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THE (a, b)-MONOCHROMATIC TRANSVERSAL GAME ON CLIQUE-HYPERGRAPHS OF POWERS OF CYCLES *, **

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Abstract. We study the (a, b)-monochromatic transversal game that is a combinatorial Maker–Breaker game where Alice and Bob alternately colour a vertices in red and b vertices in blue of a hypergraph, respectively. Either player is enabled to start the game. Alice tries to construct a hyperedge transversal, and Bob tries to prevent this. The winner is Alice if she obtains a red hyperedge transversal; otherwise, Bob wins the game if he obtains a monochromatic blue hyperedge. Maker–Breaker games were determined to be PSPACE-complete. In this work, we analyze the game played on clique-hypergraphs of powers of cycles, and we show strategies that, depending on the choice of the parameters, allow a specific player to win the game.

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1. Introduction

The concept of *hypergraphs* has been extensively studied in many areas. It generalizes the definition of graphs allowing edges to have more than two incident vertices. Hypergraph theory is also applied in modern mathematics and related research fields. For instance, Torres and Wagler [22] studied a model to the dynamics of discrete deterministic systems represented by a hypergraph; and Groshaus and Szwarcfiter [16] described polynomial time algorithms for recognizing bipartite-conformal and bipartite-Helly hypergraphs.

A *transversal* in a hypergraph is a set of vertices intersecting every hyperedge [3], which is a generalization of the concept of a vertex cover in a graph. A *clique* of a graph is a subset of its vertices that induces a complete graph. A clique-transversal of a graph G is a subset of vertices intersecting all the cliques of G. The first NP-hardness result for clique-transversals was obtained in 1992 by Erdös *et al.* [13]. In 2008, Duran *et al.* [11] proposed an algorithm for determining the minimum cardinality τ_c of a clique-transversal of a general graph G, which runs in polynomial time, whenever τ_c is fixed.

Keywords. Combinatorial games, hypergraphs, transversal.

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** *Dedicated to healthcare professionals and essential workers amid the COVID-19 pandemic.*

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A *maximal clique* is a clique that is not properly contained in any other clique. A *clique-hypergraph* of a graph G is a hypergraph with the same vertex set of G, and whose hyperedges are the maximal cliques of the graph G.

The study of clique-hypergraphs was firstly presented in 1991 by Duffus *et al.* [10]. The authors asked what is the smallest number of colours needed to colour the vertices of a clique-hypergraph so that no hyperedge of size at least two is monochromatic (*clique-chromatic number*). Later, in 2003, Gravier *et al.* [15] showed that, given a fixed graph F, there exists an integer $f(F)$ such that the clique-hypergraph of any F-free graph can be $f(F)$-coloured if and only if all components of F are paths. After that, in 2004, Bacsó *et al.* [2] proved that this number is 3 for almost all perfect graphs. In 2013, the answer to Duffus' question in the case of the clique-chromatic number of powers of cycles was presented by Campos *et al* [7], where the authors showed that such number is equal to 2, except for odd cycles of size at least 5 where the answer is 3.

Classic problems in graph theory, like colouring, labeling and domination, have been studied from the perspective of Combinatorial games (see [1,8,14,20]). For example, the combinatorial solitaire Clobber game, which is essentially an optimization problem, was investigated in the class of complete multipartite graphs by Duchêne *et al.* [9].

Combinatorial games [4], like Solitaire Clobber [9], are alternating finite two-player games of pure strategy in which all the relevant information is public to both players, as well as no randomness or luck is allowed.

A combinatorial game using the concept of transversals in hypergraphs is the *transversal game*, presented by Bujtás *et al.* [5,6]. In this game, two players, Edge-hitter and Staller, take turns choosing a vertex from $\mathcal{H}$. Each chosen vertex must hit at least one edge not hit by the vertices previously chosen. The game ends when the set of selected vertices becomes a transversal in $\mathcal{H}$. The strategy of Edge-hitter is to minimize the number of chosen vertices, while Staller wishes to maximize it. The authors defined the *game transversal number* of $\mathcal{H}$ as the number of vertices chosen when Edge-hitter starts the transversal game and both players play optimally. Using this definition, they showed that the $\frac{3}{4}$-Game Total Domination Conjecture [18] is true over the graph classes with minimum degree at least 2, and compared this parameter with its transversal number.

The area of combinatorial games also includes *Maker–Breaker games* which were introduced by Erdös and Selfridge [12] in 1973, and has been extensively studied as described in the survey of Hefetz *et al.* [17]. In these games, Maker wins if he manages to hold all the elements of a winning set, while Breaker wins if he manages to prevent this. Draws do not occur, so, in each Maker–Breaker game, either Maker or Breaker has a winning strategy. The formal definition is given as follows:

Definition 1.1 ([17])**.** Let X be a finite set and let $\mathcal{F} \subseteq 2^X$ be a family of subsets. In a Maker–Breaker game over the hypergraph $(X, \mathcal{F})$:

(i) the set X is called the board; the elements of $\mathcal{F} \subseteq 2^X$ are the winning sets;
(ii) the players are called Maker and Breaker;
(iii) during a particular play, the players alternately occupy elements of X; as a default, we set Maker to start (unless stated otherwise);
(iv) the winner is
  (a) Maker if he occupies a winning set completely by the end of the game;
  (b) Breaker if he occupies an element in every winning set.

In this work, we consider a combinatorial Maker–Breaker game over hypergraphs: the (a, b)-*monochromatic transversal game*. In this game, the set X is the set of vertices $\mathcal{V}$ of a hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$, and $\mathcal{F}$ is the set $\mathcal{E}$ of its hyperedges. Alice and Bob alternately colour a vertices in red and b vertices in blue of a hypergraph, respectively. Either player is enabled to start the game. Alice (Maker) tries to construct a hyperedge transversal, and Bob (Breaker) tries to prevent this. The winner is Alice if she obtains a red hyperedge transversal; otherwise, Bob wins the game if he obtains a monochromatic blue hyperedge. We observe that there always exists a winner of the game: for any red and blue colouring of the vertices of a hypergraph $\mathcal{H}$, there exists a monochromatic blue hyperedge if and only if the set of red vertices is not a transversal of the hyperedges of $\mathcal{H}$.

