THE \((a, b)\)-MONOCHROMATIC TRANSVERSAL GAME ON
CLIQUE-HYPERGRAPHS OF POWERS OF CYCLES\(^*, **\)

WILDER P. MENDES\(^1\), SIMONE DANTAS\(^2, *\) AND SYLVAIN GRAVIER\(^3\)

Abstract. We study the \((a, b)\)-monochromatic transversal game that is a combinatorial Maker–Breaker game where Alice and Bob alternately colour \(a\) vertices in red and \(b\) vertices in blue of a hypergraph, respectively. Either player is enabled to start the game. Alice tries to construct a hyperedge transversal, and Bob tries to prevent this. The winner is Alice if she obtains a red hyperedge transversal; otherwise, Bob wins the game if he obtains a monochromatic blue hyperedge. Maker–Breaker games were determined to be PSPACE-complete. In this work, we analyze the game played on clique-hypergraphs of powers of cycles, and we show strategies that, depending on the choice of the parameters, allow a specific player to win the game.

Mathematics Subject Classification. 05C57, 05C15, 05C65.

Received November 11, 2022. Accepted March 4, 2024.

1. Introduction

The concept of hypergraphs has been extensively studied in many areas. It generalizes the definition of graphs allowing edges to have more than two incident vertices. Hypergraph theory is also applied in modern mathematics and related research fields. For instance, Torres and Wagler [22] studied a model to the dynamics of discrete deterministic systems represented by a hypergraph; and Groshaus and Szwarcfiter [16] described polynomial time algorithms for recognizing bipartite-conformal and bipartite-Helly hypergraphs.

A transversal in a hypergraph is a set of vertices intersecting every hyperedge [3], which is a generalization of the concept of a vertex cover in a graph. A clique of a graph is a subset of its vertices that induces a complete graph. A clique-transversal of a graph \(G\) is a subset of vertices intersecting all the cliques of \(G\). The first NP-hardness result for clique-transversals was obtained in 1992 by Erdős et al. [13]. In 2008, Duran et al. [11] proposed an algorithm for determining the minimum cardinality \(\tau_c\) of a clique-transversal of a general graph \(G\), which runs in polynomial time, whenever \(\tau_c\) is fixed.

Keywords. Combinatorial games, hypergraphs, transversal.

* Submitted to RAIRO-OR Special issue Graphs, Combinatorics, Algorithms and Optimization.
** Dedicated to healthcare professionals and essential workers amid the COVID-19 pandemic.

1 DEX, Universidade do Estado de Minas Gerais, Belo Horizonte, Brazil.
2 IME, Universidade Federal Fluminense, Niterói, Brazil.
3 CNRS, Université Grenoble Alpes, Saint-Martin-d’Hères, France.
*Corresponding author: sdantas@id.uff.br

© The authors. Published by EDP Sciences, ROADEF, SMAI 2024

This is an Open Access article distributed under the terms of the Creative Commons Attribution License (https://creativecommons.org/licenses/by/4.0), which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.
A maximal clique is a clique that is not properly contained in any other clique. A clique-hypergraph of a graph $G$ is a hypergraph with the same vertex set of $G$, and whose hyperedges are the maximal cliques of the graph $G$.

The study of clique-hypergraphs was firstly presented in 1991 by Duffus et al. [10]. The authors asked what is the smallest number of colours needed to colour the vertices of a clique-hypergraph so that no hyperedge of size at least two is monochromatic (clique-chromatic number). Later, in 2003, Gravier et al. [15] showed that, given a fixed graph $F$, there exists an integer $f(F)$ such that the clique-hypergraph of any $F$-free graph can be $f(F)$-coloured if and only if all components of $F$ are paths. After that, in 2004, Bacsó et al. [2] proved that this number is 3 for almost all perfect graphs. In 2013, the answer to Duffus’ question in the case of the clique-chromatic number of powers of cycles was presented by Campos et al. [7], where the authors showed that such number is equal to 2, except for odd cycles of size at least 5 where the answer is 3.

Classic problems in graph theory, like colouring, labeling and domination, have been studied from the perspective of Combinatorial games (see [1, 8, 14, 20]). For example, the combinatorial solitaire Clobber game, which is essentially an optimization problem, was investigated in the class of complete multipartite graphs by Duchêne et al. [9].

Combinatorial games [4], like Solitaire Clobber [9], are alternating finite two-player games of pure strategy in which all the relevant information is public to both players, as well as no randomness or luck is allowed.

A combinatorial game using the concept of transversals in hypergraphs is the transversal game, presented by Bujtás et al. [5, 6]. In this game, two players, Edge-hitter and Staller, take turns choosing a vertex from $\mathcal{H}$. Each chosen vertex must hit at least one edge not hit by the vertices previously chosen. The game ends when the set of selected vertices becomes a transversal in $\mathcal{H}$. The strategy of Edge-hitter is to minimize the number of chosen vertices, while Staller wishes to maximize it. The authors defined the game transversal number of $\mathcal{H}$ as the number of vertices chosen when Edge-hitter starts the transversal game and both players play optimally. Using this definition, they showed that the $\frac{3}{2}$-Game Total Domination Conjecture [18] is true over the graph classes with minimum degree at least 2, and compared this parameter with its transversal number.

The area of combinatorial games also includes Maker–Breaker games which were introduced by Erdős and Selfridge [12] in 1973, and has been extensively studied as described in the survey of Hefetz et al. [17]. In these games, Maker wins if he manages to hold all the elements of a winning set, while Breaker wins if he manages to prevent this. Draws do not occur, so, in each Maker–Breaker game, either Maker or Breaker has a winning strategy. The formal definition is given as follows:

**Definition 1.1** ([17]). Let $X$ be a finite set and let $\mathcal{F} \subseteq 2^X$ be a family of subsets. In a Maker–Breaker game over the hypergraph $(X, \mathcal{F})$:

(i) the set $X$ is called the board; the elements of $\mathcal{F} \subseteq 2^X$ are the winning sets;
(ii) the players are called Maker and Breaker;
(iii) during a particular play, the players alternately occupy elements of $X$; as a default, we set Maker to start (unless stated otherwise);
(iv) the winner is
   (a) Maker if he occupies a winning set completely by the end of the game;
   (b) Breaker if he occupies an element in every winning set.

