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APPROXIMATE PROPER EFFICIENCIES IN NONSMOOTH SEMI-INFINITE MULTIOBJECTIVE OPTIMIZATION PROBLEMS

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Abstract. This article is devoted to studying a nonsmooth semi-infinite multiobjective optimization problem (SIMP) in terms of the Mordukhovich/limiting subdifferentials. We first establish necessary and sufficient conditions for an ε-quasi positively properly efficient solution of a problem (SIMP). We also investigate Mond–Weir type dual problems under assumptions of ε-quasi pseudo-generalized convexity. Next, we provide an application to a nonsmooth fractional semi-infinite multiobjective optimization problem. Finally, some examples are given to illustrate the obtained results. The obtained results improve or include some recent known ones.

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1. Introduction

In this paper, we consider the following semi-infinite multiobjective optimization problem:

$$\text{(SIMP)}\quad \begin{aligned} &\min f(x) := (f_1(x), \cdots, f_m(x)),\\ &\text{s.t. } x \in C := \{x \in \Omega \mid g_t(x) \leq 0, \forall t \in T\}, \end{aligned}$$

where T is a nonempty infinite index set, Ω is a nonempty locally closed (not necessarily convex) subset of $\mathbb{R}^n$, $f := (f_1, \cdots, f_m), f_k : \mathbb{R}^n \to \mathbb{R}, k = 1, \cdots, m$ and $g_t : \mathbb{R}^n \to \mathbb{R}, t \in T$ are locally Lipschitz with respect to x uniformly in t. Let $g_T := (g_t)_{t\in T}$.

The semi-infinite optimization problems could be applied in various fields such as in engineering design, mathematical physics, robotics, optimal control, transportation problems, fuzzy sets, cooperative games, etc. A semi-infinite multiobjective optimization problem (SIMP) is the simultaneous minimization with a finite number of objective functions and an infinite number of inequality constraints. Among many other interesting research, optimality conditions and duality for a problem (SIMP) have been considered numerously by many researchers. We refer the readers to the papers [1, 3, 5, 6, 12, 17, 18, 40–42], and the references therein.

On the other hand, since sometimes the exact solutions do not exist while the approximate ones do, even in the convex case, see [24, 25] and other references therein. Therefore, the study of approximate solutions is very important in optimization because, from the computational point of view, numerical algorithms usually

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generate only approximate solutions if we stop them after a finite number of step. The results on the optimality conditions/duality theorems for the approximate solutions of a multiobjective optimization problem were studied in [4]. In [7], authors investigated the optimality conditions/duality theorems/saddle point theorems for the approximate efficient solutions of a nonsmooth robust multiobjective optimization problem. By using the Clarke subdifferential, the optimality conditions/duality theorems for the approximate solutions of a nonsmooth semi-infinite programming problem were given in [23,37]. In [21,36,38], authors investigated approximate optimality conditions/approximate duality theorems/approximate saddle point theorems for a problem (SIMP). Optimality conditions/duality theorems for a approximate solution of the nonsmooth semi-infinite programming problem with data uncertainty were given in [20,39]. By using the Mordukhovich/limiting subdifferential, authors [13] established optimality conditions/duality theorems for an ε-quasi solution of the semi-infinite programming problem. In [33], authors obtained necessary conditions for approximate solutions of a fractional semi-infinite multiobjective optimization problem. The optimality conditions/duality theorems for the approximate Pareto efficient solutions of the nonsmooth semi-infinite interval-valued multiobjective optimization problems were studied in [11]. The optimality conditions/duality theorems for ε-quasi Pareto efficient solutions of the nonsmooth semi-infinite multiobjective optimization problems were given in [28].

Proper efficiencies are concepts used widely in multiobjective optimization problems to eliminate anomalous efficient points. In order to find relationships between proper efficiencies, we introduce the reader to [10]. Recently, optimality conditions/duality theorems for the positively properly efficient solutions of the multiobjective optimization problems have been investigated in [2]. Besides, there were many studies on isolated efficient solutions and properly efficient solutions for multiobjective optimization problems, see *e.g.*, [8,9,14,15,22,29,31] and the references therein. Moreover, Chuong and Yao have obtained the new results for the positively properly efficient solutions of a semi-infinite multiobjective optimization problem in [6]. Optimality conditions and duality for properly efficient solutions of fractional semi-infinite multiobjective optimization problems were investigated in [34,35]. In addition, optimality conditions and duality for isolated efficient solutions/properly efficient solutions of semi-infinite multiobjective optimization problems have been studied in [16,19,30,32]. Very recently, optimality conditions/duality theorems for properly efficient solutions of the semi-infinite multiobjective optimization problems with uncertainty data were obtained in [27]. To the best of our knowledge, so far there have been no papers studying the optimality conditions and dualilty for the ε-quasi positively properly efficient solutions of a semi-infinite multiobjective optimization problem.

Besides, new applications of the advanced tools of nonsmooth analysis to the optimization theory were introduced in [26]. The Mordukhovich/limiting subdifferential is a useful tool in nonsmooth analysis and closely related to optimality conditions of locally Lipschitzian functions in optimization theory. Note that, the Mordukhovich/limiting subdifferential and the Mordukhovich/limiting normal cone might be nonconvex. Therefore, the Mordukhovich/limiting subdifferential seems to be useful for deriving optimality conditions and dualilty for SIMP. Recently, optimality conditions and dualilty for the positively properly efficient solutions of a SIMP have been established in [6]. However, the optimality conditions and dualilty for the ε-quasi positively properly efficient solutions have not been yet considered in [6].

Inspired by the above observations, we provide some new results for the approximate optimality conditions for a positively properly efficient solution of SIMP in terms of the Mordukhovich/limiting subdifferential. The rest of the paper is organized as follows. Sections 1 and 2 present introduction notations and preliminaries. In Section 3, we investigate necessary and sufficient conditions for a positively properly approximate efficient solution of SIMP. In Section 4, we study Mond–Weir type dual problems with respect to the problem (SIMP). In Section 5, we provide an application to a nonsmooth fractional semi-infinite multiobjective optimization problem. Finally, conclusions are given in Section 6.

2. Preliminaries

Throughout the paper we use the standard notation of variational analysis in [26]. Unless otherwise specified, all spaces under consideration are assumed to be Euclidean spaces $\mathbb{R}^n$ with $n \in \mathbb{N} := \{1, 2, \cdots\}$. The inner product and the norm in $\mathbb{R}^n$ are denoted by is denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. The closed unit ball in the dual space $\mathbb{R}^n$ is denoted by $\mathbb{B}_{\mathbb{R}^n}$. The topological closure and the topological interior of a set $D \subset \mathbb{R}^n$ are denoted by cl$D$ and int$D$. As usual, the polar cone of D is the set

$$D^\circ := \{x^* \in \mathbb{R}^n \mid \langle x^*, x \rangle \leq 0, \forall x \in D\}. \tag{1}$$

Besides, the nonnegative (resp., nonpositive) orthant cone of Euclidean space $\mathbb{R}^n$ is denoted by $\mathbb{R}^n_+ = \{(x_1, \cdots, x_n) \mid x_i \geq 0, i = 1, \cdots, n\}$ (resp., $\mathbb{R}^n_-$) for $n \in \mathbb{N} := \{1, 2, \cdots\}$, while int$\mathbb{R}^n_+$ is used to indicate the topological interior of $\mathbb{R}^n_+$.

Given a set-valued map $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, we denote by

$$\operatorname*{Lim\,sup}_{x \to \bar{x}} G(x) := \{y \in \mathbb{R}^n \mid \exists \text{ sequence } x_k \to \bar{x} \text{ and } y_k \to y \text{ with } y_k \in G(x_k), \forall k \in \mathbb{N}\}$$

the sequential Painlevé–Kuratowski upper/outer limit of G as $x \to \bar{x}$.

A set $S \subset \mathbb{R}^n$ is convex if $\lambda a + (1 - \lambda) b \in S$ for all $a, b \in S$ and $0 \leq \lambda \leq 1$.

A set $S \subset \mathbb{R}^n$ is called closed around $\bar{x} \in S$ if there is a neighborhood U of $\bar{x}$ such that $S \cap \text{cl}U$ is closed. We say that S is locally closed if S is closed around x for every $x \in S$.

Let $S \subset \mathbb{R}^n$ be closed around $\bar{x} \in S$. The Fréchet/regular normal cone to S at $\bar{x} \in S$ is defined by

$$\widehat{N}(\bar{x}; S) := \left\{ x^* \in \mathbb{R}^n \mid \limsup_{x \xrightarrow{S} \bar{x}} \frac{\langle x^*, x - \bar{x} \rangle}{\| x - \bar{x} \|} \leq 0 \right\},$$

where $x \xrightarrow{S} \bar{x}$ means that $x \to \bar{x}$ with $x \in S$. If $\bar{x} \notin S$, we put $\widehat{N}(\bar{x}; S) := \emptyset$.

The Mordukhovich/limiting normal cone $N(\bar{x}; S)$ to S at $\bar{x} \in S \subset \mathbb{R}^n$ is obtained from Fréchet/regular normal cones by taking the sequential Painlevé–Kuratowski upper limit as

$$N(\bar{x}; S) := \operatorname*{Lim\,sup}_{x \xrightarrow{S} \bar{x}} \widehat{N}(x; S).$$

If $\bar{x} \notin S$, we put $N(\bar{x}; S) := \emptyset$. Specially, if S is convex, then

$$N(\bar{x}; S) = \{x^* \in \mathbb{R}^n \mid \langle x^*, x - \bar{x} \rangle \leq 0, \forall x \in S\}.$$

Let $f : \mathbb{R}^n \to \bar{\mathbb{R}} := [-\infty, +\infty]$ be an extended real-valued function. The domain, graph and epigraph of f are given by

$$\begin{aligned} \text{dom}f &:= \{x \in \mathbb{R}^n \mid |f(x)| < +\infty\}, \\ \text{gph}f &:= \{(x, \mu) \in \mathbb{R}^n \times \mathbb{R} \mid \mu = f(x)\}, \end{aligned}$$

and

$$\text{epi}f := \{(x, \mu) \in \mathbb{R}^n \times \mathbb{R} \mid \mu \geq f(x)\}.$$

Let $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ be defined on a convex set $S \subset \mathbb{R}^n$. Then f is convex on S if

$$f(\lambda x + (1 - \lambda) y) \leq \lambda f(x) + (1 - \lambda) f(y), \quad \text{for all } x, y \in S \text{ and } 0 < \lambda < 1.$$

Let $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ be finite at $\bar{x} \in \text{dom}f$, then the Mordukhovich/limiting subdifferential of f at $\bar{x}$ is defined by

$$\partial f(\overline{x}) := \{x^* \in \mathbb{R}^n \mid (x^*, -1) \in N((\bar{x}, f(\bar{x})); \text{epi}f)\}.$$

If $|f(\bar{x})| = +\infty$, then one puts $\partial f(\bar{x}) := \emptyset$.

Let $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ be a convex function with $\bar{x} \in \text{dom} f$. The subdifferential of convex function f at $\bar{x}$ is defined by

$$\partial f(\overline{x}) := \{x^* \in \mathbb{R}^n \mid (x^*, -1) \in N((\bar{x}, f(\bar{x})); \text{epi} f)\}.$$

Let $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ be finite at $\bar{x} \in \text{dom} f$, then f is lower semi-continuous at $\bar{x}$ if $\lim \inf_{x \to \bar{x}} f(x) \geq f(\bar{x})$.