From the perspective of complexity, the (a, b)-MTG is PSPACE-complete since Maker–Breaker games were determined to be PSPACE-complete by Schaefer [21], where the game was mentioned as POS CNF. We consider the game played on clique-hypergraphs of complete graphs, cycles, paths, and powers of cycles. For each of these graphs, we show strategies and bounds to the parameters a and b so that one of the players wins the game.

We organize the paper as follows. First, in Section 2, we introduce the game and present standard definitions and notation. In Section 3, we start playing the game considering that Alice and Bob colour a single vertex at a time on complete graphs, powers of cycles and paths. In Section 4, we exhibit results when Alice and Bob are allowed to colour more vertices per turn on powers of cycles C_n^k, and show that Alice has advantage on the (a, b)-MTG played on the hypergraphs of these graphs when $a \geq b$ and the number of vertices $n > 3k$, $k \geq 2$. In Section 5, we prove that, for n sufficiently large, Bob wins the game when $b > a$ on clique-hypergraphs of powers of cycles. In Section 6, we show how "small" n must be in order to guarantee Alice's victory. Finally, we present our conclusions in Section 7.

2. Description of the game

In this work, all graphs are finite, undirected and simple. A *hypergraph* $\mathcal{H}$ is a pair $(\mathcal{V}, \mathcal{E})$ where $\mathcal{V}$ is a finite vertex set, and $\mathcal{E}$ is a family of nonempty subsets of $\mathcal{V}$ called *hyperedges*.

The (a, b)-*monochromatic transversal game*, called (a, b)-MTG, is a combinatorial Maker–Breaker game where two players, Alice and Bob, alternately take turns colouring the vertices of a hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ (either player may start the game). On each turn, either Alice colours $a \in \mathbb{N}^*$ vertices in red or Bob colours $b \in \mathbb{N}^*$ vertices in blue, where $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$. Alice tries to construct a hyperedge transversal, and Bob tries to prevent this. Alice wins the game if she obtains a red hyperedge *transversal*, that is, a subset of vertices of $\mathcal{V}$ that has a nonempty intersection with every hyperedge of $\mathcal{H}$. Bob wins the game if he obtains a monochromatic blue hyperedge of $\mathcal{E}$.

The (a, b)-MTG presents properties that are inherited from the general ones established by Hefetz *et al.* [17] to the Maker–Breaker game. Next, we show their version adapted to the (a, b)-MTG notation.

Remark 2.1 ([17], Prop. 2.1.6)**.** If there exists a strategy that allows Alice (resp. Bob) to win when Bob (resp. Alice) starts playing the (a, b)-MTG on a given hypergraph, then there exists a strategy that allows Alice (resp. Bob) to win when she (resp. he) starts playing the game on that hypergraph.

Remark 2.2 ([17], Rem. 2.1.4)**.** If there exists a strategy that allows Alice (resp. Bob) to win the (a_0, b_0)-MTG played on a given hypergraph, independently of who starts it, then for any $a > a_0$ (resp. $b > b_0$) there exists a strategy that allows Alice (resp. Bob) to win the (a, b_0)-MTG (resp. (a_0, b)-MTG) played on that hypergraph, independently of who starts it.

The *clique-hypergraph* $\mathcal{H}(G) = (V, \mathcal{E})$ of a graph $G = (V, E)$ is a hypergraph such that V is the vertex set of G, and the hyperedge set $\mathcal{E}$ is the set of all *maximal cliques* in G, *i.e.*, $\mathcal{E}$ is the set of all maximal subsets of V whose vertices induce a complete graph in G. In this work, we consider clique-hypergraphs of powers of cycles.

A *k-th power of cycle of length* n, C_n^k, for $k \geq 1$, is a graph on n vertices whose vertex set is $V(C_n^k) = \{v_i : i \in \mathbb{Z}_n\}$, and whose edges $\{v_i, v_j\}$, $i, j \in \mathbb{Z}_n$ have the property that $i = j \pm r \pmod{n}$ for some $r \in \{1, 2, \ldots, k\}$. We take $(v_0, \ldots, v_{n-1})$ to be a cyclic order on the vertex set of C_n^k, and always perform arithmetic modulo n on vertex indexes. The *neighborhood* of a vertex v, denoted by $N(v)$, is the set of all the vertices adjacent to v. The *reach* of an edge $\{v_i, v_j\}$ is defined as $d_{ij} = \min\{(i - j) \bmod n, (j - i) \bmod n\}$. Observe that, with the previous definition, C_n^1 is a cycle C_n, and for $k \geq \lfloor \frac{n}{2} \rfloor$, graph C_n^k is isomorphic to the complete graph.

According to [7], the maximal cliques of powers of cycles C_n^k can be classified into two types: an *external clique*, whose vertex set is composed by $k + 1$ vertices with consecutive indexes $v_x, \ldots, v_{x+k (\text{mod } n)}$, for some $x \in \mathbb{Z}_n$, and an *internal clique* that has non-consecutive vertex indexes. Figure 1 illustrates powers of cycles with and without internal maximal cliques.

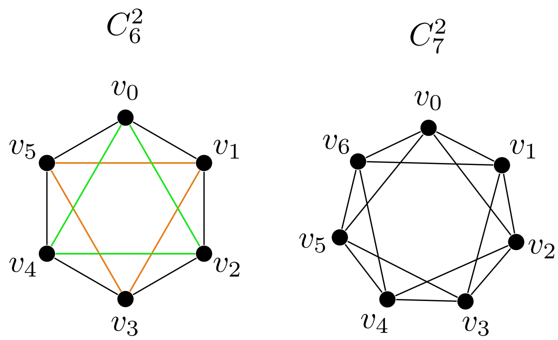

FIGURE 1. Graphs C_6^2, with internal maximal cliques in orange and green, and C_7^2, without internal maximal cliques.