In this work, we consider a combinatorial Maker–Breaker game over hypergraphs: the $(a, b)$-monochromatic transversal game. In this game, the set $X$ is the set of vertices $V$ of a hypergraph $\mathcal{H} = (V, E)$, and $\mathcal{F}$ is the set $\mathcal{E}$ of its hyperedges. Alice and Bob alternately colour $a$ vertices in red and $b$ vertices in blue of a hypergraph, respectively. Either player is enabled to start the game. Alice (Maker) tries to construct a hyperedge transversal, and Bob (Breaker) tries to prevent this. The winner is Alice if she obtains a red hyperedge transversal; otherwise, Bob wins the game if he obtains a monochromatic blue hyperedge. We observe that there always exists a winner of the game: for any red and blue colouring of the vertices of a hypergraph $\mathcal{H}$, there exists a monochromatic blue hyperedge if and only if the set of red vertices is not a transversal of the hyperedges of $\mathcal{H}$. 
From the perspective of complexity, the \((a, b)\)-MTG is PSPACE-complete since Maker–Breaker games were determined to be PSPACE-complete by Schaefer [21], where the game was mentioned as POS CNF. We consider the game played on clique-hypergraphs of complete graphs, cycles, paths, and powers of cycles. For each of these graphs, we show strategies and bounds to the parameters \(a\) and \(b\) so that one of the players wins the game.

We organize the paper as follows. First, in Section 2, we introduce the game and present standard definitions and notation. In Section 3, we start playing the game considering that Alice and Bob colour a single vertex at a time on complete graphs, powers of cycles and paths. In Section 4, we exhibit results when Alice and Bob are allowed to colour more vertices per turn on powers of cycles \(C_n^k\), and show that Alice has advantage on the \((a, b)\)-MTG played on the hypergraphs of these graphs when \(a > b\) and the number of vertices \(n > 3k\), \(k \geq 2\). In Section 5, we prove that, for \(n\) sufficiently large, Bob wins the game when \(b > a\) on clique-hypergraphs of powers of cycles. In Section 6, we show how “small” \(n\) must be in order to guarantee Alice’s victory. Finally, we present our conclusions in Section 7.

2. Description of the game

In this work, all graphs are finite, undirected and simple. A hypergraph \(\mathcal{H}\) is a pair \((\mathcal{V}, \mathcal{E})\) where \(\mathcal{V}\) is a finite vertex set, and \(\mathcal{E}\) is a family of nonempty subsets of \(\mathcal{V}\) called hyperedges.

The \((a, b)\)-monochromatic transversal game, called \((a, b)\)-MTG, is a combinatorial Maker–Breaker game where two players, Alice and Bob, alternately take turns colouring the vertices of a hypergraph \(\mathcal{H} = (\mathcal{V}, \mathcal{E})\) (either player may start the game). On each turn, either Alice colours \(a \in \mathbb{N}^*\) vertices in red or Bob colours \(b \in \mathbb{N}^*\) vertices in blue, where \(\mathbb{N}^* = \mathbb{N} \setminus \{0\}\). Alice tries to construct a hyperedge transversal, and Bob tries to prevent this. Alice wins the game if she obtains a red hyperedge transversal, that is, a subset of vertices of \(\mathcal{V}\) that has a nonempty intersection with every hyperedge of \(\mathcal{H}\). Bob wins the game if he obtains a monochromatic blue hyperedge of \(\mathcal{E}\).

The \((a, b)\)-MTG presents properties that are inherited from the general ones established by Hefetz et al. [17] to the Maker–Breaker game. Next, we show their version adapted to the \((a, b)\)-MTG notation.

Remark 2.1 ([17], Prop. 2.1.6). If there exists a strategy that allows Alice (resp. Bob) to win when Bob (resp. Alice) starts playing the \((a, b)\)-MTG on a given hypergraph, then there exists a strategy that allows Alice (resp. Bob) to win when she (resp. he) starts playing the game on that hypergraph.

Remark 2.2 ([17], Rem. 2.1.4). If there exists a strategy that allows Alice (resp. Bob) to win the \((a_0, b_0)\)-MTG played on a given hypergraph, independently of who starts it, then for any \(a > a_0\) (resp. \(b > b_0\)) there exists a strategy that allows Alice (resp. Bob) to win the \((a, b)\)-MTG (resp. \((a_0, b)\)-MTG) played on that hypergraph, independently of who starts it.

The clique-hypergraph \(\mathcal{H}(G) = (V, \mathcal{E})\) of a graph \(G = (V, E)\) is a hypergraph such that \(V\) is the vertex set of \(G\), and the hyperedge set \(\mathcal{E}\) is the set of all maximal cliques in \(G\), i.e., \(\mathcal{E}\) is the set of all maximal subsets of \(V\) whose vertices induce a complete graph in \(G\). In this work, we consider clique-hypergraphs of powers of cycles.

A \(k\)-th power of cycle of length \(n\), \(C_n^k\), for \(k \geq 1\), is a graph on \(n\) vertices whose vertex set is \(V(C_n^k) = \{v_i : i \in \mathbb{Z}_n\}\), and whose edges \(\{v_i, v_j\}, i, j \in \mathbb{Z}_n\) have the property that \(i = j \pm r\) (mod \(n\)) for some \(r \in \{1, 2, \ldots, k\}\). We take \((v_0, \ldots, v_{n-1})\) to be a cyclic order on the vertex set of \(C_n^k\), and always perform arithmetic modulo \(n\) on vertex indexes. The neighborhood of a vertex \(v\), denoted by \(N(v)\), is the set of all the vertices adjacent to \(v\). The reach of an edge \(\{v_i, v_j\}\) is defined as \(d_{ij} = \min\{(i - j) \mod n, (j - i) \mod n\}\). Observe that, with the previous definition, \(C_n^1\) is a cycle \(C_n\), and for \(k \geq \left\lfloor \frac{n}{2} \right\rfloor\), graph \(C_n^k\) is isomorphic to the complete graph.