Given $S \subset \mathbb{R}^n$ and consider the indicator function $\delta(\cdot, S)$ defined by

$$\delta(x; S) := \begin{cases} 0, & \text{if } x \in S, \\ +\infty, & \text{otherwise.} \end{cases}$$

Furthermore, we have a relation between the Mordukhovich/limiting normal cone and the Mordukhovich/limiting subdifferential of the indicator function as follows

$$N(\bar{x}; S) = \partial\delta(\bar{x}; S), \qquad \forall \bar{x} \in S.$$

Let $f : \mathbb{R}^n \to \mathbb{R}$. We say that f is locally Lipschitz at $x \in \mathbb{R}^n$, if there exist a positive constant $L > 0$ and a neighborhood U of x, such that

$$|f(x_1) - f(x_2)| \leq L\|x_1 - x_2\|, \qquad \forall x_1, x_2 \in U.$$

Moreover, for a sequence $\{f_t\}_{t \in T}$, where T is a nonempty infinite index set, we say that $\{f_t\}_{t \in T}$ is locally Lipschitz with respect to x uniformly in t if there exist a positive constant $L > 0$ and a neighborhood U of x such that

$$|f_t(x_1) - f_t(x_2)| \leq L\|x_1 - x_2\|, \qquad \forall x_1, x_2 \in U \text{ and } t \in T.$$

For a function f is locally Lipschitz at $\bar{x}$ with $L > 0$, it implies that (see [26], Cor. 1.81)

$$\|x^*\| \leq L, \qquad \forall x^* \in \partial f(\bar{x}).$$

Lemma 2.1 (See [26], Prop. 1.114)**.** *Let $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ be finite at $\bar{x} \in \mathbb{R}^n$. If $\bar{x}$ is a local minimizer of f then*

$$0 \in \partial f(\bar{x}).$$

Lemma 2.2 (See [26], Thm. 3.36)**.** *Suppose that $f_k : \mathbb{R}^n \to \bar{\mathbb{R}}, k = 1, \cdots, m$ (with $m \geq 2$) are lower semi-continuous around $\bar{x} \in \mathbb{R}^n$, and let all but one of these functions be Lipschitz continuous around $\bar{x}$. Then one has*

$$\partial(f_1 + \cdots + f_m)(\bar{x}) \subset \partial f_1(\bar{x}) + \cdots + \partial f_m(\bar{x}).$$

Lemma 2.3 (See [26], Cor. 1.111 (ii))**.** *Let $f_1, f_2 : \mathbb{R}^n \to \bar{\mathbb{R}}$ be locally Lipschitz functions around $\bar{x} \in \mathbb{R}^n$. Suppose that $f_2(\bar{x}) \neq 0$. Then one has*

$$\partial\left(\frac{f_1}{f_2}\right)(\bar{x}) \subset \frac{\partial(f_2(\bar{x})f_1)(\bar{x}) + \partial(-f_1(\bar{x})f_2)(\bar{x})}{[f_2(\bar{x})]^2}.$$

Let T be a nonempty infinite index set. Let $\mathbb{R}^{(T)}$ be the linear space given below

$$\mathbb{R}^{(T)} := \{\lambda = (\lambda_t)_{t \in T} \mid \lambda_t = 0 \text{ for all } t \in T \text{ but only finitely many } \lambda_t \neq 0\}.$$

Let $\mathbb{R}_+^{(T)}$ be the positive cone in $\mathbb{R}^{(T)}$ defined by

$$\mathbb{R}_+^{(T)} := \left\{\lambda = (\lambda_t)_{t \in T} \in \mathbb{R}^{(T)} \mid \lambda_t \geq 0 \text{ for all } t \in T\right\}.$$

With $\lambda \in \mathbb{R}^{(T)}$, its supporting set, $T(\lambda) := \{t \in T \mid \lambda_t \neq 0\}$, is a finite subset of T. Given $\{z_t\} \subset Z, t \in T, Z$ being a real linear space, we understand that

$$\sum_{t\in T} \lambda_t z_t = \begin{cases} \sum_{t\in T(\lambda)} \lambda_t z_t, & \text{if} \quad T(\lambda) \neq \emptyset, \\ 0, & \text{if} \quad T(\lambda) = \emptyset. \end{cases}$$

For $g_t, t \in T$,

$$\sum_{t\in T} \lambda_t g_t = \begin{cases} \sum_{t\in T(\lambda)} \lambda_t g_t, & \text{if} \quad T(\lambda) \neq \emptyset, \\ 0, & \text{if} \quad T(\lambda) = \emptyset. \end{cases}$$

3. Approximate optimality conditions

In this section, we establish optimality conditions for an ε-quasi positively properly efficient solution of the problem (SIMP).

Firstly, we propose the concept of an ε-quasi positively properly efficient solution for the problem (SIMP) as follows:

Definition 3.1. Let $\varepsilon := (\varepsilon_1, \cdots, \varepsilon_m) \in \mathbb{R}^m_+ \backslash \{0\}$. A point $\bar{x} \in C$ is said to be an ε-quasi positively properly efficient solution of the problem (SIMP) if there exists $\alpha := (\alpha_1, \cdots, \alpha_m) \in \text{int}\mathbb{R}^m_+$ such that

$$\langle \alpha, f(x) + \varepsilon \|x - \bar{x}\| \rangle \geq \langle \alpha, f(\bar{x}) \rangle, \qquad \forall x \in C.$$

We can rewrite Definition 3.1 as follows:

Definition 3.2. Let $\varepsilon := (\varepsilon_1, \cdots, \varepsilon_m) \in \mathbb{R}^m_+ \backslash \{0\}$. A point $\bar{x} \in C$ is said to be an ε-quasi positively properly efficient solution of the problem (SIMP) if there exists $\alpha := (\alpha_1, \cdots, \alpha_m) \in \text{int}\mathbb{R}^m_+$ such that

$$\alpha^T [f(x) + \varepsilon \|x - \bar{x}\|] \geq \alpha^T f(\bar{x}), \qquad \forall x \in C,$$

where the symbol α^T stands for vector transposition.

Next, we will introduce the definition of ε-properly efficient solutions, which is considered by Liu [22].

Definition 3.3 (See [22]). Let $\varepsilon := (\varepsilon_1, \cdots, \varepsilon_m) \in \mathbb{R}^m_+ \backslash \{0\}$. A point $\bar{x} \in C$ is said to be an ε-properly efficient solution of the problem (SIMP) if there exists $\alpha := (\alpha_1, \cdots, \alpha_m) \in \text{int}\mathbb{R}^m_+$ such that

$$\alpha^T [f(x) + \varepsilon] \geq \alpha^T f(\bar{x}), \qquad \forall x \in C,$$

where the symbol α^T stands for vector transposition.

Now, we provide an example to prove that ε-properly efficient solutions and ε-quasi positively properly efficient solutions might be different.

Example 3.4. Let $f : \mathbb{R} \to \mathbb{R}^2$ be defined by $f(x) = (f_1(x), f_2(x))$ with

$$f_1(x) = f_2(x) = x, x \in \mathbb{R}.$$

Take $T = [-2, -1]$ and let $g_t : \mathbb{R} \to \mathbb{R}$ be given by

$$g_t(x) = t\left(x + \frac{1}{2}\right), x \in \mathbb{R}, t \in T.$$

We consider the problem (SIMP) with $m = 2$ and $\Omega = (-\infty, 0] \subset \mathbb{R}$. By simple computation, one has $C = \left[-\frac{1}{2}, 0\right]$. Now, take $\bar{x} = 0 \in C, \varepsilon = (\varepsilon_1, \varepsilon_2) \in \mathbb{R}^2_+ \backslash \{0\}$ with $\varepsilon_1 = \varepsilon_2 = \frac{1}{2}$ and $\alpha = (\alpha_1, \alpha_2) \in \text{int}\mathbb{R}^2_+$ with

$\alpha_1 + \alpha_2 = 1$. Then, it is easy to show that $\bar{x}$ is an ε-properly efficient solution of the problem (SIMP). Indeed, we have

$$\sum_{k=1}^{2} \alpha_k f(x) + \sum_{k=1}^{2} \alpha_k \varepsilon_k = x + \frac{1}{2} \geq 0 = \sum_{k=1}^{2} \alpha_k f_k(\bar{x}), \qquad \forall x \in C.$$

However, $\bar{x} = 0$ is not an ε-quasi positively properly efficient solution of the problem (SIMP). Indeed, take $\hat{x} = -\frac{1}{3} \in C = \left[-\frac{1}{2}, 0\right], \varepsilon = (\varepsilon_1, \varepsilon_2) \in \mathbb{R}^2_+ \backslash \{0\}$ with $\varepsilon_1 = \varepsilon_2 = \frac{1}{2}$ and $\alpha = (\alpha_1, \alpha_2) \in \mathrm{int}\mathbb{R}^2_+$ with $\alpha_1 + \alpha_2 = 1$. Clearly,

$$\sum_{k=1}^{2} \alpha_k f(\hat{x}) + \sum_{k=1}^{2} \alpha_k \varepsilon_k \|\hat{x} - \bar{x}\| = \hat{x} + \frac{1}{2}|\hat{x}| = -\frac{1}{6} < 0 = \sum_{k=1}^{2} \alpha_k f_k(\bar{x}).$$

We recall a constraint qualification for the problem (SIMP) as follows:

Definition 3.5 (See [3])**.** Let $\bar{x} \in C$. We say that the following Mordukhovich/limiting constraint qualification (MCQ) is satisfied at $\bar{x}$ if

$$N(\bar{x}; C) \subseteq \bigcup_{\lambda \in A(\bar{x})} \left[\sum_{t \in T} \lambda_t \partial g_t(\bar{x})\right] + N(\bar{x}; \Omega),$$

where

$$A(\bar{x}) := \left\{\lambda \in \mathbb{R}^{(T)}_+ \mid \lambda_t g_t(\bar{x}) = 0, \forall t \in T\right\} \tag{2}$$

is set of active constraint multipliers at $\bar{x} \in \Omega$.

Now, we propose a necessary optimality condition for an ε-quasi positively properly efficient solution of the problem (SIMP) under the qualification condition (MCQ).

Theorem 3.6. *Let $\varepsilon := (\varepsilon_1, \cdots, \varepsilon_m) \in \mathbb{R}^m_+ \backslash \{0\}$ and let $\bar{x} \in C$ be an ε-quasi positively properly efficient solution of the problem* (SIMP)*. Suppose that the condition* (MCQ) *at $\bar{x}$ holds. Then, there exist $\alpha := (\alpha_1, \cdots, \alpha_m) \in \mathrm{int}\mathbb{R}^m_+$ and $\lambda \in A(\bar{x})$ defined in* (2) *such that*

$$0 \in \sum_{k=1}^{m} \alpha_k \partial f_k(\bar{x}) + \sum_{t \in T} \lambda_t \partial g_t(\bar{x}) + \sum_{k=1}^{m} \alpha_k \varepsilon_k \mathbb{B}_{\mathbb{R}^n} + N(\bar{x}; \Omega). \tag{3}$$

Proof. Let $\bar{x} \in C$ be an ε-quasi positively properly efficient solution of the problem (SIMP). Then there exists $\alpha := (\alpha_1, \cdots, \alpha_m) \in \mathrm{int}\mathbb{R}^m_+$ such that

$$\sum_{k=1}^{m} \alpha_k f_k(x) + \sum_{k=1}^{m} \alpha_k \varepsilon_k \|x - \bar{x}\| \geq \sum_{k=1}^{m} \alpha_k f(\bar{x}), \qquad \forall x \in C. \tag{4}$$

For any $x \in \mathbb{R}^n$, set

$$\psi(x) := \sum_{k=1}^{m} \alpha_k f_k(x) + \sum_{k=1}^{m} \alpha_k \varepsilon_k \|x - \bar{x}\|. \tag{5}$$

From (4) we deduce that $\bar{x}$ is a minimizer of the following scalar optimization problem

$$\min_{x \in C} \psi(x).$$

It follows easily that $\bar{x}$ is a minimizer of the following unconstrained optimization problem

$$\min_{x \in \mathbb{R}^n} \{\psi(x) + \delta(x; C)\}.$$

We deduce from Lemma 2.1 that

$$0 \in \partial(\psi + \delta(\cdot; C))(\bar{x}.) \tag{6}$$

Since function ψ is Lipschitz continuous around $\bar{x}$ and the function $\delta(\cdot; C)$ is lower semi-continuous around $\bar{x}$, we deduce from $\partial\delta(\bar{x}; C) = N(\bar{x}; C)$, (6) and Lemma 2.2 that

$$0 \in \partial\psi(\bar{x}) + N(\bar{x}; C). \tag{7}$$

Note further that we have $\partial(\| \cdot - \bar{x}\|)(\bar{x}) = \mathbb{B}_{\mathbb{R}^n}$ and function $\| \cdot - \bar{x}\|$ is Lipschitz continuous around $\bar{x}$. Because the qualification condition (MCQ) holds at $\bar{x} \in C$. So, one implies

$$N(\bar{x}; C) \subseteq \bigcup_{\lambda \in A(\bar{x})} \left[\sum_{t \in T} \lambda_t \partial g_t(\bar{x}) \right] + N(\bar{x}; \Omega), \tag{8}$$

where

$$A(\bar{x}) := \left\{ \lambda \in \mathbb{R}_+^{(T)} \mid \lambda_t g_t(\bar{x}) = 0, \forall v_t \in t \in T \right\}.$$

Now, applying Lemma 2.2 to the function ψ defined in (5), it yields from (7) and (8) that

$$0 \in \sum_{k=1}^{m} \alpha_k \partial f_k(\bar{x}) + \sum_{t \in T} \lambda_t \partial g_t(\bar{x}) + \sum_{k=1}^{m} \alpha_k \varepsilon_k \mathbb{B}_{\mathbb{R}^n} + N(\bar{x}; \Omega),$$

with $\alpha := (\alpha_1, \cdots, \alpha_m) \in \text{int}\mathbb{R}_+^m$ and $\lambda \in A(\bar{x})$ defined in (2). The proof is complete. □

The following simple example shows that the condition (MCQ) is essential in Theorem 3.6.

