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CONGESTION: RELATION BETWEEN TRADITIONAL DEA CONGESTION AND TWO-STAGE PROCESSES CONGESTION

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Abstract. Congestion is a kind of inefficiency because it causes the decision-making units (DMUs) to be inefficient and reduces their output. Identifying inefficient DMUs and determining the cause of their inefficiencies has been one of the most important reasons for referring to the internal structure of DMUs and analyzing the effect of intermediate products on the subDMU's (or stages) performance. In this paper, to characterize the cause of the DMU's congestion, we refer to its internal structure as a two-stage network data envelopment analysis (DEA) and decompose the DMU's congestion into black-box (BB) and two-stage structure congestion. Also, inputs have two separate and simultaneous roles; black-box inputs role and stage1's input role, so three congestion types occur. Thus, we sought to analyze the relation between two types of initial inputs congestion, intermediate products congestion, and express their effects on BB congestion. Finally, we define three congestion definitions and model the relation between two types of input congestion, intermediate products congestion, and BB inputs congestion. Finally, a practical example illustrates the proposed method.

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1. Introduction

Data Envelopment Analysis (DEA) is a technique that has been widely used for the past three decades to evaluate efficiency and has become a powerful tool for measuring the performance and efficiency of various decision-making units (DMUs) as management science has advanced both theoretically and practically. However, it is worth noting that traditional DEA has certain limitations when it comes to quantifying the efficiency frontier and determining the extent of congestion within DMU inputs. To address this issue, non-parametric DEA has emerged as a useful tool for discerning the presence of congestion by employing the production possibility set (PPS) instead of the production frontier. This approach relies on the notion that PPS is based on data points, which is pivotal in evaluating efficiency. Consequently, most DEA studies regard the existence of congestion as a manifestation in the inputs of assessed DMUs.

Congestion can occur when certain inputs are increased, leading to a decrease in outputs without any improvement in other inputs and outputs, or *vice versa*. This type of inefficiency is a central factor in efficiency assessment and represents an extreme form of inefficiency. Färe *et al.* [5] were the first to introduce an implementable method for analyzing congestion quantitatively. Färe *et al.* [8] discussed models and methods related to data

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envelopment analysis for production efficiency evaluation, including the FGL approach. Among these, Tone and Sahoo's [21] model has proven to be particularly impactful. This model classifies DMUs into inefficient and efficient categories based on congestion-based data envelopment analysis. The focus of this paper is on a two-stage network DEA structure, to define and model subDMUs congestion. Through an analysis of sub-DMUs, we have examined congestion at initial input values and within intermediate products. Additionally, we have investigated congestion within the black-box (BB) structure and the two-stage network, intending to explore potential relationships between them. This decomposition has allowed us to identify congestion within a two-stage network, providing valuable insight into the connection between total congestion (BB structure congestion) and subDMUs congestion.

Given that a DMU's performance hinges on that of its subDMUs, identifying inefficiencies must extend to sub-DMUs' underperformance. This paper posits that the congestion of a DMU stems from congestion within its sub-DMUs. This congestion can arise from three sources: congestion in the first stage's inputs (stage1 or subDMU$_o^{(1)}$), congestion within intermediate products, or congestion in both stage1 inputs and intermediate products. An intriguing aspect revealed in basic network DEA (NDEA) is the ability to measure two types of congestion for BB inputs due to the equivalence of stage1 inputs and BB inputs. This article provides a comprehensive analysis of this nuanced point, elucidating distinct sizes and independence of these two congestion types. Furthermore, it furnishes conditions for the equality of these two congestion types.

Since Färe *et al.* [5] introduced congestion levels in the economy, so researchers have proposed methods for congestion measurement in DEA, which included those proposed by Färe *et al.* [5–8], Cook *et al.* [1], Cooper *et al.* [2–4], Odeck [15], Sueyoshi and Sekitani [19], Noura *et al.* [14], Wei and Yan [22] and Tone and Sahoo [20]. Tone and Sahoo [21] is one of the most effective papers. In their model, first, under the convexity and strong output disposability assumptions, one congestion-based technology is modeled. The DMUs are classified into inefficient and efficient categories using the congestion-based data envelopment analysis model concerning congestion-based technology. According to their definition, DMU$_o$ has weak congestion if there is an increase in one or some inputs but not in all, reducing one or some outputs without changing others. DMU$_o$ has strong congestion if the increase in all inputs reduces all outputs. Sueyoshi and Sekitani [19] provided an insight into broad congestion and expanded its definition. Although their method is based on multiple projections, it is unable to detect weak and strong congestion. Khoveyni *et al.* [10] suggested a new method to identify weak and strong congestion with non-negative and negative data. As Mehdiloozad *et al.* [12] expressed, this approach fails to identify the correct strong congestion. According to Mehdiloozad *et al.* [12], the seventh method proposes a one-stage LP model to detect the maximum projection needed to identify the weak and strong congestion of DMUs *via* DEA.

Also, the papers authored by Shadab *et al.* [17] explore congestion measurement in sustainable supply chains using DEA. In their work published in "Neural Computing and Applications," the authors propose a method to measure congestion in sustainable supply chains based on DEA. They aim to assess inefficiencies caused by congestion in the supply chain context. Similarly, in Shadab *et al.* [18], the authors focus on detecting congestion in DEA through a single model, offering insights into identifying and addressing inefficiencies due to congestion. Also, Shadab *et al.* [16] introduces a different approach using anchor points in DEA to measure congestion. The papers collectively contribute to the understanding of congestion's impact on supply chain efficiency and offer various methods to measure and detect congestion within the DEA framework. Moosavi and Bagherzadeh [13] present two new insights into a congestion area and non-congestion area for production technology and two new mathematical definitions of congestion based on the PPS properties and detecting the weak and strong congestion status of DMUs. Also, Kassaei *et al.* [9] defined the concept of congestion for systems with a two-stage structure. Then, they have proposed a single linear programming model to check and measure the congestion of each stage as well as the congestion of the entire system.

Congestion in the operational context stands as a significant manifestation of inefficiency, precipitating sub-optimal functioning within DMUs and concomitantly attenuating their productive capacity. The identification and elucidation of inefficacious DMUs, coupled with the discernment of the origins of their inefficiencies, have conventionally driven a thorough exploration of DMU internal structures. Concurrently, an investigation into

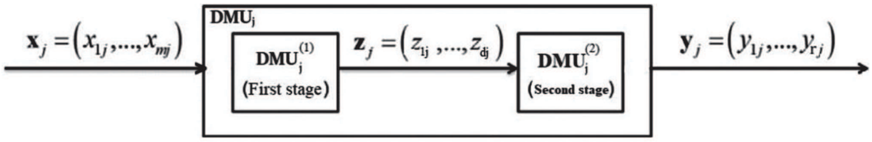

FIGURE 1. The visual description of two-stage network structure.

the repercussions of intermediate products on the performance of subDMUs has been instrumental. This study endeavors to anatomize the underlying causes of DMU congestion by adopting a methodological framework rooted in a two-stage NDEA. This approach facilitates the deconstruction of DMU congestion into distinct forms, namely BB congestion and two-stage structural congestion. Remarkably, inputs demonstrate a dual and simultaneous role, being pertinent to both the BB context and stage1. This duality engenders three distinct classifications of congestion.

Consequently, the research embarks on an analysis of the intricate interrelationships between two types of initial input congestion and the congestion stemming from intermediate products. These dynamics collectively exert influences on BB congestion. The study culminates in the formulation of three discrete congestion definitions and the establishment of a comprehensive model that captures the interplay between different variants of input congestion, intermediate product-related congestion, and BB input congestion. So, this work not only establishes that inefficiency begets congestion but also positions congestion as an extreme form of inefficiency. It's crucial to recognize that congestion characterizes the production frontier, rather than the DMU itself. To provide clarity, this paper proposes three definitions for the congestion concept in the presence of intermediate products, contributing an alternative perspective to ongoing discourse. The subsequent sections outline the definitions, assumptions, and common congestion measurement methods (Sects. 2 and 3, respectively). Our proposed method and two-stage network congestion are detailed in Section 4. The complete congestion analysis for the two-stage network is expounded. Our model is presented in Section 5, followed by numerical examples and comparison with existing methods in Section 6. The paper concludes in Section 7.

2. Preliminary assumption and definitions

Suppose that there exists n homogenous DMUs, $\mathrm{DMU}_j (j = 1, \dots, n)$ in a constant time interval. $X_j = (x_{1j}, \dots, x_{mj})^t$ and $Y_j = (y_{1j}, \dots, y_{sj})^t$ are the input and output vectors of DMU_j. Such that each DMU has two subDMUs (or stages). A lack of attention paid to each DMU's internal structure prompts the consideration of the BB structure. We call the stage1 ($\mathrm{subDMU}_o^{(1)}$) and the stage2 ($\mathrm{subDMU}_o^{(2)}$). As mentioned in the introduction, congestion is a production-related situation that can be viewed as a severe technical inefficiency case. Before proceeding to the main discussion, we need to define two important concepts:

Technical efficiency 2.1. A $\mathrm{DMU}_o (o \in j = 1, \dots, n)$ is technically efficient in VRS technology if the optimal solution of the following BCC model (1) be $\varphi^* = 1$ in evaluating it.

$$\varphi^* = \max \varphi$$
$$\text{s.t.} \begin{cases} \sum_{j=1}^{n} \lambda_j x_j \le x_{io} & i = 1, 2, \dots, m \\ \sum_{j=1}^{n} \lambda_j y_j \ge \varphi y_{ro} & r = 1, 2, \dots, s \\ \sum_{j=1}^{n} \lambda_j = 1 \\ \lambda_j \ge 0. \end{cases} \tag{1}$$

There is congestion in the inputs when increasing some or all inputs reduces some or all the outputs without improving the other inputs and outputs. In other words, decreasing one or more inputs that increase some or all the outputs while worsening no other input or output.

2.1. Technology set

The technology T is defined as the following,

$$T = \{(x, y)|x \text{ can produce } y\}. \tag{2}$$

Notice that the technologies T, T_c and T_v were defined in the upper section. Assume that the outputs and inputs are non-negative. Consider the following two definitions of technology:

(1) T_v is the decussating of all technologies $T \subset R$ that satisfy the syllogism of (i) involving all DMUs, (ii) strong disposability of outputs, (iii) strong disposability of inputs, and (iv) convexity of PPS.
(2) T_{convex} is the decussating of all technologies $T \subset R$ that satisfy the syllogism of (i) involving DMUs, (ii) strong disposability of outputs, and (iv) the convexity of PPS.

The congestion measurement is generally discussed in the context of output-oriented DEA models. The congestion concept does not fit traditional DEA literature because of the economic researcher's congestion concepts. Also, congestion is modeled and measured by deleting the axiom of strong input disposability. Therefore, to model congestion, this principle must be omitted from the fundamental principles of DEA. A PPS called T_{convex} should be constructed to act as the reference for the congestion measurement in DEA. The explicit DEA-base representations of T_v and T_{convex} are defined as follows:

$$T_v = \left\{ (x_o, y_o) \middle| \exists\lambda : \sum_{j=1}^{n} \lambda_j x_j \leq x_o, \sum_{j=1}^{n} \lambda_j y_j \geq y_o, \sum_{j=1}^{n} \lambda_j = 1, \lambda_j \geq 0; j = 1, \ldots, n \right\}, \tag{3}$$

$$T_{\text{convex}} = \left\{ (x_o, y_o) \middle| \exists\lambda : \sum_{j=1}^{n} \lambda_j x_j = x_o, \sum_{j=1}^{n} \lambda_j y_j \geq y_o, \sum_{j=1}^{n} \lambda_j = 1, \lambda_j \geq 0; j = 1, \ldots, n \right\}. \tag{4}$$

Technology T_{convex} satisfies all the syllogism that defines technology T_v. Because T_v is the smallest technology that satisfies this syllogism, so $T_{\text{convex}} \subseteq T_v$. So, to formulate T_v (model (5)) and T_{convex} (model (6)) respectively as follows:

$$\max \varphi + \varepsilon \left(\sum_{i=1}^{m} s_{io}^{-} + \sum_{r=1}^{s} s_{ro}^{+} \right)$$
$$\text{s.t.} \begin{cases} \sum_{j=1}^{n} \lambda_j x_{ij} + s_{io}^{-} = x_{io} & i = 1, \ldots, m \\ \sum_{j=1}^{n} \lambda_j y_{rj} - s_{ro}^{+} = \varphi y_{ro} & r = 1, \ldots, s \\ \sum_{j=1}^{n} \lambda_j = 1 \\ \lambda_j \geq 0, s_i^{-} \geq 0, s_r^{+} \geq 0 \end{cases} \tag{5}$$

$$\max \varphi + \varepsilon \left(\sum_{r=1}^{s} s_{ro}^{+} \right)$$

TABLE 1. Assumption of reference technologies, definition, and analysis in stage1 and stage2.