We observe that if $b > k$, then the (a, b)-MTG played on clique-hypergraphs of powers of cycles becomes trivial because, on his first turn, Bob colours in blue all the vertices of an external maximal clique. Similarly, if $a \geq \lceil \frac{n}{k+1} \rceil$ and C_n^k has no internal maximal cliques, then Alice colours in red all the vertices of a transversal.

Furthermore, observe that if Bob can not colour a clique of size k' while playing the (a, b)-MTG on a clique-hypergraph $\mathcal{H}(C_n^{k'})$, then he can not colour a clique of size k for $k > k'$. By this observation we obtain the following remark:

Remark 2.3. Let C_n^k be a power of cycle with no internal maximal cliques and let k' be such that $2 \leq k' < k$. Any clique of C_n^k contains a clique of $C_n^{k'}$ and so any transversal of $C_n^{k'}$ is also a transversal of C_n^k. If there exists a strategy that allows Alice to win the (a, b)-MTG played on the clique-hypergraph $\mathcal{H}(C_n^{k'})$, then there exists a strategy that allows Alice to win the (a, b)-MTG played on the clique-hypergraph $\mathcal{H}(C_n^k)$.

3. FIRST PLAYS

We begin this section by analyzing the $(1, 1)$-MTG played on clique-hypergraphs of powers of cycles C_n^k. First, we show strategies that can be used when k has either its smallest or largest possible value, that is, C_n (when $k = 1$) or $C_n^k = K_n$ (when $k \geq \lfloor \frac{n}{2} \rfloor$).

Proposition 3.1. *If K_n is a complete graph with $n \geq 2$, then there exists a strategy that allows Alice to win the $(1, 1)$-MTG played on the clique-hypergraph $\mathcal{H}(K_n)$, independently of who starts playing the game.*

Proof. Suppose that Bob starts playing the game. We observe that the clique-hypergraph $\mathcal{H}(K_n)$ of the complete graph K_n on n vertices contains a unique hyperedge with n vertices. Therefore, Alice obtains a red transversal on her first turn and wins. The result follows by Remark 2.1. $\square$

Now we analyze the game played on the clique-hypergraph of a cycle. Since C_3 is isomorphic to the complete graph K_3 (Prop. 3.1), we consider $n \geq 4$.

Proposition 3.2. *If C_n is a cycle of length $n \geq 4$, then there exists a strategy that allows Bob to win the $(1, 1)$-MTG played on the clique-hypergraph $\mathcal{H}(C_n)$, independently of who starts playing the game.*

Proof. First, we observe that the hyperedges of $\mathcal{H}(C_n)$ are the edges of C_n. Suppose that Alice starts the game and colours the vertex v_i red. Bob wins the game colouring a vertex v_j blue that is not adjacent to v_i. Indeed, since $n \geq 4$, independently of which vertex Alice colours in red on her next turn, she does not obtain a transversal and, on his next turn, Bob obtains a monochromatic blue hyperedge colouring a non-coloured vertex v_l adjacent to v_j, with $l \in \{j - 1 \pmod n, j + 1 \pmod n\}$. The result follows by Remark 2.1. $\square$

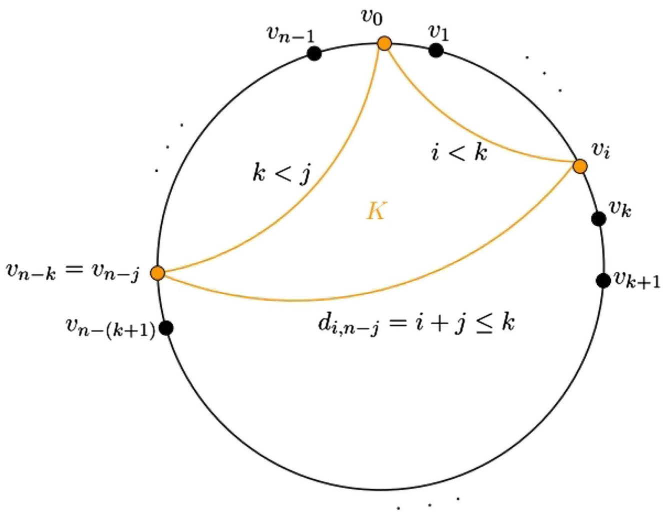

FIGURE 2. Idea of the proof of Lemma 4.1.

The $(1,1)$-MTG played on clique-hypergraphs $\mathcal{H}(P_n)$ of paths P_n on n vertices, $n \geq 3$, is similar to the $(1,1)$-MTG over $\mathcal{H}(C_n)$, $n \geq 4$. If $n = 2$, since P_2 is isomorphic to K_2, then the result follows by Proposition 3.1. If $3 \leq n \leq 5$, then there exists a strategy that allows the player who had the first turn to win the game. Indeed, if Bob starts playing, he colours vertex $v_{\lfloor \frac{n}{2} \rfloor}$ blue. Now, Alice must colour $v_{\lfloor \frac{n}{2} \rfloor - 1}$ or $v_{\lfloor \frac{n}{2} \rfloor + 1}$. On his next turn, Bob obtains a blue hyperedge colouring $v_{\lfloor \frac{n}{2} \rfloor + 1}$ or $v_{\lfloor \frac{n}{2} \rfloor - 1}$. If Alice starts playing, she colours vertex $v_{\lfloor \frac{n}{2} \rfloor}$ red. On the next turns, independently of which vertex Bob colours blue, Alice wins the game colouring in red the vertex adjacent to Bob's last coloured vertex, and prevents him from making a monochromatic blue P_2. If $n \geq 6$, an argument analogous to the proof of Proposition 3.2 shows that there exists a strategy that allows Bob to win the game, independently of who starts playing the game.