Example 3.7. Let $f : \mathbb{R} \to \mathbb{R}^2$ be defined by $f(x) = (f_1(x), f_2(x))$, where

$$f_1(x) = f_2(x) = x, x \in \mathbb{R}.$$

and let $g_t : \mathbb{R} \to \mathbb{R}$ be given by

$$g_t(x) = t^3 x^2, x \in \mathbb{R}, t \in T = [1, 2].$$

We consider the problem (SIMP) with $m = 2$ and $\Omega = (-\infty, 0] \subset \mathbb{R}$. By simple computation, one has $C = \{0\}$. Now, take $\bar{x} = 0 \in C, \varepsilon = (\varepsilon_1, \varepsilon_2) \in \mathbb{R}_+^2 \setminus \{0\}$ with $\varepsilon_1 = \varepsilon_2 = \frac{1}{2}$ and $\alpha = (\alpha_1, \alpha_2) \in \text{int}\mathbb{R}_+^2$ with $\alpha_1 + \alpha_2 = 1$. Then, it is easy to show that $\bar{x}$ is an ε-quasi positively properly efficient solution of the problem (SIMP) because $C = \{0\}$. Indeed, we have

$$\sum_{k=1}^{2} \alpha_k f(x) + \sum_{k=1}^{2} \alpha_k \varepsilon_k \|x - \bar{x}\| = x + \frac{1}{2}|x| \geq 0 = \sum_{k=1}^{2} \alpha_k f_k(\bar{x}), \qquad \forall x \in C.$$

On the other hand, we have $N(\bar{x}; \Omega) = N(\bar{x}; (-\infty, 0]) = [0, +\infty)$ and $\partial f_k(\bar{x}) = \{1\}, k = 1, 2, \partial g_t(\bar{x}) = \{0\}$, for any $t \in T$,

$$\bigcup_{\lambda \in A(\bar{x})} \left[\sum_{t \in T} \lambda_t \partial g_t(\bar{x}) \right] + N(\bar{x}; \Omega) = [0, +\infty).$$

Moreover, $N(\bar{x}; C) = N(\bar{x}; \{0\}) = \mathbb{R}$. Clearly, the qualification condition (MCQ) is not satisfied at $\bar{x} = 0$. On the other hand, take $\bar{x} = 0, \varepsilon = (\varepsilon_1, \varepsilon_2) \in \mathbb{R}_+^2 \setminus \{0\}$ with $\varepsilon_1 = \varepsilon_2 = \frac{1}{2}, \alpha = (\alpha_1, \alpha_2) \in \text{int}\mathbb{R}_+^2$ with $\alpha_1 + \alpha_2 = 1$ and $\mathbb{B}_{\mathbb{R}} = [-1, 1]$. It is easy to see that

$$0 \notin \left[\frac{1}{2}, +\infty\right) = \{1\} + \left[-\frac{1}{2}, \frac{1}{2}\right] + [0, +\infty) = \sum_{k=1}^{2} \alpha_k \partial f_k(\bar{x}) + \sum_{t \in T} \lambda_t \partial g_t(\bar{x}) + \sum_{k=1}^{2} \alpha_k \varepsilon_k \mathbb{B}_{\mathbb{R}} + N(\bar{x}; \Omega),$$

$\lambda_t g_t(\bar{x}) = 0, \forall t \in T$. This implies that (3) is not satisfied at $\bar{x} = 0$. Thus, the assertion of Theorem 3.6 is not valid because the condition (MCQ) does not hold.

Remark 3.8. Theorem 3.6 improves Theorem 3.3 in [6], Theorem 2 in [27].

Now, we introduce a concept of the approximate (KKT) condition for the problem (SIMP).

Definition 3.9. Let $\varepsilon := (\varepsilon_1, \cdots, \varepsilon_m) \in \mathbb{R}^m_+ \backslash \{0\}$. A point $\bar{x} \in C$ is said to satisfy the approximate (KKT) condition with respect to the problem (SIMP) if there exist $\alpha := (\alpha_1, \cdots, \alpha_m) \in \text{int}\mathbb{R}^m_+$ and $\lambda \in A(\bar{x})$ defined in (2) such that

$$0 \in \sum_{k=1}^{m} \alpha_k \partial f_k(\bar{x}) + \sum_{t \in T} \lambda_t \partial g_t(\bar{x}) + \sum_{k=1}^{m} \alpha_k \varepsilon_k \mathbb{B}_{\mathbb{R}^n} + N(\bar{x}; \Omega).$$

The following simple example proves that, in general, a feasible point may satisfy the qualification condition (MCQ), but if this point is not an ε-quasi positively properly efficient solution of the problem (SIMP), then the approximate (KKT) condition does not hold at this point.

Example 3.10. Let $f : \mathbb{R} \to \mathbb{R}^2$ be defined by $f(x) = (f_1(x), f_2(x))$, where

$$f_1(x) = f_2(x) = 2x, x \in \mathbb{R}$$

and let $g_t : \mathbb{R} \to \mathbb{R}$ be given by

$$g_t(x) = -t^3 x^2, x \in \mathbb{R}, t \in T = [1, 2].$$

We consider the problem (SIMP) with $m = 2$ and $\Omega = (-\infty, 0] \subset \mathbb{R}$. By simple computation, one has $C = (-\infty, 0]$. Besides, take $\bar{x} = 0 \in C$, it is easy to follow that $\partial f_k(\bar{x}) = \{2\}, k = 1, 2, \partial g_t(\bar{x}) = \{0\}, \forall t \in T$. Therefore, we have

$$\bigcup_{\lambda \in A(\bar{x})} \left[\sum_{t \in T} \lambda_t \partial g_t(\bar{x}) \right] + N(\bar{x}; \Omega) = [0, +\infty).$$

Moreover, we have $N(\bar{x}; \Omega) = N(\bar{x}; (-\infty, 0]) = [0, +\infty)$. Clearly, the qualification condition (MCQ) holds at $\bar{x}$. On the other hand, take $\bar{x} = 0, \varepsilon = (\varepsilon_1, \varepsilon_2) \in \mathbb{R}^2_+ \backslash \{0\}$, where $\varepsilon_1 = \varepsilon_2 = \frac{1}{2}, \alpha = (\alpha_1, \alpha_2) \in \text{int}\mathbb{R}^2_+$ with $\alpha_1 + \alpha_2 = 1$ and $\mathbb{B}_{\mathbb{R}} = [-1, 1]$. It is easy to see that

$$0 \notin \left[\frac{3}{2}, +\infty\right) = \{2\} + \left[-\frac{1}{2}, \frac{1}{2}\right] + [0, +\infty) = \sum_{k=1}^{2} \alpha_k \partial f_k(\bar{x}) + \sum_{t \in T} \lambda_t \partial g_t(\bar{x}) + \sum_{k=1}^{2} \alpha_k \varepsilon_k \mathbb{B}_{\mathbb{R}} + N(\bar{x}; \Omega),$$

$\lambda_t g_t(\bar{x}) = 0, \forall t \in T$. From this it follows that the approximate (KKT) condition does not hold at $\bar{x}$. The reason is that $\bar{x} = 0$ is not an ε-quasi positively properly efficient solution of the problem (SIMP) with $\varepsilon = (\varepsilon_1, \varepsilon_2) \in \mathbb{R}^2_+ \backslash \{0\}$, where $\varepsilon_1 = \varepsilon_2 = \frac{1}{2}$. Indeed, we can choose $x = -1 \in C = (-\infty, 0]$ and $\alpha = (\alpha_1, \alpha_2) \in \text{int}\mathbb{R}^2_+$ with $\alpha_1 + \alpha_2 = 1$. Clearly,

$$\sum_{k=1}^{2} \alpha_k f_k(x) + \sum_{k=1}^{2} \alpha_k \varepsilon_k \|x - \bar{x}\| = 2x + \frac{1}{2}|x| = -\frac{3}{2} < 0 = \sum_{k=1}^{2} \alpha_k f_k(\bar{x}).$$

Before we discuss sufficient condition for the ε-quasi positively properly efficient solution of the problem (SIMP), we introduce the concepts of convexity, which are inspired by Long *et al.* [23].

Definition 3.11. The locally Lipschitz functions $g_t : \mathbb{R}^n \to \mathbb{R}, t \in T$ are said to be quasi convex on Ω at $\bar{x} \in \Omega$ if for all $x \in \Omega$,

$$g_t(x) \leq g_t(\bar{x}) \Rightarrow \langle x_t^*, x - \bar{x} \rangle \leq 0, \qquad \forall x_t^* \in \partial g_t(\bar{x}), \quad t \in T.$$

Definition 3.12. Let $\varepsilon := (\varepsilon_1, \cdots, \varepsilon_m) \in \mathbb{R}^m_+ \backslash \{0\}$. We say that f is ε-quasi pseudo-convex on Ω at $\bar{x} \in \Omega$ if for all $x \in \Omega$, there exist $x_k^* \in \partial f_k(\bar{x}), k = 1, \cdots, m$ such that

$$\langle x_k^*, x - \bar{x} \rangle + \varepsilon_k \|x - \bar{x}\| \geq 0 \Rightarrow f_k(x) + \varepsilon_k \|x - \bar{x}\| \geq f_k(\bar{x}), \qquad k = 1, \cdots, m.$$

Remark 3.13. Suppose that Ω is convex set. If $f_k, k=1,\cdots,m$ are convex functions on Ω then f is ε-quasi pseudo-convex on Ω at $\bar{x}\in\Omega$.

The next example shows that the class of ε-quasi pseudo-convex functions is properly larger than the one of convex functions.

Example 3.14. Let $f:\mathbb{R}\to\mathbb{R}^2$ be defined by $f(x)=(f_1(x),f_2(x))$, where

$$f_1(x)=f_2(x)=-|x|, x\in\mathbb{R}.$$

Let $\varepsilon=(\varepsilon_1,\varepsilon_2)\in\mathbb{R}^2_+\backslash\{0\}$ with $\varepsilon_1=\varepsilon_2=1$ and $\Omega=(-\infty,0]\subset\mathbb{R}$. By simple computation, one has

$$\partial f_1(\bar{x})=\partial f_2(\bar{x})=\{-1,1\}.$$

It is easy to see that f is ε-quasi pseudo-convex on Ω at $\bar{x}=0$. Indeed, for any $x\in\Omega$, take $x_k^*\in\partial f_k(\bar{x})=\{-1,1\}, k=1,2$, one has

$$\begin{aligned}&\langle x_k^*, x-\bar{x}\rangle+\varepsilon_k\|x-\bar{x}\|=\pm x+|x|\geq 0, && k=1,2\\ &\Rightarrow f_k(x)+\varepsilon_k\|x-\bar{x}\|=-|x|+|x|\geq 0=f_k(\bar{x}), && k=1,2.\end{aligned}$$

However, f is not a convex function.

Now, we will give a sufficient condition for an ε-quasi positively properly efficient solution of the problem (SIMP).

Theorem 3.15. *Let $\varepsilon:=(\varepsilon_1,\cdots,\varepsilon_m)\in\mathbb{R}^m_+\backslash\{0\}$ and let Ω be a convex set. Assume that $\bar{x}\in C$ satisfies the approximate* (KKT) *condition. If f is ε-quasi pseudo-convex on Ω at $\bar{x}$ and $g_t, t\in T$ are quasi-convex on Ω at $\bar{x}$, then $\bar{x}\in C$ is an ε-quasi positively properly efficient solution of the problem* (SIMP).

Proof. Since $\bar{x}\in C$ satisfies the approximate (KKT) condition, there exist $\alpha:=(\alpha_1,\cdots,\alpha_m)\in \text{int}\mathbb{R}^m_+, \lambda\in\mathbb{R}^{(T)}_+$ and $x_k^*\in\partial f_k(\bar{x}), k=1,\cdots,m, x_t^*\in\partial g_t(\bar{x}), t\in T$, as well as $b^*\in\mathbb{B}_{\mathbb{R}^n}, w^*\in N(\bar{x};\Omega)$ such that

$$\sum_{k=1}^m\alpha_k x_k^*+\sum_{t\in T}\lambda_t x_t^*+\sum_{k=1}^m\alpha_k\varepsilon_k b^*+w^*=0 \tag{9}$$

and

$$\lambda_t g_t(\bar{x})=0, \forall t\in T.$$

It follows from (9) that (for such $x\in C$)

$$\left\langle\sum_{k=1}^m\alpha_k x_k^*, x-\bar{x}\right\rangle+\left\langle\sum_{t\in T}\lambda_t x_t^*, x-\bar{x}\right\rangle+\left\langle\sum_{k=1}^m\alpha_k\varepsilon_k b^*, x-\bar{x}\right\rangle+\langle w^*, x-\bar{x}\rangle=0. \tag{10}$$

Since Ω is a convex set and $w^*\in N(\bar{x};\Omega)$, it follows that, for any $x\in\Omega$,

$$\langle w^*, x-\bar{x}\rangle\leq 0.$$

Now, take arbitrarily $x\in C$. Then there exists $b^*\in\mathbb{R}_{\mathbb{R}^n}$ such that

$$\|x-\bar{x}\|=\langle b^*, x-\bar{x}\rangle.$$

From (10) it follows that

$$\left\langle\sum_{k=1}^m\alpha_k x_k^*, x-\bar{x}\right\rangle+\left\langle\sum_{t\in T}\lambda_t x_t^*, x-\bar{x}\right\rangle+\sum_{k=1}^m\alpha_k\varepsilon_k\|x-\bar{x}\|\geq 0,$$

which means that

$$\left\langle \sum_{k=1}^{m} \alpha_k x_k^*, x - \bar{x} \right\rangle + \sum_{k=1}^{m} \alpha_k \varepsilon_k \|x - \bar{x}\| \geq -\left\langle \sum_{t \in T} \lambda_t x_t^*, x - \bar{x} \right\rangle. \tag{11}$$

Moreover, for any $x \in C$, then $\lambda_t g_t(x) \leq 0$ for any $t \in T$. It follows from $\lambda_t g_t(\bar{x}) = 0, \forall t \in T$ that

$$\lambda_t g_t(x) \leq 0 = \lambda_t g_t(\bar{x}), \qquad \forall t \in T.$$

By g_t is quasi-convex on Ω at $\bar{x}$ and $x_t^* \in \partial g_t(\bar{x}), \forall t \in T$, we obtain

$$\langle \lambda_t x_t^*, x - \bar{x} \rangle \leq 0, \qquad \forall t \in T.$$

Therefore, it is easy to yield that

$$\left\langle \sum_{t \in T} \lambda_t x_t^*, x - \bar{x} \right\rangle \leq 0. \tag{12}$$

Combining (11) and (12), we can assert that

$$\left\langle \sum_{k=1}^{m} \alpha_k x_k^*, x - \bar{x} \right\rangle + \sum_{k=1}^{m} \alpha_k \varepsilon_k \|x - \bar{x}\| \geq 0.$$

Since f is ε-quasi pseudo-convex on Ω at $\bar{x}$, it follows that

$$\sum_{k=1}^{m} \alpha_k f_k(x) + \sum_{k=1}^{m} \alpha_k \varepsilon_k \|x - \bar{x}\| \geq \sum_{k=1}^{m} \alpha_k f_k(\bar{x}).$$

Therefore, $\bar{x}$ is an ε-quasi positively properly efficient solution of the problem (SIMP). □

Now, we present an example to show the importance of the ε-quasi pseudo-convexity in Theorem 3.15.