Analysis, Section, Page	Stage1	Stage2	Definition
First analysis, 4.1, page 1880	$T^{(1)}_{\text{convex}}$	$T^{(2)}_{v}$ or $T^{(2)}_{c}$	Definition 4.1, page 1884
The second analysis, 4.2, page 1882	$T^{(1)}_{c}$	$T^{(2)}_{\text{convex}}$	Definition 4.2, page 1885
The third analysis, 4.3, page 1883	$T^{(1)}_{v}$	$T^{(2)}_{\text{convex}}$	Definition 4.3, page 1885
Fourth analysis, 4.5, page 1885	$T^{(1)}_{\text{convex}}$	$T^{(2)}_{\text{convex}}$	Impossible

$$\text{s.t.} \begin{cases} \sum_{j=1}^{n} \lambda_j x_{ij} = x_{io} & i = 1, \ldots, m \\ \sum_{j=1}^{n} \lambda_j y_{rj} = \varphi y_{ro} + s^{+}_{ro} & r = 1, \ldots, s \\ \sum_{j=1}^{n} \lambda_j = 1 \\ \lambda_j \geq 0, s^{+}_{r} \geq 0. \end{cases} \tag{6}$$

Note that $\varepsilon > 0$ is a non-Archimedean element, namely, it is not a real number and defined to be is a non-Archimedean element; namely, it is not an actual number and determined to be smaller than any positive actual number. By solving model (15) of Jahanshahloo and Khodabakhshi (2004) can be avoided assigning a value to the ε.

So, we have a definition as below:

Definition 2.2. Assume that $(\varphi^*, s^{-^*}_{io}, s^{+^*}_{ro}, \lambda^*)$ is an optimum solution of the model (5). We called DMU_o is efficient if and only if $\varphi^* = 1, (s^{-^*}_{io}, s^{+^*}_{ro}) = (0, 0)$. Otherwise, DMU_o is inefficient. In model (6), we called that DMU_o is efficient if and only if $\varphi^* = 1, (s^{+^*}_{ro}) = (0_s)$.

3. Background of congestion

This section provides a concise overview of the methodologies employed by Noura *et al.* [14], Färe *et al.* [6], and Cooper and Tone [2, 4], while also suggesting further readings in Mehdiloozad *et al.* [12] and Tone and Sahoo [21] for comprehensive insights. Mehdiloozad *et al.* [12] emphasized the significance of multiple optimal projections and the involvement of negative data. Their work tackled the intricacies of multiple projections by establishing that if strong congestion is present at any DMU spanning a face, then at least one DMU within the same face experiences weak congestion. They demonstrated that the congestion of inefficient DMUs possessing non-zero output slacks could be identified through their reference DMUs. Contrary to these prior approaches, our proposed framework conceives congestion as a fundamental concept akin to returns to scale. Consequently, any discourse on its assessment is inherently linked to solely efficient DMUs. Within this paradigm, we define the congestion of an inefficient DMU based on its efficient projection, as determined by the congestion-based DEA model. Like the work by Tone and Sahoo [21], our approach operates under the assumption that the set of possible projections exists as a singleton, thus ensuring the mathematical integrity of this concept. Notably, Tone and Sahoo's [21] methodology presumes positive input-output data without negative implications.

To illustrate the distinctions arising from varying assumptions regarding reference technologies in stage J (where J can be 1 or 2), a comparative summary is provided in the subsequent Table 1. This comprehensive framework expounds on the nuances inherent in the treatment of reference technologies across different segments of the paper.

3.1. Detecting the candidate DMUs in a two-stage network

In this part, we identify the positions of the $\text{subDMU}_o^{(1)}$ and $\text{subDMU}_o^{(2)}$ (second stage or stage2) in the first and second stages frontier technology as $T_v^{(1)}$ and $T_{\text{convex}}^{(2)}$ respectively. To do this, we consider the following slack-base DEA model in the presence of both negative and non-negative data.

$$
\begin{aligned}
&\max\ R_o=\sum_{i=1}^{m}\alpha_{io}+\sum_{h=1}^{k}\beta_{ho}+\sum_{r=1}^{k}\gamma_{ro}\\
&\text{stage1:}\begin{cases}\sum_{j=1}^{n}\lambda_j x_{ij}=x_{io}-\alpha_{io} & i=1,\ldots,m \qquad \text{(a)}\\ \sum_{j=1}^{n}\lambda_j z_{hj}=z_{ho}+\beta_{ho} & h=1,\ldots,k \qquad \text{(b)}\\ \sum_{j=1}^{n}\lambda_j=1 & \end{cases}\\
&\text{stage2:}\begin{cases}\sum_{j=1}^{n}\mu_j z_{hj}=z_{ho} & h=1,\ldots,k \qquad \text{(c)}\\ \sum_{j=1}^{n}\mu_j y_{rj}=y_{ro}+\gamma_{ro} & r=1,\ldots,s \qquad \text{(d)}\\ \sum_{j=1}^{n}\mu_j=1 & \qquad \text{(e)}\\ \{\lambda_j\geq 0,\mu_j\geq 0,\alpha_{io}\geq 0,\beta_{ho}\geq 0,\gamma_{ro}\geq 0. & \qquad \text{(f)}\end{cases}
\end{aligned}
\tag{7}
$$

Note that for all i, α_{io} indicates that the slack variable corresponds to the inputs. $\forall h:\beta_{ho}$ indicates that the slack variable corresponds to the intermediate products. The slack variable corresponding to the final outputs indicates $\forall r:\gamma_{ro}$. Constraint (c) indicates that stage2 has congestion and that this constraint does not have corresponding slacks. This model is feasible because the following solution is a feasible solution for the model (7): $R_o=0,\lambda_o=\mu_o=1,(\forall j:j\neq o)\lambda_j=\mu_j=0,(\forall i):\alpha_{io}=0,(\forall h):\beta_{ho}=0$.

By using a model (7), we can find the $\text{subDMU}_o^{(1)}$ and $\text{subDMU}_o^{(2)}$ in the first and second stages of the technical efficient frontier. $\text{subDMU}_o^{(1)}$ is on the stage1 technically efficient frontier if and only if $(\forall h):\beta^*_{ho}=0$ and $\text{subDMU}_o^{(2)}$ is on the stage2 technically efficient frontier if and only if $(\forall i):\alpha^*_{io}=0,(\forall r):\gamma^*_{ro}=0$. DMU_0 is referred to as candidate DMU if and only if it applies to set a $\Lambda=\{j\in\{1,\ldots,n\}\big|((\forall i):\alpha^*_{io}=0,(\forall r):\gamma^*_{ro}=0)\cup((\forall h):\beta^*_{ho}=0)\}$. The members of set Λ are the subDMUs on the frontier of the first and second stages simultaneously. For the following explanation, note Figures 2b and 2c. Figure 2c is BB Figure's of candidate DMU's of Figure 2b. So, D_9 and D_{11} not exist in 2c because at least one of the stages not on the technical efficiency frontier of 2b.

In elucidating the depictions presented in Figures 2b and 2c, it is imperative to underscore that these visual representations encapsulate distinct aspects of the internal structures of the considered DMUs. When contemplating the PPS in the context of the two-stage network configuration, Figure 2b emerges as a result. Conversely, Figure 2c materializes when evaluating the PPS within the framework of the black-box structure. Evidently, the DMUs featured in Figure 2b align with those within Figure 2c, albeit diverging in terms of their contextual structures – one pertains to the network configuration while the other corresponds to the black-box arrangement.

In essence, these two representations, the black-box structure and the network structure, collectively represent a specific cohort of observations. The analytical process also involves a concise contemplation of Figure 2c, which visually expounds on the black-box configuration of the observations depicted in Figure 2b. Similarly, Figure 2b provides a visual representation of the two-stage network structure portrayed in Figure 2c. Given that the definition of congestion is contingent upon the PPS frontier, the division of units encompasses two overarching categories: inefficient DMUs and candidate DMUs.

Within the realm of inefficiency, two distinct scenarios emerge for the DMUs. First, there exists a subset where stage1 of DMU_o exhibits inefficiency, while stage2 attains the pinnacle of output. Second, another subset demonstrates inefficiency in stage1 of DMU_o, accompanied by a stage2 that fails to achieve the maximum output. Consequently, this bifurcation can be formulated as follows.

(a) DMU_o features inefficiency in its stage1, whereas the stage realizes maximal output for at least one output component. (b) Turning attention to candidate DMU_o, where both stages occupy positions on the technical efficiency frontier (shown on Fig. 2b), we consider stage2's potential to optimize the final output by augmenting the input of $\text{subDMU}_o^{(1)}$. By examining the indicative pattern illustrated in Figures 2a–2c, DMU_o can be identified as a "Congestion Starting Point" (CSP) (as D_9 in Fig. 2c) under the provision that there exists at least one congested observation (as D_7 in Fig. 2c).

* BB means that the internal structure and DMU's intermediate products are not considered. The DMU's subunit that makes up the DMU's internal structure is related to the intermediate products. The congestion of the subDMUs in the DMUs congestion analysis shows that the goal is to identify the subDMUs that are related to the DMU congestion and the two-stage network. The basic structure of the two-stage network has been shown in Figure 1. Also, it should be noted that Figure 2b delineates the boundary encompassing T_v and T_{convex} within the context of the BB state. The distinctive demarcations are indicated by shades of yellow and black. Figure 2c encompasses units wherein the subunits of both the initial and second stages are situated along the technically efficient frontier. Consequently, units 9 and 11 are absent from Figure 2c, as one of the subunits within each of these units does not align with the frontier.

In essence, the inclusion of units is contingent upon their conformity to this border-related criterion. Assume that we consider all stages (in stage1 technically efficient frontier or outside of this frontier). Note that if we have an inefficient $\text{subDMU}_p^{(1)}$ in the stage1 such that second $\text{subDMU}_p^{(2)}$ has maximum output as $y_r^{\max}$, then their input is more than the input of the CSP (assume that candidate DMU_o is CSP without considering inefficient subDMUs). This inefficient $\text{subDMU}_p^{(1)}$ is a new CSP (($y_r^{\max} = \max_j\{y_{rj}\}$ in a single output. If there are multiple outputs $y_r^{\max} = (\left(y_{1j}^{\max}, y_{2j}^{\max}, \ldots, y_{rj}^{\max}\right)$, which $y_{ij}^{\max}(i = 1, \ldots, m)$ are the maximum of each component or $y_r^{\max}$ represents the maximum value of the r-th output of the different DMUs, this definition indicate that it is a set formed by the indexes of the DMUs for which that maximum output level is attained. The same happens in the case of $z_r^{\max}$.)). All DMUs with inputs higher than the CSP's input show congestion in the BB structure. If we have multiple inputs where all input components of the inefficient $\text{subDMU}_p^{(1)}$ are less than CSP's input (subDMU_o), then DMU_o is CSP. If all input components of the inefficient $\text{subDMU}_p^{(1)}$ are higher than the CSP's input (subDMU_o), then DMU_p is a new CSP. If some components are less and some are higher, then we do not have one special CSP and instead have multiple CSPs. The critical point is that a lack of attention to inefficient subDMUs may negatively affect our calculations and analysis. Therefore, subDMUs with maximum output has been included in our calculations. Because subDMUs with an output that is less than the maximum does not affect the frontier, they have been excluded from the calculations. If these DMUs have congestion, then only their non-radial distance from the congestion frontier is calculated, and their congestion value is thus obtained.

In this case, the BB frontier changes under the influence of this inefficient $\text{subDMU}_o^{(1)}$ (stage1 of the candidate DMU). This inefficient $\text{subDMU}_o^{(1)}$ the BB is non-congested (if the input of this inefficient subDMU is the largest input of the observations) if the intermediate products are congested. Note that if the input of this inefficient $\text{subDMU}_o^{(1)}$ is smaller than the CSPs input, then it will not affect the BB frontiers.