4. Alice's dream

Now, we consider the (a,b)-MTG played on the clique-hypergraph $\mathcal{H}(C_n^k)$. We recall that the maximal cliques of powers of cycles can be classified into external and internal cliques. The next result presents a lower bound to the size of n that guarantees the non-existence of internal maximal cliques.

Lemma 4.1. *If $n > 3k$ and $k \geq 2$, then C_n^k has no internal maximal clique.*

Proof. Let K be a maximal clique of C_n^k. Without loss of generality we may assume that $v_0 \in K$. Let $i, j \in \mathbb{N}^*$, $i \neq j$, where: (1) $i \leq k$ and $j \geq k$ are largest as possible and such that (2) $v_{n-j}, v_i \in K$. Since $n > 3k$, and by (2), the reach between the vertices v_{n-j} and v_i is $d_{i,n-j} = i + j \leq k$. By (1), $v_s \notin K$ for each $s \in]i, k+1[$ and $v_{n-s} \notin K$ for each $s \in]k+1, j[$. Moreover, for each $s \in [k+1, n-(k+1)]$, since $v_0 \in K$ and $d_{0,s} > k$, we have that $v_s \notin K$. Now, since the subgraph H induced by $\{v_{n-j}, \ldots, v_0, \ldots, v_i\}$ is a clique, it follows that $K = H$ and $i + j = k$. □

Now, the following theorem shows that the condition $a \geq b$ ensures that Alice wins the game when C_n^k has no internal maximal cliques.

Theorem 4.2. *Let $n > 3k$ and $k \geq 2$. If $a \geq b$ and $b < k$, then there exists a strategy that allows Alice to win the (a,b)-MTG played on the clique-hypergraph $\mathcal{H}(C_n^k)$, independently of who starts playing.*

Proof. Suppose that Bob starts playing. Bob's turns consist of colouring in blue ℓ disjoint paths P^j in the cycle C_n, for $j = 0, \ldots, \ell - 1$ and $\ell \leq b$. By definition, we have that $\sum_{j=0}^{\ell-1} |P^j| \leq b < k$. Alice's strategy consists in applying the following rules:

(=1) If P^j has one vertex, say v_i, then Alice colours v_{i+1} red if is not coloured yet; otherwise she colours v_{i-1} red.

(>1) If P^j has two distinct extremities v_i and v_s with $i < s$, then Alice colours vertices v_{i-1} and v_{s+1} red, if they are not coloured yet.

First, observe that Alice colours at least b vertices (and at most a vertices) per turn. Second, since $b < k$, Bob must play at least twice in a set of $k + 1$ consecutive vertices (interval) to colour all its vertices blue. Suppose that, at some turn, for the sake of contradiction, Bob succeeds to colour in blue an interval I of length $k + 1$. We show that at some turn Alice colours in red a vertex belonging to I. Let t be the first turn Bob plays in I. Let $P^0, \ldots, P^j$ be the ordered clockwise disjoint paths intersecting I at turn t. If $j > 0$ then let v_i be the right (clockwise) extremity of P^0. By Alice's strategy, the vertex $v_{i+1} \in I$ is coloured in red at turn $t+1$, which yields a contradiction. So $j = 0$. Now, if $|P^0| \geq 2$ then let v_i and v_s with $i < s$ be the extremities of P^0. Since $b < k$, then v_{i-1} or v_{s+1} belongs to I. Now, by rule (>1), Alice colours them at turn $t + 1$, which give again a contradiction. Hence, P^0 is reduced to a single vertex v_i and, by rule (=1), v_i is the right extremity of I.

Let t' be the second turn Bob plays in I. Let v_r be the vertex coloured in blue at turn t' closest to v_i. If $r < i - 1$ then, by Alice's strategy, v_{r+1} is coloured in red at turn $t' + 1$, a contradiction. So, $r = i - 1$. Let P^r be the blue path coloured at turn t' containing v_r, and let v_l be the left extremity of P^r. Remark that, since $|P^r| \leq b + 1 \leq k$, vertex $v_{l-1} \in I$. By Alice's strategy, v_{l-1} is coloured in red at turn $t' + 1$ which gives again a contradiction with the existence of I. The result follows by Remark 2.1. $\square$

The following result shows that case $b = k$ forces a to be higher to ensure Alice's victory. In fact, if $b = k$ then, when n is large enough, Alice wins only if $a \geq 2b$.

Theorem 4.3. *Let $a \geq b = k \geq 2$ and $n \geq (2k+1)k$. Bob wins the (a, b)-MTG played on the clique-hypergraph $\mathcal{H}(C_n^k)$ when he starts playing if and only if $a < 2b$.*

Proof. Suppose that $a \geq 2b$. Alice applies rule (>1) of the proof of Theorem 4.2, even for path with only one vertex. So, for each path coloured blue by Bob, at some turn t, with extremities v_i and v_s with $i \leq s$, Alice colours, at turn $t+1$, the vertices v_{i-1} and v_{s+1} red. This ensures that there exist at most $b = k$ blue consecutive vertices. Hence, Alice wins.

Now, suppose that $a < 2b$ and $n \geq (2k + 1)k$. Bob starts playing and colours blue the vertices of indices in $\{(2k + 1)p \mid p = 0, \ldots, k - 1\}$. Now, Alice colours in red at most $a < 2b$ vertices among the vertices of indices belonging to the $2b$ intervals $](2k+1)p, (2k+1)p+k] \cup [(2k+1)p-k, (2k+1)p[$, for each $p = 0, \ldots, k-1$. Hence, on the second turn, there exists $q \in \{0, \ldots, k-1\}$ such that none of the vertices of index in $](2k+1)q, (2k+1)q+k]$ or $[(2k + 1)q - k, (2k + 1)q[$ is coloured red. Let I be such interval with no red vertex. Since I is a set of k vertices, at the third turn, Bob colours all of them and wins with hyperedge $v_{(2k+1)q} \cup \{v_i \mid i \in I\}$ coloured blue. $\square$

5. Bob is the winner

In the previous sections, we have shown that Alice has advantage with respect to Bob when they play the (a, b)-MTG with $a \geq b$ and $b < k$ on clique-hypergraphs of powers of cycles $\mathcal{H}(C_n^k)$. In this section, we consider $1 \leq a < b$ and define conditions so that Bob wins the (a, b)-MTG played on these hypergraphs.