Example 3.16. Let $f : \mathbb{R} \to \mathbb{R}^2$ be defined by $f(x) = (f_1(x), f_2(x))$ with

$$f_1(x) = f_2(x) = x^5, x \in \mathbb{R}.$$

Let $g_t : \mathbb{R} \to \mathbb{R}$ be given by

$$g_t(x) = t^3 x^2, x \in \mathbb{R}, t \in T = [-2, -1].$$

We consider the problem (SIMP) with $m = 2$ and $\Omega = (-\infty, 0] \subset \mathbb{R}$. By selecting $\bar{x} = 0 \in \Omega$ and by simple computation, one has $C = (-\infty, 0]$, $N(\bar{x}; \Omega) = N(\bar{x}; (-\infty, 0]) = [0, +\infty)$,

$$\partial f_k(\bar{x}) = \{0\}, k = 1, 2 \text{ and } \partial g_t(\bar{x}) = \{0\}, t \in T.$$

It is easy to follow that $g_t, t \in T$ is quasi-convex on Ω at $\bar{x}$. Indeed, one has

$$\begin{aligned} g_t(x) = t^3 x^2 \leq 0 = g_t(\bar{x}), &\qquad t \in [-2, -1] \\ \Rightarrow \langle x_t^*, x - \bar{x} \rangle = 0 \leq 0, &\qquad \forall x_t^* \in \partial g_t(\bar{x}) = \{0\}, \quad \forall x \in \Omega, t \in T. \end{aligned}$$

Now, take $\varepsilon = (\varepsilon_1, \varepsilon_2) \in \mathbb{R}^2_+ \backslash \{0\}$ with $\varepsilon_1 = \varepsilon_2 = \frac{1}{2}$. Then, it implies that $\bar{x} = 0 \in C$ satisfies the approximate (KKT) condition. Indeed, let us select $\bar{x} = 0, \alpha = (\alpha_1, \alpha_2) \in \text{int}\mathbb{R}^2_+$ with $\alpha_1 + \alpha_2 = 1$ and $\mathbb{B}_{\mathbb{R}} = [-1, 1]$. Then,

$$0 \in \left[-\frac{1}{2}, +\infty\right) = \left[-\frac{1}{2}, \frac{1}{2}\right] + [0, +\infty) = \sum_{k=1}^{2} \alpha_k \partial f_k(\bar{x}) + \sum_{t \in T} \lambda_t \partial g_t(\bar{x}) + \sum_{k=1}^{2} \alpha_k \varepsilon_k \mathbb{B}_{\mathbb{R}} + N(\bar{x}; \Omega),$$

$\lambda_t g_t(\bar{x}) = 0, \forall t \in T$. However, $\bar{x}$ is not an ε-quasi positively properly efficient of the problem (SIMP). In order to see this, let us take $\hat{x} = -1 \in C$ and $\varepsilon = (\varepsilon_1, \varepsilon_2) \in \mathbb{R}^2_+ \backslash \{0\}, \varepsilon_1 = \varepsilon_2 = \frac{1}{2}$ and $\alpha = (\alpha_1, \alpha_2) \in \text{int}\mathbb{R}^2_+$ with $\alpha_1 + \alpha_2 = 1$. Then,

$$\sum_{k=1}^{2} \alpha_k f_k(\hat{x}) + \sum_{k=1}^{2} \alpha_k \varepsilon_k \|\hat{x} - \bar{x}\| = -1 + \frac{1}{2} = -\frac{1}{2} < 0 = \sum_{k=1}^{2} \alpha_k f_k(\bar{x}).$$

Reason is that f is not ε-quasi pseudo-convex on Ω at $\bar{x} = 0$. Indeed, take $x = -2 \in \Omega, x_k^* \in \partial f_k(\bar{x}) = \{0\}, k = 1, 2$ and $\varepsilon = (\varepsilon_1, \varepsilon_2) \in \mathbb{R}^2_+ \backslash \{0\}, \varepsilon_1 = \varepsilon_2 = \frac{1}{2}$. Clearly,

$$f_k(x) + \varepsilon_k \|x - \bar{x}\| = x^5 + \frac{1}{2}|x| = -32 + 1 = -31 < 0 = f_k(\bar{x}), \qquad k = 1, 2.$$

However,

$$\langle x_k^*, x - \bar{x} \rangle + \varepsilon_k \|x - \bar{x}\| = 1 > 0, \qquad k = 1, 2.$$

Remark 3.17. Theorem 3.15 improves Theorem 3 in [27].

Motivated by the definition of generalized convexity due to [4], we will introduce a concept of generalized convexity as follows:

Definition 3.18. We say that (f, g_T) is generalized convex on Ω at $\bar{x} \in \Omega$, if for any $x \in \Omega$, $x_k^* \in \partial f_k(\bar{x}), k = 1, \cdots, m$ and $x_t^* \in \partial g_t(\bar{x}), t \in T$, there exists $w \in N(\bar{x}; \Omega)^\circ$ such that

$$\begin{aligned} f_k(x) - f_k(\bar{x}) &\geq \langle x_k^*, w \rangle, \qquad k = 1, \cdots, m, \\ g_t(x) - g_t(\bar{x}) &\geq \langle x_t^*, w \rangle, \qquad \forall t \in T \end{aligned}$$

and

$$\langle b^*, w \rangle \leq \|x - \bar{x}\|, \qquad \forall b^* \in \mathbb{B}_{\mathbb{R}^n}.$$

Remark 3.19. Note that, if Ω is convex and $f_k(\cdot), k = 1, \cdots, m, g_t(\cdot), t \in T$ are convex, then (f, g_T) is generalized convex on Ω at any $\bar{x} \in \Omega$ with $w := x - \bar{x}$ for each $x \in \Omega$.

The next example shows that the class of generalized convex functions is properly larger than the one of convex functions.

Example 3.20. Let $f : \mathbb{R} \to \mathbb{R}^2$ be defined by $f(x) = (f_1(x), f_2(x))$, where

$$f_1(x) = f_2(x) = x^6 + x^4 + 1, x \in \mathbb{R}.$$

Let $g_t : \mathbb{R} \to \mathbb{R}$ be given by

$$g_t(x) = -t|x|, \qquad \forall t \in T = [1, 2].$$

Consider $\mathbb{B}_{\mathbb{R}} = [-1, 1], \Omega = \mathbb{R}, \bar{x} = 0 \in \Omega$, one has

$$\begin{aligned} &N(\bar{x}; \Omega) = N(\bar{x}; \mathbb{R}) = \{0\}, N(\bar{x}; \Omega)^\circ = N(\bar{x}; \mathbb{R})^\circ = \mathbb{R}, \\ &\partial f_k(\bar{x}) = \{0\}, k = 1, 2, \partial g_t(\bar{x}) = \{-t, t\}, \qquad \forall t \in T. \end{aligned}$$

It is easy to see that (f, g_T) is generalized convex on Ω at $\bar{x}$. Indeed, for any $x \in \Omega, x_k^* \in \partial f_k(\bar{x}) = \{0\}, k = 1, 2$ and $x_t^* \in \partial g_t(\bar{x}) = \{-t, t\}, t \in T$, taking $w = \frac{g_t(x) - g_t(\bar{x})}{x_t^*}$, one has $w \in N(\bar{x}; \Omega)^\circ = \mathbb{R}$,

$$\begin{aligned} f_k(x) - f_k(\bar{x}) &= x^6 + x^4 \geq 0 = x_k^* \cdot w = \langle x_k^*, w \rangle, \qquad k = 1, 2, \\ g_t(x) - g_t(\bar{x}) &= x_t^* \cdot w \geq x_t^* \cdot w = \langle x_t^*, w \rangle, \qquad \forall t \in T \end{aligned}$$

and

$$\langle b^*, w \rangle = b^* \cdot w \leq |x - \bar{x}|, \qquad \forall b^* \in [-1, 1].$$

However, g is not a convex function.

In the line of [7], we will introduce a concept of ε-quasi pseudo-generalized convexity as follows:

Definition 3.21. Let $\varepsilon := (\varepsilon_1, \cdots, \varepsilon_m) \in \mathbb{R}^m_+ \backslash \{0\}$. We say that (f, g_T) is ε-quasi pseudo-generalized convex on Ω at $\bar{x} \in \Omega$, if for any $x \in \Omega$, $x_k^* \in \partial f_k(\bar{x}), k = 1, \cdots, m$ and $x_t^* \in \partial g_t(\bar{x}), t \in T$, there exists $w \in N(\bar{x}; \Omega)^\circ$ such that

$$\begin{aligned} \langle x_k^*, w\rangle + \varepsilon_k \|x - \bar{x}\| \geq 0 \Rightarrow f_k(x) + \varepsilon_k \|x - \bar{x}\| \geq f_k(\bar{x}), &\qquad k = 1, \cdots, m, \\ g_t(x) \leq g_t(\bar{x}) \Rightarrow \langle x_t^*, w\rangle \leq 0, &\qquad \forall t \in T \end{aligned}$$

and

$$\langle b^*, w\rangle \leq \|x - \bar{x}\|, \qquad \forall b^* \in \mathbb{B}_{\mathbb{R}^n}.$$

Remark 3.22. If (f, g_T) is generalized convex on Ω at $\bar{x} \in \Omega$, then (f, g_T) is ε-quasi pseudo-generalized convex on Ω at $\bar{x} \in \Omega$.

The next example shows that the class of ε-quasi pseudo-generalized convex functions is properly larger than the one of generalized convex functions.

Example 3.23. Let $f : \mathbb{R} \to \mathbb{R}^2$ be defined by $f(x) = (f_1(x), f_2(x))$, where

$$f_1(x) = f(x_2) = \begin{cases} \frac{x}{2}, & \text{if } x \geq 0, \\ \frac{2x}{3}, & \text{if } x < 0. \end{cases}$$

Take $T = [1, 2]$. Let $g_t : \mathbb{R} \to \mathbb{R}$ be given by

$$g_t(x) = -tx, x \in \mathbb{R}, t \in T = [1, 2].$$

Consider $\Omega = \mathbb{R}, \bar{x} = 0 \in \Omega$ and $\mathbb{B}_{\mathbb{R}} = [-1, 1]$, one has

$$\begin{aligned} &N(\bar{x}; \Omega) = N(\bar{x}; \mathbb{R}) = \{0\}, N(\bar{x}; \Omega)^\circ = N(\bar{x}; \mathbb{R})^\circ = \mathbb{R}, \\ &\partial f_k(\bar{x}) = \left\{\frac{1}{2}, \frac{2}{3}\right\}, \quad k = 1, 2, \quad \partial g_t(\bar{x}) = \{-t\}, \quad \forall t \in T. \end{aligned}$$

It easy to see that (f, g_T) is ε-quasi pseudo-generalized convex on Ω at $\bar{x}$. Indeed, for any $x \in \Omega, x_k^* \in \partial f_k(\bar{x}), k = 1, 2$ and $x_t^* \in \partial g_t(\bar{x})$, taking $w = 0 \in N(\bar{x}; \Omega)^\circ = \mathbb{R}, \varepsilon = (\varepsilon_1, \varepsilon_2) \in \mathbb{R}^2_+ \backslash \{0\}$ with $\varepsilon_1 = \varepsilon_2 = \frac{3}{4}$, it follows that

$$\begin{aligned} \langle x_k^*, w\rangle + \varepsilon_k \|x - \bar{x}\| = \frac{3}{4}|x| \geq 0, &\qquad \forall x \in \Omega, \quad k = 1, 2 \\ \Rightarrow f_k(x) + \varepsilon_k \|x - \bar{x}\| = f_k(x) + \frac{3}{4}|x| \geq 0 = f_k(\bar{x}), &\qquad \forall x \in \Omega, \quad k = 1, 2, \\ g_t(x) = -tx \leq 0 = g_t(\bar{x}), &\qquad \forall x \in \Omega \\ \Rightarrow \langle x_t^*, w\rangle = 0 \leq 0, &\qquad \forall t \in T \end{aligned}$$

and

$$\langle b^*, w\rangle = 0 \leq |x|, \qquad \forall b^* \in \mathbb{B}_{\mathbb{R}}, \quad \forall x \in \Omega.$$

Meanwhite, (f, g_T) is not generalized convex function on Ω at $\bar{x} = 0$. Indeed, taking $\hat{x} = 1 \in \Omega, x_k^* = \frac{2}{3} \in \partial f_k(\bar{x}), k = 1, 2$ and $x_t^* = -t \in \partial g_t(\bar{x}), \forall t \in T$ and assume that there exists $w \in N(\bar{x}; \Omega)^\circ = \mathbb{R}$ such that

$$\begin{aligned} f_k(\hat{x}) - f_k(\bar{x}) \geq \langle x_k^*, w\rangle, &\qquad k = 1, \cdots, m, \\ g_t(\hat{x}) - g_t(\bar{x}) \geq \langle x_t^*, w\rangle, &\qquad \forall t \in T \end{aligned}$$

and

$$\langle b^*, w\rangle \leq \|\hat{x} - \bar{x}\|, \qquad \forall b^* \in \mathbb{B}_{\mathbb{R}}.$$

It yields that

$$\begin{aligned} &\frac{1}{2} \geq \frac{2}{3}w, \\ &-t \geq -tw, \qquad \forall t \in [1,2]. \end{aligned}$$

Thus, one has $1 \leq w \leq \frac{3}{4}$, a contradiction.