To explain the visual description of the stage1 technical efficient frontier (green color), the stage2 technical efficient frontier (blue color), and the Candidate two-stage DMUs in one input-intermediate product-output, we have considered the set of 11 DMUs from the study by Khoveyni *et al.* [11]. According to the above analysis and calculations, D_4 is a CSP in the BB structure (excluding inefficient subDMUs). As the inputs increase from this point, the final output decreases from the maximum level as the intermediate product increases. It seems that any subDMU$_o^{(1)}$ that has more inputs than $D_4^{(1)}$ has congestion. The congestion frontier starts from the modeling and is a fault.

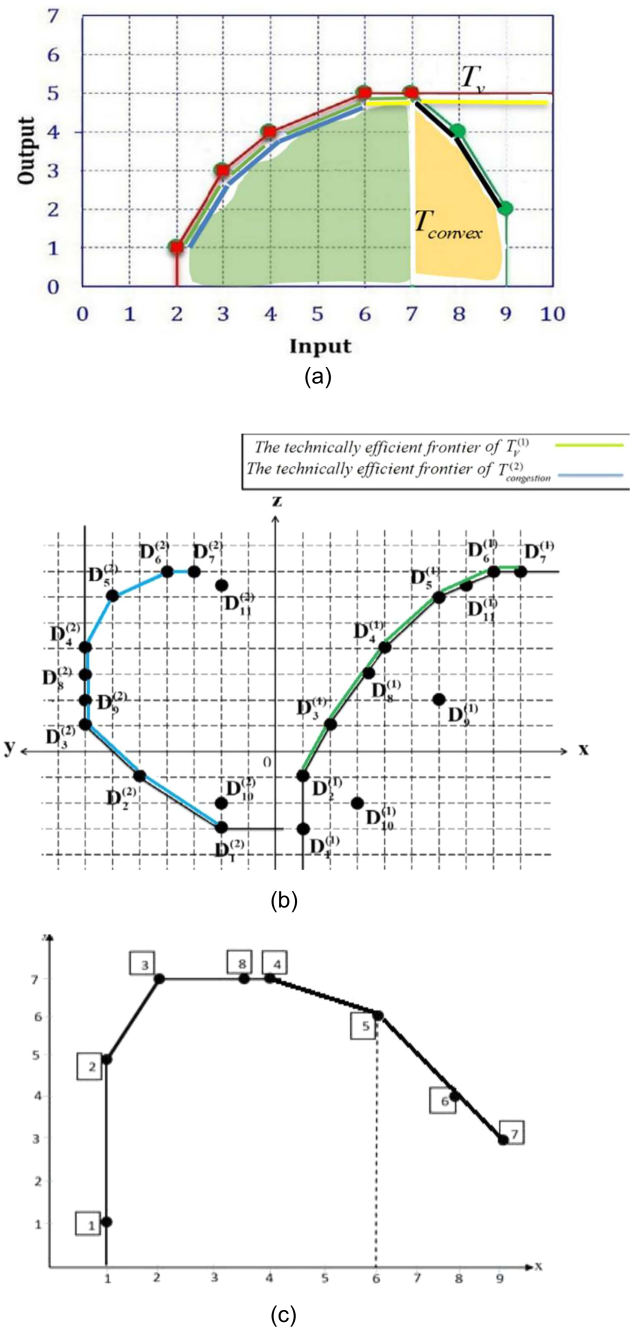

FIGURE 2. (a) Technologies T_{convex} (green line) and T_v (red line). (b) The visual description of first stage technical efficient frontier (green color) and second stage technical efficient frontier (blue color). (c) The visual description of BB frontier of candidate DMUs of Figure 2b.

Note that $D_9^{(1)}$ is inefficient in stage1 but with maximum output. There is a subDMU$_o^{(1)}$ with more input than the input of $D_4^{(1)}$ which produces maximum output. $D_4^{(1)}$ is not the CSP and $D_5^{(1)}$ and $D_9^{(1)}$ in the BB structure are the CSPs. D_5 is an inefficient DMU in the BB structure located at the frontier of the congestion and non-congestion areas. D_5 has no congestion (meaning that the inefficient DMU in the BB structure is a CSP).

Remark 3.1. The stage1 on the first stage's technically efficient frontier is called the candidate stage1, and stage2 on the second stage's technically efficient frontier is called the candidate stage2. If both stages (subDMU$_o^{(1)}$ and subDMU$_o^{(2)}$) are on the technically efficient frontier of each stage; then, the DMU$_o$ is referred to as candidate DMU$_o$.

Remark 3.2. If at least one of the output components of the subDMU$_o^{(2)}$ has maximum output, then the DMU$_o$ is technically efficient in the BB structure (the theorem of DEA).

Corollary 3.3. *If at least one of the intermediate products (as stage1's output) components has a maximum value, then stage1 is technically efficient.*

Proof. According to Remark 3.2. □

Theorem 3.4. *If at least one of the final output's components has a maximum value, then stage2 is technically efficient.*

Proof. According to Remark 3.2. □

4. Two-stage congestion analysis for candidate DMUs

The critical questions that need to be answered: what is the relation between subDMUs congestion and BB congestion? Furthermore, is BB congestion affected by subDMUs congestion or *vice versa*? The answers to these questions are the motivation behind this research. After analyzing the congestion in the network and its conditions, by modeling the analysis presented, we measured the value of the two-stage system congestion (initial input and intermediate product congestion). Finally, we analyzed the relationship between two-stage network congestion and BB congestion. As mentioned earlier, congestion in the absence of strong disposability of inputs occurs. It means that by increasing the input of the DMU$_j$ $(j \in 1, \ldots, n)$, the surplus inputs not only cause inefficiency in the DMU$_j$ $(j \in 1, \ldots, n)$ but also reduce the output of the DMU$_j$ $(j \in 1, \ldots, n)$. Suppose that DMU$_j$ $(j \in 1, \ldots, n)$ consists of two subDMUs (or subunits), namely stage1 and stage2 (as in Fig. 1). BB DMU$_j$ $(j \in 1, \ldots, n)$ produces y_{rj} using x_{ij}.

Now we consider the internal structure of the DMU$_j$. The stage1 uses input x_{ij} to generate the intermediate products $z_{dj}(d = 1, \ldots, D)$ as the stage1's output. Stage2 uses intermediate products z_{dj} to produce the final output $y_{rj}(r = 1, \ldots, s)$. For the DMU$_j$ $(j \in 1, \ldots, n)$ in BB structure, the network structure has been shown in Figure 3b. To begin our analysis, we considered DMU$_j$ $(j \in 1, \ldots, n)$ in BB structure. Suppose that the DMU$_j$ lacks strong disposability of inputs and $X_j = (x_1, \ldots, x_p, \ldots, x_k, \ldots, x_m)$ and $Y_j = (y_1, \ldots, y_r)$ represent the input-output vector of the DMU$_j$.

Figure 3a describes the congestion in the BB structure. It is evident that increasing stage1's input from x_k to x_n means that the output of the DMU$_j$ decreases to the $\acute{y}$. This is due to the dominant congestion of production technology. We also assume that $s^c_{\mathrm{BB}}(s^c)$ is the congestion of the DMU$_j$ $(j \in 1, \ldots, n)$ in BB structure. This is measurable using one of the congestion calculation models by Noura *et al.* [14]. Now consider the internal structure of the same DMU$_j$ $(j \in 1, \ldots, n)$ (which has two subDMUs in a series) (Fig. 3b). According to the first case, the stage1 generates the intermediate products $Z_j = (z_1, \ldots, z_d)$ using input $X_j = (x_1, \ldots, x_m)$. By using these intermediate products as the stage2 input, it produces the final output of the DMU$_j$ $(j \in 1, \ldots, n)$ as well as the stage2 $Y_j = (y_1, \ldots, y_r)$. In other words, it is the internal structure of the DMU$_j$ $(j \in 1, \ldots, n)$ that produces the output $Y_j = (y_1, \ldots, y_r)$ using inputs $X_j = (x_1, \ldots, x_m)$ without change. In other words, to study

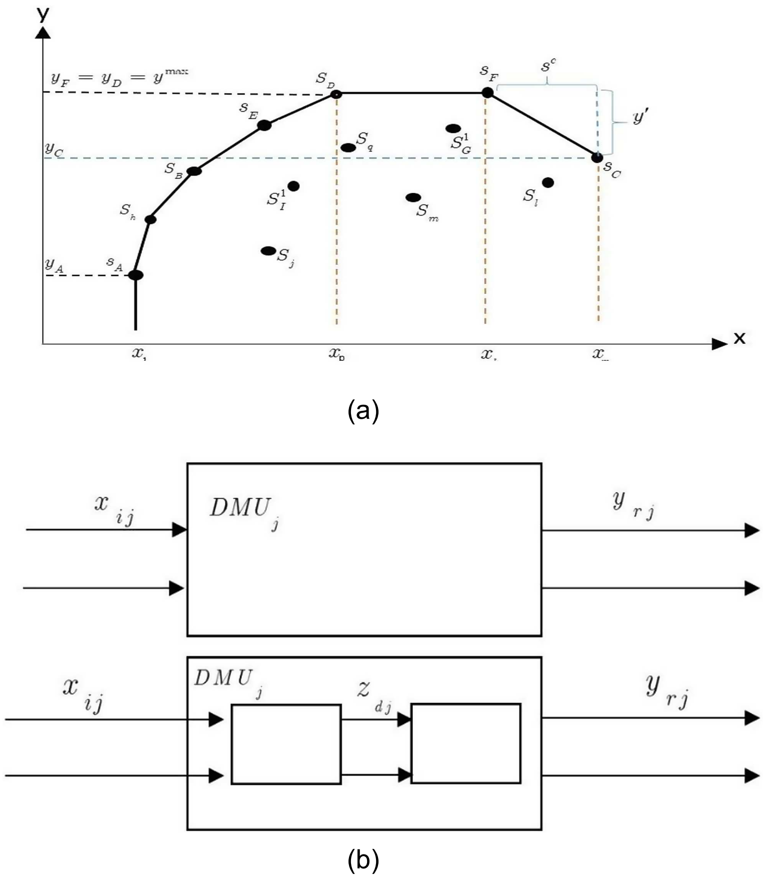

FIGURE 3. (a) Congestion in the black-box structure. (b) The visual description of the DMU_j in black-box and two-stage structure.

the DMU_j $(j \in 1, \ldots, n)$ under evaluation concerning congestion, we have to examine its internal structure. In all subsequent analyses, suppose that the DMU_j $(j \in 1, \ldots, n)$ has one input and one intermediate product. One output can be expanded to multiple inputs and intermediate products and outputs.

4.1. First analysis for candidate DMUs

Suppose that stage1 has congestion and stage2 has no congestion. Note Figure 4a, which illustrates this structure. Suppose s_{BB}^{c} is a congestion of the DMU_j $(j \in 1, \ldots, n)$ in the BB structure, and s^{1c} represents the stage1 congestion and s^{2c} represents stage2's congestion in the two-stage processes. We also show the input changes that cause congestion using s_x and the changes in the intermediate product using s_z.