First we present the following definition. A $(k+1)$-*matching of cardinality* m of a hypergraph $\mathcal{H}$ is the disjoint union of m hyperedges of size $k + 1$.

Lemma 5.1. *Let a, b and k be such that $1 \leq a < b \leq k + 1$ and $k \geq 2$. Let $p = \lfloor \frac{b}{a+1} \rfloor$ and α be the positive integer such that $b + (\alpha - 1)p < k + 1 \leq b + \alpha p$. Bob wins the (a, b)-MTG played on a hypergraph $\mathcal{H}$ that contains $(k + 1)$-matching of cardinality $(a + 1)^\alpha$ when he starts playing.*

Proof. Let a, b and k be such that $1 \leq a < b \leq k+1$ and $k \geq 2$. Let $p = \lfloor \frac{b}{a+1} \rfloor$ and α be the positive integer such that $b + (\alpha - 1)p < k+1 \leq b + \alpha p$. Let $\mathcal{M}$ be a $(k+1)$-matching of cardinality $(a+1)^\alpha$.

The idea of Bob's strategy is that he always chooses $a+1$ hyperedges and colours blue p vertices in each hyperedge that has no red vertex (except on his last turn). Bob aims to have a hyperedge with αp vertices blue and no red ones to play on his last turn, say $2t_\alpha + 1$. Applying this basic strategy, Bob needs at least $a+1$ hyperedges with $(\alpha-1)p$ blue vertices and no red one at turn $2t_\alpha - 1$ otherwise Alice will be able to color one vertex on each hyperedge with αp blue vertices at turn $2t_\alpha$. The rest of the proof consists in generalizing this last argument: for any $0 < q \leq \alpha$, how many hyperedges having $(q-1)p$ blue vertices and no red ones, Bob needs at some turn $2t_q - 1$ for ensuring that at turn $2t_q + 1$, he has "enough" hyperedges having qp blue vertices and no red ones?

We define $t_0 = 0$ and $t_q = \sum_{j=\alpha-q}^{\alpha-1} (a+1)^j$ for $0 < q \leq \alpha$. Let $\mathcal{M}_q$ be the subset of hyperedges in $\mathcal{M}$ containing qp blue vertices and no red vertex after $2t_q$ turns. We claim that:

Claim 5.2. Bob has a strategy that ensures that, for all $q \in \{0, \dots, \alpha\}$, we have that $|\mathcal{M}_q| \geq (a+1)^{\alpha - q}$.

Proof. For a fixed $q \in \{1, \dots, \alpha\}$, at turn $2t+1$ with $t \in \{t_{q-1}, \dots, t_q - 1\}$, Bob's strategy consists in selecting $a+1$ hyperedges of $\mathcal{M}_q$ and colouring p vertices of the ones that do not contain a red vertex. Moreover, hyperedges selected at turn $2t+1$ must not have been previously selected at some turn $2t' + 1 < 2t + 1$ and $t' \geq t_{q-1}$.

Let $t \in \{t_{q-1}, \dots, t_q - 1\}$, at turn $2t+1$, Bob colours at most p vertices of $(a+1)$ edges which is allowed since $b \geq (a+1)p$. Also, any selected hyperedge with no red vertex has qp blue vertices after $2t+1$ turns.

Now, we are ready to prove the claim by induction on q. By definition, we have that $|\mathcal{M}_0| \geq (a+1)^\alpha$. First, suppose that, for some $j > 0$, we have $|\mathcal{M}_q| \geq (a+1)^{\alpha-q}$ holds for all $q < j$.

For all turns $2t+1$, when $t \in \{t_{j-1}, \dots, t_j - 1\}$, Bob selects $(t_j - t_{j-1})(a+1) = (a+1)^{\alpha-j+1}$ hyperedges of $\mathcal{M}_{j-1}$ (which is possible since, by induction hypothesis, $|\mathcal{M}_{j-1}| \geq (a+1)^{\alpha-j+1}$), while Alice could not colour more than $(t_j - t_{j-1})a = (a+1)^{\alpha-j}a$ hyperedges for all turns $2t+2$, when $t \in \{t_{j-1}, \dots, t_j - 1\}$. Therefore, $|\mathcal{M}_j| \geq (a+1)^{\alpha-j+1} - a(a+1)^{\alpha-j} = (a+1)^{\alpha-j}$. □

The previous claim shows that $|\mathcal{M}_\alpha| \geq 1$. Moreover, any hyperedge in $\mathcal{M}_\alpha$ has αp blue vertices after $2t_\alpha$ turns and contains no red one. So, at turn $2t_\alpha + 1$, Bob selects one hyperedge of $\mathcal{M}_\alpha$ and colours the remaining $(k+1) - \alpha p$ vertices which are less than b. □

By the proof of Lemma 5.1, Bob wins the game in at most

$$2\frac{(a+1)^\alpha - 1}{a} + 1 \text{ turns.}$$

Moreover, we remark that if $n \geq m(k+1)$, then the clique-hypergraph $\mathcal{H}(C_n^k)$ contains a $(k+1)$-matching of cardinality m. So, the following theorem is a direct consequence of Lemma 5.1.

Theorem 5.3. *Let a, b, and k be such that $1 \leq a < b \leq k+1$ and $k \geq 2$. Let $p = \lfloor \frac{b}{a+1} \rfloor$ and α be the positive integer such that $b + (\alpha-1)p < k+1 \leq b + \alpha p$. If $n \geq (a+1)^\alpha (k+1)$ then Bob wins the (a,b)-MTG played on a clique-hypergraph $\mathcal{H}(C_n^k)$ when he starts playing.*

Now, we consider the case when Alice starts.