Now, we will give a sufficient condition for an ε-quasi positively properly efficient solution of the problem (SIMP).

Theorem 3.24. *Let $\varepsilon := (\varepsilon_1, \cdots, \varepsilon_m) \in \mathbb{R}^m_+ \backslash \{0\}$. Assume that $\bar{x} \in C$ satisfies the approximate* (KKT) *condition. If (f, g_T) is ε-quasi pseudo-generalized convex on Ω at $\bar{x}$, then $\bar{x}$ is an ε-quasi positively properly efficient solution of the problem* (SIMP).

Proof. Since $\bar{x} \in C$ satisfies the approximate (KKT) condition, there exist $\alpha := (\alpha_1, \cdots, \alpha_m) \in \text{int}\mathbb{R}^m_+, \lambda \in A(\bar{x})$ defined in (2) and $x^*_k \in \partial f_k(\bar{x}), k = 1, \cdots, m, x^*_t \in \partial g_t(\bar{x}), t \in T$, as well as $b^* \in \mathbb{B}_{\mathbb{R}^n}$ such that

$$-\left(\sum_{k=1}^m \alpha_k x^*_k + \sum_{t\in T} \lambda_t x^*_t + \sum_{k=1}^m \alpha_k \varepsilon_k b^*\right) \in N(\bar{x}; \Omega). \tag{13}$$

Suppose on contrary that $\bar{x} \in C$ is not an ε-quasi properly efficient solution of the problem (SIMP). For such $\alpha := (\alpha_1, \cdots, \alpha_m) \in \text{int}\mathbb{R}^m_+$, it then follows that there exists $x \in C$ such that

$$\sum_{k=1}^m \alpha_k f_k(x) + \sum_{k=1}^m \alpha_k \varepsilon_k \|x - \bar{x}\| < \sum_{k=1}^m \alpha_k f_k(\bar{x}). \tag{14}$$

On the other hand, if $\lambda \in A(\bar{x})$, then $\lambda_t g_t(\bar{x}) = 0, \forall t \in T$. Note that for any $x \in C$, then $\lambda_t g_t(x) \leq 0$ for any $t \in T$. It follows that

$$\lambda_t g_t(x) \leq 0 = \lambda_t g_t(\bar{x}), \qquad \text{for any } t \in T. \tag{15}$$

By the ε-quasi pseudo-generalized convexity of (f, g_T) on Ω at $\bar{x} \in \Omega$ and (14), (15), for such $x \in C \subseteq \Omega, x^*_k \in \partial f_k(\bar{x}), k = 1, \cdots, m, x^*_t \in \partial g_t(\bar{x}), t \in T$, there exists $w \in N(\bar{x}; \Omega)^\circ$ such that

$$\sum_{k=1}^m \alpha_k \langle x^*_k, w\rangle + \sum_{k=1}^m \alpha_k \varepsilon_k \|x - \bar{x}\| < 0, \tag{16}$$

$$\lambda_t \langle x^*_t, w\rangle \leq 0, \qquad \forall t \in T, \tag{17}$$

$$\langle b^*, w\rangle \leq \|x - \bar{x}\|, \qquad \forall b^* \in \mathbb{B}_{\mathbb{R}^n}. \tag{18}$$

On the other hand, we can rewrite (17) as

$$\sum_{t\in T} \lambda_t \langle x^*_t, w\rangle \leq 0. \tag{19}$$

Combining (16) with (19), we can assert that

$$\sum_{k=1}^m \alpha_k \langle x^*_k, w\rangle + \sum_{k=1}^m \alpha_k \varepsilon_k \|x - \bar{x}\| + \sum_{t\in T} \lambda_t \langle x^*_t, w\rangle < 0.$$

Applying (18), it follows that

$$\sum_{k=1}^{m} \alpha_k \langle x_k^*, w\rangle + \sum_{k=1}^{m} \alpha_k \varepsilon_k \langle b^*, w\rangle + \sum_{t\in T} \lambda_t \langle x_t^*, w\rangle < 0. \tag{20}$$

On the other hand, by the definition of polar cone (1), it yields from (13) and the relation $w \in N(\bar{x};\Omega)^\circ$ that

$$\sum_{k=1}^{m} \alpha_k \langle x_k^*, w\rangle + \sum_{k=1}^{m} \alpha_k \varepsilon_k \langle b^*, w\rangle + \sum_{t\in T} \lambda_t \langle x_t^*, w\rangle \geq 0,$$

which contradicts (20). This follows that $\bar{x} \in C$ is an ε-quasi positively properly efficient solution of the problem (SIMP). The proof is complete. □

Finally in this section, we will provide an example to see the importance of ε-quasi pseudo-generalized convexity of (f, g_T) in Theorem 3.24.

Example 3.25. Let $x \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}^2$ be defined by $f(x) = (f_1(x), f_2(x))$, where

$$f_k(x) = \begin{cases} x^2 \cos \frac{1}{x}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases}$$

$k = 1, 2$. Take $T = [-2, -1]$ and let $g_t : \mathbb{R} \to \mathbb{R}$ be given by

$$g_t(x) = t(x + 1), x \in \mathbb{R}, t \in T.$$

We consider problem (SIMP) with $m = 2$ and $\Omega = (-\infty, 0] \subset \mathbb{R}$. By simple computation, one has $C = [-1, 0]$. By selecting $\bar{x} = 0 \in C$, one has $N(\bar{x};\Omega) = N(\bar{x}; (-\infty, 0]) = [0, +\infty)$ and $N(\bar{x};\Omega)^\circ = N(\bar{x}; (-\infty, 0])^\circ = (-\infty, 0]$,

$$\partial f_k(\bar{x}) = [-1, 1], k = 1, 2 \text{ and } \partial g_t(\bar{x}) = \{t\}, t \in T.$$

Now, take $\varepsilon = (\varepsilon_1, \varepsilon_2) \in \mathbb{R}^2_+\backslash\{0\}$ with $\varepsilon_1 = \varepsilon_2 = \frac{1}{4\pi}$. Then, it implies that $\bar{x} = 0 \in C$ satisfies the approximate (KKT) condition. Indeed, let us select $\alpha = (\alpha_1, \alpha_2) \in \text{int}\mathbb{R}^2_+, \alpha_1 + \alpha_2 = 1, \lambda_t = 0, t \in T$ and $\mathbb{B}_\mathbb{R} = [-1, 1]$. Then,

$$\begin{aligned} 0 \in \left[-1 - \frac{1}{4\pi}, +\infty\right) &= [-1, 1] + [0, +\infty) + \left[-\frac{1}{4\pi}, \frac{1}{4\pi}\right] \\ &= \sum_{k=1}^{2} \alpha_k \partial f_k(\bar{x}) + \sum_{t\in T} \lambda_t \partial g_t(\bar{x}) + \sum_{k=1}^{2} \alpha_k \varepsilon_k \mathbb{B}_\mathbb{R} + N(\bar{x};\Omega), \end{aligned}$$

$\lambda_t g(\bar{x}) = 0, \forall t \in T$. However, $\bar{x}$ is not an ε-quasi positively properly efficient solution of the problem (SIMP). In order to see this, let us take $\hat{x} = -\frac{1}{\pi} \in C = [-1, 0], \varepsilon = (\varepsilon_1, \varepsilon_2) \in \mathbb{R}^2_+\backslash\{0\}$ with $\varepsilon_1 = \varepsilon_2 = \frac{1}{4\pi}$ and $\alpha = (\alpha_1, \alpha_2) \in \text{int}\mathbb{R}^2_+$ with $\alpha_1 + \alpha_2 = 1$. Then,

$$\sum_{k=1}^{2} \alpha_k f_k(\hat{x}) + \sum_{k=1}^{2} \alpha_k \varepsilon_k \|\hat{x} - \bar{x}\| = -\frac{1}{\pi^2} + \frac{1}{4\pi} \cdot \frac{1}{\pi} = -\frac{3}{4\pi^2} < 0 = \sum_{k=1}^{2} \alpha_k f_k(\bar{x}).$$

The reason is that (f, g_T) is not ε-quasi pseudo-generalized convex on Ω at $\bar{x}$. Indeed, take $x = -\frac{1}{3\pi} \in \Omega = (-\infty, 0], \varepsilon = (\varepsilon_1, \varepsilon_2) \in \mathbb{R}^2_+\backslash\{0\}$ with $\varepsilon_1 = \varepsilon_2 = \frac{1}{4\pi}$ and $x_k^* = 0 \in \partial f_k(\bar{x}) = [-1, 1], k = 1, 2$. Clearly,

$$\langle x_k^*, w\rangle + \varepsilon_k \|x - \bar{x}\| = \frac{1}{4\pi} \cdot \frac{1}{3\pi} = \frac{1}{12\pi^2} > 0, \qquad k = 1, 2, \quad \forall w \in N(\bar{x};\Omega)^\circ = (-\infty, 0].$$

However,

$$f_k(x) + \varepsilon_k \|x - \bar{x}\| = -\frac{1}{9\pi^2} + \frac{1}{4\pi} \cdot \frac{1}{3\pi} = -\frac{1}{36\pi^2} < 0 = f_k(\bar{x}), \qquad k = 1, 2.$$

Remark 3.26. Theorem 3.24 improves Theorem 3 in [6].

4. Approximate duality theorems

In this section, we consider the Mond–Weir type dual problem (MWD) with respect to the problem (SIMP).

For $x \in \mathbb{R}^n, \Omega$ is a nonempty locally closed (not necessarily convex) subset of $\mathbb{R}^n$, $\alpha := (\alpha_1, \cdots, \alpha_m) \in \operatorname{int}\mathbb{R}^m_+$ with $\sum_{k=1}^m \alpha_k = 1$ and $\lambda \in \mathbb{R}^{(T)}_+, f := (f_1, \cdots, f_m), g_T := (g_t)_{t\in T}$, let us denote a vector function $L := (L_1, \cdots, L_m)$ by

$$L(x, \alpha, \lambda) := f(x).$$

For $\varepsilon := (\varepsilon_1, \cdots, \varepsilon_m) \in \mathbb{R}^m_+\backslash\{0\}$, we consider the Mond–Weir type dual problem (MWD) with respect to the primal problem (SIMP) as follows:

$$\text{(MWD)} \quad \begin{cases} \max & L(y, \alpha, \lambda) \\ \text{s.t.} & 0 \in \displaystyle\sum_{k=1}^m \alpha_k \partial f_k(y) + \sum_{t\in T} \lambda_t \partial g_t(y) + \sum_{k=1}^m \alpha_k \varepsilon_k \mathbb{B}_{\mathbb{R}^n} + N(y; \Omega), \\ & \displaystyle\sum_{t\in T} \lambda_t g_t(y) \geq 0, \\ & y \in \Omega, \alpha := (\alpha_1, \cdots, \alpha_m) \in \operatorname{int}\mathbb{R}^m_+, \displaystyle\sum_{k=1}^m \alpha_k = 1, \lambda \in \mathbb{R}^{(T)}_+. \end{cases}$$

The feasible set of the problem (MWD) is defined by

$$C_{\text{MWD}} := \Bigg\{ (y, \alpha, \lambda) \in \Omega \times \operatorname{int}\mathbb{R}^m_+ \times \mathbb{R}^{(T)}_+ \mid 0 \in \sum_{k=1}^m \alpha_k \partial f_k(y) + \sum_{t\in T} \lambda_t \partial g_t(y) + \sum_{k=1}^m \alpha_k \varepsilon_k \mathbb{B}_{\mathbb{R}^n} + N(y; \Omega), \sum_{t\in T} \lambda_t g_t(y) \geq 0, \sum_{k=1}^m \alpha_k = 1 \Bigg\}.$$

In what follows, we use the following notation for convenience:

$$u \preceq v \Leftrightarrow u - v \in -\mathbb{R}^m_+\backslash\{0\}, \quad u \npreceq v \text{ is the negation of } u \preceq v.$$

Now, we will introduce the definition of an ε-quasi efficient solution/an ε-quasi positively properly efficient solution for the problem (MWD).

Definition 4.1. A point $(\bar{y}, \bar{\alpha}, \bar{\lambda}) \in C_{\text{MWD}}$ is said to be

(i) an ε-quasi efficient solution of the problem (MWD) if

$$L\big(\bar{y}, \bar{\alpha}, \bar{\lambda}\big) \npreceq L(y, \alpha, \lambda) - \varepsilon\|\bar{y} - y\|, \qquad \forall (y, \alpha, \lambda) \in C_{\text{MWD}}.$$

(ii) an ε-quasi positively properly efficient solution of the problem (MWD) if there exists $\theta := (\theta_1, \cdots, \theta_m) \in -\operatorname{int}\mathbb{R}^m_+$ such that

$$\langle \theta, L(y, \alpha, \lambda) + \varepsilon\|\bar{y} - y\| \rangle \geq \big\langle \theta, L(\bar{y}, \bar{\alpha}, \bar{\lambda}) \big\rangle, \qquad \forall (y, \alpha, \lambda) \in C_{\text{MWD}}.$$

Now, we establish the following weak duality theorem, which describes relation between the problem (SIMP) and the problem (MWD).