By increasing the stage1's input from x_1 to x_B (step 1 in Fig. 4a), the production of the intermediate product increases (step 2) from z_1 to $z_{B=Q}$. This increase in the intermediate product has caused the stage2's output to increase from y_o to $y^{\max}$ (step 3). In the following, by decreasing the stage1's input from x_B to x_p (step 4) the production of the intermediate product increases from $z_{B=Q}$ to $z^{\max}$ (step 5). Also, the stage2's output has remained unchanged in $y^{\max}$. It is worth observing that a portion of the primary input utilization in stage 4 encompasses the surplus consumption of primary inputs. This is due to the circumstance where, commencing from a certain stage onward, the final output remains invariant despite an augmentation in input magnitude. The comprehensive analysis of these specific instances constitutes a separate focus, inviting future research

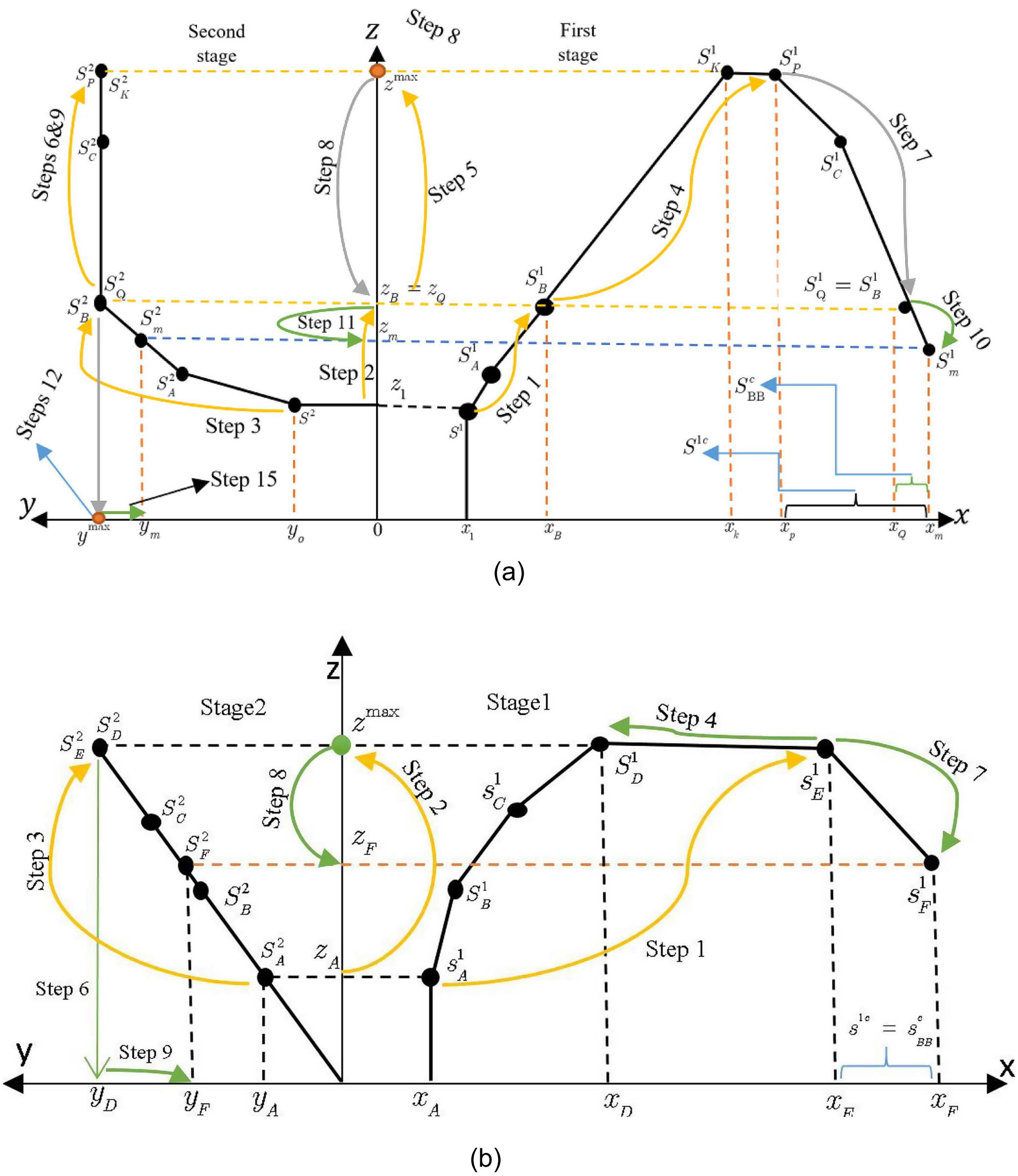

FIGURE 4. (a) and (b) The visual description of the candidate two-stage DMUs of case (first analysis).

endeavors by scholars. It means, by decreasing input from x_p to x_k, and then x_k to x_B first, the intermediate product's production and output unchanged in $z^{\max}$ and $y^{\max}$, second the intermediate product's production decreases from $z^{\max}$ to z_B. But in this phase, the output of the stage2 has remained unchanged in $y^{\max}$. In the following purpose of this paper, by increasing the inputs from level x_p to x_Q (step 7), the production of the intermediate product decreases (step 8) from $z^{\max}$ to z_p. Also, the output of the stage2 has remained unchanged in $y^{\max}$ yet (step 9). During phase 7-8-9, an observed trend emerges wherein an escalation in inputs corresponds to a reduction in intermediate values. This phenomenon aligns with the conceptualization of congestion. As per this trend, congestion is evident within the inputs of the initial stage. However, it's imperative to acknowledge that congestion is limited to the inputs pertaining solely to the initial stage. This distinction is notable due to the consistent nature of the final output. Consequently, it is important to emphasize that the initial inputs, within the construct of Black Boxes, do not manifest any congestion in this context. In the following, by increasing the stage1's input from x_Q to x_{m} (step 10), not only does the production of the intermediate product decrease (step 11) from $z_{B=Q}$ to z_m. But also the final output decreases from $y^{\max}$ to y_m. In this specific phase, an observable phenomenon comes to the fore: an augmentation in inputs corresponds to a decrease in intermediate products and subsequently results in a reduction in the final output. This confluence of effects implies that both stage 1 and the BB configuration exhibit congestion, with inputs showing two distinct forms of congestion. It is crucial to underscore that neither intermediate products nor the second stage display any signs of congestion, as a reduction in intermediate products corresponds to a decline in outputs. Within this context, it is significant to highlight that the absence of congestion in the second stage engenders a discernible consequence. Specifically, the final output of DMU_j, influenced by the diminishing intermediate product (a consequence of increased initial inputs), initiates a decline from its maximal production level ($y^{\max}$). This outcome serves as a clear indicator that DMU_j is characterized by congestion within the BB configuration. Crucially, it is worth acknowledging that upon augmenting the input beyond x_p, a state of congestion emerges specifically within the context of the initial stage (stage1), rather than the broader BB framework. This prompts an inquiry into the pivotal juncture at which this increase attains critical significance. Specifically, can all incremental increments in the initial input (exceeding x_p) be systematically calculated, resulting in congestion within the stage1 input component of the broader BB congestion?

The response to this query is negative. Illustrated in Figure 4a, elevating the initial input level from x_p to x_Q yields a sole consequence: a reduction in the intermediate product, with no discernible effect on the final output. Notably, this alteration does not exert an influence on the latter. Yet, further raising the initial input from x_Q entails a dual impact, causing both the intermediate product and the final output to decline concurrently. This nuanced outcome has specific implications for the assessment of congestion within the BB framework and the larger stage1 context.

A pertinent observation emerges only a fraction of the initial input congestion manifests as the initial BB congestion, while not all aspects of stage1 congestion are mirrored in the BB congestion. From this standpoint, it can be asserted that the congestion threshold x_o within the initial stage input (within the DMU_j's two-stage configuration) surpasses the corresponding congestion threshold x_o within the BB configuration of DMU_j. This applies to any DMU under evaluation as DMU_o $s_o^c = s_{\mathrm{BB}}^c \leq s^{1c} = s_o^{1c}$.

4.2. Second analysis for candidate DMUs

Leveraging the foundational tenets elucidated in the preceding analysis, a consequential inference can be drawn in the eventuality of stage2 operating under the constant returns to scale technology. In such a scenario, the analytical circumstances coalesce with those of the initial analysis, underpinning a fundamental equivalence, yet distinguished by a singular dissimilarity: $s^{1c} = s_{\mathrm{BB}}^c$. This structural portrayal is adeptly depicted in Figure 4b. It is worth emphasizing that a surge in the initial input magnitude precipitates a concomitant decrease in the intermediate product's manifestation as a product of stage1, further affording itself as a constituent input to stage2. The subsequent curtailment of the intermediate product when serving as stage2's input engenders a concomitant contraction in the ultimate output of stage2, a phenomenon that also resonates in the curtailed manifestation of the final output. It is crucial to underscore that the magnitude of the reduction in the final output aligns seamlessly with the quantum of output curtailment experienced within the context of the congested BB configuration.

Despite the absence of congestion within stage2, it may initially appear that the decline in output would only materialize once congestion conditions are encountered within the BB. Nonetheless, owing to the congestion observed within stage1, an intricate interplay unfolds whereby the augmentation of the initial input level prompts a concomitant reduction in the intermediate product tier. Consequently, this decrement in the intermediate product level reverberates in the diminution of the stage2 final output, substantiating its dependency on the

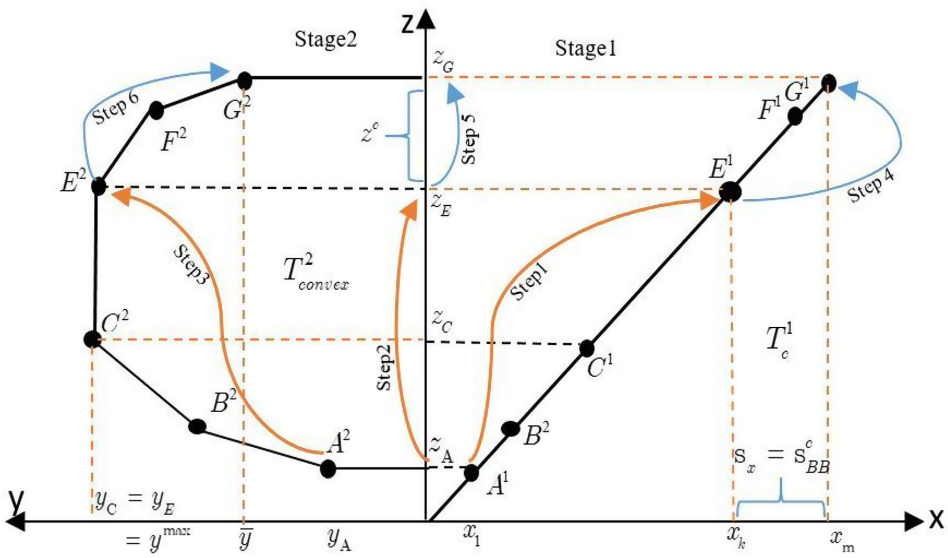

FIGURE 5. The visual description of the Candidate two-stage DMUs of case second analysis.

intermediate product's attenuation. Succinctly put, it becomes evident that stage2 remains devoid of congestion. However, this façade is swiftly dispelled upon discerning that the moderation of the intermediate product inexorably engenders a decrement in the final output, thereby invoking congestion within the BB. Thus, we can assert that both stage1 and DMU_j within the BB structure harbor congestion. In the context of the second analytical assessment, concerning the intricate relationship between BB congestion and two-stage network congestion, along with their respective magnitudes, it is apt to stipulate that BB congestion corresponds to the value of stage1 congestion, thereby articulated as $s^{1c} = s^c_{\text{BB}}$.

Within the context of Constant Returns to Scale (CRS) technology and the terminal phase, it is imperative to underscore that an augmentation in the intermediate product tier triggers a commensurate decrement in the stage2 output, attributed to a concurrent shift in the initial input threshold leading to escalation. Let us hypothetically regard this augmentation as a perturbation to the initial input, herein labeled as s_x. Notably, the attenuation in the BB output experiences a twofold impact: an initial influence imparted by the intermediate product and subsequently, the interplay of intermediate product congestion. Furthermore, the BB congestion encounters the intermediary product's impact as a third contributing factor. Nonetheless, the aforementioned augmentation in the initial input (interpreted as an augmentation in the initial input s_x) precisely equates to the BB congestion. While the stage1 process remains unburdened by congestion, the stage2 final output undergoes diminishment amounting to s_x due to the perturbations in the initial input. The BB congestion value aligns with $s_x = s^c_{\text{BB}}$, and correspondingly, the stage2 congestion value equates to z^c. Thus, the occurrence of congestion within the initial input transpires not within the two-stage procedure but rather under the dominion of the BB's intermediate product influence, wherein congestion materializes. It is pertinent to note that Figure 5 provides a visual exposition of the analogous state depicted within the configuration of the prospective two-stage DMUs, as pertains to the second analytical case.

