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ALGORITHMIC ASPECT ON TOTAL ROMAN {2}-DOMINATION OF CARTESIAN PRODUCTS OF PATHS AND CYCLES

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Abstract. A total Roman {2}-dominating function (TR2DF) on a graph G with vertex set V is a function $f : V \to \{0, 1, 2\}$ having the property that for every vertex v with $f(v) = 0$, $\sum_{u \in N(v)} f(u) \geq 2$, where $N(v)$ represents the open neighborhood of v, and the subgraph of G induced by the set of vertices with $f(v) > 0$ has no isolated vertex. The weight of a TR2DF f is the value $w(f) = \sum_{v \in V} f(v)$, and the minimum weight of a TR2DF of G is the total Roman {2}-domination number $\gamma_{\mathrm{tR2}}(G)$. The total Roman {2}-domination problem (TR2DP) is to determine the value $\gamma_{\mathrm{tR2}}(G)$. In this paper, we first propose an integer linear programming (ILP) formulation for the TR2DP. Furthermore, we apply the discharging approach to determine the total Roman {2}-domination number for some Cartesian products of paths and cycles.

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1. Introduction

Through out this paper we only consider simple graph $G = (V, E)$ with vertex set $V = V(G)$ and edge set $E = E(G)$. The *open neighborhood* of a vertex $v \in V$ is the set $N(v) = \{u \in V | uv \in E\}$, while its *closed neighborhood* is $N[v] = \{v\} \cup N(v)$. We use P_n and C_n to denote a path and a cycle of order n, respectively.

Consider a function $f : V \to \{0, 1, 2\}$ and the ordered partition (V_0, V_1, V_2) of V induced by f, where $V_i = \{v \in V | f(v) = i\}$ for $i \in \{0, 1, 2\}$. Since there is a one-to-one correspondence between the function f and the ordered partition (V_0, V_1, V_2) of V, we will identify f with the ordered partition (V_0, V_1, V_2) of V and write $f = (V_0, V_1, V_2)$.

A function $f : V \to \{0, 1, 2\}$ is a *Roman dominating function* on G if every vertex $u \in V$ for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The *weight* of a Roman dominating function is the value $w(f) = \sum_{v \in V} f(v)$. *The Roman domination number* $\gamma_R(G)$ is the minimum weight among all Roman dominating functions on G. The concept of a Roman domination was formally proposed by Cockayne *et al.* [13] and was motivated by the work of ReVelle and Rosing [25], and Stewart [27]. Since then, Roman domination has been extensively studied by researchers, see for example [7, 14, 22, 24]. For further details on Roman domination and its variations we refer to the reader the book chapters [9, 11] and survey [10].

Keywords. Total Roman {2}-domination, Cartesian product graph, double domination.

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In 2016, Chellali *et al.* [8] introduced a variant of Roman dominating function. For a graph $G = (V, E)$, a *Roman* $\{2\}$*-dominating function* (also called Italian dominating function in [18] and weak $\{2\}$-dominating function in [3,21]) is a function $f = (V_0, V_1, V_2)$ satisfying the condition that for every vertex $v \in V_0$, $f(N(v)) \geq 2$, that is, v is either adjacent to a vertex u with $f(u) = 2$ or to two vertices x and y with $f(x) = f(y) = 1$.

As a new variant of the Roman domination, the *total Roman* $\{2\}$*-dominating function* (TR2DF) of a graph G with no isolated vertex is a Roman $\{2\}$-dominating function $f = (V_0, V_1, V_2)$ on G with the additional property that the subgraph induced by $V_1 \cup V_2$ has no isolated vertex. The *total Roman* $\{2\}$*-domination number* $\gamma_{\mathrm{tR2}}(G)$ is the minimum weight among all TR2DFs on G. A TR2DF of weight $\gamma_{\mathrm{tR2}}(G)$ is called a $\gamma_{\mathrm{tR2}}(G)$-function. This concept was independently introduced in [2, 15], and further studied in [1, 20, 26]. The *total Roman* $\{2\}$*-domination problem* (TR2DP) is an optimization problem of finding the TR2DF of minimum weight.

Recall that a set $S \subseteq V$ is a *double dominating set* of G if for every vertex $v \in V$, $|N[v] \cap S| \geq 2$. The minimum cardinality of a double dominating set of G is the *double domination number* $\gamma_{\times 2}(G)$. By definition, it is clear that $\gamma_{\times 2}(G) \geq \gamma_{\mathrm{tR2}}(G)$. The problem of characterizing all graphs with $\gamma_{\times 2}(G) = \gamma_{\mathrm{tR2}}(G)$ remains open. The authors in [2, 15, 23] observed that $\gamma_{\times 2}(G) = \gamma_{\mathrm{tR2}}(G)$ if there exists a TR2DF $f = (V_0, V_1, V_2)$ of G such that $V_2 = \emptyset$. By this observation, they proved that $\gamma_{\times 2}(G) = \gamma_{\mathrm{tR2}}(G)$ when G is a path or a cycle.

It has been shown in [2, 15] that the associated decision problem for total Roman $\{2\}$-domination is NP-complete even when restricted to bipartite and chordal graphs. But it is possible to compute this parameter in polynomial time for bounded treewidth graphs, threshold graphs and chain graphs [6], which are subclasses of bipartite graphs and chordal graphs. One common approach to designing exact algorithms or approximation algorithms for an NP-complete problem is to model it using integer linear programming (ILP). Some integer linear programming formulations have been described for several variants of domination, such as Roman domination [4], weak Roman domination [19] and double Roman domination [5]. Motivated by this, we propose an ILP model for TR2DP.

The Cartesian product of two graphs G and H is the graph with vertex set $V(G) \times V(H)$ and two vertices (v_1, v_2) and (u_1, u_2) are adjacent whenever $v_1u_1 \in E(G)$ and $v_2 = u_2$, or $v_2u_2 \in E(H)$ and $v_1 = u_1$. When considering the Cartesian product graph of paths and cycles, we denote $V(G\Box H) = \{x_{i,j} | 1 \leq i \leq |V(G)|, 1 \leq j \leq |V(H)|\}$. The set of vertices $\mathcal{C}^i = \{x_{i,j} | 1 \leq j \leq |V(H)|\}$ is called the ith column of $G\Box H$. In what follows, when considering the vertices $x_{i,j}$ in graph $C_n\Box C_m$, we use the arithmetic operations of the index i over modulo n and j over modulo m, respectively. Let f be a TR2DF of a Cartesian product of paths and cycles, for any column $\mathcal{C}$, we denote the weight of $\mathcal{C}$ by $w_f(\mathcal{C}) = \sum_{v \in V(\mathcal{C})} f(v)$. In the history of domination problem, a lot of work has been done to study the class of Cartesian product graphs, see, for example [12, 16, 21]. We use the standard notation $[n] = \{1, 2, \ldots, n\}$.