According to [7], the maximal cliques of powers of cycles \(C_n^k\) can be classified into two types: an external clique, whose vertex set is composed by \(k + 1\) vertices with consecutive indexes \(v_x, \ldots, v_{x+k(mod n)}\), for some \(x \in \mathbb{Z}_n\), and an internal clique that has non-consecutive vertex indexes. Figure 1 illustrates powers of cycles with and without internal maximal cliques.
We observe that if $b > k$, then the $(a, b)$-MTG played on clique-hypergraphs of powers of cycles becomes trivial because, on his first turn, Bob colours in blue all the vertices of an external maximal clique. Similarly, if $a \geq \lceil \frac{n}{k+1} \rceil$ and $C^k_n$ has no internal maximal cliques, then Alice colours in red all the vertices of a transversal.

Furthermore, observe that if Bob can not colour a clique of size $k'$ while playing the $(a, b)$-MTG on a clique-hypergraph $\mathcal{H}(C^{k'}_n)$, then he can not colour a clique of size $k$ for $k > k'$. By this observation we obtain the following remark:

**Remark 2.3.** Let $C^k_n$ be a power of cycle with no internal maximal cliques and let $k'$ be such that $2 \leq k' < k$. Any clique of $C^k_n$ contains a clique of $C^{k'}_n$ and so any transversal of $C^{k'}_n$ is also a transversal of $C^k_n$. If there exists a strategy that allows Alice to win the $(a, b)$-MTG played on the clique-hypergraph $\mathcal{H}(C^{k'}_n)$, then there exists a strategy that allows Alice to win the $(a, b)$-MTG played on the clique-hypergraph $\mathcal{H}(C^k_n)$.

### 3. First Plays

We begin this section by analyzing the $(1, 1)$-MTG played on clique-hypergraphs of powers of cycles $C^k_n$. First, we show strategies that can be used when $k$ has either its smallest or largest possible value, that is, $C^k_n$ (when $k = 1$) or $C^k_n = K_n$ (when $k \geq \lceil \frac{n}{2} \rceil$).

**Proposition 3.1.** If $K_n$ is a complete graph with $n \geq 2$, then there exists a strategy that allows Alice to win the $(1, 1)$-MTG played on the clique-hypergraph $\mathcal{H}(K_n)$, independently of who starts playing the game.

**Proof.** Suppose that Bob starts playing the game. We observe that the clique-hypergraph $\mathcal{H}(K_n)$ of the complete graph $K_n$ on $n$ vertices contains a unique hyperedge with $n$ vertices. Therefore, Alice obtains a red transversal on her first turn and wins. The result follows by Remark 2.1.

Now we analyze the game played on the clique-hypergraph of a cycle. Since $C_3$ is isomorphic to the complete graph $K_3$ (Prop. 3.1), we consider $n \geq 4$.

**Proposition 3.2.** If $C_n$ is a cycle of length $n \geq 4$, then there exists a strategy that allows Bob to win the $(1, 1)$-MTG played on the clique-hypergraph $\mathcal{H}(C_n)$, independently of who starts playing the game.

**Proof.** First, we observe that the hyperedges of $\mathcal{H}(C_n)$ are the edges of $C_n$. Suppose that Alice starts the game and colours the vertex $v_l$ red. Bob wins the game colouring a vertex $v_j$ blue that is not adjacent to $v_l$. Indeed, since $n \geq 4$, independently of which vertex Alice colours in red on her next turn, she does not obtain a transversal and, on his next turn, Bob obtains a monochromatic blue hyperedge colouring a non-coloured vertex $v_l$ adjacent to $v_j$, with $l \in \{j - 1 \mod n, j + 1 \mod n\}$. The result follows by Remark 2.1.
The (1,1)-MTG played on clique-hypergraphs \( \mathcal{H}(P_n) \) of paths \( P_n \) on \( n \) vertices, \( n \geq 3 \), is similar to the (1,1)-MTG over \( \mathcal{H}(C_n) \), \( n \geq 4 \). If \( n = 2 \), since \( P_2 \) is isomorphic to \( K_2 \), then the result follows by Proposition 3.1. If \( 3 \leq n \leq 5 \), then there exists a strategy that allows the player who had the first turn to win the game. Indeed, if Bob starts playing, he colours vertex \( v_{\lfloor n/2 \rfloor} \) blue. Now, Alice must colour \( v_{\lfloor n/2 \rfloor-1} \) or \( v_{\lfloor n/2 \rfloor+1} \). On his next turn, Bob obtains a blue hyperedge colouring \( v_{\lfloor n/2 \rfloor+1} \) or \( v_{\lfloor n/2 \rfloor-1} \). If Alice starts playing, she colours vertex \( v_{\lfloor n/2 \rfloor} \) red. On the next turns, independently of which vertex Bob colours blue, Alice wins the game colouring in red the vertex adjacent to Bob’s last coloured vertex, and prevents him from making a monochromatic blue \( P_2 \). If \( n \geq 6 \), an argument analogous to the proof of Proposition 3.2 shows that there exists a strategy that allows Bob to win the game, independently of who starts playing the game.

4. Alice’s dream

Now, we consider the \((a,b)\)-MTG played on the clique-hypergraph \( \mathcal{H}(C^k_n) \). We recall that the maximal cliques of powers of cycles can be classified into external and internal cliques. The next result presents a lower bound to the size of \( n \) that guarantees the non-existence of internal maximal cliques.