4.3. Third analysis for candidate DMUs

Suppose in the unit under evaluation that the first subunit lacks congestion (T^1_v), and stage2 has congestion, as Figure 6 illustrates. By increasing the stage1's input from x_A to x_D, the production of the intermediate product increases from z_A to z_D. Stage2 consumes this intermediate product, and the final output from the

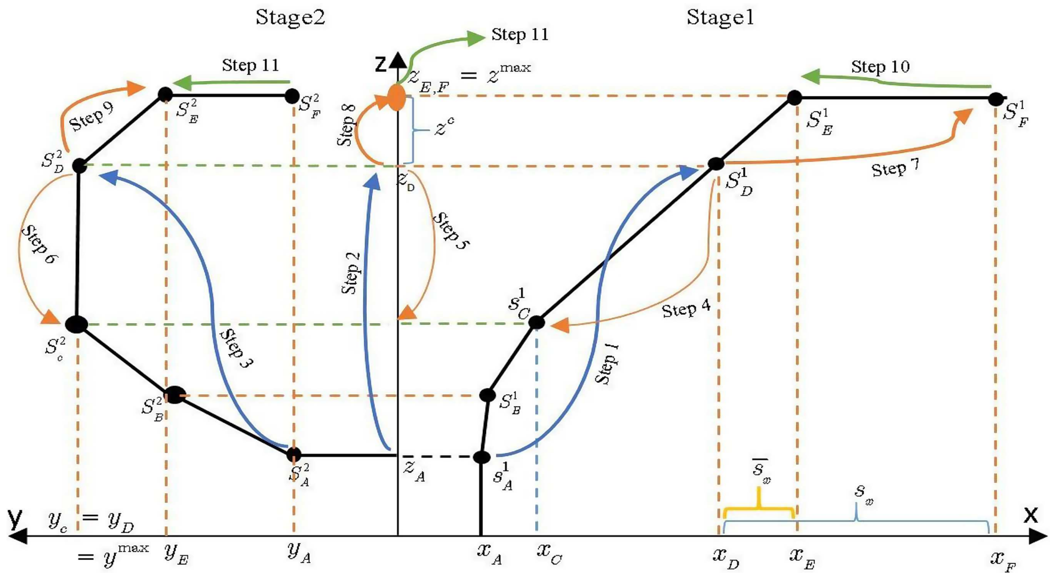

FIGURE 6. The visual description of the Candidate two-stage DMUs of case third analysis.

level goes from y_A to $y_C = y_D$ (maximum output). Critical point in this step is: As the level of the stage1's input decreases from x_D to x_C the intermediate product production decreases. However, the final output production remains unchanged at the $y_C = y_D = y^{\max}$ level. In the first phase, input resources are wasted because the final output remains unchanged by increasing the input value.

In the following of our analysis except of critical point, by increasing the stage1's input from x_D, the intermediate product production increases, but the final output production decreases. Note that in this case, as the intermediate product increases, the output of stage2 decreases, indicating congestion in the intermediate product. Therefore, given that stage1 lacks congestion, the final output is affected by increasing the intermediate product. There is congestion at the intermediate product because (by increasing initial input) by increasing intermediate product production, the final output decreases (from the maximum level).

So, considering to DMU$_j$ as a BB structure, we can say that; As the initial inputs increase, the final output starts to decrease, so BB does have congestion (by increasing inputs from x_D) and x_D is a CSP. So, inputs in the role of BB does has congestion.

4.4. The economic interpretation of congestion in the two-stage network

Consider the unit under evaluation as the DMU$_o$ with a two-stage structure in which stage1 produces the intermediate products z_o by consuming inputs x_o the stage2 consumes these intermediate products (as the input of stage2) and produces the final outputs of the DMU$_o$. We can say that DMU$_o$ has congestion whenever one of the following situations occurs:

(1) If there is an increase in some (or all) of the input components in the stage1, it decreases some (or all) intermediate product components. This reduces some (or all) of the final output components. In this case, stage1 and the DMU$_o$ under evaluation in the BB structure have weak (strong) congestion.

Definition 4.1 (Congestion in two-stage NDEA-1)**.** DMU$_o = (x_o, z_o, y_o)$ with two-stage structure as stated in the first analysis has congestion if and only if there exist $(\bar{x}_o, \bar{z}_o) \in T^1_{\text{convex}}$ and $(\bar{z}_o, \bar{y}_o) \in T^2_v$ or $(\bar{z}_o, \bar{y}_o) \in T^2_c$ such that $\begin{cases} \bar{x}_o = x_o + s^- \\ \bar{z}_o = z_o - s^z \\ \bar{y}_o = y_o - s^+ \end{cases}$ which $s^- \underset{\neq}{\geq} 0, s^z \underset{\neq}{\geq} 0, s^+ \underset{\neq}{\geq} 0$ and $s^- = s^c_{\text{BB}}$. Also we have $s_x > s^-$ also, if $(\bar{z}_o, \bar{y}_o) \in T^2_c$

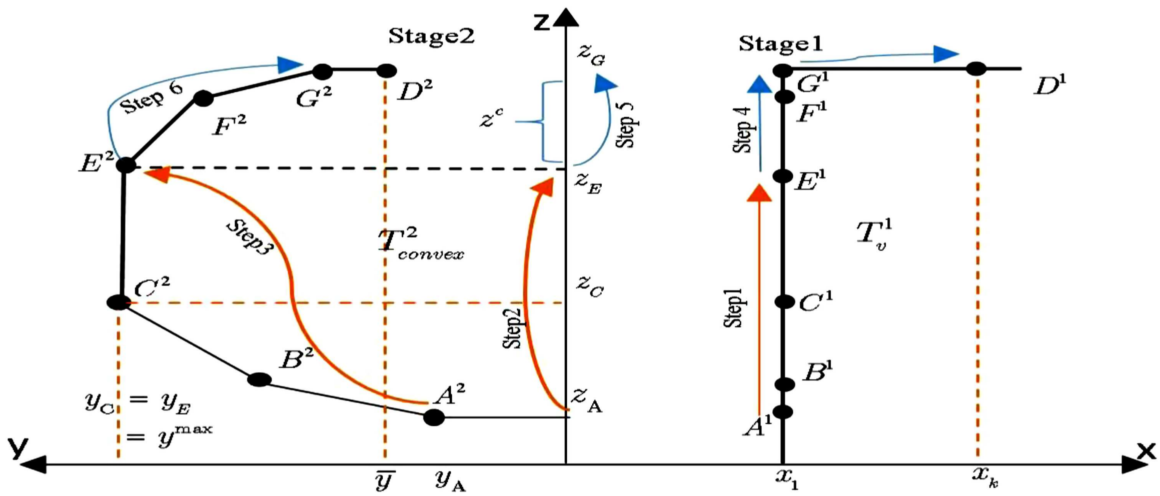

FIGURE 7. The visual description of the candidate two-stage DMUs of case Definition 4.3.

so $s^c_{\mathrm{BB}} = s_x = s^-$. Also stage one has congestion if and only if there exist $(\bar{x}_o, \bar{z}_o) \in T^1_{\text{convex}}$ such that $\begin{cases} \bar{x}_o = x_o + s_x \\ \bar{z}_o = z_o - s_z \end{cases}$ which $s_x \underset{\neq}{\geq} 0, s_z \underset{\neq}{\geq} 0$.

(2) If an increase in some (or all) of the stage1's input components affected an increase in some (or all) of the intermediate product's components some (or all) of the final output's components decrease. In this case, the stage2 and the DMUs under evaluation in the BB structure have weak (strong) congestion.

Definition 4.2 (Congestion in two-stage NDEA-2)**.** $\mathrm{DMU}_o = (x_o, z_o, y_o)$ with two-stage structure as stated in the second analysis has congestion if and only if there exist $(\bar{x}_o, \bar{z}_o) \in T^1_v$ and $(\bar{z}_o, \bar{y}_o) \in T^2_{\text{convex}}$ such that $\begin{cases} \bar{x}_o = x_o - s^- \\ \bar{z}_o = z_o - s^z \\ \bar{y}_o = y_o + s^+ \end{cases}$ which $s^- \underset{\neq}{\geq} 0, s^z \underset{\neq}{\geq} 0, s^+ \underset{\neq}{\geq} 0$ and $s^- = s^c_{\mathrm{BB}}$. Also stage two has congestion if and only if $\begin{cases} \bar{z}_o = z_o + s_z \\ \bar{y}_o = y_o - s_y \end{cases}$ which $s_y \underset{\neq}{\geq} 0, s_z \underset{\neq}{\geq} 0$.

(3) By keeping the stage1's input (not changing the inputs to the stage1), if some (or all) of the components of the intermediate products are increased, and some (or all) of the components of the output are reduced, in this situation, the stage2 has weak (strong) congestion while the stage1 lacks congestion.

Definition 4.3 (Congestion in two-stage NDEA-3)**.** Consider a $\mathrm{DMU}_o = (x_o, z_o, y_o)$ two-stage structure, as stated in the second analysis. The stage2 (intermediate products) has congestion if and only if there exists $(\bar{x}_o, \bar{z}_o) \in T^1_v$ and $(\bar{z}_o, \bar{y}_o) \in T^2_{\text{convex}}$ such that $\begin{cases} \bar{x}_o = x_o \\ \bar{z}_o = z_o + s^z \\ \bar{y}_o = y_o - s^+ \end{cases}$ which $s^z \underset{\neq}{\geq} 0, s^+ \underset{\neq}{\geq} 0$. in this state, the BB has no congestion (Fig. 7).

4.5. Fourth analysis for candidate DMUs

Suppose that the first and second stages are congested, which Figure 8 illustrate in term.

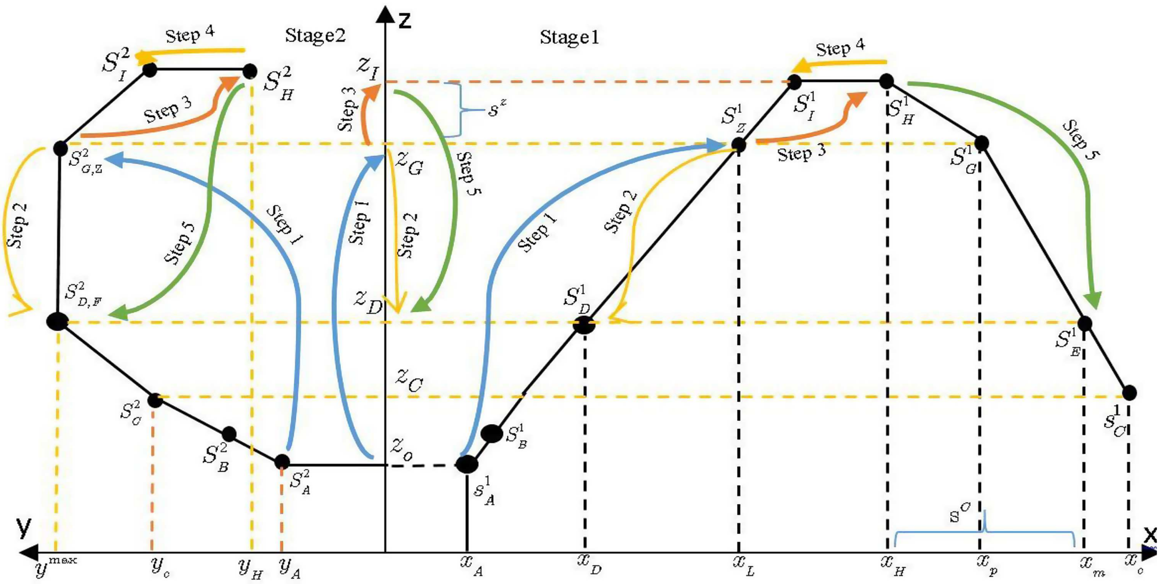

FIGURE 8. The visual description of the candidate two-stage DMUs of case four analysis.

Intermediate product as a stage1's output, it increases from z_o to z_G. Furthermore, by consuming these intermediate products, final output increases from y_A to $y_r^{\max}$. This production is upward and continues to its maximum level.

By increasing the stage1's input from x_l to x_H, the intermediate products increase from z_G to the $z_d^{\max}$. However, through the consumption of these intermediate products, the final output starts to decrease from the $y_r^{\max}$. Moreover, this reduction in the final output causes congestion in the intermediate product, meaning that stage2 is congested. By increasing the input of stage1 from x_H to x_p, the intermediate product as the output of the stage1 begins to decrease from $z_d^{\max}$ to z_G levels. This means that the stage1, more precisely the initial input, has congestion, but with this decrease in the intermediate product as the stage2 input, the output level increases and continues to the $y_r^{\max}$. However, the exciting thing is that increasing the stage1 input from x_p to x_m, the intermediate product continues to decline while the final output remains at the $y_r^{\max}$. By increasing the stage1 input from x_m to x_c the intermediate product continues to decline. But the final output starts to decline again from $y_r^{\max}$ level. This analysis's critical point is that congestion will occur in stage2 and the BB structure by increasing the initial input to the two-stage system. However, by increasing the input to stage1, the stage2 congestion is eliminated, but so is the BB congestion. This is inconsistent with the DEA principles and the definition of congestion.

