$	$(0.12, 0.16, 0.18)$	$\rho^{gr} = 0.12 + 0.04\mu + 0.06(1-\mu)\alpha_\rho$
$\widetilde{\beta}$	$(0.22, 0.25, 0.27)$	$\beta^{gr} = 0.22 + 0.03\mu + 0.05(1-\mu)\alpha_\beta$
$\widetilde{\delta}$	$(0.02, 0.03, 0.05)$	$\delta^{gr} = 0.02 + 0.01\mu + 0.03(1-\mu)\alpha_\delta$
$\widetilde{v}$	$(22, 25, 27)$	$v^{gr} = 22 + 3\mu + 5(1-\mu)\alpha_v$
$\widetilde{u}$	$(35, 40, 45)$	$u^{gr} = 35 + 5\mu + 10(1-\mu)\alpha_u$

Then the eigenvalues of the above matrix is $\frac{1}{2}(-b+\sqrt{b^2-4h})$ and $\frac{1}{2}(-b-\sqrt{b^2-4h})$, where $b = \beta^{gr}(\mu, \alpha_\beta)$, and $h = \frac{q^{gr}(\mu,\alpha_q)v^{gr}(\mu,\alpha_v)\rho^{gr}(\mu,\alpha_\rho)X^{gr}(\mu,\alpha_x,t)}{1+v^{gr}(\mu,\alpha_v)} > 0$. Then if $\sqrt{b^2-4h} < 0$ then $\sqrt{b^2-4h}$ is an imaginary part and real part of the eigenvalues are negative. And if $\sqrt{b^2-4h} > 0$ then also all the eigenvalues of the given system are real and negative because $\sqrt{b^2-4h} < b$. Therefore the given system is stable.

By using the numerical data from Table 4 the eigenvalues are $-0.1265 + 0.25594i$ and $-0.1265 - 0.25594i$. Since the real part of the eigenvalues are negative. Therefore the steady state solution are stable.

In the input data Table 1 the horizontal membership function corresponding to fuzzy variables are obtained by using Definition 3 and Note 1.

In the input data Table 4 the horizontal membership function corresponding to fuzzy variables are obtained by using Definition 3 and Note 1.

8. Discussion

In this paper we have analyzed a prey–predator model in different environment such as crisp, fuzzy, Z-number, and also consider all the parameters are imprecise in imprecise environment. Also, we have assumed the parameters namely growth rate of prey–predation rate of the prey population, increase rate and decay rate of the predator population as imprecise parameters. The eigenvalues are obtained from different cases of different fuzzy dynamical systems. From these results, it is observed that the systems are either unstable or stable. Using interval mathematics, all the fuzzy dynamical system converted into crisp dynamical system and check the stability analysis. For interval mathematics we observed that some eigenvalues are positive, which show the fuzzy dynamical system is unstable. The interval mathematics fails for fuzzy dynamical system. So the granular differentiability is the best technique to check the stability of fuzzy dynamical system. We check the local stability of the prey–predator system, which is done with the help of eigenvalues of Jacobian matrices. It can be observed with the help of eigenvalues that the systems are stable in different environments. By using the numerical data of Tables 1 and 3, we observed that the eigenvalues have negative real part, so the steady state solution of the system is stable. It is revealed that the decision makers not only get better knowledge about preservation amount, but also get the proper amount of money that can be spent on preservation technology. Any organization can improve their total operational costs for preserving items, using this model to maximize the profit. The Figures 1 and 2 shows the prey population and predator population respectively with the help of numerical data Table 3. And also we check the stability of production inventory model in imprecise environment under granular differentiability concept. By using the numerical data of Table 4 we observed that all the eigenvalues of production inventory system have negative real part, so the steady state solution of the production inventory system is stable. It can be observed from both the figures that the graphical solutions are at a good agrement with the numerical values.