Corollary 5.4. *Let a, b, and k be such that $1 \leq a < b \leq k+1$. Let $p = \lfloor \frac{b}{a+1} \rfloor$ and α be the positive integer such that $b + (\alpha-1)p < k+1 \leq b + \alpha p$. Bob wins the (a,b)-MTG played on a $(k+1)$-matching of cardinality at least $a + (a+1)^\alpha$ independently of who starts playing.*

Proof. By Lemma 5.1, it is enough to consider the case when Alice starts. After Alice's first turn, it remains at least $(a+1)^\alpha$ hyperedges. Now, it is Bob's turn, and again, by Lemma 5.1, Bob wins. □

This result implies the following corollary:

Corollary 5.5. *Let a, b, and k be such that $1 \le a < b \le k+1$. Let $p = \lfloor \frac{b}{a+1} \rfloor$ and α be the positive integer such that $b + (\alpha - 1)p < k+1 \le b + \alpha p$.*

If $n \ge (a + (a+1)^\alpha)(k+1)$ then Bob wins the (a,b)-MTG played on a clique-hypergraph $\mathcal{H}(C_n^k)$ independently of who starts playing.

Finally, we observe that, for fixed α, the bound for n given in Theorem 5.3 may be improved as shown in Corollary 6.3 when $\alpha = 1$.

6. Alice's revenge

From previous results, the only way for Alice to win the game is to consider a "small" n. But how "small" n can be? The following theorem and corollary give an element of the answer.

First, we present a new way of seeing the evolution of the game throughout the turns. Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a hypergraph and set $\mathcal{V}_0 = \mathcal{V}$ and $\mathcal{E}_0 = \mathcal{E}$. For each $t \ge 1$, $\mathcal{B}_t$ (resp. $\mathcal{V} \setminus \mathcal{V}_t$) is the set of blue (resp. red) vertices after t turns. We assume that $\mathcal{B}_0 = \emptyset$. Let $S_t \subseteq (\mathcal{V}_{t-1} \setminus \mathcal{B}_{t-1})$ be the set of vertices coloured at turn t.

Definition 6.1. A hypergraph $\mathcal{H}_t = (\mathcal{V}_t, \mathcal{E}_t)$ is the hypergraph at turn t, where $\mathcal{V}_t$ and $\mathcal{E}_t$ are obtained according to the following rules:

(1) If turn t is played by Alice then $\mathcal{V}_t = \mathcal{V}_{t-1} \setminus S_t$, $\mathcal{E}_t$ is the subset of hyperedges e in $\mathcal{E}_{t-1}$ such that $e \cap S_t = \emptyset$, and $\mathcal{B}_t = \mathcal{B}_{t-1}$.
(2) If turn t is played by Bob then $\mathcal{V}_t = \mathcal{V}_{t-1}$, $\mathcal{E}_t = \mathcal{E}_{t-1}$, and $\mathcal{B}_t = \mathcal{B}_{t-1} \cup S_t$.

The game ends at some t^* such that $\mathcal{V}_{t^*} = \mathcal{B}_{t^*}$. Finally, Alice wins if and only if $\mathcal{E}_{t^*} = \emptyset$.

For example, let $t = 0$ and consider the $(2,2)$-MTG on the clique-hypergraph $\mathcal{H}_0 = \mathcal{H}(C_{10}^2) = (\mathcal{V}_0, \mathcal{E}_0)$, where $\mathcal{V}_0 = \{v_0, v_1, \ldots, v_9\}$, and $\mathcal{E}_0 = \{e_0, e_2, \ldots, e_9\}$ such that $e_0 = \{v_0, v_1, v_2\}$, $e_1 = \{v_1, v_2, v_3\}$, $e_2 = \{v_2, v_3, v_4\}$, $e_3 = \{v_3, v_4, v_5\}$, $e_4 = \{v_4, v_5, v_6\}$, $e_5 = \{v_5, v_6, v_7\}$, $e_6 = \{v_6, v_7, v_8\}$, $e_7 = \{v_7, v_8, v_9\}$, $e_8 = \{v_8, v_9, v_0\}$, and $e_9 = \{v_9, v_0, v_1\}$. Note that, $\mathcal{B}_0 = \emptyset$. We refer to Figure 3.

Suppose that, in $t = 1$, Bob colours vertices v_3 and v_4 in blue, thus $\mathcal{V}_1 = \mathcal{V}_0$, $\mathcal{E}_1 = \mathcal{E}_0$, and $\mathcal{B}_1 = \mathcal{B}_0 \cup S_1 = \emptyset \cup \{v_3, v_4\} = \{v_3, v_4\}$. Therefore, $\mathcal{H}_1 = (\mathcal{V}_1, \mathcal{E}_1)$ and $\mathcal{B}_1 = \{v_3, v_4\}$.

Now, in $t = 2$, if Alice colours vertices v_2 and v_5 red, then $S_2 = \{v_2, v_5\}$, $\mathcal{V}_2 = \mathcal{V}_1 \setminus \{v_2, v_5\}$, and $\mathcal{E}_2 = \{e_6, e_7, e_8, e_9\}$, since $e_i \cap S_2 = \emptyset$ for $i \in \{6, 7, 8, 9\}$. Therefore, $\mathcal{H}_2 = (\mathcal{V}_2, \mathcal{E}_2)$ and $\mathcal{B}_2 = \mathcal{B}_1$ (see Fig. 3).

Now we are ready to state the following result.