In this paper, we first propose an integer linear programming (ILP) formulation for the total Roman $\{2\}$-domination problem. Furthermore, we determine the total Roman $\{2\}$-domination number for some Cartesian products of paths and cycles by discharging method. Additionally, we obtain that the equality $\gamma_{\times 2}(G) = \gamma_{\mathrm{tR2}}(G)$ holds for the graphs under our consideration.

We close this section with two useful lemmas.

Lemma 1.1. *Let f be a TR2DF of $P_n\Box H$, where H is a path P_m or a cycle C_m. Let $\mathcal{C}^{i-1}$, $\mathcal{C}^i$, $\mathcal{C}^{i+1}$ be three consecutive columns of $P_n\Box H$.*

(i) *For $2 \leq i \leq n-1$, if $w_f(\mathcal{C}^i) = 0$, then $w_f(\mathcal{C}^{i-1} \cup \mathcal{C}^{i+1}) \geq 2m$; if $w_f(\mathcal{C}^i) = 1$, then $w_f(\mathcal{C}^{i-1} \cup \mathcal{C}^{i+1}) \geq 2m-3$.*
(ii) *For $i = 1$ or n, if $w_f(\mathcal{C}^i) = 0$, then the weight of the adjacent column of $\mathcal{C}^i$ is $2m$; if $w_f(\mathcal{C}^i) = 1$, then the weight of the adjacent column of $\mathcal{C}^i$ is at least $2m-3$.*

Proof. (i) If $w_f(\mathcal{C}^i) = 0$, then each vertex $x_{i,j} \in \mathcal{C}^i$ should be Italian dominated by $x_{i-1,j}$ and $x_{i+1,j}$. Hence, we have $f(x_{i-1,j}) + f(x_{i+1,j}) \geq 2$, and further $w_f(\mathcal{C}^{i-1} \cup \mathcal{C}^{i+1}) \geq 2m$. If $w_f(\mathcal{C}^i) = 1$, we distinguish two cases.

– $f(x_{i,j}) = 1$ for some $j \in \{2, \ldots, m-1\}$. To total Roman $\{2\}$-dominate vertices in $\mathcal{C}^i$, we have $f(x_{i-1,k}) + f(x_{i+1,k}) \geq 2$ for $k \notin \{j-1, j, j+1\}$, and $f(x_{i-1,k}) + f(x_{i+1,k}) \geq 1$ for $k \in \{j-1, j, j+1\}$. Therefore, $w_f(\mathcal{C}^{i-1} \cup \mathcal{C}^{i+1}) \geq 2(m-3) + 3 = 2m-3$.

- $H = P_m$ and $f(x_{i,j}) = 1$ for some $j \in \{1, m\}$. Without loss of generality, let $f(x_{i,1}) = 1$. Then to total Roman $\{2\}$-dominate vertices in $\mathcal{C}^i$, we have $f(x_{i-1,k}) + f(x_{i+1,k}) \geq 2$ for $k \in \{3, \dots, m\}$, and $f(x_{i-1,k}) + f(x_{i+1,k}) \geq 1$ for $k \in \{1, 2\}$. Therefore, $w_f(\mathcal{C}^{i-1} \cup \mathcal{C}^{i+1}) \geq 2(m-2) + 2 = 2m - 2 > 2m - 3$.

(ii) The proof of part (ii) is analogous to that of part (i).

□

Lemma 1.2. *Let f be a TR2DF of $C_n \square C_m$, and $\mathcal{C}^{i-1}$, $\mathcal{C}^i$, $\mathcal{C}^{i+1}$ be three consecutive columns of $C_n \square C_m$.*

(i) *If $w_f(\mathcal{C}^i) = 0$, then $w_f(\mathcal{C}^{i-1} \cup \mathcal{C}^{i+1}) \geq 2m$.*
(ii) *If $w_f(\mathcal{C}^i) = 1$, then $w_f(\mathcal{C}^{i-1} \cup \mathcal{C}^{i+1}) \geq 2m - 3$.*

Proof. The proof follows immediately from the definition of a TR2DF. □

2. ILP model

In this section, an ILP model for the TR2DP together with the proof of its correctness will be presented. This model uses two vectors of binary variables, $\mathbf{x}$ and $\mathbf{y}$ to find a TR2DF of minimum weight. The interpretation is that $x_v = 1$ if and only if $f(v) = 1$, and $y_v = 1$ if and only if $f(v) = 2$.

The TR2DP can be formulated as follows:

$$\min \sum_{v \in V} x_v + 2 \sum_{v \in V} y_v$$

$$\text{s.t. } x_v + y_v + \frac{1}{2} \sum_{u \in N(v)} x_u + \sum_{u \in N(v)} y_u \geq 1, \qquad \forall v \in V \tag{1}$$

$$\sum_{u \in N(v)} x_u + \sum_{u \in N(v)} y_u \geq x_v + y_v, \qquad \forall v \in V \tag{2}$$

$$x_v + y_v \leq 1, \qquad \forall v \in V \tag{3}$$

$$x_v, y_v \in \{0, 1\}, \qquad \forall v \in V. \tag{4}$$

Theorem 2.1. *The optimal objective function value of the TR2DP formulation is equal to the total Roman $\{2\}$-domination number γ_{tR2}.*

Proof. Given a feasible solution $(\mathbf{x}, \mathbf{y})$ of the TR2DP formulation, we construct a function $f : V \to \{0, 1, 2\}$ as follows:

- if $x_v = y_v = 0$, then set $f(v) = 0$;
- if $x_v = 1$, $y_v = 0$, then set $f(v) = 1$;
- if $x_v = 0$, $y_v = 1$, then set $f(v) = 2$.

Observe that constraint set (1) ensures that if $f(v) = 0$, then vertex v must be adjacent to at least two neighbors assigned 1 under f or at least one neighbor assigned 2 under f. Constraint set (2) ensures that if $f(v) \geq 1$, then vertex v must have at least one neighbor u with $f(u) \geq 1$. We conclude that f is a TR2DF.

Conversely, we argue that for any TR2DF f, there is a corresponding solution $(\mathbf{x}, \mathbf{y})$ satisfying constraint sets (1)–(4):

- if $f(v) = 0$, then set $x_v = y_v = 0$;
- if $f(v) = 1$, then set $x_v = 1$, $y_v = 0$;
- if $f(v) = 2$, then set $x_v = 0$, $y_v = 1$.

Note that if $f(v) = 0$, v must have at least one neighbor assigned 2 under f or at least two neighbors assigned 1 under f, which means there exists at least one vertex $u \in N(v)$ with $y_u = 1$ or two vertices $u_1, u_2 \in N(v)$

with $x_{u_1} = x_{u_2} = 1$. Hence, constraint set (1) is satisfied. If $f(v) \geq 1$, then one of x_v and y_v equals 1. It is clear that constraint set (1) is satisfied.

Consider constraint set (2). If $f(v) = 0$, then $x_v = y_v = 0$, so constraint set (2) is trivial. If $f(v) \geq 1$, then $x_v + y_v = 1$. By the definition of TR2DF, v must be adjacent to a vertex u with $f(u) \geq 1$, which means $x_u = 1$ or $y_u = 1$. It follows that constraint set (2) is satisfied. Finally, constraint sets (3) and (4) naturally hold from our construction.