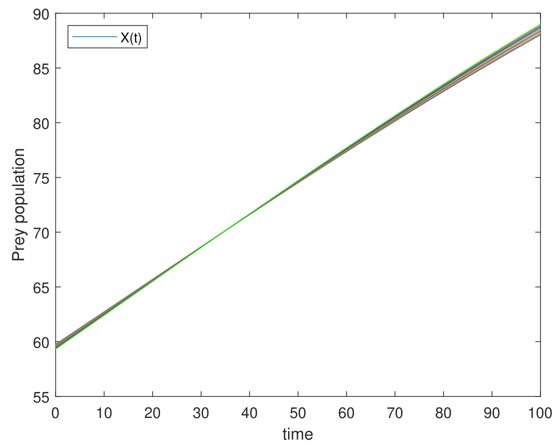

FIGURE 1. μ-level set of Prey population X(t), corresponding to ten values of μ for some $\mu \in [0, 1]$.

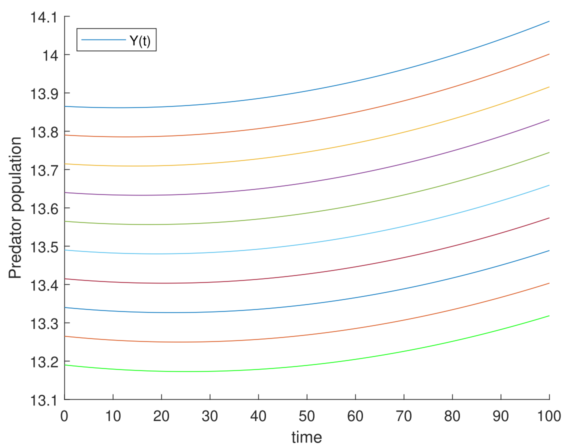

FIGURE 2. μ-level set of Predator population X(t), corresponding to ten values of μ for some $\mu \in [0, 1]$.

9. Managerial implication

In this paper we have presented a fuzzy prey–predator model under granular differentiability, z-differentiability approach. We have appreciated different decision variables for prey–predator model, production inventory model are fuzzy nature. The stability analysis of fuzzy model in fuzzy dynamical system is more important that the model is realistic or not. Now a day, some researchers developed the fuzzy model in fuzzy dynamical system. In this paper, we developed a fuzzy prey–predator model and fuzzy production inventory model and also check the stability analysis of the proposed model. By checking stability analysis, we ensure that this proposed model is realistic. Also by checking the stability analysis of fuzzy dynamical system for production inventory model under granular differentiability approach we ensure that the granular differentiability approach is more suitable to convert the fuzzy dynamical system into the crisp system than interval mathematics concept.

10. Conclusion

Due to Global warming and climate change the natural parameters are uncertain in nature. Some researchers exhibited some uncertain models considering the parameters to be imprecise. These uncertain parameters converted to interval numbers depending upon different parametric conditions and the problems are solved. In this paper the parameters like predator population decay rate, predator population increase rate of prey, prey population predation rate, prey population growth rate are assumed to be fuzzy nature.

At first time, we developed stability analysis theoretically to convert the imprecise model into deterministic model form using granular differentiability approach. The Section 4.1 presents the method for checking the stability analysis of autonomous non-homogeneous linear dynamical system with constant coefficient. The Section 4.2 presents the method for checking the stability analysis of fuzzy dynamical system with interval mathematics and granular differentiability concept. Where the interval mathematics presents the system is unstable and the granular differentiability concept presents the system is stable. This paper explore that theoretically the granular differentiability approach is more stable than the interval mathematics to convert the fuzzy dynamical system into crisp dynamical-system. By checking the stability analysis of fuzzy prey–predator modal and fuzzy production inventory model using the granular differentiability approach to convert the fuzzy dynamical system into crisp system, we ensure that the theoretical approach of stability analysis is practically implemented in different fuzzy dynamical system such as bio-mathematics, production inventory, control system, bio-medical, pollution control model, and also in many real life problem. This model also presented numerically and graphically with the help of some numerical example. This model also provides the lower and upper boundaries of the stable solutions for prey and predator population rather than single solution curve. This model can be expanded to a prey–predator harvesting model with budget constraints, prey–predator harvesting model etc. The results of this study confirm these trends in market dynamics. In general, predator-prey models for analyzing and predicting various aspects of the struggle for existence in the markets of operating systems for electronic devices showed satisfactory predictive ability and suitability for practical use. In future, the said stability techniques can be used to check the stability of fuzzy bio-mathematical model like diseases model, fishery model etc. This technique can also be expanded to check the stability fuzzy non-linear differential equation.

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Data availability. The data set generated during the current study are available from corresponding author on reasonable request.

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