**Lemma 4.1.** If \( n > 3k \) and \( k \geq 2 \), then \( C^k_n \) has no internal maximal clique.

**Proof.** Let \( K \) be a maximal clique of \( C^k_n \). Without loss of generality we may assume that \( v_0 \in K \). Let \( i, j \in \mathbb{N}^* \), \( i \neq j \), where: (1) \( i \leq k \) and \( j \geq k \) are largest as possible and such that (2) \( v_{n-j}, v_i \in K \). Since \( n > 3k \), and by (2), the reach between the vertices \( v_{n-j} \) and \( v_i \) is \( d_{i,n-j} = i + j \leq k \). By (1), \( v_s \notin K \) for each \( s \in [k+1, n-(k+1)] \) and \( v_{n-s} \notin K \) for each \( s \in [k+1, n-(k+1)] \). Moreover, for each \( s \in [k+1, n-(k+1)] \), since \( v_0 \in K \) and \( d_{0,s} > k \), we have that \( v_s \notin K \). Now, since the subgraph \( H \) induced by \( \{v_{n-j}, \ldots, v_0, \ldots, v_i\} \) is a clique, it follows that \( K = H \) and \( i + j = k \). \( \square \)

Now, the following theorem shows that the condition \( a \geq b \) ensures that Alice wins the game when \( C^k_n \) has no internal maximal cliques.

**Theorem 4.2.** Let \( n > 3k \) and \( k \geq 2 \). If \( a \geq b \) and \( b < k \), then there exists a strategy that allows Alice to win the \((a,b)\)-MTG played on the clique-hypergraph \( \mathcal{H}(C^k_n) \), independently of who starts playing.
Proof. Suppose that Bob starts playing. Bob’s turns consist of colouring in blue $\ell$ disjoint paths $P_j$ in the cycle $C_n$, for $j = 0, \ldots, \ell - 1$ and $\ell \leq b$. By definition, we have that $\sum_{j=0}^{\ell-1}|P_j| \leq b < k$. Alice’s strategy consists in applying the following rules:

1. If $P_j$ has one vertex, say $v_i$, then Alice colours $v_i+1$ red if it is not coloured yet; otherwise she colours $v_{i-1}$ red.

2. If $P_j$ has two distinct extremities $v_i$ and $v_s$ with $i < s$, then Alice colours vertices $v_{i-1}$ and $v_{s+1}$ red, if they are not coloured yet.

First, observe that Alice colours at least $b$ vertices (and at most $a$ vertices) per turn. Second, since $b < k$, Bob must play at least twice in a set of $k+1$ consecutive vertices (interval) to colour all its vertices blue. Suppose that, at some turn, for the sake of contradiction, Bob succeeds to colour in blue an interval $I$ of length $k + 1$. We show that at some turn Alice colours in red a vertex belonging to $I$. Let $t$ be the first turn Bob plays in $I$. Let $P^0, \ldots, P^j$ be the ordered clockwise disjoint paths intersecting $I$ at turn $t$. If $j > 0$ then let $v_i$ be the right (clockwise) extremity of $P^0$. By Alice’s strategy, the vertex $v_{i+1}$ is coloured in red at turn $t+1$, which yields a contradiction. So $j = 0$. Now, if $|P^0| \geq 2$ then let $v_i$ and $v_s$ with $i < s$ be the extremities of $P^0$. Since $b < k$, then $v_{i-1}$ or $v_{s+1}$ belongs to $I$. Now, by rule (2), Alice colours them at turn $t + 1$, which give again a contradiction. Hence, $P^0$ is reduced to a single vertex $v_i$, and, by rule (1), $v_i$ is the right extremity of $I$.

Let $t'$ be the second turn Bob plays in $I$. Let $v_t$ be the vertex coloured in blue at turn $t'$ closest to $v_i$. If $r < i - 1$ then, by Alice’s strategy, $v_{r+1}$ is coloured in red at turn $t' + 1$, a contradiction. So, $r = i - 1$. Let $P^r$ be the blue path coloured at turn $t'$ containing $v_r$, and let $v_t$ be the left extremity of $P^r$. Remark that, since $|P^r| \leq b + 1 \leq k$, vertex $v_{i-1} \in I$. By Alice’s strategy, $v_{i-1}$ is coloured in red at turn $t' + 1$ which gives again a contradiction with the existence of $I$. The result follows by Remark 2.1.

The following result shows that case $b = k$ forces $a$ to be higher to ensure Alice’s victory. In fact, if $b = k$ then, when $n$ is large enough, Alice wins only if $a \geq 2b$.

Theorem 4.3. Let $a \geq b = k \geq 2$ and $n \geq (2k + 1)k$. Bob wins the $(a, b)$-MTG played on the clique-hypergraph $\mathcal{H}(C_n^k)$ when he starts playing if and only if $a \geq 2b$.

Proof. Suppose that $a \geq 2b$. Alice applies rule (1) of the proof of Theorem 4.2, even for path with only one vertex. So, for each path coloured blue by Bob, at some turn $t$, with extremities $v_i$ and $v_s$ with $i \leq s$, Alice colours, at turn $t+1$, the vertices $v_{i-1}$ and $v_{s+1}$ red. This ensures that there exist at most $b = k$ blue consecutive vertices. Hence, Alice wins.

Now, suppose that $a < 2b$ and $n \geq (2k + 1)k$. Bob starts playing and colours blue the vertices of indices in $\{(2k + 1)p \mid p = 0, \ldots, k - 1\}$. Now, Alice colours in red at most $a < 2b$ vertices among the vertices of indices belonging to the $2b$ intervals $[(2k + 1)p, (2k + 1)p + k] \cup [(2k + 1)p - k, (2k + 1)p]$, for each $p = 0, \ldots, k - 1$. Hence, on the second turn, there exists $q \in \{0, \ldots, k - 1\}$ such that none of the vertices of index in $[(2k + 1)q, (2k + 1)q + k]$ or $[(2k + 1)q - k, (2k + 1)q]$ is coloured red. Let $I$ be such interval with no red vertex. Since $I$ is a set of $k$ vertices, at the third turn, Bob colours all of them and wins with hyperedge $v_{(2k + 1)q} \cup \{v_i \mid i \in I\}$ coloured blue.

5. Bob is the winner

In the previous sections, we have shown that Alice has advantage with respect to Bob when they play the $(a, b)$-MTG with $a \geq b$ and $b < k$ on clique-hypergraphs of powers of cycles $\mathcal{H}(C_n^k)$. In this section, we consider $1 \leq a < b$ and define conditions so that Bob wins the $(a, b)$-MTG played on these hypergraphs.