We have shown a bijection between the set of feasible solutions of TR2DP and the set of all total Roman $\{2\}$-dominating functions of G. So the optimal objective function value of the TR2DP formulation is equal to the minimum weight among all TR2DFs on G. □

3. Total Roman $\{2\}$-domination number of $P_n \Box P_2$ and $C_n \Box P_2$

Lemma 3.1. *For $n \geq 2$, $\gamma_{\mathrm{tR2}}(P_n \Box P_2) \geq n + 1$.*

Proof. First, we prove that $\gamma_{\mathrm{tR2}}(P_n \Box P_2) \geq n$. Let f be a TR2DF of $P_n \Box P_2$. We assign an initial charge $ch(\mathcal{C}) = w_f(\mathcal{C})$ for each column $\mathcal{C}$ of $P_n \Box P_2$. Then we design a discharging rule to redistribute the charge accordingly.

R1. Every column $\mathcal{C}$ with $ch(\mathcal{C}) \geq 2$ sends $\frac{ch(\mathcal{C})-1}{2}$ charge to each adjacent column with initial charge 0.

Let $ch'(\mathcal{C})$ be the new charge after the discharging process is finished according to **R1**. We will show that $ch'(\mathcal{C}) \geq 1$ for all columns of $P_n \Box P_2$.

- For each column $\mathcal{C}$ with $ch(\mathcal{C}) \geq 2$, since it sends charges to at most two adjacent columns, the new charge $ch'(\mathcal{C}) \geq ch(\mathcal{C}) - 2 \times \frac{ch(\mathcal{C})-1}{2} = 1$.
- For each column $\mathcal{C}^i$ with $ch(\mathcal{C}^i) = 0$, if $i = 1$ or n, then the initial charge of the adjacent column of $\mathcal{C}^i$ is 4, so $\mathcal{C}^i$ receives $\frac{4-1}{2} = 1.5$ charge from the adjacent column. If $2 \leq i \leq n-1$, then by Lemma 1.1, $w_f(\mathcal{C}^{i-1} \cup \mathcal{C}^{i+1}) \geq 4$. Hence, $\mathcal{C}^i$ is either adjacent to a column $\mathcal{C}^a$ with $ch(\mathcal{C}^a) \geq 3$ or adjacent to two columns $\mathcal{C}^b$ and $\mathcal{C}^c$ with $ch(\mathcal{C}^b) \geq 2$, $ch(\mathcal{C}^c) \geq 2$. So $\mathcal{C}^i$ receives $\frac{ch(\mathcal{C}^a)-1}{2} \geq 1$ charge from $\mathcal{C}^a$ or $\frac{ch(\mathcal{C}^b)-1}{2} + \frac{ch(\mathcal{C}^c)-1}{2} \geq \frac{4-2}{2} = 1$ charge from $\mathcal{C}^b$ and $\mathcal{C}^c$. Therefore, the new charge of $\mathcal{C}^i$ is $ch'(\mathcal{C}^i) \geq 1$.
- For each column $\mathcal{C}$ with $ch(\mathcal{C}) = 1$, no rule is applied to $ch(\mathcal{C})$, so $ch'(\mathcal{C}) = 1$.

From the above discussion, we know that the new charge of each column of $P_n \Box P_2$ is at least 1. Since no charge is lost in the discharging procedure, we have $\sum_{v \in V(P_n \Box P_2)} f(v) = \sum_{i=1}^{n} ch(\mathcal{C}^i) = \sum_{i=1}^{n} ch'(\mathcal{C}^i) \geq n$.

Next, we prove that $\gamma_{\mathrm{tR2}}(P_n \Box P_2) \geq n + 1$. Suppose, to the contrary, that $\gamma_{\mathrm{tR2}}(P_n \Box P_2) = n$. Then by the analysis above, we can obtain that $ch'(\mathcal{C}) = 1$ for each column $\mathcal{C}$ of $P_n \Box P_2$. In particular, $ch'(\mathcal{C}^1) = ch'(\mathcal{C}^2) = 1$ requires $ch(\mathcal{C}^1) = 1$ and $ch(\mathcal{C}^2) \leq 1$. However, since f is a TR2DF, $ch(\mathcal{C}^1) = 1$ implies $ch(\mathcal{C}^2) \geq 2$, a contradiction. This completes the proof. □

Theorem 3.2. *For $n \geq 2$, $\gamma_{\mathrm{tR2}}(P_n \Box P_2) = n + 1$.*

Proof. It suffices to provide a TR2DF f of $P_n \Box P_2$ with $w(f) = n+1$. Define a function $f : V(P_n \Box P_2) \rightarrow \{0, 1, 2\}$ as follows: $f(x_{1,1}) = 1$, $f(x_{i,1}) = f(x_{i,2}) = 1$ if $i \equiv 0 \pmod 2$ and $f(x_{i,j}) = 0$ otherwise. If n is odd, redefine f by setting $f(x_{n,2}) = 1$. It is easy to check that $\sum_{v \in V(P_n \Box P_2)} f(v) = n + 1$. Note that each vertex $x_{i,j}$ with $f(x_{i,j}) = 0$ has exactly two neighbors assigned 1 and each vertex $x_{i,j}$ with $f(x_{i,j}) = 1$ has at least one neighbor assigned 1. Hence, f is a TR2DF of $P_n \Box P_2$ with weight $n + 1$. This completes the proof. □

Since $P_n \Box P_2$ is a spanning subgraph of $C_n \Box P_2$, it is easy to see that $\gamma_{\mathrm{tR2}}(P_n \Box P_2) \geq \gamma_{\mathrm{tR2}}(C_n \Box P_2)$. The next result determines the total Roman $\{2\}$-domination number for $C_n \Box P_2$.

Theorem 3.3. *For $n \geq 3$, $\gamma_{\mathrm{tR2}}(C_n \Box P_2) = 2\lceil \frac{n}{2} \rceil$.*

Proof. First, we show that $\gamma_{\text{tR2}}(C_n\Box P_2) \geq 2\lceil \frac{n}{2} \rceil$. Using a similar argument as in the proof of Lemma 3.1, we can get $\gamma_{\text{tR2}}(C_n\Box P_2) \geq n$. Thus, if n is even then we are done. When n is odd, suppose to the contrary that $\gamma_{\text{tR2}}(C_n\Box P_2) = n$. Let f be a TR2DF of $C_n\Box P_2$ with weight $w(f) = n$. Let $ch(\mathcal{C}) = w_f(\mathcal{C})$ be the initial charge of the column $\mathcal{C}$ and $ch'(\mathcal{C})$ the new charge after the discharging process is finished according to **R1**. Hence, $ch'(\mathcal{C}) = 1$ for each column $\mathcal{C}$ of $C_n\Box P_2$. By the analysis in the proof of Lemma 3.1, we know that $ch(\mathcal{C}) \leq 3$ for each column $\mathcal{C}$, and if $ch(\mathcal{C}^i) \geq 2$ for some column $\mathcal{C}^i$, then $ch(\mathcal{C}^{i-1}) = ch(\mathcal{C}^{i+1}) = 0$. We consider two cases.