Theorem 6.2. *Let $1 \le a \le b < k+1$ and $\beta = \lceil \frac{k+1}{b} \rceil \ge 2$. If $n \in \mathbb{N}^*$ such that $3k < n \le a(\beta - 1)(k+1)$, then there exists a strategy that allows Alice to win the (a,b)-monochromatic transversal game played on the clique-hypergraph $\mathcal{H}(C_n^k)$, independently of who starts playing.*

Proof. Suppose that Bob starts playing and assume that $n \le a(\beta - 1)(k+1)$. Let $\mathcal{H}_t$ be the hypergraph as in Definition 6.1. We recall that, for $t \ge 1$, S_{2t} is the set of vertices coloured red by Alice at turn $2t$ (Alice plays in even turns); and $\mathcal{B}_{2t-1}$ is the set of vertices coloured blue by Bob after $2t - 1$ turns (Bob plays in the odd turns). Without loss of generality, one may assume that $v_0 \notin \mathcal{B}_1$.

At turn $2t$ and $1 \le t < \beta$, Alice's strategy is to colour red a vertices which, once deleted, create a partition of the hypergraph $\mathcal{H}_{2t-1}$ into $ta - 1$ parts defined by at most k consecutive isolated vertices, and one part R_t defined by at most $n - (ta(k+1) - tb)$ consecutive vertices.

Therefore, at turn $2t$ with $1 \le t < \beta$, Alice colours in red the vertices $v_i \in S_{2t}$ with $i \in \{i_{(t-1)a}, \ldots, i_{ta-1}\}$ defined by:

(i) $i_0 = 0$.
(ii) If $j > 0$ then the integer i_j is the largest index $i \le i_{j-1} + k + 1$ such that $v_i \notin \mathcal{B}_{2t-1}$.

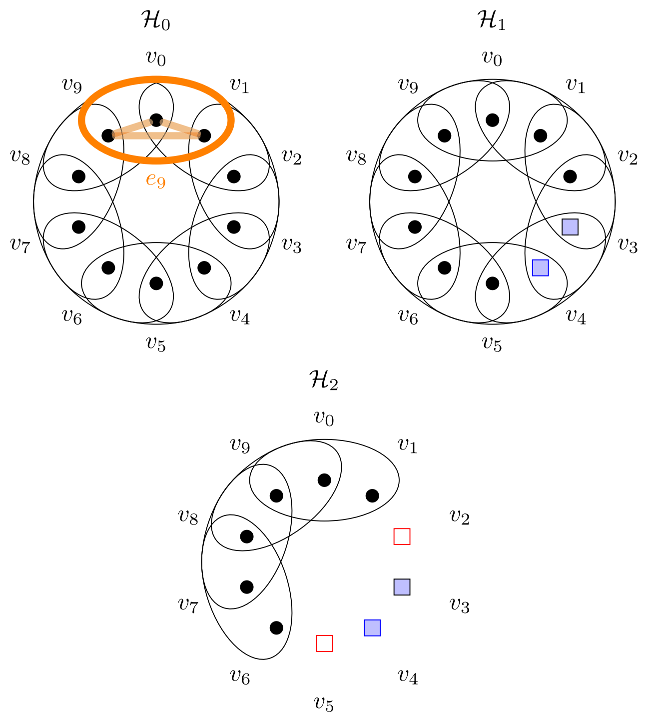

FIGURE 3. Hypergraphs: $\mathcal{H}_0 = \mathcal{H}(C^2_{10})$ (hyperedge e_9 highlighted in orange with a maximal clique in C^2_{10}); $\mathcal{H}_1$ after Bob colours blue v_3 and v_4 in $\mathcal{H}_0$ (blue squares); and $\mathcal{H}_2$ after Alice colours in red v_2 and v_5 in $\mathcal{H}_1$ (empty red squares represent deleted vertices).

For $0 < j \leq ta - 1$ then the vertices of indices in $\{i_{j-1} + 1, i_j - 1\}$ are isolated vertices in $\mathcal{H}_{2t-1}$ since $i_j - i_{j-1} - 1 \leq k$. Moreover, $i_{ta-1} \geq ta(k + 1) - u$ where u is the number of vertices in $\mathcal{B}_{2t-1}$ with indices belonging to $[1, ta(k+1)]$. Therefore $i_{ta-1} \geq ta(k+1) - tb$ and so the remaining set of vertices R_t of $\mathcal{H}_{2t-1}$ has at most $n - (ta(k + 1) - tb)$ vertices.

Hence $|R_{\beta-1}| \leq (\beta-1)b < k+1$ since $n \leq a(\beta-1)(k+1)$. Therefore, Alice wins the game since the hyperedge set $\mathcal{E}_{2(\beta-1)} = \emptyset$. The result follows by Remark 2.1. □

We note that in Theorem 6.2 the bound on n is linear in a despite of the one of Theorem 5.3 which is in a^α. Next result shows that the two bounds are close whenever $\alpha = \beta - 1 = 1$.

Corollary 6.3. *Let a, b, p and k be integers such that $b \geq p.a$ and $k + 1 = b + p$. There exists a strategy that allows Alice to win the (a, b)-monochromatic transversal game played on the clique-hypergraph $\mathcal{H}(C^k_n)$ when Bob starts playing if and only if $n < a(k + 1) + 1$.*

Proof. First assume that $n \geq a(k+1)+1$. Bob's strategy is a slight improvement of the one given in Theorem 5.3. For $i = 0, \dots, a-1$, let e_i be the hyperedge of $\mathcal{H}(C^k_n)$ induced by vertices of indexes $\{i(k+1), \dots, i(k+1)+k\}$. Bob's first turn consists to colour blue the p vertices of indices $\{i(k+1), \dots, i(k+1)+p-1\}$ for all $i = 0, \dots, a-1$. Now, for $i = 0, \dots, a - 1$, Alice has to colour red at least one vertex on each e_i, otherwise, Bob colours blue all the $k + 1 - p$ remaining vertices in the third turn. Since there exists a such hyperedges then Alice colours exactly one vertex on each e_i. Let $v_{i(k+1)+r_i}$ be the red vertex in e_i. By definition $r_i \geq p$ for each i.