First we present the following definition. A $(k+1)$-matching of cardinality $m$ of a hypergraph $\mathcal{H}$ is the disjoint union of $m$ hyperedges of size $k + 1$.

Lemma 5.1. Let $a, b$ and $k$ be such that $1 \leq a < b \leq k + 1$ and $k \geq 2$. Let $p = \lfloor \frac{b}{a + 1} \rfloor$ and $\alpha$ be the positive integer such that $b + (\alpha - 1)p < k + 1 \leq b + \alpha p$. Bob wins the $(a, b)$-MTG played on a hypergraph $\mathcal{H}$ that contains $(k + 1)$-matching of cardinality $(a + 1)\alpha$ when he starts playing.
Proof. Let \(a, b\) and \(k\) be such that \(1 < a < b \leq k + 1\) and \(k \geq 2\). Let \(p = \left\lfloor \frac{b}{a+1} \right\rfloor\) and \(\alpha\) be the positive integer such that \(b + (\alpha - 1)p < k + 1 \leq b + \alpha p\). Let \(\mathcal{M}\) be a \((k+1)\)-matching of cardinality \((a+1)\alpha\).

The idea of Bob's strategy is that he always chooses \(a + 1\) hyperedges and colours blue \(p\) vertices in each hyperedge that has no red vertex (except on his last turn). Bob aims to have a hyperedge with \(\alpha p\) vertices blue and no red ones to play on his last turn, say \(2t_\alpha + 1\). Applying this basic strategy, Bob needs at least \(a + 1\) hyperedges with \((\alpha - 1)p\) blue vertices and no red one at turn \(2t_\alpha - 1\) otherwise Alice will be able to color one vertex on each hyperedge with \(\alpha p\) blue vertices at turn \(2t_\alpha\). The rest of the proof consists in generalizing this last argument: for any \(0 < q \leq \alpha\), how many hyperedges having \((q - 1)p\) blue vertices and no red ones, Bob needs at some turn \(2t_q - 1\) for ensuring that at turn \(2t_q + 1\), he has "enough" hyperedges having \(q\) blue vertices and no red ones?

We define \(t_0 = 0\) and \(t_q = \sum_{j=\alpha-q}^{\alpha-1} (a+1)^j\) for \(0 < q \leq \alpha\). Let \(\mathcal{M}_q\) be the subset of hyperedges in \(\mathcal{M}\) containing \(q\) blue vertices and no red vertex after \(2t_q\) turns. We claim that:

**Claim 5.2.** Bob has a strategy that ensures that, for all \(q \in \{0, \ldots, \alpha\}\), we have that \(|\mathcal{M}_q| \geq (a+1)^{\alpha-q}\).

Proof. For a fixed \(q \in \{1, \ldots, \alpha\}\), at turn \(2t + 1\) with \(t \in \{t_{q-1}, \ldots, t_q - 1\}\), Bob's strategy consists in selecting \(a + 1\) hyperedges of \(\mathcal{M}_q\) and colouring \(p\) vertices of the ones that do not contain a red vertex. Moreover, hyperedges selected at turn \(2t + 1\) must not have been previously selected at some turn \(2t' + 1 < 2t + 1\) and \(t' \geq t_{q-1}\).

Let \(t \in \{t_{q-1}, \ldots, t_q - 1\}\), at turn \(2t + 1\), Bob colours at most \(p\) vertices of \((a+1)\) edges which is allowed since \(b \geq (a+1)p\). Also, any selected hyperedge with no red vertex has \(qp\) blue vertices after \(2t + 1\) turns.

Now, we are ready to prove the claim by induction on \(q\). By definition, we have that \(|\mathcal{M}_0| \geq (a+1)^\alpha\). First, suppose that, for some \(j > 0\), we have \(|\mathcal{M}_j| \geq (a+1)^{\alpha-j}\) holds for all \(q < j\).

For all turns \(2t + 1\), when \(t \in \{t_{j-1}, \ldots, t_j - 1\}\), Bob selects \((t_j - t_{j-1})(a+1) = (a+1)^{\alpha-j+1}\) hyperedges of \(\mathcal{M}_{j-1}\) (which is possible since, by induction hypothesis, \(|\mathcal{M}_{j-1}| \geq (a+1)^{\alpha-j+1}\)), while Alice could not colour more than \((t_j - t_{j-1})a = (a+1)^{\alpha-j}a\) hyperedges for all turns \(2t + 2\), when \(t \in \{t_{j-1}, \ldots, t_j - 1\}\). Therefore, \(|\mathcal{M}_j| \geq (a+1)^{\alpha-j+1} - a(a+1)^{\alpha-j} = (a+1)^{\alpha-j}\).

The previous claim shows that \(|\mathcal{M}_\alpha| \geq 1\). Moreover, any hyperedge in \(\mathcal{M}_\alpha\) has \(\alpha p\) blue vertices after \(2t_\alpha\) turns and contains no red one. So, at turn \(2t_\alpha + 1\), Bob selects one hyperedge of \(\mathcal{M}_\alpha\) and colours the remaining \((k+1) - \alpha p\) vertices which are less than \(b\).

By the proof of Lemma 5.1, Bob wins the game in at most

\[
\frac{2(a+1)^\alpha - 1}{a} + 1 \text{ turns.}
\]

Moreover, we remark that if \(n \geq m(k+1)\), then the clique-hypergraph \(\mathcal{H}(C^k_n)\) contains a \((k+1)\)-matching of cardinality \(m\). So, the following theorem is a direct consequence of Lemma 5.1.

**Theorem 5.3.** Let \(a, b,\) and \(k\) be such that \(1 < a < b \leq k + 1\) and \(k \geq 2\). Let \(p = \left\lfloor \frac{b}{a+1} \right\rfloor\) and \(\alpha\) be the positive integer such that \(b + (\alpha - 1)p < k + 1 \leq b + \alpha p\). If \(n \geq (a+1)^\alpha(k+1)\) then Bob wins the \((a, b)\)-MTG played on a clique-hypergraph \(\mathcal{H}(C^k_n)\) when he starts playing.