Case 1. There is some column of $C_n\Box P_2$ with weight 1. By symmetry, we may assume $ch(\mathcal{C}^2) = 1$, $f(x_{2,1}) = 0$ and $f(x_{2,2}) = 1$. So we have $ch(\mathcal{C}^1) \leq 1$ and $ch(\mathcal{C}^3) \leq 1$. Since f is total, we can obtain that either $f(x_{1,2}) = 1$ or $f(x_{3,2}) = 1$, say $f(x_{1,2}) = 1$. Then $f(x_{1,1}) = f(x_{3,2}) = 0$ and $f(x_{3,1}) = 1$. Since $ch(\mathcal{C}^1) = ch(\mathcal{C}^3) = 1$, we further obtain $ch(\mathcal{C}^n) \leq 1$ and $ch(\mathcal{C}^4) \leq 1$. Continuing in this way, we get that $ch(\mathcal{C}) = 1$ for each column of $C_n\Box P_2$, and $f(x_{i,1}) = 1$ if $i \equiv 0, 3 \pmod 4$, $f(x_{i,2}) = 1$ if $i \equiv 1, 2 \pmod 4$. However, it is easy to check that for $n \equiv 3 \pmod 4$, $x_{n,1}$ is an isolated vertex in $V_1 \cup V_2$, and for $n \equiv 1 \pmod 4$, $f(x_{1,1}) = 0$ but $f(N(x_{1,1})) = 1$, a contradiction.

Case 2. There is no column of $C_n\Box P_2$ with weight 1. Since n is odd, at most $\frac{n-1}{2}$ columns of $C_n\Box P_2$ can receive positive weight. Then there are two adjacent columns with weight 0, say $w_f(\mathcal{C}^1) = w_f(\mathcal{C}^2) = 0$. So $f(x_{3,1}) = f(x_{3,2}) = 2$, and $ch(\mathcal{C}^3) = 4$, a contradiction.

Hence, $\gamma_{\text{tR2}}(C_n\Box P_2) \geq n$ if n is even and $\gamma_{\text{tR2}}(C_n\Box P_2) \geq n+1$ if n is odd. Next, we prove the upper bounds. For odd n, since $P_n\Box P_2$ is a spanning subgraph of $C_n\Box P_2$, we have $\gamma_{\text{tR2}}(C_n\Box P_2) \leq \gamma_{\text{tR2}}(P_n\Box P_2) = n+1 = 2\lceil \frac{n}{2} \rceil$. For even n, we provide a TR2DF f of $C_n\Box P_2$ as follows. Let $V_1 = \bigcup_{i=1}^{\frac{n}{2}} \mathcal{C}^{2i-1}$, $V_0 = V(C_n\Box P_2) \setminus V_1$ and $V_2 = \emptyset$. Note that each vertex assigned 0 has exactly two neighbors assigned 1 and one neighbor assigned 0. Consequently, $f = (V_0, V_1, V_2)$ is a TR2DF of $C_n\Box P_2$ with weight n. ☐

4. Total Roman {2}-domination number of $P_n\Box P_3$ and $P_n\Box P_4$

In this section, we give the exact value of the total Roman {2}-domination number for $P_n\Box P_3$, and give the estimation of this number with respect to $P_n\Box P_4$.

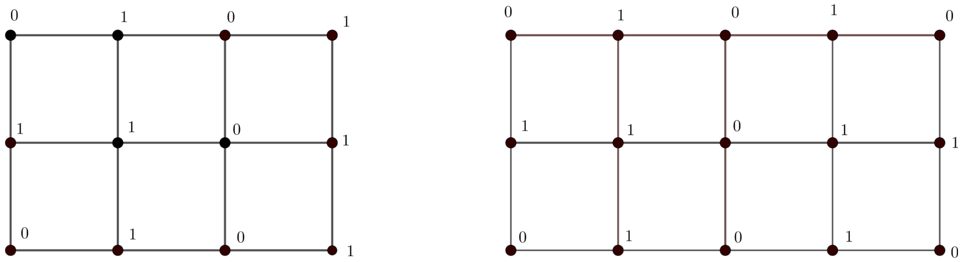
Lemma 4.1. *For $n \geq 3$, $\gamma_{\text{tR2}}(P_n\Box P_3) \geq \lceil \frac{3n+1}{2} \rceil$.*

Proof. First, we prove that $\gamma_{\text{tR2}}(P_n\Box P_3) \geq \lceil \frac{3n}{2} \rceil$. Let f be a TR2DF of $P_n\Box P_3$. We assign an initial charge $ch(\mathcal{C}) = w_f(\mathcal{C})$ for each column $\mathcal{C}$ of $P_n\Box P_3$. Then we design a discharging rule to redistribute the charge accordingly.

R2. Every column $\mathcal{C}$ with $ch(\mathcal{C}) \geq 2$ sends $\frac{ch(\mathcal{C})-1.5}{2}$ charge to each adjacent column with initial charge at most 1.

Let $ch'(\mathcal{C})$ be the new charge after the discharging process is finished according to **R2**. Now we show that the average new charge of the columns of $P_n\Box P_3$ is at least 1.5.

- For each column $\mathcal{C}$ with $ch(\mathcal{C}) \geq 2$, since it sends charges to at most two adjacent columns, the new charge $ch'(\mathcal{C}) \geq ch(\mathcal{C}) - 2 \times \frac{ch(\mathcal{C})-1.5}{2} = 1.5$.
- For each column $\mathcal{C}^i$ with $ch(\mathcal{C}^i) = 0$, if $i = 1$ or n, then by Lemma 1.1 the adjacent column of $\mathcal{C}^i$ has weight 6, so $\mathcal{C}^i$ receives $\frac{6-1.5}{2} = 2.25$ charge from the adjacent column. If $2 \leq i \leq n-1$, then by Lemma 1.1, $\mathcal{C}^i$ is either adjacent to a column $\mathcal{C}^a$ with $ch(\mathcal{C}^a) \geq 5$ or adjacent to two columns $\mathcal{C}^b$ and $\mathcal{C}^c$ with $ch(\mathcal{C}^b) \geq 2$, $ch(\mathcal{C}^c) \geq 2$ and $ch(\mathcal{C}^b)+ch(\mathcal{C}^c) \geq 6$. So $\mathcal{C}^i$ receives $\frac{ch(\mathcal{C}^a)-1.5}{2} \geq 1.75$ charge from $\mathcal{C}^a$ or $\frac{ch(\mathcal{C}^b)-1.5}{2} + \frac{ch(\mathcal{C}^c)-1.5}{2} \geq \frac{6-3}{2} = 1.5$ charge from $\mathcal{C}^b$ and $\mathcal{C}^c$. Therefore, the new charge of $\mathcal{C}^i$ is $ch'(\mathcal{C}^i) \geq 1.5$.
- For each column $\mathcal{C}^i$ with $ch(\mathcal{C}^i) = 1$, if $i = 1$ or n, $\mathcal{C}^i$ is adjacent to a column with initial charge at least 3. So the new charge of $\mathcal{C}^i$ is $ch'(\mathcal{C}^i) \geq 1 + \frac{3-1.5}{2} = 1.75$. If $2 \leq i \leq n-1$, by Lemma 1.1, $ch(\mathcal{C}^{i-1}) + ch(\mathcal{C}^{i+1}) \geq 3$. If one of $ch(\mathcal{C}^{i-1})$ and $ch(\mathcal{C}^{i+1})$ is at least 3, then as discussed in the case where $i = 1$ or n, we have $ch'(\mathcal{C}^i) \geq 1.75$. If both of $ch(\mathcal{C}^{i-1})$ and $ch(\mathcal{C}^{i+1})$ are equal to 2, then $ch'(\mathcal{C}^i) = ch(\mathcal{C}^i) + \frac{ch(\mathcal{C}^{i-1})-1.5}{2} + \frac{ch(\mathcal{C}^{i+1})-1.5}{2} =$