Another point is that by increasing the initial input from the x_c level, the congestion returns to the BB again. This is inconsistent with all DEA principles. Therefore, as a fundamental prerequisite and a principle concerning two-stage network congestion and BB, we must accept that both stages should not be congested simultaneously because the DMU congestion is quenched. The following is an important conclusion from the analysis presented above.

Remark. In the two-stage network, if the unit under evaluation DMU_o in the BB structure has congestion; thus, only one of the two subDMUs will be congested. In other words, the two subDMUs do not experience congestion simultaneously. On the contrary, if one of the two stages has congestion, then the DMU_o has congestion.

So, in this case and according to the structure of Figure 9:

(1) If S1 has congestion then, the other subDMUs cannot be congested.

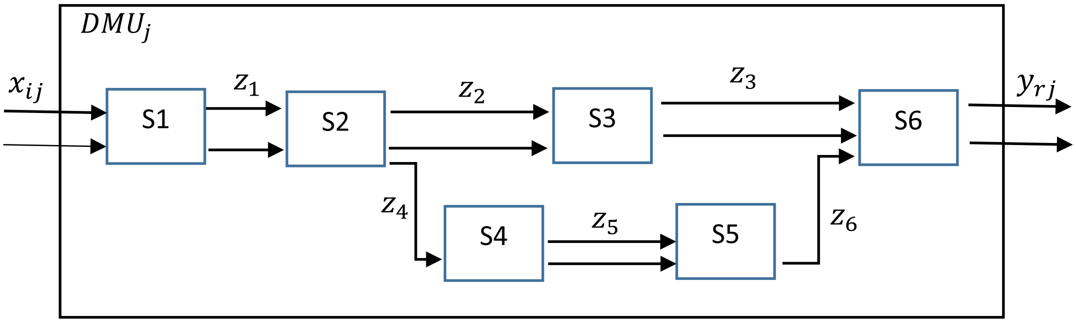

Figure 9. Complex network of DMU_j.

(2) If S2 has congestion, then the other subDMUs cannot be congested.
(3) If S3 has congestion, then S4 or S5 can only be congested.
(4) If S4 has congestion, then S3 can only be congested.
(5) If S5 has congestion, then S3 can only be congested.
(6) If S6 has congestion, then the other subDMUs cannot be congested.

5. Modeling the proposed method for candidate DMUs

We note that to define the concept of congestion in BB structure and the two-stage network, we must first eliminate the strong disposability of inputs (x) and strong disposability of the intermediate product (as input role for stage2). However, according to the preceding section's analysis, these two cannot be eliminated simultaneously. The PPS $T_v^{\text{two-stage}}$ and $T_{\text{convex}}^{\text{two-stage}}$ will be as follows:

$$T_{\text{convex}}^{\text{two-stage}} = \left\{ (x,z,y) \middle| \exists\lambda : x = \sum_{j=1}^{n} \lambda_j x_{ij}, z \le \sum_{j=1}^{n} \lambda_j z_{dj}, z = \sum \mu_j z_j, y \le \sum_{j=1}^{n} \mu_j y_{rj}, \sum_{j=1}^{n} \lambda_j = 1, \sum_{j=1}^{n} \mu_j = 1, \right.$$
$$\left. \lambda_j \ge 0, \mu_j \ge 0, i = 1,\dots,m, d = 1,\dots,D, r = 1,\dots,s \right\}$$
$$T_v^{\text{two-stage}} = \left\{ (x,z,y) \middle| \exists\lambda : x \ge \sum_{j=1}^{n} \lambda_j x_{ij}, z \le \sum_{j=1}^{n} \lambda_j z_{dj}, z \ge \sum \mu_j z_j, y \le \sum_{j=1}^{n} \mu_j y_{rj}, \sum_{j=1}^{n} \lambda_j = 1, \sum_{j=1}^{n} \mu_j = 1, \right.$$
$$\left. \lambda_j \ge 0, \mu_j \ge 0, i = 1,\dots,m,\ d = 1,\dots,D,\ r = 1,\dots,s \right\}. \tag{8}$$

Suppose that $\mathrm{DMU}_o = (x_o, z_o, y_o); (o \in \{1,\dots,n\})$ is a candidate DMU, *i.e.*, $o \in \Lambda$. Also, assume that $(x_o, z_o) \in T_{\text{convex}}^1$ and $(z_o, y_o) \in T_v^2$. To perform the calculation related to the first analysis and calculate stage1 congestion and BB congestion and analyze the relationship between the two congestion types, we consider the method in Figure 9. Notice that with the increasing level of the initial input from x_o to x_p, the production of the intermediate product increases from z_o to $z_d^{\max}$. This increase in the intermediate products also increases the final output from y_o to $y_r^{\max}$. To calculate this state, consider the following slack-base non-radial DEA model:

$$\min \bar{w}_o = \sum_{i=1}^{m} s_{io}^- + \sum_{d=1}^{D} \tilde{s}_{do} + \sum_{r=1}^{s} s_{ro}^+ \qquad \text{(Phase I)}$$

$$
\begin{aligned}
&\text{stage1:}\begin{cases}
\sum_{j=1}^{n}\lambda_j x_{ij} - s_{io}^{-} = x_{io} & i=1,\dots,m \qquad \text{(a)}\\
\sum_{j=1}^{n}\lambda_j z_{dj} - \tilde{s}_{do} = z_{do} & d=1,\dots,D \qquad \text{(b)}\\
\sum_{j=1}^{n}\lambda_j = 1 & \qquad\qquad\qquad \text{(c)}
\end{cases}\\
&\text{stage2:}\begin{cases}
\sum_{j=1}^{n}\mu_j z_{dj} - \tilde{s}_{do} = z_{do} & d=1,\dots,D \qquad \text{(d)}\\
\sum_{j=1}^{n}\mu_j y_{rj} - s_{ro}^{+} = y_{ro} & r=1,\dots,S \qquad \text{(e)}\\
z_d^{\max} = \tilde{s}_{do} + z_{do} & d=1,\dots,D \qquad \text{(f)}\\
y_r^{\max} = s_{ro}^{+} + y_{ro} & r=1,\dots,S \qquad \text{(g)}\\
\sum_{j=1}^{n}\mu_j = 1 & \qquad\qquad\qquad \text{(h)}\\
\lambda_j \geq 0, \mu_j \geq 0, s_{do}^{+} \geq 0, \tilde{s}_{do} \geq 0, s_{io}^{-} \geq 0. &
\end{cases}
\end{aligned}
\tag{9}
$$

We define the set of $\Omega = \{j \in \Lambda |$ Phase (I) in evaluating DMU_j is infeasible$\}$, and $E = \{j \in \Lambda |$ Phase (I) in evaluating DMU_j is feasible$\}$, note that if the model (9) is feasible, *i.e.*, $E \neq \emptyset$ so we can obtain the projection point of DMU_o as $(\bar{x}_o, \bar{z}_o, y_r^{\max})$.

$$
\begin{cases}
\bar{x}_{io} = x_{io} + s_{io}^{-^*}\\
\bar{z}_{do} = z_o + \tilde{s}_{do}^{*} = z_d^{\max}\\
y_r^{\max} = y_{ro} + s_{io}^{+^*}.
\end{cases}
\tag{10}
$$

So, $y_r^{\max} = \{p \in R | y_{rp} = \max_{j=1,\dots n}(y_{rj}), \forall r; y_r^{\max} \neq \emptyset\}$, $z_d^{\max} = \{p \in R | z_{dp} = \max_{j=1,\dots n}(z_{dj} | \exists r (j \in y_r^{\max}))\}$, and $\forall d; z_d^{\max} \neq \emptyset, j^{\max} = \{j | \exists d; (j \in z_d^{\max})\}$ (the largest intermediate product that produces $y_r^{\max}$) are not decision variables. In other words, $y_r^{\max}$ represents the maximum value of the r-th output of the different DMUs, this definition indicates that it is a set formed by the indexes of the DMUs for which that maximum output level is attained. The same happens in the case of $z_r^{\max}$.

However, they guarantee that the input slack does not exceed its exact value. It should be noted that the members of set Ω are congested DMUs. We evaluated these DMUs in Phase three. Relation (10) refers to the frontiers between the congestion zone and the non-congestion zone. We can say that it is a congestion starting point (CSP) if the model (9) is not feasible at least one point. It is important to note that if it is not possible to increase the intermediate products or, if $y_{ro} \neq y_r^{\max}$, then, stage1 will be infeasible. However, this Phase's important thing is that this stage is a waste of resources (stage1 input). With the increasing inputs from the levels, not only do the intermediate products remain unchanged, but the final output also remains unchanged at the $y_r^{\max}$ level. For projection point (10) and for calculating wastage inputs, we set up the following non-radial slack-base model:

$$
\text{Max } w_o'' = \sum_{i=1}^{m} s_{io}'' \qquad \text{(Phase II)}
$$

$$
\begin{array}{ll}
\text{stage1:}\left\{\begin{array}{lll}
\sum_{j=1}^{n}\lambda_j x_{ij}=\bar{x}_{io}-s''_{io} & i=1,\dots,m & \text{(a)}\\
\sum_{j=1}^{n}\lambda_j z_{dj}=\bar{z}^{*}_{do} & d=1,\dots,D & \text{(b)}\\
\sum_{j=1}^{n}\lambda_j=1,\lambda_j\geq 0, & j=1,\dots,n & \text{(g)}
\end{array}\right. & \\
\text{stage2:}\left\{\begin{array}{lll}
\sum_{j=1}^{n}\mu_j z_{dj}=z^{\max}_{d} & d=1,\dots,D & \text{(e)}\\
\sum_{j=1}^{n}\mu_j y_{rj}=y^{\max}_{r} & r=1,\dots,S & \text{(f)}\\
\sum_{j=1}^{n}\mu_j=1,\mu_j\geq 0 & & \text{(g)}\\
\{\mu_j\geq 0, 0\leq s''_{io}\leq s^{-*}_{io}, & j=1,\dots,n,\quad i=1,\dots,m. & \text{(h)}
\end{array}\right. & (11)
\end{array}
$$

s''_{io} is the amount of input resources wasted in the ith component and $\bar{z}^{*}_{do}=z^{\max}_{d}$. But, by increasing the stage1's input from x_p to x_k, the intermediate product production begins to decrease and decreases from $z^{\max}_{d}$ to z_k (expressed as the stage1 congestion). That is, with the increase in input, a decrease in the output of stage1 (intermediate products) occurs. If this increase in input and decrease in intermediate product occurs across all components, strong congestion occurs. If the increase in some inputs decreases some of the intermediate products while remaining unchanged in others, then weak congestion occurs in stage1. However, the final output level remains at the same level, meaning that neither the stage2 nor the BB in this Phase is congested. To calculate the congestion of the stage1, consider the following slack-base model (it should be noted that this model is running for relation (10) and the member of set Ω):

$$
\max\ \omega'_o=\sum_{i=1}^{m}\bar{\bar{s}}_{io}+\sum_{d=1}^{D}\tilde{\tilde{s}}_{do} \qquad \text{(Phase III)}
$$

$$
\begin{array}{ll}
\text{stage1:}\left\{\begin{array}{lll}
\sum_{j=1}^{n}\lambda_j x_{ij}-\bar{\bar{s}}_{io}=\bar{x}_{io} & i=1,\dots,m & \text{(a)}\\
\sum_{j=1}^{n}\lambda_j z_{dj}+\tilde{\tilde{s}}_{do}=\bar{z}_{do} & d=1,\dots,D & \text{(b)}\\
\sum_{j=1}^{n}\lambda_j=1 & &
\end{array}\right. & \\
\text{stage2:}\left\{\begin{array}{lll}
\sum_{j=1}^{n}\mu_j z_{dj}+\tilde{\tilde{s}}_{do}=\bar{z}_{do} & d=1,\dots,D & \text{(e)}\\
\sum_{j=1}^{n}\mu_j y_{rj}=y^{\max}_{r} & r=1,\dots,S & \text{(f)}\\
\sum_{j=1}^{n}\mu_j=1 & & \text{(g)}\\
\{\lambda_j\geq 0,\mu_j\geq 0,\bar{\bar{s}}_{io}\geq s''^{*}_{io},\tilde{\tilde{s}}_{do}\geq 0 & & \text{(h)}
\end{array}\right. & (12)
\end{array}
$$