Now, we consider the case when Alice starts.

**Corollary 5.4.** Let \(a, b,\) and \(k\) be such that \(1 < a < b \leq k + 1\). Let \(p = \left\lfloor \frac{b}{a+1} \right\rfloor\) and \(\alpha\) be the positive integer such that \(b + (\alpha - 1)p < k + 1 \leq b + \alpha p\). Bob wins the \((a, b)\)-MTG played on a \((k+1)\)-matching of cardinality at least \(a + (a+1)^\alpha\) independently of who starts playing.

Proof. By Lemma 5.1, it is enough to consider the case when Alice starts. After Alice's first turn, it remains at least \((a+1)^\alpha\) hyperedges. Now, it is Bob's turn, and again, by Lemma 5.1, Bob wins. \(\square\)
This result implies the following corollary:

**Corollary 5.5.** Let $a$, $b$, and $k$ be such that $1 \leq a < b \leq k + 1$. Let $p = \lfloor \frac{b}{a} \rfloor$ and $\alpha$ be the positive integer such that $b + (\alpha - 1)p < k + 1 \leq b + \alpha p$.

If $n \geq (a + (a + 1)^p)(k + 1)$ then Bob wins the $(a, b)$-MTG played on a clique-hypergraph $\mathcal{H}(C_n^k)$ independently of who starts playing.

Finally, we observe that, for fixed $\alpha$, the bound for $n$ given in Theorem 5.3 may be improved as shown in Corollary 6.3 when $a = 1$.

6. **Alice’s revenge**

From previous results, the only way for Alice to win the game is to consider a “small” $n$. But how “small” $n$ can be? The following theorem and corollary give an element of the answer.

First, we present a new way of seeing the evolution of the game throughout the turns. Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a hypergraph and set $\mathcal{V}_0 = \mathcal{V}$ and $\mathcal{E}_0 = \mathcal{E}$. For each $t \geq 1$, $B_t$ (resp. $\mathcal{V} \setminus \mathcal{V}_t$) is the set of blue (resp. red) vertices after $t$ turns. We assume that $B_0 = \emptyset$. Let $S_t \subseteq (\mathcal{V}_{t-1} \setminus B_{t-1})$ be the set of vertices coloured at turn $t$.

**Definition 6.1.** A hypergraph $\mathcal{H}_t = (\mathcal{V}_t, \mathcal{E}_t)$ is the hypergraph at turn $t$, where $\mathcal{V}_t$ and $\mathcal{E}_t$ are obtained according to the following rules:

1. If turn $t$ is played by Alice then $\mathcal{V}_t = \mathcal{V}_{t-1} \setminus S_t$, $\mathcal{E}_t$ is the subset of hyperedges $e$ in $\mathcal{E}_{t-1}$ such that $e \cap S_t = \emptyset$, and $B_t = B_{t-1}$.
2. If turn $t$ is played by Bob then $\mathcal{V}_t = \mathcal{V}_{t-1}$, $\mathcal{E}_t = \mathcal{E}_{t-1}$, and $B_t = B_{t-1} \cup S_t$.

The game ends at some $t^*$ such that $\mathcal{V}_{t^*} = B_{t^*}$. Finally, Alice wins if and only if $\mathcal{E}_{t^*} = \emptyset$.

For example, let $t = 0$ and consider the $(2, 2)$-MTG on the clique-hypergraph $\mathcal{H}_0 = \mathcal{H}(C_{10}^2) = (\mathcal{V}_0, \mathcal{E}_0)$, where $\mathcal{V}_0 = \{v_0, v_1, \ldots, v_9\}$ and $\mathcal{E}_0 = \{e_0, e_2, \ldots, e_9\}$ such that $e_0 = \{v_0, v_1, v_2\}$, $e_1 = \{v_1, v_2, v_3\}$, $e_2 = \{v_2, v_3, v_4\}$, $e_3 = \{v_3, v_4, v_5\}$, $e_4 = \{v_4, v_5, v_6\}$, $e_5 = \{v_5, v_6, v_7\}$, $e_6 = \{v_6, v_7, v_8\}$, $e_7 = \{v_7, v_8, v_9\}$, $e_8 = \{v_8, v_9, v_0\}$, and $e_9 = \{v_9, v_0, v_1\}$. Note that, $B_0 = \emptyset$. We refer to Figure 3.

Suppose that, in $t = 1$, Bob colours vertices $v_3$ and $v_4$ in blue, thus $\mathcal{V}_1 = \mathcal{V}_0$, $\mathcal{E}_1 = \mathcal{E}_0$, and $B_1 = B_0 \cup S_1 = \emptyset \cup \{v_3, v_4\} = \{v_3, v_4\}$. Therefore, $\mathcal{H}_1 = (\mathcal{V}_1, \mathcal{E}_1)$ and $B_1 = \{v_3, v_4\}$.

Now, in $t = 2$, if Alice colours vertices $v_2$ and $v_5$ red, then $S_2 = \{v_2, v_5\}$, $\mathcal{V}_2 = \mathcal{V}_1 \setminus \{v_2, v_5\}$, and $\mathcal{E}_2 = \{e_6, e_7, e_8, e_9\}$, since $e_i \cap S_2 = \emptyset$ for $i \in \{6, 7, 8, 9\}$. Therefore, $\mathcal{H}_2 = (\mathcal{V}_2, \mathcal{E}_2)$ and $B_2 = B_1$ (see Fig. 3).

Now we are ready to state the following result.

**Theorem 6.2.** Let $1 \leq a \leq b < k + 1$ and $\beta = \lceil \frac{k+1}{b} \rceil \geq 2$. If $n \in \mathbb{N}^*$ such that $3k < n \leq a(\beta - 1)(k + 1)$, then there exists a strategy that allows Alice to win the $(a, b)$-monochromatic transversal game played on the clique-hypergraph $\mathcal{H}(C_n^k)$, independently of who starts playing.