$P_4 \square P_3$ $P_5 \square P_3$

FIGURE 1. A TR2DF of $P_4 \square P_3$ and $P_5 \square P_3$, respectively.

$1 + \frac{4-3}{2} = 1.5$. It remains to consider the case that $ch(\mathcal{C}^{i-1}) \geq 1$, $ch(\mathcal{C}^{i+1}) \geq 1$ and $ch(\mathcal{C}^{i-1}) + ch(\mathcal{C}^{i+1}) = 3$. In this case, we have $f(x_{i,2}) = 1$, $f(x_{i,1}) = f(x_{i,3}) = 0$. Without loss of generality, assume $ch(\mathcal{C}^{i-1}) = 1$ and $ch(\mathcal{C}^{i+1}) = 2$.

(1) If $V_1 \cap \mathcal{C}^{i-1} = \{x_{i-1,1}\}$ or $V_1 \cap \mathcal{C}^{i-1} = \{x_{i-1,3}\}$, then $f(x_{i-2,1}) + f(x_{i-2,3}) \geq 3$, that is, $ch(\mathcal{C}^{i-2}) \geq 3$. After the discharging process, we have $ch'(\mathcal{C}^{i-2}) \geq 1.5$, $ch'(\mathcal{C}^{i-1}) \geq 1.75$, $ch'(\mathcal{C}^{i}) \geq 1.25$, $ch'(\mathcal{C}^{i+1}) \geq 1.5$. Now, the average value of $ch'(\mathcal{C}^{i-2})$, $ch'(\mathcal{C}^{i-1})$, $ch'(\mathcal{C}^{i})$ and $ch'(\mathcal{C}^{i+1})$ is at least 1.5.

(2) If $V_1 \cap \mathcal{C}^{i-1} = \{x_{i-1,2}\}$, then $V_1 \cap \mathcal{C}^{i+1} = \{x_{i+1,1}, x_{i+1,3}\}$. By the definition of TR2DF, we have $f(x_{i+2,1}) \geq 1$ and $f(x_{i+2,3}) \geq 1$. So $ch(\mathcal{C}^{i+2}) \geq 2$. On the other hand, since $f(x_{i-1,1}) = f(x_{i-1,3}) = 0$, then $f(x_{i-2,1}) \geq 1$ and $f(x_{i-2,3}) \geq 1$. If $ch(\mathcal{C}^{i-2}) \geq 3$, as discussed in (1), the average value of $ch'(\mathcal{C}^{i-2})$, $ch'(\mathcal{C}^{i-1})$, $ch'(\mathcal{C}^{i})$ and $ch'(\mathcal{C}^{i+1})$ is at least 1.5. So assume $ch(\mathcal{C}^{i-2}) = 2$, and then $f(x_{i-2,1}) = f(x_{i-2,3}) = 1$, $f(x_{i-2,2}) = 0$. This implies that $f(x_{i-3,1}) \geq 1$ and $f(x_{i-3,3}) \geq 1$. Hence, $ch(\mathcal{C}^{i-3}) \geq 2$. After the discharging process, we have $ch'(\mathcal{C}^{i-3}) \geq 1.5$, $ch'(\mathcal{C}^{i-2}) \geq 1.75$, $ch'(\mathcal{C}^{i-1}) \geq 1.25$, $ch'(\mathcal{C}^{i}) \geq 1.25$, $ch'(\mathcal{C}^{i+1}) \geq 1.75$ and $ch'(\mathcal{C}^{i+2}) \geq 1.5$. Now, the average value of $ch'(\mathcal{C}^{i-3})$, $ch'(\mathcal{C}^{i-2})$, $ch'(\mathcal{C}^{i-1})$, $ch'(\mathcal{C}^{i})$, $ch'(\mathcal{C}^{i+1})$ and $ch'(\mathcal{C}^{i+2})$ is at least 1.5.

From the above discussion, we know that the average new charge of each column of $P_n \square P_3$ is at least 1.5. Since no charge is lost in the discharging procedure, we have $\sum_{v \in V(P_n \square P_3)} f(v) = \sum_{i=1}^{n} ch(\mathcal{C}^i) = \sum_{i=1}^{n} ch'(\mathcal{C}^i) \geq \lceil \frac{3n}{2} \rceil$.

Next, we prove that $\gamma_{\text{tR2}}(P_n \square P_3) \geq \lceil \frac{3n+1}{2} \rceil$. If n is odd, then $\gamma_{\text{tR2}}(P_n \square P_3) \geq \lceil \frac{3n}{2} \rceil = \lceil \frac{3n+1}{2} \rceil$, as required. We consider the case when n is even. Suppose, to the contrary, that $\gamma_{\text{tR2}}(P_n \square P_3) = \lceil \frac{3n}{2} \rceil = \frac{3n}{2}$. By the above, the average new charge for each column of $P_n \square P_3$ is equal to 1.5. Hence, we deduce that $ch(\mathcal{C}^1) = 2$, $ch(\mathcal{C}^2) = ch(\mathcal{C}^3) = 1$ and $ch(\mathcal{C}^4) = ch(\mathcal{C}^5) = 2$ (It corresponds to the case (2) above). It can be checked that $ch(\mathcal{C}^1) = 2$ and $ch(\mathcal{C}^2) = ch(\mathcal{C}^3) = 1$ only if $V_1 \cap \mathcal{C}^2 = \{x_{2,2}\}$ and $f(x_{1,i}) + f(x_{3,i}) = 1$ for $i = 1, 2, 3$. So $V_2 \cap (\mathcal{C}^1 \cup \mathcal{C}^3) = \emptyset$. If $V_1 \cap \mathcal{C}^1 = \{x_{1,1}, x_{1,3}\}$, then $x_{1,1}$ is an isolated vertex in $V_1 \cup V_2$, a contradiction. If $V_1 \cap \mathcal{C}^1 = \{x_{1,1}, x_{1,2}\}$ or $\{x_{1,3}, x_{1,2}\}$, say $V_1 \cap \mathcal{C}^1 = \{x_{1,3}, x_{1,2}\}$, then $f(x_{1,1}) = 0$ and $f(N(x_{1,1})) = 1$, a contradiction. Therefore, when n is even, $\gamma_{\text{tR2}}(P_n \square P_3) \geq \lceil \frac{3n}{2} \rceil + 1 = \lceil \frac{3n+1}{2} \rceil$. □