Theorem 4.2. *Suppose that $x \in C$ and $(y, \alpha, \lambda) \in C_{\text{MWD}}$. If (f, g_T) is ε-quasi pseudo-generalized convex on Ω at y, then*

$$f(x) \npreceq L(y, \alpha, \lambda) - \varepsilon\|x - y\|.$$

Proof. Since $(y,\alpha,\lambda) \in C_{\mathrm{MWD}}$, there exist $x_k^* \in \partial f_k(y), k = 1,\cdots,m, \alpha \in \mathrm{int}\mathbb{R}_+^m$ with $\sum_{k=1}^m \alpha_k = 1$ and $x_t^* \in \partial g_t(y), t \in T$, $\lambda \in \mathbb{R}_+^{(T)}$, as well as $b^* \in \mathbb{B}_{\mathbb{R}^n}$ such that

$$-\left(\sum_{k=1}^m \alpha_k x_k^* + \sum_{t\in T} \lambda_t x_t^* + \sum_{k=1}^m \alpha_k \varepsilon_k b^*\right) \in N(y;\Omega) \tag{21}$$

and

$$\sum_{t\in T} \lambda_t g_t(y) \geq 0. \tag{22}$$

Let $x \in C$. Suppose on contrary that

$$f(x) \preceq L(y,\alpha,\lambda,) - \varepsilon\|x-y\|.$$

Hence, $\langle \alpha, (f(x) - f(y) + \varepsilon\|x-y\|\rangle < 0$ due to $\alpha := (\alpha_1,\cdots,\alpha_m) \in \mathrm{int}\mathbb{R}_+^m$. Thus,

$$\sum_{k=1}^m \alpha_k f_k(x) + \sum_{k=1}^m \alpha_k \varepsilon_k \|x-y\| < \sum_{k=1}^m \alpha_k f_k(y). \tag{23}$$

Note that, for $x \in C$ we have $g_t(x) \leq 0$ for any $t \in T$. It yields that

$$\sum_{t\in T} \lambda_t g_t(x) \leq 0. \tag{24}$$

From (22) and (24), it follows that

$$\sum_{t\in T} \lambda_t g_t(x) \leq \sum_{t\in T} \lambda_t g_t(y). \tag{25}$$

By the ε-quasi pseudo-generalized convexity of (f, g_T) on Ω at $y \in \Omega$ and (23), (25) for such $x \in C \subseteq \Omega, x_k^* \in \partial f_k(y), k = 1,\cdots,m, x_t^* \in \partial g_t(y), t \in T$, there exists $w \in N(y;\Omega)^\circ$ such that

$$\sum_{k=1}^m \alpha_k \langle x_k^*, w\rangle + \sum_{t\in T} \alpha_k \varepsilon_k \|x-y\| < 0, \tag{26}$$

$$\sum_{t\in T} \lambda_t \langle x_t^*, w\rangle \leq 0, \tag{27}$$

$$\langle b^*, w\rangle \leq \|x-y\|, \qquad \forall b^* \in \mathbb{B}_{\mathbb{R}^n}. \tag{28}$$

Combining (26)–(28), we can assert that

$$\sum_{k=1}^m \alpha_k \langle x_k^*, w\rangle + \sum_{t\in T} \alpha_k \varepsilon_k \langle b^*, w\rangle + \sum_{t\in T} \lambda_t \langle x_t^*, w\rangle < 0. \tag{29}$$

On the other side, we deduce from (1) and the relation $w \in N(y;\Omega)^\circ$ that

$$\sum_{k=1}^m \alpha_k \langle x_k^*, w\rangle + \sum_{t\in T} \alpha_k \varepsilon_k \langle b^*, w\rangle + \sum_{t\in T} \lambda_t \langle x_t^*, w\rangle \geq 0,$$

which contradicts (29). Thus, $f(x) \npreceq L(y,\alpha,\lambda) - \varepsilon\|x-y\|$. □

The following example shows that the ε-quasi pseudo-generalized convexity of (f, g_T) on Ω imposed in Theorem 4.2 cannot be removed.

Example 4.3. Let $f : \mathbb{R} \to \mathbb{R}^2$ be defined by

$$f(x) = (f_1(x), f_2(x)),$$

where $f_1(x) = f_2(x) = x^3$ and let $g_t : \mathbb{R} \to \mathbb{R}$ be given by

$$g_t(x) = -tx^2, x \in \mathbb{R}, t \in T = [1, 2].$$

We consider the problem (SIMP) with $m = 2$ and $\Omega = [-1, 0] \subset \mathbb{R}$. By simple computation, one has $C = [-1, 0]$. Now, consider the dual problem (MWD). By choosing $\bar{y} = 0 \in \Omega, \bar{\lambda} \in \mathbb{R}^{(T)}_+, \bar{\alpha} = (\bar{\alpha}_1, \bar{\alpha}_2) \in \mathrm{int}\mathbb{R}^2_+$ with $\bar{\alpha}_1 + \bar{\alpha}_2 = 1$ and $\varepsilon = (\varepsilon_1, \varepsilon_2) \in \mathbb{R}^2_+ \backslash \{0\}$ with $\varepsilon_1 = \varepsilon_2 = \frac{1}{5}, \mathbb{B}_{\mathbb{R}} = [-1, 1]$, we have $N(\bar{y}; \Omega) = N(\bar{y}; [-1, 0]) = [0, +\infty)$ and $\partial f_1(\bar{y}) = \partial f_2(\bar{y}) = \{0\}, \partial g_t(\bar{y}) = \{0\}, \forall t \in T$. It is easy to see that

$$0 \in \left[-\frac{1}{5}, +\infty\right) = \left[-\frac{1}{5}, \frac{1}{5}\right] + [0, +\infty) = \sum_{k=1}^{2} \bar{\alpha}_k \partial f_k(\bar{y}) + \sum_{t \in T} \bar{\lambda}_t \partial g_t(\bar{y}) + \sum_{k=1}^{2} \bar{\alpha}_k \varepsilon_k \mathbb{B}_{\mathbb{R}} + N(\bar{y}; \Omega)$$

and $\bar{\alpha}_1 + \bar{\alpha}_2 = 1, \sum_{t \in T} \bar{\lambda}_t g_t(\bar{y}) = 0 \geq 0$. Thus, $(\bar{y}, \bar{\alpha}, \bar{\lambda}) \in C_{\mathrm{MWD}}$. However, by choosing $\bar{x} = -1 \in C = [-1, 0]$, it follows that

$$\begin{aligned} f(\bar{x}) &= (f_1(\bar{x}), f_2(\bar{x})) = (-1, -1) \\ &\preceq \left(-\frac{1}{5}, -\frac{1}{5}\right) \\ &= \left(L_1(\bar{y}, \bar{\alpha}, \bar{\lambda}) - \varepsilon_1 \|\bar{x} - \bar{y}\|, L_2(\bar{y}, \bar{\alpha}, \bar{\lambda}) - \varepsilon_2 \|\bar{x} - \bar{y}\|\right) \\ &= L(\bar{y}, \bar{\alpha}, \bar{\lambda}) - \varepsilon \|\bar{x} - \bar{y}\|. \end{aligned}$$

The reason is that (f, g_T) is not ε-quasi pseudo-generalized convex on Ω at $\bar{y} = 0$. To see this, we can choose $y = -\frac{1}{2} \in \Omega$ and $x^*_k \in \partial f_k(\bar{y}) = \{0\}, k = 1, 2, \varepsilon_1 = \varepsilon_2 = \frac{1}{5}$. Then, it is easy to see that $N(\bar{y}; \Omega)^\circ = N(\bar{y}; [-1, 0])^\circ = (-\infty, 0]$ and

$$\langle x^*_k, w\rangle + \varepsilon_k \|y - \bar{y}\| = \frac{1}{5} \cdot \frac{1}{2} = \frac{1}{10} > 0, \qquad \forall w \in N(\bar{y}; \Omega)^\circ, \quad k = 1, 2.$$

However,

$$f_k(y) + \varepsilon_k \|y - \bar{y}\| = -\frac{1}{8} + \frac{1}{5} \cdot \frac{1}{2} = -\frac{1}{40} < 0 = f_k(\bar{y}), \qquad k = 1, 2.$$

Remark 4.4. Theorem 4.2 improves Theorem 4 in [27].

Now, we establish the following strong duality theorem, which describes relation between the problem (SIMP) and the problem (MWD).

Theorem 4.5. *Suppose that $\bar{x} \in C$ is an ε-quasi positively properly efficient solution of the problem* (SIMP) *such that the condition* (MCQ) *is satisfied at $\bar{x}$. Then there exists $(\bar{\alpha}, \bar{\lambda}) \in \mathrm{int}\mathbb{R}^m_+ \times \mathbb{R}^{(T)}_+$ such that $(\bar{x}, \bar{\alpha}, \bar{\lambda}) \in C_{\mathrm{MWD}}$ and $f(\bar{x}) = L(\bar{x}, \bar{\alpha}, \bar{\lambda})$. If in addition (f, g_T) is ε-quasi pseudo-generalized convex on Ω at $y \in \Omega$, then $(\bar{x}, \bar{\alpha}, \bar{\lambda})$ is an ε-quasi efficient solution of the problem* (MWD).

Proof. According to Theorem 3.6, there exist $\alpha := (\alpha_1, \cdots, \alpha_m) \in \mathrm{int}\mathbb{R}^m_+$ and $\lambda \in A(\bar{x})$ defined in (2) such that

$$0 \in \sum_{k=1}^{m} \alpha_k \partial f_k(\bar{x}) + \sum_{t \in T} \lambda_t \partial g_t(\bar{x}) + \sum_{k=1}^{m} \alpha_k \varepsilon_k \mathbb{B}_{\mathbb{R}^n} + N(\bar{x}; \Omega). \tag{30}$$

Putting

$$\bar{\alpha}_k := \frac{\alpha_k}{\sum_{k=1}^{m} \alpha_k}, k = 1, \cdots, m, \bar{\lambda}_t := \frac{\lambda_t}{\sum_{k=1}^{m} \alpha_k}, t \in T,$$

one has $\bar{\alpha} := (\bar{\alpha}_1, \cdots, \bar{\alpha}_m) \in \text{int}\mathbb{R}^m_+$ with $\sum_{k=1}^m \bar{\alpha}_k = 1$, $\bar{\lambda} := (\bar{\lambda}_t)_{t\in T} \in \mathbb{R}^{(T)}_+$. Furthermore, the assertion in (30) is also valid when α_k's and λ_t's are replaced by $\bar{\alpha}_k$'s and $\bar{\lambda}_t$'s, respectively. Clearly, $(\bar{x}, \bar{\alpha}, \bar{\lambda}) \in C_{\text{MWD}}$. Besides, since $\lambda \in A(\bar{x})$, one has $\lambda_t g_t(\bar{x}) = 0, \forall t \in T$. It implies that $\sum_{t\in T} \bar{\lambda}_t g_t(\bar{x}) = 0$. Therefore, one has

$$f(\bar{x}) = L\big(\bar{x}, \bar{\alpha}, \bar{\lambda}\big).$$

Because (f, g_T) is ε-quasi pseudo-generalized convex on Ω at any $y \in \Omega$, so we apply the result of Theorem 4.2 to deduce that

$$L\big(\bar{x}, \bar{\alpha}, \bar{\lambda}\big) = f(\bar{x}) \not\preceq L(y, \alpha, \lambda) - \varepsilon\|\bar{x} - y\|,$$

for any $(y, \alpha, \lambda) \in C_{\text{MWD}}$. This means that $(\bar{x}, \bar{\alpha}, \bar{\lambda})$ is an ε-quasi efficient solution of the problem (MWD). The proof is complete. □

The next example asserts the importance of the qualification condition (MCQ) imposed in Theorem 4.5. More precisely, if $\bar{x}$ is an ε-quasi positively properly efficient solution of the problem (SIMP) at which the qualification condition (MCQ) is not satisfied, then we may not find out a pair $(\bar{\alpha}, \bar{\lambda}) \in \text{int}\mathbb{R}^m_+ \times \mathbb{R}^{(T)}_+$ such that $(\bar{x}, \bar{\alpha}, \bar{\lambda})$ belongs to the feasible set C_{MWD} of the dual problem (MWD).