**Proof.** Suppose that Bob starts playing and assume that $n \leq a(\beta - 1)(k + 1)$. Let $\mathcal{H}_t$ be the hypergraph as in Definition 6.1. We recall that, for $t \geq 1$, $S_{2t}$ is the set of vertices coloured red by Alice at turn $2t$ (Alice plays in even turns); and $B_{2t-1}$ is the set of vertices coloured blue by Bob after $2t - 1$ turns (Bob plays in the odd turns). Without loss of generality, one may assume that $v_0 \notin B_1$.

At turn $2t$ and $1 \leq t < \beta$, Alice’s strategy is to colour red $a$ vertices which, once deleted, create a partition of the hypergraph $\mathcal{H}_{2t-1}$ into $ta - 1$ parts defined by at most $k$ consecutive isolated vertices, and one part $R_t$ defined by at most $n - (ta(k+1) - tb)$ consecutive vertices.

Therefore, at turn $2t$ with $1 \leq t < \beta$, Alice colours in red the vertices $v_i \in S_{2t}$ with $i \in \{i_{(t-1)a}, \ldots, i_{ta-1}\}$ defined by:

(i) $i_0 = 0$.

(ii) If $j > 0$ then the integer $i_j$ is the largest index $i \leq i_{j-1} + k + 1$ such that $v_i \notin B_{2t-1}$. 
Figure 3. Hypergraphs: $\mathcal{H}_0 = \mathcal{H}(C_{10}^2)$ (hyperedge $e_9$ highlighted in orange with a maximal clique in $C_{10}^2$); $\mathcal{H}_1$ after Bob colours blue $v_3$ and $v_4$ in $\mathcal{H}_0$ (blue squares); and $\mathcal{H}_2$ after Alice colours in red $v_2$ and $v_5$ in $\mathcal{H}_1$ (empty red squares represent deleted vertices).

For $0 < j \leq ta - 1$ then the vertices of indices in $\{ij_{j-1} + 1, ij_{j-1}\}$ are isolated vertices in $\mathcal{H}_{2t-1}$ since $i_j - i_j - 1 - 1 \leq k$. Moreover, $i_{ta-1} \geq ta(k + 1) - u$ where $u$ is the number of vertices in $\mathcal{B}_{2t-1}$ with indices belonging to $[1, ta(k + 1)]$. Therefore $i_{ta-1} \geq ta(k + 1) - tb$ and so the remaining set of vertices $R_t$ of $\mathcal{H}_{2t-1}$ has at most $n - (ta(k + 1) - tb)$ vertices.

Hence $|R_{\beta-1}| \leq (\beta - 1)b < k + 1$ since $n \leq a(\beta - 1)(k + 1)$. Therefore, Alice wins the game since the hyperedge set $\mathcal{E}_2(\beta-1) = \emptyset$. The result follows by Remark 2.1.

We note that in Theorem 6.2 the bound on $n$ is linear in $a$ despite of the one of Theorem 5.3 which is in $a^\alpha$. Next result shows that the two bounds are close whenever $\alpha = \beta - 1 = 1$.

**Corollary 6.3.** Let $a, b, p$ and $k$ be integers such that $b \geq p.a$ and $k + 1 = b + p$. There exists a strategy that allows Alice to win the $(a, b)$-monochromatic transversal game played on the clique-hypergraph $\mathcal{H}(C_n^k)$ when Bob starts playing if and only if $n < a(k + 1) + 1$.

**Proof.** First assume that $n \geq a(k + 1) + 1$. Bob’s strategy is a slight improvement of the one given in Theorem 5.3. For $i = 0, \ldots, a - 1$, let $e_i$ be the hyperedge of $\mathcal{H}(C_n^k)$ induced by vertices of indexes $\{i(k + 1), \ldots, i(k + 1) + k\}$. Bob’s first turn consists to colour blue the $p$ vertices of indices $\{i(k + 1), \ldots, i(k + 1) + p - 1\}$ for all $i = 0, \ldots, a - 1$. Now, for $i = 0, \ldots, a - 1$, Alice has to colour red at least one vertex on each $e_i$, otherwise, Bob colours blue all the $k + 1 - p$ remaining vertices in the third turn. Since there exists $a$ such hyperedges then Alice colours exactly one vertex on each $e_i$. Let $v_{i(k + 1) + r_i}$ be the red vertex in $e_i$. By definition $r_i \geq p$ for each $i$.

We claim that if $0 \leq i < j \leq a - 1$ then $r_i \geq r_j$. Indeed, it is enough to observe that if there exists some $i < a - 1$ with $r_i < r_{i+1}$, then the subset of vertices with indexes in $\{i(k + 1) + r_i + 1, \ldots, (i + 1)(k + 1) + r_i\}$
is an hyperedge $e$ of $\mathcal{H}(C_n^k)$ with no red vertex and containing the $p$ blue vertices of $e_{i+1}$, so in the third turn, Bob wins colouring blue the $b$ remaining vertices of $e$.

Now, since $r_0 \geq r_{a-1}$ and $n \geq a(k+1)+1$, then $n-(k+1-r_{a-1}) > (a-1)(k+1)+r_{a-1}$. Hence, the set of vertices with indexes in $\{n-(k+1-r_{a-1}), \ldots, 0, \ldots, r_{a-1}-1\}$ is an hyperedge $e$ of $\mathcal{H}(C_n^k)$ with no red vertex and, since $r_{a-1} \geq p$, $e$ contains the $p$ blue vertices of $e_0$. So, again, in the third turn, Bob wins colouring blue the $b$ remaining vertices of $e$.