Theorem 4.2. *For $n \geq 3$, $\gamma_{\text{tR2}}(P_n \square P_3) = \lceil \frac{3n+1}{2} \rceil$.*

Proof. In view of Lemma 4.1, it suffices to provide a TR2DF f of $P_n \square P_3$ with $w(f) = \lceil \frac{3n+1}{2} \rceil$. Define a function $f : V(P_n \square P_3) \to \{0, 1, 2\}$ as follows. If n is odd, let $V_1 = \{x_{1,2}, x_{n,2}\} \cup \mathcal{C}^2 \cup \mathcal{C}^4 \cup \mathcal{C}^6 \cup \cdots \cup \mathcal{C}^{n-1}$, $V_0 = V(P_n \square P_3) \setminus V_1$, $V_2 = \emptyset$. If n is even, let $V_1 = \{x_{1,2}\} \cup \mathcal{C}^2 \cup \mathcal{C}^4 \cup \mathcal{C}^6 \cup \cdots \cup \mathcal{C}^n$, $V_0 = V(P_n \square P_3) \setminus V_1$, $V_2 = \emptyset$.

Note that for each vertex $x_{i,j}$ with $f(x_{i,j}) = 0$, we have $f(N(x_{i,j})) = 2$ and for each vertex $x_{i,j}$ with $f(x_{i,j}) = 1$, we have $f(N(x_{i,j})) \geq 1$. Hence, f is a TR2DF of $P_n \square P_3$ with weight $\lceil \frac{3n+1}{2} \rceil$. A TR2DF of $P_4 \square P_3$ and $P_5 \square P_3$, respectively, is shown in Figure 1. This completes the proof. □

In the remaining part of this section, we establish an upper bound of the total Roman $\{2\}$-domination number of $P_n \square P_4$. Moreover, we show that this bound is attained for $n = 4, 5, 6$. In fact, we do believe that this upper bound provides the exact value for all $n \geq 7$. The following symbol will be used to construct a TR2DF of $P_n \square P_4$. Let A_i, A_j be two arrays with size $m \times s$, $m \times t$, respectively. We use the notation $A_i A_j$ to denote the concatenation of A_i and A_j which forms an $m \times (s+t)$ array. In particular, let A_i^k denote the concatenation of arrays A_i for k times which forms an $m \times ks$ array. We also need the following observation to show the sharpness of Theorem 4.4 below.

Observation 4.3. For $P_n \square P_4$ $(n \geq 4)$, if $w_f(\mathcal{C}^1) = 2$, then $w_f(\mathcal{C}^2) \geq 2$, with equality if and only if $V_1 \cap (\mathcal{C}^1 \cup \mathcal{C}^2) = \{x_{1,2}, x_{1,3}, x_{2,1}, x_{2,4}\}$ and $V_2 \cap (\mathcal{C}^1 \cup \mathcal{C}^2) = \emptyset$. So if $w_f(\mathcal{C}^1) = w_f(\mathcal{C}^2) = 2$, then $w_f(\mathcal{C}^3) \geq 2$, furthermore, if $w_f(\mathcal{C}^3) = 2$, then $V_1 \cap \mathcal{C}^3 = \{x_{3,1}, x_{3,4}\}$, $V_2 \cap \mathcal{C}^3 = \emptyset$ and $w_f(\mathcal{C}^4) \geq 2$.

Theorem 4.4. *For $n \geq 4$,*

$$\gamma_{\mathrm{tR2}}(P_n \square P_4) \leq \begin{cases} 2n+1, & n=5,6; \\ 2n, & \text{otherwise.} \end{cases}$$

This bound is sharp for $n = 4, 5, 6$.

Proof. Denote $n = 3k + t$, where $k \geq 1$ and $0 \leq t \leq 2$. Let

$$A_1 = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \qquad D = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

– If $t = 0$, when $n = 6$, the array A_1 provides a pattern of TR2DF of $P_6 \square P_4$ with weight 13; when $n = 9$, the array A_2 provides a pattern of TR2DF of $P_9 \square P_4$ with weight 18; when $n \geq 12$, the array $C^3 B^{k-4}$ provides a pattern of TR2DF of $P_n \square P_4$ with weight $2n$.
– If $t = 1$, then the array CB^{k-1} provides a pattern of TR2DF of $P_n \square P_4$ with weight $2n$.
– If $t = 2$, when $n = 5$, the array D provides a pattern of TR2DF of $P_5 \square P_4$ with weight 11; when $n \geq 8$, the array $C^2 B^{k-2}$ provides a pattern of TR2DF of $P_n \square P_4$ with weight $2n$.

Therefore, the desired upper bound is established. We now proceed to show the sharpness of the bound when $n = 4, 5, 6$.

We first consider the case $n = 4$. It is sufficient to prove that $\gamma_{\mathrm{tR2}}(P_4 \square P_4) \geq 8$. Suppose, to the contrary, that f is a TR2DF of $P_4 \square P_4$ with weight $w(f) \leq 7$. Then by Lemma 1.1, it can be seen that $w_f(\mathcal{C}^i) \geq 1$ for $1 \leq i \leq 4$, and $w_f(\mathcal{C}^i) \geq 2$ for $i = 1, 4$. If $w_f(\mathcal{C}^1) = 2$, then observation 4.3 leads to $w_f(\mathcal{C}^2) \geq 2$. Since $w_f(\mathcal{C}^3) \geq 1$ and $w_f(\mathcal{C}^4) \geq 2$, we can get $w_f(\mathcal{C}^2) = 2$, $w_f(\mathcal{C}^3) = 1$ and $w_f(\mathcal{C}^4) = 2$, contradicting to Observation 4.3. Hence, $w_f(\mathcal{C}^1) \geq 3$ and $w_f(\mathcal{C}^4) \geq 3$. Then $\sum_{i=1}^{4} w_f(\mathcal{C}^i) \geq 8$, a contradiction. Hence, $\gamma_{\mathrm{tR2}}(P_n \square P_4) \geq 8$, and then $\gamma_{\mathrm{tR2}}(P_n \square P_4) = 8$.

Then we consider the case $n = 5$. It is sufficient to prove that $\gamma_{\mathrm{tR2}}(P_5 \square P_4) \geq 11$. Suppose, to the contrary, that f is a TR2DF of $P_5 \square P_4$ with weight $w(f) \leq 10$.

Claim 1. $w_f(\mathcal{C}^i) \geq 1$ for $1 \leq i \leq 5$.