Example 4.6. Let $f : \mathbb{R} \to \mathbb{R}^2$ be defined by $f(x) = (f_1(x), f_2(x))$, where

$$f_1(x) = f_2(x) = 3x, x \in \mathbb{R}.$$

Let $g_t : \mathbb{R} \to \mathbb{R}$ be given by

$$g_t(x) = t^2 x^4, x \in \mathbb{R}, t \in T = [-5, -1].$$

We consider problem (SIMP) with $m = 2$ and $\Omega = (-\infty, 0] \subset \mathbb{R}$. By simple computation, one has $C = \{0\}$. Now, take $\bar{x} = 0 \in C, \varepsilon = (\varepsilon_1, \varepsilon_2) \in \mathbb{R}^2_+\backslash\{0\}$ with $\varepsilon_1 = \varepsilon_2 = \frac{1}{2}$ and $\bar{\alpha} = (\bar{\alpha}_1, \bar{\alpha}_2) \in \text{int}\mathbb{R}^2_+$ with $\bar{\alpha}_1 + \bar{\alpha}_2 = 1$. Then, it is easy to show that $\bar{x}$ is an positively properly efficient solution of the problem (SIMP). Indeed, we have

$$\sum_{k=1}^2 \bar{\alpha}_k f_k(x) + \sum_{k=1}^2 \bar{\alpha}_k \varepsilon_k \|x - \bar{x}\| = 3x + \frac{1}{2}|x| \geq 0 = \sum_{k=1}^2 \bar{\alpha}_k f_k(\bar{x}), \qquad \forall x \in C.$$

Now, consider the dual problem (MWD). By choosing $\bar{x} = 0 \in \Omega, \bar{\lambda} \in \mathbb{R}^{(T)}_+, \mathbb{B}_{\mathbb{R}} = [-1, 1], \bar{\alpha} = (\bar{\alpha}_1, \bar{\alpha}_2) \in \text{int}\mathbb{R}^2_+$ with $\bar{\alpha}_1 + \bar{\alpha}_2 = 1$ and $\varepsilon = (\varepsilon_1, \varepsilon_2) \in \mathbb{R}^2_+\backslash\{0\}$ with $\varepsilon_1 = \varepsilon_2 = \frac{1}{2}$, we have $N(\bar{x}; \Omega) = N(\bar{x}; (-\infty, 0]) = [0, +\infty)$ and $\partial f_k(\bar{x}) = \{3\}, k = 1, 2, \partial g_t(\bar{x}) = \{0\}, \forall t \in T$. It is easy to see that

$$0 \notin \left[\frac{5}{2}, +\infty\right) = \{3\} + \left[-\frac{1}{2}, \frac{1}{2}\right] + [0, +\infty) = \sum_{k=1}^2 \bar{\alpha}_k \partial f_k(\bar{x}) + \sum_{t\in T} \bar{\lambda}_t \partial g_t(\bar{x}) + \sum_{k=1}^2 \bar{\alpha}_k \varepsilon_k \mathbb{B}_{\mathbb{R}} + N(\bar{x}; \Omega).$$

Thus, $(\bar{x}, \bar{\alpha}, \bar{\lambda}) \notin F_{\text{MWD}}$. The reason is that the qualification condition (MCQ) is not satisfied at $\bar{x} \in C$. Indeed, we have $N(\bar{x}; \Omega) = N(\bar{x}; (-\infty, 0]) = [0, +\infty)$ and $\partial g_t(\bar{x}) = \{0\}$ for any $t \in T$,

$$\bigcup_{\lambda \in A(\bar{x})} \left[\sum_{t\in T} \lambda_t \partial g_t(\bar{x})\right] + N(\bar{x}; \Omega) = [0, +\infty).$$

Besides, one has $N(\bar{x}; C) = \mathbb{R}$. Therefore, the qualification condition (MCQ) is not satisfied at $\bar{x}$.

Remark 4.7. Theorem 4.5 improves Theorem 5 in [27].

Finally, we establish the following converse duality theorem, which describes relation between the problem (SIMP) and the problem (MWD).

Theorem 4.8. *Suppose that* $(\bar{x}, \bar{\alpha}, \bar{\lambda}) \in C_{\mathrm{MWD}}$. *If* $\bar{x} \in C$ *and* (f, g_T) *is* ε*-quasi pseudo-generalized convex on* Ω *at* $\bar{x}$*, then* $\bar{x}$ *is an* ε*-quasi positively properly efficient solution of the problem* (SIMP).

Proof. Since $(\bar{x}, \bar{\alpha}, \bar{\lambda}) \in C_{\mathrm{MWD}}$, there exist $x_k^* \in \partial f_k(\bar{x}), k = 1, \cdots, m, \bar{\alpha} \in \mathrm{int}\mathbb{R}_+^m$ with $\sum_{k=1}^m \bar{\alpha}_k = 1$ and $x_t^* \in \partial g_t(\bar{x}), t \in T$, $\bar{\lambda} := (\bar{\lambda}_t)_{t\in T} \in \mathbb{R}_+^{(T)}$, as well as $b^* \in \mathbb{B}_{\mathbb{R}^n}$ such that

$$-\left(\sum_{k=1}^m \bar{\alpha}_k x_k^* + \sum_{t\in T} \bar{\lambda}_t x_t^* + \sum_{k=1}^m \bar{\alpha}_k \varepsilon_k b^*\right) \in N(\bar{x}; \Omega) \tag{31}$$

and

$$\sum_{t\in T} \bar{\lambda}_t g_t(\bar{x}) \geq 0. \tag{32}$$

Suppose on contrary that $\bar{x} \in C$ is not an ε-quasi positively properly efficient solution of the problem (SIMP). For such $\bar{\alpha} := (\bar{\alpha}_1, \cdots, \bar{\alpha}_m) \in \mathrm{int}\mathbb{R}_+^m$, it then follows that there exists $\hat{x} \in C$ satisfying

$$\sum_{k=1}^m \bar{\alpha}_k f_k(\hat{x}) + \sum_{k=1}^m \bar{\alpha}_k \varepsilon_k \|\hat{x} - \bar{x}\| < \sum_{k=1}^m \bar{\alpha}_k f_k(\bar{x}). \tag{33}$$

Note that, for any $\hat{x} \in C$, we have $g_t(\hat{x}) \leq 0$ for any $t \in T$. It yields that

$$\sum_{t\in T} \bar{\lambda}_t g_t(\hat{x}) \leq 0. \tag{34}$$

From (32) together with (34), it follows that

$$\sum_{t\in T} \bar{\lambda}_t g_t(\hat{x}) \leq \sum_{t\in T} \bar{\lambda}_t g_t(\bar{x}). \tag{35}$$

By the ε-quasi pseudo-generalized convexity of (f, g_T) on Ω at $\bar{x} \in \Omega$ and (33), (35) for such $x \in C \subseteq \Omega, x_k^* \in \partial f_k(\bar{x}), k = 1, \cdots, m, x_t^* \in \partial g_t(\bar{x}), t \in T$, there exists $w \in N(y; \Omega)^\circ$ such that

$$\sum_{k=1}^m \bar{\alpha}_k \langle x_k^*, w\rangle + \sum_{k=1}^m \bar{\alpha}_k \varepsilon_k \|\hat{x} - \bar{x}\| < 0, \tag{36}$$

$$\sum_{t\in T} \bar{\lambda}_t \langle x_t^*, w\rangle \leq 0, \tag{37}$$

$$\langle b^*, w\rangle \leq \|\hat{x} - \bar{x}\|, \qquad \forall b^* \in \mathbb{B}_{\mathbb{R}^n}. \tag{38}$$

Combining (36) and (37), we can assert that

$$\sum_{k=1}^m \bar{\alpha}_k \langle x_k^*, w\rangle + \sum_{t\in T} \bar{\lambda}_t \langle x_t^*, w\rangle + \sum_{k=1}^m \bar{\alpha}_k \varepsilon_k \|\hat{x} - \bar{x}\| < 0. \tag{39}$$

Applying (38), it follows that

$$\sum_{k=1}^m \bar{\alpha}_k \langle x_k^*, w\rangle + \sum_{t\in T} \bar{\lambda}_t \langle x_t^*, w\rangle + \sum_{k=1}^m \bar{\alpha}_k \varepsilon_k \langle b^*, w\rangle < 0. \tag{40}$$

On the other hand, by the definition of polar cone (1), we yield from (31) and the relation $w \in N(\bar{x}; \Omega)^\circ$ that

$$\sum_{k=1}^m \bar{\alpha}_k \langle x_k^*, w\rangle + \sum_{t\in T} \bar{\lambda}_t \langle x_t^*, w\rangle + \sum_{k=1}^m \bar{\alpha}_k \varepsilon_k \langle b^*, w\rangle \geq 0,$$

which contradicts (40). This means that $\bar{x} \in C$ is an ε-quasi positively properly efficient solution of the problem (SIMP). The proof is complete. ◻

The following example shows that the ε-quasi pseudo-generalized convexity of (f, g_T) on imposed in the Theorem 4.8 cannot be removed.

Example 4.9. Let $f : \mathbb{R} \to \mathbb{R}^2$ be defined by

$$f(x) = (f_1(x), f_2(x)),$$

where $f_1(x) = f_2(x) = x^5, x \in \mathbb{R}$. Let $g_t : \mathbb{R} \to \mathbb{R}$ be given by

$$g_t(x) = -tx^2, x \in \mathbb{R}, t \in T = [1, 2].$$

We consider problem (SIMP) with $m = 2$ and $\Omega = [-1, 0] \subset \mathbb{R}$. By simple computation, one has $C = [-1, 0]$. Now, consider the dual problem (MWD). By choosing $\bar{y} = 0 \in C, \bar{\alpha} = (\bar{\alpha}_1, \bar{\alpha}_2) \in \text{int}\mathbb{R}^2_+$ with $\bar{\alpha}_1 + \bar{\alpha}_2 = 1$ and $\varepsilon = (\varepsilon_1, \varepsilon_2) \in \mathbb{R}^2_+ \backslash \{0\}$ with $\varepsilon_1 = \varepsilon_2 = \frac{1}{17}, \mathbb{B}_{\mathbb{R}} = [-1, 1], \bar{\lambda} \in \mathbb{R}^{(T)}_+$, we have $N(\bar{y}; \Omega) = N(\bar{y}; [-1, 0]) = [0, +\infty)$ and $\partial f_1(\bar{y}) = \partial f_2(\bar{y}) = \{0\}, \partial g_t(\bar{y}) = \{0\}, \forall t \in T$. It is easy to see that

$$0 \in \left[-\frac{1}{17}, +\infty\right) = \left[-\frac{1}{17}, \frac{1}{17}\right] + [0, +\infty) = \sum_{k=1}^{2} \bar{\alpha}_k \partial f_k(\bar{y}) + \sum_{t \in T} \bar{\lambda}_t \partial g_t(\bar{y}) + \sum_{k=1}^{2} \bar{\alpha}_k \varepsilon_k \mathbb{B}_{\mathbb{R}} + N(\bar{y}; \Omega)$$

and $\bar{\alpha}_1 + \bar{\alpha_2} = 1, \sum_{t \in T} \bar{\lambda}_t g_t(\bar{y}) = 0 \geq 0$. However, $\bar{x}$ is not an ε-quasi positively properly efficient solution of the problem (SIMP). In order to see this, let us take $\hat{x} = -1 \in C = [-1, 0], \varepsilon = (\varepsilon_1, \varepsilon_2) \in \mathbb{R}^2_+ \backslash \{0\}$ with $\varepsilon_1 = \varepsilon_2 = \frac{1}{17}$ and $\bar{\alpha} = (\bar{\alpha}_1, \bar{\alpha}_2) \in \text{int}\mathbb{R}^2_+$ with $\bar{\alpha}_1 + \bar{\alpha}_2 = 1$. Then,

$$\sum_{k=1}^{2} \bar{\alpha}_k f_k(\hat{x}) + \sum_{k=1}^{2} \bar{\alpha}_k \varepsilon_k \|\hat{x} - \bar{x}\| = -1 + \frac{1}{17} = -\frac{16}{17} < 0 = \sum_{k=1}^{2} \bar{\alpha}_k f_k(\bar{x}).$$

The reason is that (f, g_T) is not ε-quasi pseudo-generalized convex on Ω at $\bar{y} = 0$. To see this, we can choose $y = -\frac{1}{2} \in \Omega = [-1, 0], \varepsilon = (\varepsilon_1, \varepsilon_2) \in \mathbb{R}^2_+ \backslash \{0\}$ with $\varepsilon_1 = \varepsilon_2 = \frac{1}{17}$ and $x^*_k \in \partial f_k(\bar{x}), k = 1, 2$. Then, it is easy to see that $N(\bar{y}; \Omega)^\circ = N(0; [-1, 0])^\circ = (-\infty, 0]$ and

$$\langle x^*_k, w\rangle + \varepsilon_k \|y - \bar{y}\| = \frac{1}{17} \cdot \frac{1}{2} = \frac{1}{34} > 0, \qquad \forall w \in N(\bar{y}; \Omega)^\circ = (-\infty, 0], \quad k = 1, 2.$$

However,

$$f_k(x) + \varepsilon_k \|y - \bar{y}\| = -\frac{1}{32} + \frac{1}{17} \cdot \frac{1}{2} = -\frac{1}{544} < 0 = f_k(\bar{y}), \quad k = 1, 2.$$

Remark 4.10. Theorem 4.8 improves Theorem 6 in [27].

5. Application in fractional semi-infinite multiobjective optimization problem

In this section, we consider the following fractional semi-infinite multiobjective optimization problem:

$$\text{(FSIMP)} \quad \begin{aligned} &\min f(x) := \left(\frac{p_1(x)}{q_1(x)}, \cdots, \frac{p_m(x)}{q_m(x)}\right), \\ &\text{s.t. } x \in C := \{x \in \Omega \mid g_t(x) \leq 0, \forall t \in T\}, \end{aligned}$$

where T be a nonempty infinite index set, Ω is a nonempty locally closed (not necessarily convex) subset of $\mathbb{R}^n$, $p_k, q_k : \mathbb{R}^n \to \mathbb{R}, k = 1, \cdots, m$ and $g_t : \mathbb{R} \to \mathbb{R}, t \in T$ are locally Lipschitz functions. For the sake of convenience, we further assume that $q_k(x) > 0, k = 1, \cdots, m$ for all $x \in \Omega$ and that $p_k(\bar{x}) \leq 0, k = 1, \cdots, m$ for the reference point $\bar{x} \in \Omega$. In what follows, we also use the notation $f := (f_1, \cdots, f_m)$ with $f_k := \frac{p_k}{q_k}, k = 1, \cdots, m$.