We claim that if $0 \leq i < j \leq a - 1$ then $r_i \geq r_j$. Indeed, it is enough to observe that if there exists some $i < a - 1$ with $r_i < r_{i+1}$, then the subset of vertices with indexes in $\{i(k + 1) + r_i + 1, \dots, (i + 1)(k + 1) + r_i\}$

is an hyperedge e of $\mathcal{H}(C_n^k)$ with no red vertex and containing the p blue vertices of e_{i+1}, so in the third turn, Bob wins colouring blue the b remaining vertices of e.

Now, since $r_0 \geq r_{a-1}$ and $n \geq a(k+1)+1$, then $n-(k+1-r_{a-1}) > (a-1)(k+1)+r_{a-1}$. Hence, the set of vertices with indexes in $\{n-(k+1-r_{a-1}), \dots, 0, \dots, r_{a-1}-1\}$ is a hyperedge e of $\mathcal{H}(C_n^k)$ with no red vertex and, since $r_{a-1} \geq p$, e contains the p blue vertices of e_0. So, again, in the third turn, Bob wins colouring blue the b remaining vertices of e.

Finally, Theorem 6.2 shows that Alice wins whenever $n \leq a(k+1)$. □

7. Conclusion

Combining Remark 2.2 with Proposition 3.1, Proposition 3.2 and by Theorem 4.2, it is possible to complete the following table of results:

Hypergraph	Value of a	Value of b	Who wins (indep. of who started)
$\mathcal{H}(K_n)$	$1 \leq a \leq n-1$	$b < n$	Alice
$\mathcal{H}(C_n)$, $n \geq 4$	1	$b < n$	Bob
$\mathcal{H}(P_n)$, $n \geq 6$	1	1	Bob
$\mathcal{H}(C_n^k)$, $n > 3k$, $k \geq 2$	$a \geq b$	$b < k$	Alice

Furthermore, by Corollary 5.5, taking $p = \lfloor \frac{b}{a+1} \rfloor$ and $\alpha \in \mathbb{N}^*$ such that $b+(\alpha-1)p < k+1 \leq b+\alpha p$, we have:

Hypergraph	Value of a	Value of b	Who wins (indep. of who started)
$\mathcal{H}(C_n^k)$, $n \geq a+(a+1)^\alpha(k+1)$	$1 \leq a < b$	$b \leq k$	Bob

Also, by Theorem 6.2 taking $\beta = \lceil \frac{k+1}{b} \rceil \geq 2$ for $n \in \mathbb{N}^*$, we have:

Hypergraph	Value of a	Value of b	Who wins (indep. of who started)
$\mathcal{H}(C_n^k)$, $3k < n \leq a(\beta-1)(k+1)$	$1 \leq a \leq b$	$b \leq k$	Alice

We observe that Theorem 6.2 improves Theorem 5.3 when $\alpha = 1$. We conjecture that, when b is close to a and far from k, Alice can win even for "large" n.

Generally speaking, Theorem 5.1 shows that, for $1 \leq a < b \leq k+1$, Bob wins the (a,b)-MTG played on a hypergraph that contains a $(k+1)$-matching of cardinality $(a+1)^\alpha$ when he starts playing, where α is the positive integer such that $b+(\alpha-1)p < k+1 \leq b+\alpha p$ with $p = \lfloor \frac{b}{a+1} \rfloor$.

The case when C_n^k has internal maximal cliques seems challenging. Next result illustrates a way to study this case. It consists of giving an upper bound on a transversal and so, if a is larger than this bound, this guarantees Alice's victory whenever she starts.

Theorem 7.1. *Let $n = 3k-\varepsilon$ with $k \geq 3$ and $0 \leq \varepsilon < k-1$. The hypergraph $\mathcal{H}(C_n^k)$ admits a transversal of size $k-\varepsilon$.*

Proof. Set $T = \mathcal{K} \cup \{v_{2k-1-\varepsilon}\}$ with $\mathcal{K} = \{v_0, \ldots, v_{k-2-\varepsilon}\}$. We prove that T is a transversal of $\mathcal{H}(C_n^k)$. Let K be a maximal clique such that $K \cap \mathcal{K} = \emptyset$. Since $n = 3k - \varepsilon$, we have that $N(v_{2k-1-\varepsilon}) \supseteq V(C_n^k) \setminus \mathcal{K}$. Thus, K must contain the red vertex $v_{2k-1-\varepsilon}$. Therefore, any hyperedge of $\mathcal{H}(C_n^k)$ has at least one vertex belonging to T, *i.e.*, T is a transversal of $\mathcal{H}(C_n^k)$ of size

$$\underbrace{((k-2+\varepsilon)-(0)+1)}_{\text{vertices in } \kappa} + \underbrace{(1)}_{v_{2k-1-\varepsilon}} = k - \varepsilon.$$

□

For the case $n = 3k$ and $k \geq 2$, the gap ε is vanished, *i.e.*, $\varepsilon = 0$, which proves that this result is optimal. First, we recall a key lemma of Meidanis [19]:

Lemma 7.2 ([19]). *Let E^k be the set of edges of C_n^k with reach k. If $n = 3k$, $k \geq 2$, then the subgraph induced by E^k has k connected components, each one being a cycle of length* 3.

Theorem 7.3. *Let $n = 3k$ with $k \geq 2$, and let $b \geq 3$. There exists a strategy that allows Alice to win the (a, b)-MTG played on the clique-hypergraph $\mathcal{H}(C_n^k)$ if and only if Alice starts playing and $a \geq k$.*

Proof. By Lemma 7.2, C_{3k}^k admits k disjoint triangles. It is easy to prove that these triangles are internal maximal cliques. Therefore, when $b \geq 3$, Alice loses if $a < k$ or Bob starts. Indeed, wherever Alice plays, she misses at least one of these triangles, and in both cases, in Bob's turn, he colours blue 3 vertices of one of the k triangles, which is a maximal clique, that ensures his victory. The converse follows from applying Theorem 7.1 with $\varepsilon = 0$. □

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