If $w_f(\mathcal{C}^1) = 0$ or $w_f(\mathcal{C}^2) = 0$, then by Lemma 1.1, $\sum_{i=1}^{3} w_f(\mathcal{C}^i) \geq 8$. Hence $w_f(\mathcal{C}^4) + w_f(\mathcal{C}^5) \leq 2$. By Lemma 1.1(ii) and Observation 4.3, $w_f(\mathcal{C}^5)$ cannot be 0, 1 or 2, a contradiction. Hence, $w_f(\mathcal{C}^1) \geq 1$ and $w_f(\mathcal{C}^2) \geq 1$, then by symmetry, $w_f(\mathcal{C}^5) \geq 1$ and $w_f(\mathcal{C}^4) \geq 1$.

If $w_f(\mathcal{C}^3) = 0$, then $w_f(\mathcal{C}^2) + w_f(\mathcal{C}^4) \geq 8$. Hence, $w_f(\mathcal{C}^1) = w_f(\mathcal{C}^5) = 1$. Now by Lemma 1.1, $w_f(\mathcal{C}^1) + w_f(\mathcal{C}^2) \geq 6$ and $w_f(\mathcal{C}^4) + w_f(\mathcal{C}^5) \geq 6$, contrary to $w(f) \leq 10$.

Claim 2. $w_f(\mathcal{C}^i) \geq 2$ for $i = 1, 5$.

Suppose $w_f(\mathcal{C}^1) = 1$. Then $w_f(\mathcal{C}^2) \geq 5$. If $w_f(\mathcal{C}^2) \geq 6$, we have $w_f(\mathcal{C}^3) = w_f(\mathcal{C}^4) = w_f(\mathcal{C}^5) = 1$ by Claim 1. However, according to Lemma 1.1, $w_f(\mathcal{C}^3) + w_f(\mathcal{C}^5) \geq 5$ if $w_f(\mathcal{C}^4) = 1$, a contradicton. So $w_f(\mathcal{C}^2) = 5$ and $\sum_{i=3}^{5} w_f(\mathcal{C}^i) \leq 4$. Then we have that $w_f(\mathcal{C}^5) = 2$ and $w_f(\mathcal{C}^3) = w_f(\mathcal{C}^4) = 1$, which is impossible by Lemma 1.1. Hence, $w_f(\mathcal{C}^1) \geq 2$ and $w_f(\mathcal{C}^5) \geq 2$.

Claim 3. $w_f(\mathcal{C}^1) \geq 3$ or $w_f(\mathcal{C}^5) \geq 3$.

By Claims 1 and 2, we have $w_f(\mathcal{C}^2) + w_f(\mathcal{C}^4) \leq 5$. So one of $w_f(\mathcal{C}^2)$ and $w_f(\mathcal{C}^4)$ is at most 2, say $w_f(\mathcal{C}^2) \leq 2$. In addition, by Observation 4.3 and Lemma 1.1, we have $w_f(\mathcal{C}^4) + w_f(\mathcal{C}^5) \geq 4$. If $w_f(\mathcal{C}^1) = 2$, then $w_f(\mathcal{C}^2) = w_f(\mathcal{C}^3) = 2$. By Observation 4.3, we further obtain that $V_1 \cap \mathcal{C}^4 = \{x_{4,2}, x_{4,3}\}$, $V_2 \cap \mathcal{C}^4 = \emptyset$ and $w_f(\mathcal{C}^5) = 2$. However, it is easy to check that in this case f is not a TR2DF of $P_5 \square P_4$, a contradiction. Hence, $w_f(\mathcal{C}^1) \geq 3$ when $w_f(\mathcal{C}^2) \leq 2$, and $w_f(\mathcal{C}^5) \geq 3$ when $w_f(\mathcal{C}^4) \leq 2$.

By Claim 3, without loss of generality, we may assume $w_f(\mathcal{C}^5) \geq 3$. Since $w_f(\mathcal{C}^3) = 1$ implies $w_f(\mathcal{C}^2) + w_f(\mathcal{C}^4) \geq 5$ and $w_f(\mathcal{C}^3) = 2$ implies $w_f(\mathcal{C}^2) + w_f(\mathcal{C}^4) \geq 3$, we have $\sum_{i=2}^{4} w_f(\mathcal{C}^i) \geq 5$. Combining Claim 2, we can get $w_f(\mathcal{C}^1) = 2$, $\sum_{i=2}^{4} w_f(\mathcal{C}^i) = 5$ and $w_f(\mathcal{C}^5) = 3$. So $w_f(\mathcal{C}^2) = w_f(\mathcal{C}^3) = 2$ and $w_f(\mathcal{C}^4) = 1$, contradicting Observation 4.3. By the above analysis, we conclude that $\gamma_{\mathrm{tR2}}(P_5 \square P_4) \geq 11$, and hence $\gamma_{\mathrm{tR2}}(P_5 \square P_4) = 11$.

Finally, we consider the case $n = 6$. It is sufficient to prove that $\gamma_{\mathrm{tR2}}(P_6 \square P_4) \geq 13$. Suppose, to the contrary, that f is a TR2DF of $P_6 \square P_4$ with weight $w(f) \leq 12$.

Claim 4. $w_f(\mathcal{C}^i) \geq 1$ for $1 \leq i \leq 6$.

If $w_f(\mathcal{C}^1) = 0$ or $w_f(\mathcal{C}^2) = 0$, then $\sum_{i=1}^{3} w_f(\mathcal{C}^i) \geq 8$ and $\sum_{i=4}^{6} w_f(\mathcal{C}^i) \leq 4$. By Lemma 1.1 and Observation 4.3, $w_f(\mathcal{C}^6) = 3$. No matter $w_f(\mathcal{C}^5) = 1$ or $w_f(\mathcal{C}^5) = 0$, we both have $w_f(\mathcal{C}^4) + w_f(\mathcal{C}^6) \geq 5$, a contradiction. Hence, $w_f(\mathcal{C}^1) \geq 1$, $w_f(\mathcal{C}^2) \geq 1$ and by symmetry, $w_f(\mathcal{C}^6) \geq 1$, $w_f(\mathcal{C}^5) \geq 1$.

If $w_f(\mathcal{C}^3) = 0$, then $w_f(\mathcal{C}^2) + w_f(\mathcal{C}^4) \geq 8$. Obviously, $w_f(\mathcal{C}^4) \geq 1$, otherwise $w_f(\mathcal{C}^2) + w_f(\mathcal{C}^5) \geq 8 + 8 = 16$, a contradiction. If $w_f(\mathcal{C}^6) = 1$, then $w_f(\mathcal{C}^5) \geq 5$ and $\sum_{i=2}^{6} w_f(\mathcal{C}^i) \geq 14$, a contradiction. Hence, $w_f(\mathcal{C}^6) \geq 2$. Since $w_f(\mathcal{C}^i) \geq 1$ for $i = 1, 2, 5, 6$, we have $w_f(\mathcal{C}^1) = w_f(\mathcal{C}^5) = 1$, $w_f(\mathcal{C}^6) = 2$. However, by symmetry of $\mathcal{C}^1$ and $\mathcal{C}^6$, according to Observation 4.3, it is impossible. Hence, $w_f(\mathcal{C}^3) \geq 1$ and by symmetry $w_f(\mathcal{C}^4) \geq 1$.