We define the set $\Gamma = \{j \in \Omega \mid \text{Phase(III) in evaluating DMU}_j \text{ is infeasible}\}$. The purpose of introducing this model is to calculate stage1's input congestion. Then, same as a relation (7), we have:

$$\begin{cases} \bar{\bar{x}}_{io} = \bar{x}_{io} + \bar{\bar{s}}_{io}^{-^*} \\ \bar{\bar{z}}_{do} = \bar{z}_o - \tilde{\tilde{s}}_{do}^* \\ y_r^{\max} = \bar{y}_{ro} \end{cases} \tag{13}$$

where $\bar{\bar{s}}_{io}^{-^*}$ is a congestion value for the ith component of the inputs (in the stage1's input role) that reduces the production of the dth component of the intermediate products (*i.e.*, $\tilde{\tilde{s}}_{do}^*$). If $\bar{\bar{s}}_{io}^{-^*} > 0$, an increase in all input components reduces all components of the intermediate products (strong congestion). If this increase occurs in some of the $\bar{\bar{s}}_{io}^{-^*}$ components, such that some of the intermediate product's components decrease while others remain unchanged, weak congestion occurs. However, since this increase in input up to x_k level causes a decrease in the intermediate product but does not decrease the final output. This congestion in stage1 does not cause congestion in the BB structure. However, by increasing the input in the stage1 from x_k, then the production of the intermediate products continues to decrease from z_k to $z''(z_k < z'')$. However, this Phase increases the initial input, reduces the intermediate products, and reduces the final outputs. That is, in the BB structure, increasing the inputs reduces the outputs. In other words, stage1 is congested, but stage1 congestion congests the BB in turn. In this Phase, consider the following slack-base model to calculate the stage1 congestion (beyond the BB congestion) and the BB congestion itself. Note that this Phase is running for relation (13) and the members of set Γ.

$$\text{Max } \tilde{w}_o = \sum_{i=1}^{m} \hat{s}_{io}^- + \sum_{d=1}^{D} \hat{\hat{s}}_{do} + \sum_{j=1}^{n} \hat{s}_{ro}^+ \qquad \text{(Phase IV)}$$

$$\text{stage1:} \begin{cases} \sum_{j=1}^{n} \lambda_j x_{ij} - \hat{s}_{io}^- = \bar{\bar{x}}_{io} & i = 1, \ldots, m & \text{(a)} \\ \sum_{j=1}^{n} \lambda_j z_{dj} + \hat{\hat{s}}_{do} = \bar{\bar{z}}_{do} & d = 1, \ldots, D & \text{(b)} \\ \sum_{j=1}^{n} \lambda_j = 1, \lambda_j \geq 0 & & \text{(c)} \end{cases}$$

$$\text{stage2:} \begin{cases} \sum_{j=1}^{n} \mu_j z_{dj} + \hat{\hat{s}}_{do} = \bar{\bar{z}}_{do} & d = 1, \ldots, D, & \text{(d)} \\ \sum_{j=1}^{n} \mu_j y_{rj} + \hat{s}_{ro}^+ = y_r^{\max} & r = 1, \ldots, S, & \text{(e)} \\ y_r^{\min} \leq y_r^{\max} - \hat{s}_{ro}^+ & & \text{(f)} \\ \sum_{j=1}^{n} \mu_j = 1, \mu_j \geq 0 & & \text{(g)} \\ \hat{s}_{ro}^+ \geq 0, \hat{s}_{io}^- \geq 0, & r = 1, \ldots, s, \quad i = 1, \ldots, m, & \text{(h)} \\ \hat{\hat{s}}_{do} \geq 0, & d = 1, \ldots, D. & \text{(t)} \end{cases} \tag{14}$$

Note that $y_r^{\min} = \min_{j\in\Omega}(y_{rj})$ then, we have:

$$\begin{cases} \bar{\bar{\bar{x}}}_{io} = \bar{\bar{x}}_{io} + \hat{s}_{io}^{-^*} \\ \bar{\bar{\bar{z}}}_{do} = \bar{\bar{z}}_o - \hat{\hat{s}}_{do}^{*} \\ \tilde{y} = y_r^{\max} - \hat{s}_{ro}^{+^*}. \end{cases} \tag{15}$$

Thus $\hat{s}_{io}^{-^*}$ is the value of the congestion of the ith component of the BB structure inputs, as well as $s_{ix,\text{stage-one}}^{c} = \hat{s}_{io}^{-^*} + \bar{\bar{s}}_{io}^{-^*}$ is the congestion of the ith component of the stage1 inputs.

Definition 5.1 (Congestion in the two-stage network DEA)**.** Suppose that DMU_o; $(x_o, z_o, y_o)(o \in \{1, \ldots, n\})$ such that $(x_o, z_o) \in T^1_{\text{convex}}$ and $(z_o, y_o) \in T_v^2$. So, DMU_o has congestion:

$$\begin{cases} \bar{x}_{io} = x_{io} + s_{io}^{-} \\ \bar{z}_{do} = z_{do} - s_{do}^{z} \\ \bar{y}_{ro} = y_r^{\max} - s_{ro}^{+}. \end{cases} \tag{16}$$

This definition also indicates stage1's input congestion. Note that the stage1 input is the same as the BB inputs. Given the intermediate product's influence on the two-stage network, input congestion in the BB structure is different from the same input value in stage1. By analyzing the congestion, it is possible to obtain a precise definition of congestion that represents both the two-stage network structure and the BB structure:

$$(\bar{x}_{io}, \bar{z}_{do}, \bar{y}_{ro}) = \begin{cases} \bar{x}_{io} = x_{io} + s_{io}^{-} \\ \bar{z}_{do} = z_{do} - s_{do}^{z} \\ \bar{y}_{ro} = y_r^{\max} - s_{ro}^{+} \begin{cases} s_{ro}^{+} = 0, & \text{stage one has congestion} \\ s_{ro}^{+} > 0, & \text{stage one and BB have congestion.} \end{cases} \end{cases} \tag{17}$$

In a simple comparison of the values obtained from the above models, it can be easily seen that for the corresponding components ith, dth, and rth:

$$\begin{cases} x_{io} \le \bar{x}_{io} \le \bar{\bar{x}}_{io} \le \tilde{\bar{x}}_{io} \\ z_{do} \le \bar{z}_{do} \le \bar{\bar{z}}_{do} \le \tilde{\bar{z}}_{do} \le z_{do}^{\max} \\ y_{ro} \le \tilde{y}_{ro} \le y_r^{\max}. \end{cases} \tag{18}$$

Suppose that $\mathrm{DMU}_o = (x_o, z_o, y_o); (o \in \{1, \ldots, n\})$ is a candidate DMU, *i.e.*, $o \in \Lambda$. Additionally, assume that $(x_o, z_o) \in T_v^1$ and $(z_o, y_o) \in T^2_{\text{convex}}$. To perform the calculations related to the first analysis, calculate the stage1 congestion, and the BB congestion, and analyze the relationship between these two congestions, we need to consider the following model (19). Notice that by increasing the initial input level from x_o to x_p, the production of the intermediate product increases from z_o to $\bar{z}_o$. This increase in the intermediate product also increases the final product from y_o to $y_r^{\max}$. To calculate this state, consider the following slack-base non-radial DEA model:

$$\text{Max } \bar{w}_o = \sum_{i=1}^{m} s_{io}^{-} + \sum_{d=1}^{D} \tilde{s}_{do} + \sum_{j=1}^{n} s_{ro}^{+} \qquad \text{(Step I)}$$

$$\text{stage1: } \begin{cases} \sum_{j=1}^{n} \lambda_j x_{ij} - s_{io}^{-} = x_{io} & i = 1, \ldots, m & \text{(a)} \\ \sum_{j=1}^{n} \lambda_j z_{dj} - \tilde{s}_{do} = z_{do} & d = 1, \ldots, D & \text{(b)} \\ \sum_{j=1}^{n} \lambda_j = 1 & & \text{(g)} \end{cases}$$

$$
\text{stage2: } \begin{cases} \sum_{j=1}^{n} \mu_j z_{dj} - \tilde{s}_{do} = z_{do} & d = 1, \ldots, D & \text{(e)} \\ \sum_{j=1}^{n} \mu_j y_{rj} - s_{ro}^{+} = y_{ro} & r = 1, \ldots, S & \text{(f)} \\ y_r^{\max} = s_{ro}^{+} + y_{ro} & & \\ \sum_{j=1}^{n} \mu_j = 1 & & \text{(g)} \\ \lambda_j \geq 0, s_{do}^{+} \geq 0, \mu_j \geq 0, s_{io}^{-} \geq 0, \tilde{s}_{do} \geq 0. & & \text{(h)} \end{cases} \tag{19}
$$

We define the set $\mho = \{j|\text{step(I) in evaluate DMU}_j \text{ is infeasible}\}$. Note that if the model (19) is feasible, we can obtain the projection point of DMU_o as $(\bar{x}_o, \bar{z}_o, y^{\max})$.

$$
\begin{cases} \bar{x}_{io} = x_{io} + s_{io}^{-^*} \\ \bar{z}_{do} = z_o + \tilde{s}_{do}^{*} \\ y_r^{\max} = y_{ro} + s_{io}^{+^*}. \end{cases} \tag{20}
$$

Note that (20) is a frontier between the congestion zone and the non-congestion zone. It is important to note that if it is not possible to increase the intermediate products or, if $y_{ro} \neq y_r^{\max}$, then stage1 would be infeasible. Nevertheless, this Phase's critical thing is that; this stage has a waste of resources (in inputs). By increasing inputs from specific values, the intermediate products remained constant (unchanged), and the final output remains unchanged. Here $s_{ro}^{+^*} > 0$, so to calculate this value, we run the following slack-base model in step two. To ensure the accuracy of the projection point obtained in step 3, we consider the following model (21)

$$
\begin{aligned}
&\text{Max } w_o'' = \sum_{i=1}^{m} s_{io}'' + \sum_{d=1}^{D} s_{do}^{z} && \text{(Step II)} \\
&\text{stage1: } \begin{cases} \sum_{j=1}^{n} \lambda_j x_{ij} = \bar{x}_{io} - s_{io}'' & i = 1, \ldots, m & \text{(a)} \\ \sum_{j=1}^{n} \lambda_j z_{dj} = \bar{z}_{do} - s_{do}^{z} & d = 1, \ldots, D & \text{(b)} \\ \sum_{j=1}^{n} \lambda_j = 1 & & \text{(g)} \end{cases} \\
&\text{stage2: } \begin{cases} \sum_{j=1}^{n} \mu_j z_{dj} = \bar{z}_{do} - s_{do}^{z} & d = 1, \ldots, D & \text{(e)} \\ \sum_{j=1}^{n} \mu_j y_{rj} = y_r^{\max} & r = 1, \ldots, S & \text{(f)} \\ \sum_{j=1}^{n} \mu_j = 1 & & \text{(g)} \\ \begin{cases} \lambda_j \geq 0,\ \mu_j \geq 0 \\ 0 \leq s_{io}^{z} \leq \tilde{s}_{io}^{*} \\ 0 \leq s_{io}'' \leq s_{io}^{-^*} \end{cases} & j = 1, \ldots, n, \quad i = 1, \ldots, m. & \text{(h)} \end{cases}
\end{aligned} \tag{21}
$$

So we can obtain the projection point of DMU$_o$ as $(\bar{\bar{x}}_o, \bar{\bar{z}}_o, \bar{\bar{y}}_{ro} = y_r^{\max})$.

$$\begin{cases} \bar{\bar{x}}_{io} = \bar{x}_{io} - s_{io}^{\prime\prime *} \\ \bar{\bar{z}}_{do} = \bar{z}_o - s_{do}^{z*} \\ \bar{\bar{y}}_{ro} = y_{ro} + s_{ro}^{+^*} = y_r^{\max}. \end{cases} \tag{22}$$