Definition 5.1. Let $\varepsilon := (\varepsilon_1, \cdots, \varepsilon_m) \in \mathbb{R}^m_+ \backslash \{0\}$ and $f := (f_1, \cdots, f_m)$ with $f_k := \frac{p_k}{q_k}, k = 1, \cdots, m$. A point $\bar{x} \in C$ is said to be an ε-quasi positively properly efficient solution of the problem (FSIMP) if there exists $\alpha := (\alpha_1, \cdots, \alpha_m) \in \text{int}\mathbb{R}^m_+$ such that

$$\langle \alpha, f(x) + \varepsilon\|x - \bar{x}\|\rangle \geq \langle \alpha, f(\bar{x})\rangle, \qquad \forall x \in C.$$

Theorem 5.2. *Let $\varepsilon := (\varepsilon_1, \cdots, \varepsilon_m) \in \mathbb{R}^m_+ \backslash \{0\}$ and let $\bar{x} \in C$ be an ε-quasi positively properly efficient solution of the problem* (FSIMP). *Suppose that the qualification condition* (MCQ) *at $\bar{x}$ holds. Then, there exist $\alpha := (\alpha_1, \cdots, \alpha_m) \in \text{int}\mathbb{R}^m_+$ and $\beta := (\beta_1, \cdots, \beta_m) \in \text{int}\mathbb{R}^m_+, \lambda \in A(\bar{x})$ defined in* (2) *such that*

$$\begin{aligned} 0 \in \sum_{k=1}^{m} \beta_k \left(\partial p_k(\bar{x}) - \frac{p_k(\bar{x})}{q_k(\bar{x})} \partial q_k(\bar{x}) \right) + \sum_{t \in T} \lambda_t \partial g_t(\bar{x}) \\ + \sum_{k=1}^{m} \alpha_k \varepsilon_k \mathbb{B}_{\mathbb{R}^n} + N(\bar{x}; \Omega), \beta_k := \frac{\alpha_k}{q_k(\bar{x})}, \qquad k = 1, \cdots, m. \end{aligned} \tag{41}$$

Proof. Suppose that $\bar{x} \in C$ is an ε-quasi positively properly efficient solution of the problem (FSIMP), one has $\bar{x}$ is an ε-quasi positively properly efficient solution of the problem (SIMP) with $f_k := \frac{p_k}{q_k}, k = 1, \cdots, m$. According to Theorem 3.6, there exist $\alpha := (\alpha_1, \cdots \alpha_m) \in \mathbb{R}^m_+$ and $\lambda \in A(\bar{x})$ defined in (2) such that

$$0 \in \sum_{k=1}^{m} \alpha_k \partial f_k(\bar{x}) + \sum_{t \in T} \lambda_t \partial g_t(\bar{x}) + \sum_{k=1}^{m} \alpha_k \varepsilon_k \mathbb{B}_{\mathbb{R}^n} + N(\bar{x}; \Omega). \tag{42}$$

Thanks to Lemma 2.3, for $k = 1, \cdots, m$, one has

$$\begin{aligned} \partial f_k(\bar{x}) = \partial \left(\frac{p_k}{q_k} \right)(\bar{x}) \subset \frac{\partial(q_k(\bar{x})p_k)(\bar{x}) + \partial(-p_k(\bar{x})q_k)(\bar{x})}{[q_k(\bar{x})]^2} = \frac{q_k(\bar{x})\partial p_k(\bar{x}) - p_k(\bar{x})\partial q_k(\bar{x})}{[q_k(\bar{x})]^2} \\ = \frac{1}{q_k(\bar{x})} \left(\partial p_k(\bar{x}) - \frac{p_k(\bar{x})}{q_k(\bar{x})} \partial q_k(\bar{x}) \right). \end{aligned} \tag{43}$$

Combining (42) with (43), we can assert that

$$0 \in \sum_{k=1}^{m} \frac{\alpha_k}{q_k(\bar{x})} \left(\partial p_k(\bar{x}) - \frac{p_k(\bar{x})}{q_k(\bar{x})} \partial q_k(\bar{x}) \right) + \sum_{t \in T} \lambda_t \partial g_t(\bar{x}) + \sum_{k=1}^{m} \alpha_k \varepsilon_k \mathbb{B}_{\mathbb{R}^n} + N(\bar{x}; \Omega).$$

Now, by letting $\beta_k := \frac{\alpha_k}{q_k(\bar{x})}$ for $k = 1, \cdots, m$, we get

$$0 \in \sum_{k=1}^{m} \beta_k \left(\partial p_k(\bar{x}) - \frac{p_k(\bar{x})}{q_k(\bar{x})} \partial q_k(\bar{x}) \right) + \sum_{t \in T} \lambda_t \partial g_t(\bar{x}) + \sum_{k=1}^{m} \alpha_k \varepsilon_k \mathbb{B}_{\mathbb{R}^n} + N(\bar{x}; \Omega),$$

where $\beta := (\beta_1, \cdots, \beta_m) \in \text{int}\mathbb{R}^m_+$ and $\lambda \in A(\bar{x})$ defined in (2). The proof of Theorem 5.2 is complete. □

The following simple example shows that the qualification condition (MCQ) is essential in Theorem 5.2.

Example 5.3. Let $f : \mathbb{R} \to \mathbb{R}^2$ be defined by

$$f(x) = \left(\frac{p_1(x)}{q_1(x)}, \frac{p_2(x)}{q_2(x)} \right),$$

where $p_1(x) = p_2(x) = x, q_1(x) = q_2(x) = x^4 + 1, x \in \mathbb{R}$ and let $g_t : \mathbb{R} \to \mathbb{R}$ be given by

$$g_t(x) = t^3 x^4, x \in \mathbb{R}, t \in T = [1, 2].$$

We consider the problem (FSIMP) with $m = 2$ and $\Omega = (-\infty, 0] \subset \mathbb{R}$. By simple computation, one has $C = \{0\}$. Now, take $\bar{x} = 0, \varepsilon = (\varepsilon_1, \varepsilon_2) \in \mathbb{R}^2_+ \backslash \{0\}$ with $\varepsilon_1 = \varepsilon_2 = \frac{1}{2}$ and $\alpha = (\alpha_1, \alpha_2) \in \text{int}\mathbb{R}^2_+$ with $\alpha_1 + \alpha_2 = 1$. Then, it is easy to show that $\bar{x} = 0 \in C$ is an ε-quasi positively properly efficient solution of the problem (FSIMP). Indeed, we have

$$\sum_{k=1}^{2} \alpha_k f_k(x) + \sum_{k=1}^{2} \alpha_k \varepsilon_k \|x - \bar{x}\| = \frac{x}{x^4 + 1} + \frac{1}{2}|x| \geq 0 = \sum_{k=1}^{2} \alpha_k f_k(\bar{x}), \qquad \forall x \in C.$$

On the other hand, we have $N(\bar{x}; \Omega) = N(\bar{x}; (-\infty, 0]) = [0, +\infty)$ and $\partial g_t(\bar{x}) = \{0\}$, for any $t \in T$,

$$\bigcup_{\lambda \in A(\bar{x})} \left[\sum_{t \in T} \lambda_t \partial g_t(\bar{x}) \right] + N(\bar{x}; \Omega) = [0, +\infty).$$

Moreover, $N(\bar{x}; C) = N(\bar{x}; \{0\}) = \mathbb{R}$. Clearly, the qualification condition (MCQ) is not satisfied at $\bar{x}$. We have $\partial p_1(\bar{x}) = \partial p_2(\bar{x}) = \{1\}, \partial q_1(\bar{x}) = \partial q_2(\bar{x}) = \{0\}$. On the other hand, take $\varepsilon = (\varepsilon_1, \varepsilon_2) \in \mathbb{R}^2_+ \backslash \{0\}$ with $\varepsilon_1 = \varepsilon_2 = \frac{1}{2}$, $\alpha = (\alpha_1, \alpha_2) \in \text{int}\mathbb{R}^2_+$ with $\alpha_1 + \alpha_2 = 1$ and $\mathbb{B}_{\mathbb{R}} = [-1, 1]$. It is easy to see that $\beta_k = \frac{\alpha_k}{q_k(\bar{x})} = \alpha_k, k = 1, 2$ and

$$\begin{aligned} 0 \notin \left[\frac{1}{2}, +\infty\right) = \{1\} + \left[-\frac{1}{2}, \frac{1}{2}\right] + [0, +\infty) &= \sum_{k=1}^{2} \beta_k \left(\partial p_k(\bar{x}) - \frac{p_k(\bar{x})}{q_k(\bar{x})} \partial q_k(\bar{x}) \right) + \sum_{t \in T} \lambda_t \partial g_t(\bar{x}) \\ &+ \sum_{k=1}^{2} \alpha_k \varepsilon_k \mathbb{B}_{\mathbb{R}} + N(\bar{x}; \Omega), \end{aligned}$$

for any $\lambda \in A(\bar{x})$. Thus, the assertion of Theorem 5.2 is not valid because the qualification condition (MCQ) does not hold at $\bar{x} = 0$.

The following simple example proves that, in general, a feasible point of the problem (FSIMP) satifying condition (41) is not necessarily to be an ε-quasi positively properly efficient solution of the problem (FSIMP) even in the smooth case.

Example 5.4. Let $f : \mathbb{R} \to \mathbb{R}^2$ be defined by

$$f(x) = \left(\frac{p_1(x)}{q_1(x)}, \frac{p_2(x)}{q_2(x)} \right)$$

where $p_1(x) = p_2(x) = x^5, q_1(x) = q_2(x) = x^4 + 1, x \in \mathbb{R}$ and let $g_t : \mathbb{R} \to \mathbb{R}$ be given by

$$g_t(x) = -t^3 x^4, x \in \mathbb{R}, t \in T = [1, 2].$$

We consider problem (FSIMP) with $m = 2$ and $\Omega = (-\infty, 0] \subset \mathbb{R}$. By simple computation, one has $C = (-\infty, 0]$. Let us check $\bar{x} = 0 \in C$. Note that $N(\bar{x}; \Omega) = N(\bar{x}; (-\infty, 0]) = [0, +\infty)$ and $\partial p_1(\bar{x}) = \partial p_2(\bar{x}) = \{0\}, \partial q_1(\bar{x}) = \partial q_2(\bar{x}) = \{0\}, \partial g_t(\bar{x}) = \{0\}, \forall t \in T$. On the other hand, take $\varepsilon = (\varepsilon_1, \varepsilon_2) \in \mathbb{R}^2_+ \backslash \{0\}$ with $\varepsilon_1 = \varepsilon_2 = \frac{1}{2}$ and $\alpha = (\alpha_1, \alpha_2) \in \text{int}\mathbb{R}^2_+$ with $\alpha_1 + \alpha_2 = 1, \mathbb{B}_{\mathbb{R}} = [-1, 1]$. It is easy to see that $\beta_k = \frac{\alpha_k}{q_k(\bar{x})} = \alpha_k, k = 1, 2$ and

$$\begin{aligned} 0 \in \left[-\frac{1}{2}, +\infty\right) = \left[-\frac{1}{2}, \frac{1}{2}\right] + [0, +\infty) &= \sum_{k=1}^{2} \beta_k \left(\partial p_k(\bar{x}) - \frac{p_k(\bar{x})}{q_k(\bar{x})} \partial q_k(\bar{x}) \right) + \sum_{t \in T} \lambda_t \partial g_t(\bar{x}) \\ &+ \sum_{k=1}^{2} \alpha_k \varepsilon_k \mathbb{B}_{\mathbb{R}} + N(\bar{x}; \Omega), \end{aligned}$$

for any $\lambda \in A(\bar{x})$. Thus, condition (41) is satisfied at $\bar{x} = 0$. However, $\bar{x} = 0$ is not an ε-quasi positively properly efficient solution of the problem (FSIMP). To see this, we can choose $x = -2 \in C, \varepsilon = (\varepsilon_1, \varepsilon_2) \in \mathbb{R}^2_+ \backslash \{0\}$ with $\varepsilon_1 = \varepsilon_2 = \frac{1}{2}$ and $\alpha = (\alpha_1, \alpha_2) \in \mathrm{int}\mathbb{R}^2_+$ with $\alpha_1 + \alpha_2 = 1$. Then, it is easy to see that

$$\sum_{k=1}^{2} \alpha_k f_k(x) + \sum_{k=1}^{2} \alpha_k \varepsilon_k \|x - \bar{x}\| = \frac{x^5}{x^4+1} + \frac{1}{2}|x| = -\frac{15}{17} < 0 = \sum_{k=1}^{2} \alpha_k f_k(\bar{x}).$$

Remark 5.5. Theorem 5.2 improves Theorem 7 in [27] and Theorem 3.3 in [34].

6. Conclusion

In this paper, we applied some advanced tools of variational analysis and generalized differentiation to establish the optimality conditions for an ε-quasi positively properly efficient solution of the problem (SIMP). Besides, we also investigated Mond–Weir type dual problems with respect to the problem (SIMP). On the other hand, an application to the problem (FSIMP) was derived.

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