Finally, Theorem 6.2 shows that Alice wins whenever $n \leq a(k+1)$. □

7. Conclusion

Combining Remark 2.2 with Proposition 3.1, Proposition 3.2 and by Theorem 4.2, it is possible to complete the following table of results:

<table>
<thead>
<tr>
<th>Hypergraph</th>
<th>Value of $a$</th>
<th>Value of $b$</th>
<th>Who wins (indep. of who started)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{H}(K_n)$</td>
<td>$1 \leq a \leq n-1$</td>
<td>$b &lt; n$</td>
<td>Alice</td>
</tr>
<tr>
<td>$\mathcal{H}(C_n)$, $n \geq 4$</td>
<td>$1$</td>
<td>$b &lt; n$</td>
<td>Bob</td>
</tr>
<tr>
<td>$\mathcal{H}(P_n)$, $n \geq 6$</td>
<td>$1$</td>
<td>$1$</td>
<td>Bob</td>
</tr>
<tr>
<td>$\mathcal{H}(C_n^k)$, $n &gt; 3k$, $k \geq 2$</td>
<td>$a \geq b$</td>
<td>$b &lt; k$</td>
<td>Alice</td>
</tr>
</tbody>
</table>

Furthermore, by Corollary 5.5, taking $p = \lfloor \frac{b}{a+1} \rfloor$ and $\alpha \in \mathbb{N}^*$ such that $b + (\alpha - 1)p < k + 1 \leq b + \alpha p$, we have:

<table>
<thead>
<tr>
<th>Hypergraph</th>
<th>Value of $a$</th>
<th>Value of $b$</th>
<th>Who wins (indep. of who started)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{H}(C_n^k)$, $n \geq a + (a+1)\alpha(k+1)$</td>
<td>$1 \leq a &lt; b$</td>
<td>$b \leq k$</td>
<td>Bob</td>
</tr>
</tbody>
</table>

Also, by Theorem 6.2 taking $\beta = \lceil \frac{k+1}{b} \rceil \geq 2$ for $n \in \mathbb{N}^*$, we have:

<table>
<thead>
<tr>
<th>Hypergraph</th>
<th>Value of $a$</th>
<th>Value of $b$</th>
<th>Who wins (indep. of who started)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{H}(C_n^k)$, $3k &lt; n \leq a(\beta-1)(k+1)$</td>
<td>$1 \leq a \leq b$</td>
<td>$b \leq k$</td>
<td>Alice</td>
</tr>
</tbody>
</table>

We observe that Theorem 6.2 improves Theorem 5.3 when $\alpha = 1$. We conjecture that, when $b$ is close to $a$ and far from $k$, Alice can win even for “large” $n$.

Generally speaking, Theorem 5.1 shows that, for $1 \leq a < b \leq k+1$, Bob wins the $(a,b)$-MTG played on a hypergraph that contains a $(k+1)$-matching of cardinality $(a+1)\alpha$ when he starts playing, where $\alpha$ is the positive integer such that $b + (\alpha - 1)p < k + 1 \leq b + \alpha p$ with $p = \lfloor \frac{b}{a+1} \rfloor$.

The case when $C_n^k$ has internal maximal cliques seems challenging. Next result illustrates a way to study this case. It consists of giving an upper bound on a transversal and so, if $a$ is larger than this bound, this guarantees Alice’s victory whenever she starts.

Theorem 7.1. Let $n = 3k - \varepsilon$ with $k \geq 3$ and $0 \leq \varepsilon < k-1$. The hypergraph $\mathcal{H}(C_n^k)$ admits a transversal of size $k-\varepsilon$. 

(a,b)-MONOCHROMATIC TRANSVERSAL GAME ON CLIQUE-HYPERGRAPHS

1769

Proof. Set $T = K \cup \{v_{2k-1-\varepsilon}\}$ with $K = \{v_0, \ldots, v_{k-2-\varepsilon}\}$. We prove that $T$ is a transversal of $\mathcal{H}(C_n^k)$. Let $K$ be a maximal clique such that $K \cap K = \emptyset$. Since $n = 3k - \varepsilon$, we have that $N(v_{2k-1-\varepsilon}) \supseteq V(C_n^k) \setminus K$. Thus, $K$ must contain the red vertex $v_{2k-1-\varepsilon}$. Therefore, any hyperedge of $\mathcal{H}(C_n^k)$ has at least one vertex belonging to $T$, i.e., $T$ is a transversal of $\mathcal{H}(C_n^k)$ of size

$$\frac{(k-2+\varepsilon) - (0) + 1}{\text{vertices in } \kappa} + \frac{1}{v_{2k-1-\varepsilon}} = k - \varepsilon.$$ 

For the case $n = 3k$ and $k \geq 2$, the gap $\varepsilon$ is vanished, i.e., $\varepsilon = 0$, which proves that this result is optimal.

First, we recall a key lemma of Meidanis [19]:

Lemma 7.2 ([19]). Let $E^k$ be the set of edges of $C_n^k$ with reach $k$. If $n = 3k$, $k \geq 2$, then the subgraph induced by $E^k$ has $k$ connected components, each one being a cycle of length 3.

Theorem 7.3. Let $n = 3k$ with $k \geq 2$, and let $b \geq 3$. There exists a strategy that allows Alice to win the $(a, b)$-MTG played on the clique-hypergraph $\mathcal{H}(C_n^k)$ if and only if Alice starts playing and $a \geq k$.

Proof. By Lemma 7.2, $C_{3k}^k$ admits $k$ disjoint triangles. It is easy to prove that these triangles are internal maximal cliques. Therefore, when $b \geq 3$, Alice loses if $a < k$ or Bob starts. Indeed, wherever Alice plays, she misses at least one of these triangles, and in both cases, in Bob’s turn, he colours blue 3 vertices of one of the $k$ triangles, which is a maximal clique, that ensures his victory. The converse follows from applying Theorem 7.1 with $\varepsilon = 0$.

Acknowledgements
This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior – Brasil (CAPES) – Finance Code 001, CAPES-PrInt project number 88881.310248/2018-01, CNPq and FAPERJ.

References

Please help to maintain this journal in open access!

This journal is currently published in open access under the Subscribe to Open model (S2O). We are thankful to our subscribers and supporters for making it possible to publish this journal in open access in the current year, free of charge for authors and readers.

Check with your library that it subscribes to the journal, or consider making a personal donation to the S2O programme by contacting subscribers@edpsciences.org.

More information, including a list of supporters and financial transparency reports, is available at [https://edpsciences.org/en/subscribe-to-open-s2o](https://edpsciences.org/en/subscribe-to-open-s2o).