Which $s_{io}^{-^*}$, $\tilde{s}_{do}^*$ and $s_{ro}^{+^*}$ are the optimal solutions from the model (19). Notice that $y_r^{\max} = \max_{j=1,\dots,n}(y_{rj})$. But, by increasing the stage1's input from x_p to x_k, the intermediate product production's increase from $\bar{z}_o$ to $z^{\max}$ but the final output decreases form $y_r^{\max}$ to $\tilde{y}$. That is, with the increase in the intermediate products, the final outputs are decreased. This means that the stage2 and intermediate products both have congestion. If this increase in the intermediate products and decrease in the final products occurs in all (some) components, then strong (weak) congestion is present. With BB, increasing the input reduces the final output, and this means that BB mode has congestion. In other words, not only the stage2 is congested, but also the BB is congested. In this Phase, consider the following slack-base model to calculate stage2 congestion beyond the BB congestion and the BB congestion itself. Note that the initial inputs of the two-stage network are non-congestive. Still, changes to this input result in the intermediate products' congestion, and a decrease in final output is considered BB congestion. To calculate the congestion of stage2, consider the following slack-base model. Note that this step is running for relation (20) and the members of the set $\mho$.

$$\max\ \omega_o' = \sum_{i=1}^{m} \bar{\bar{s}}_{io}^- + \sum_{d=1}^{D} \tilde{\bar{s}}_{do} + \sum_{d=1}^{D} \bar{\bar{s}}_{do}^+ \qquad \text{(Step III)}$$

$$\begin{aligned}
&\text{stage1:} \begin{cases} \sum_{j=1}^{n} \lambda_j x_{ij} - \bar{\bar{s}}_{io}^- = \bar{x}_{io} & i = 1,\dots,m & \text{(a)} \\ \sum_{j=1}^{n} \lambda_j z_{dj} - \tilde{\bar{s}}_{do} = \bar{z}_{do} & d = 1,\dots,D & \text{(b)} \\ \sum_{j=1}^{n} \lambda_j = 1 & & \end{cases} \\
&\text{stage2:} \begin{cases} \sum_{j=1}^{n} \mu_j z_{dj} - \tilde{\bar{s}}_{do} = \bar{z}_{do} & d = 1,\dots,D & \text{(e)} \\ \sum_{j=1}^{n} \mu_j y_{rj} = y_r^{\max} - \bar{\bar{s}}_{ro}^+ & r = 1,\dots,S & \text{(f)} \\ y_r^{\min} \le y_r^{\max} - \bar{\bar{s}}_{ro}^+ & r = 1,\dots,S & \\ \sum_{j=1}^{n} \mu_j = 1 & & \text{(g)} \\ \{\lambda_j \ge 0, \mu_j \ge 0, \bar{\bar{s}}_{io} \ge 0, \tilde{\bar{s}}_{do} \ge 0, \bar{\bar{s}}_{ro}^+ \ge 0. & & \text{(h)} \end{cases}
\end{aligned} \tag{23}$$

This model aims to calculate the amount of stage2 congestion (congestion of intermediate products) and BB's initial inputs. Note that $y_r^{\min} = \min_{j\in\Omega}(y_{rj})$. Then, we have:

$$\begin{cases} \bar{\bar{\bar{x}}}_{io} = \bar{x}_{io} + \bar{\bar{s}}_{io}^{-^*} \\ \bar{\bar{\bar{z}}}_{do} = \bar{z}_o - s_{do}^{z*} \\ \tilde{y} = y_r^{\max} - \bar{\bar{s}}_{ro}^{+^*}. \end{cases} \tag{24}$$

Thus $\bar{\bar{s}}_{io}^{-^*} = s_i^c$ is the value of the congestion of the ith component of the initial inputs of the BB structure (stage1 has no congestion). Finally, s_{do}^{z*} is the congestion value of the dth components of intermediate products.

If $\bar{\bar{s}}_{io}^{-*} > 0$, increasing all input components reduces all intermediate product components (strong congestion), and if this increase is occurring in some of the $\bar{\bar{s}}_{io}^{-*}$ components such that some of the intermediate product components decrease, others remain unchanged, as weak congestion is occurring.

Definition 5.2 (Congestion in the two-stage network DEA)**.** Suppose $\mathrm{DMU}_o = (x_o, z_o, y_o)(o \in \{1, \ldots, n\})$ (and $\mathrm{DMU}_o = (x_o, y_o)$ in BB structure) is a DMU such that $(x_o, z_o) \in T_v^1$ or T_c^1 and $(z_o, y_o) \in T_{\text{convex}}^2$. So, DMU_o has congestion if there is exist $(\bar{x}_o, \bar{z}_o, \bar{y}_o)$

$$\begin{cases} \bar{x}_{io} = x_{io} + s_{io}^- \\ \bar{z}_{do} = z_{do} + s_{do}^z \\ \bar{y}_{ro} = y_r^{\max} - s_{ro}^+. \end{cases} \tag{25}$$

This definition also indicates the congestion of intermediate products (stage2). The stage1 inputs have no congestion, but BB's initial inputs have congestion given the intermediate product's influence.

6. An empirical study

In this section, we use data from the real estate sector of Chinese in 20 counties. The data of Chinese state industries that we used in our paper are extracted from China's industry/energy yearbook 2013. This industry has two-stage with a single input and a single intermediate product and single final output. In this example, if the residential and commercial zone's rental zone is too developed, it will reduce overall profits. Therefore, avoiding potential congestion in the real estate industry is particularly important. The first stage, stage 1, constitutes a critical phase in which resources and inputs play a pivotal role in shaping subsequent outcomes. Within this stage, the primary resources identified include (x) completed investment of the present fiscal year. This stage also witnesses the emergence of essential intermediate products, characterized by (Z) rental housing area. Ultimately, the culmination of Stage 1 yields a fundamental output parameter, denoted as (y) total profit. Table 2 listed, presented the data set. First, we use model (14) to determine the efficiency score of stage two subDMUs. Table 3 summarizes the result of the model (14). In this example, the stage1 has T_v^1 technology and the stage2 has T_{convex}^2 technology. Readers can make another example that chooses the stage1 as T_{convex}^1 technology and the stage2 as T_v^2 technology.

First we use model (5) to determine the candidate province. So, $w^* = 0$ (for $j = 1, 10$) hence, DMU1 and DMU10 are candidate province. So, candidate province set is $L = \{1, 10\}$. Second, use model (9) in Phase 1 for the provinces. The results obtained from model (9) and relation (10) have been stated in Tables 3 and 4. Also, model (9) is infeasible in evaluating Provinces $2, 8, 15, 17$, and 19. According to Table 4, we get all the DMUs that are not congested. In other words, DMUs $1, 3, 4, 5, 6, 7, 10, 11, 12, 13, 14, 16, 18$, and 20 have no congestion neither in network structure nor in BB structure. DMUs $2, 8, 9, 15, 17$, and 19 have congestion in stage1 or the BB at this phase. If DMU_o is congested, so stage1 is congested, but the opposite is not true. That is, if DMU is congested in stage1, it may not be congested in the BB. For DMUs $1, 3, 4, 5, 6, 7, 10, 11, 12, 13, 14, 16, 18$, and 20 we run model (11) to calculate the slack required by model (12).

Then, we apply the model (12) and relation (13) for provinces $2, 8, 9, 15, 17$, and 19 (congested DMUs). Tables 5 and 6 show the model (12) and relation (13) obtained results. According to Table 5 and model (12), it is clear that DMU8 and DMU9 have congestion in stage1. In other words, the stage1 of DMU8 has congestion $s^c = 10.6$, and the first stage of DMU9 has $s^c = 4.5$. This congestion is not present in their BB structure (DMU8 and DMU9 have no congestion in the BB structure).

Then, we apply the model (14) and relation (15) for Provinces $2, 15, 17$, and 19. Tables 7 show the model (14) and relation (15) obtained results. According to Tables 7, DMUs 2, 15, 17, and 19 have congestion in the first stage and the BB structure. In other words, the congestion measure of the DMU2 in the BB structure is $s^c = 11.1$ and congestion measure for the first stage is $s^{1c} = 18.7$. DMU15 has $s^c = 5$ and $s^{1c} = 12.6$. DMU17 has $s^c = 12.5$ and $s^{1c} = 20.01$. DMU19 has $s^c = 2.5$ and $s^{1c} = 10.01$. So, we have two types of congestion for

Table 2. The dataset for 20 DMUs real estate sector of the Chinese.

Province	x	y	z
DMU1	51	100.5	40.07
DMU2	78.6	41	5.6
DMU3	26.6	27.6	2.7
DMU4	21	13.5	3.8
DMU5	18.7	21.2	1.03
DMU6	34.8	50.2	2.5
DMU7	15	16.3	1.5
DMU8	67.5	60	40.07
DMU9	61.8	61.8	9.1
DMU10	56.9	100.5	40.07
DMU11	47.6	64.7	14.8
DMU12	33	89	9.5
DMU13	32.6	37.5	14.7
DMU14	30.5	32.3	17.25
DMU15	72.5	73.5	13.25
DMU16	36.1	84.5	4.6
DMU17	80	20	5.2
DMU18	45	36.1	12.03
DMU19	70	80	20
DMU20	50	40	30

Table 3. Results of step (I) corresponding to the Table 2 data.

Province	${s^-}^*$	$\tilde{s}^*$	${s^+}^*$
DMU1	5.9	0	0
DMU2	No feasible solution		
DMU3	30.3	72.9	37.37
DMU4	35.9	87	36.27
DMU5	38.2	79.3	39.04
DMU6	22.1	50.3	37.57
DMU7	41.9	84.2	38.57
DMU8	No feasible solution		
DMU9	No feasible solution		
DMU10	0	0	0
DMU11	9.3	35.8	25.27
DMU12	23.9	11.5	30.57
DMU13	24.3	63	25.37
DMU14	26.4	68.2	22.82
DMU15	No feasible solution		
DMU16	20.8	16	35.47
DMU17	No feasible solution		
DMU18	11.9	64.4	28.04
DMU19	No feasible solution		
DMU20	6.9	60.5	10.07

Table 4. Congestion and non-congestion area's Frontier point from relation (10).

$\bar{x}$	$\bar{z}$	$y^{\max}$
56.9	100.5	40.07

Table 5. Results of step (III) and model (12).

Province	$\bar{\bar{s}}^{-*}$	$\tilde{\bar{s}}^{*}$	$\bar{\bar{s}}^{+*}$
DMU2	No feasible		
DMU8	10.6	40.5	0
DMU9	4.9	38.7	30.97
DMU15	No feasible		
DMU17	No feasible		
DMU19	No feasible		

Table 6. The boundary point between the BB congestion and the first stage congestion and relation (13).

Province	$\bar{\bar{x}}^{*}$	$\bar{\bar{z}}^{*}$	$y^{\max}$
DMU8	72.5	73.5	40.07
DMU9	72.5	73.5	40.07

Table 7. The results of step (IV) and DMUs which have congestion in BB structure.

Province	BB mod congestion s^{c}_{BB}	First stage congestion s^{c}_{1}
DMU2	11.1	18.7
DMU15	5	12.6
DMU17	12.5	20.01
DMU19	2.5	10.01

initial inputs: inputs of the first stage and BB structure inputs. Moreover, except for specific conditions, these two types of congestion are different.

7. Conclusion

Given the established correlation between congestion and inefficiency within DEA units (DMUs), leading to diminished output, the imperative arises to discern and evaluate DMU congestion within the broader economic framework. Prior research has primarily focused on the identification and measurement of DMU congestion. However, in parallel, the internal composition of DMUs has been explored to pinpoint the sources of DMU inefficiency. As congestion embodies a state of inefficiency, a novel avenue is pursued to investigate the origins of DMU congestion by investigating both the internal DMU structure and the interrelationships among sub-DMUs, also referred to as intermediate products. This study undertakes an exhaustive analysis of DMU congestion, delineating its conditions in conjunction with intermediate products, and scrutinizes the interplay between bottleneck (BB) congestion and network congestion. Furthermore, the concept of network congestion is introduced, revealing its multifaceted nature in contrast to BB congestion, which entails a singular definition. Network congestion definitions are tailored to align with the technological attributes of subunits, thus elucidating varying perspectives in a two-stage network setup.

Acknowledging the multifarious roles of initial inputs, divided into stage-1 inputs and BB inputs, a comprehensive partition of input congestion emerges, segregating BB input congestion, contextual to its role in the BB input framework, from first input congestion, aligned with its role in stage-1 input processes. This dissection facilitates the explication of the interrelationship between these two congestion types, highlighting their distinctiveness while acknowledging their potential equality under specific conditions. This study advances by presenting models for the identification of DMUs and sub-DMUs congestion, accompanied by corresponding quantitative congestion assessments. The culminating endeavor involves a comparative analysis of BB and network structure congestion, thereby furnishing a comprehensive exploration of the diverse facets of DMU congestion within DEA frameworks.

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