Claim 5. $w_f(\mathcal{C}^i) \geq 2$ for $i = 1, 6$.

If $w_f(\mathcal{C}^1) = 1$, then $w_f(\mathcal{C}^2) \geq 5$. By Claim 4, we have $1 \leq w_f(\mathcal{C}^6) \leq 3$ and $w_f(\mathcal{C}^5) + w_f(\mathcal{C}^6) \leq 4$. Then $w_f(\mathcal{C}^6) \neq 1$ by Lemma 1.1. If $w_f(\mathcal{C}^6) = 2$, then $w_f(\mathcal{C}^5) = 2$, and by Observation 4.3, $w_f(\mathcal{C}^4) \geq 2$. In this case, $\sum_{i=1}^{6} w_f(\mathcal{C}^i) \geq 13$, a contradiction. Hence, $w_f(\mathcal{C}^6) = 3$, and further, we have $w_f(\mathcal{C}^3) = w_f(\mathcal{C}^4) = w_f(\mathcal{C}^5) = 1$, which contradicts Lemma 1.1 when $i = 4$. Therefore, $w_f(\mathcal{C}^1) \geq 2$ and by symmetry $w_f(\mathcal{C}^6) \geq 2$.

Claim 6. $w_f(\mathcal{C}^i) \geq 2$ for $i = 3, 4$.

If $w_f(\mathcal{C}^4) = 1$, then $\sum_{i=3}^{5} w_f(\mathcal{C}^i) \geq 6$. By Claim 5 we have $w_f(\mathcal{C}^2) \leq 2$. When $w_f(\mathcal{C}^2) = 2$, then $w_f(\mathcal{C}^1) = w_f(\mathcal{C}^6) = 2$ and $\sum_{i=3}^{5} w_f(\mathcal{C}^i) = 6$. So both of $w_f(\mathcal{C}^3)$ and $w_f(\mathcal{C}^5)$ are at least 2, and then $\{w_f(\mathcal{C}^3), w_f(\mathcal{C}^5)\} = \{2, 3\}$. According to Observation 4.3, no matter $w_f(\mathcal{C}^3) = 2$ or $w_f(\mathcal{C}^5) = 2$, we both have $w_f(\mathcal{C}^4) \geq 2$, a contradiction. Hence, $w_f(\mathcal{C}^4) \geq 2$ and by symmetry $w_f(\mathcal{C}^3) \geq 2$.

Claim 7. $w_f(\mathcal{C}^i) \geq 2$ for $i = 2, 5$.

If $w_f(\mathcal{C}^5) = 1$, then $\sum_{i=4}^{6} w_f(\mathcal{C}^i) \geq 6$. Combining the Claims 4–6, we have $w_f(\mathcal{C}^1) = w_f(\mathcal{C}^2) = w_f(\mathcal{C}^3) = 2$ and $\sum_{i=4}^{6} w_f(\mathcal{C}^i) = 6$. So by Observation 4.3, $w_f(\mathcal{C}^4) = 2$ and $w_f(\mathcal{C}^6) = 3$. We deduce that $V_1 \cap \mathcal{C}^5 = \{x_{5,2}\}$ or $\{x_{5,3}\}$, and $\mathcal{C}^6 \cap (V_1 \cup V_2) = \{x_{6,1}, x_{6,4}\}$. However, $x_{6,1}$ is an isolated vertex in the subgraph induced by $V_1 \cup V_2$, a contradiction.

By Claims 4–7, we have $w_f(\mathcal{C}^i) = 2$ for $1 \le i \le 6$. Hence, we deduce that $V_1 \cap \mathcal{C}^4 = \{x_{4,2}, x_{4,3}\}$ by Observation 4.3. On the other hand, interchanging the roles of $\mathcal{C}^1$ and $\mathcal{C}^6$ in Observation 4.3, we can get $V_1 \cap \mathcal{C}^4 = \{x_{4,1}, x_{4,4}\}$, a contradiction. By the above analysis, we conclude that $\gamma_{\rm tR2}(P_6 \Box P_4) \ge 13$, and hence $\gamma_{\rm tR2}(P_6 \Box P_4) = 13$.

Therefore, the upper bound in theorem is sharp for $n = 4, 5, 6$. This completes the proof of Theorem 4.4. □

5. Total Roman {2}-domination number of $C_n \Box C_m$ for $3 \le m \le 5$

Theorem 5.1. *For $n \ge 3$, $\gamma_{\rm tR2}(C_n \Box C_3) = \lceil \frac{4n}{3} \rceil$.*

Proof. Let f be a TR2DF of $C_n \Box C_3$. First, we assign an initial charge $ch(\mathcal{C}) = w_f(\mathcal{C})$ for each column $\mathcal{C}$ of $C_n \Box C_3$. Next, we discharge the initial charge of columns with weight at least 2 to their adjacent columns according to the following rule.

R3. Every column $\mathcal{C}$ with $ch(\mathcal{C}) \ge 2$ sends $\frac{ch(\mathcal{C}) - \frac{4}{3}}{2}$ charge to each adjacent column with initial charge at most 1.

Let $ch'(\mathcal{C})$ be the new charge after the discharging process is finished according to **R3**. Now we show that $ch'(\mathcal{C}) \ge \frac{4}{3}$ for all columns of $C_n \Box C_3$.

- For each column $\mathcal{C}$ with $ch(\mathcal{C}) \ge 2$, since it sends charges to at most two adjacent columns, the new charge $ch'(\mathcal{C}) \ge ch(\mathcal{C}) - 2 \times \frac{ch(\mathcal{C}) - \frac{4}{3}}{2} = \frac{4}{3}$.
- For each column $\mathcal{C}$ with $ch(\mathcal{C}) = 1$, by Lemma 1.2, $\mathcal{C}$ must be adjacent to a column with initial charge at least 2. So the new charge of $\mathcal{C}$ is $ch'(\mathcal{C}) \ge 1 + \frac{2 - \frac{4}{3}}{2} = \frac{4}{3}$.
- For each column $\mathcal{C}$ with $ch(\mathcal{C}) = 0$, by Lemma 1.2, $\mathcal{C}$ is either adjacent to a column $\mathcal{C}^a$ with $ch(\mathcal{C}^a) \ge 5$ or adjacent to two columns $\mathcal{C}^b$ and $\mathcal{C}^c$ with $ch(\mathcal{C}^b) \ge 2$, $ch(\mathcal{C}^c) \ge 2$ and $ch(\mathcal{C}^b) + ch(\mathcal{C}^c) \ge 6$. So $\mathcal{C}$ receives $\frac{ch(\mathcal{C}^a) - \frac{4}{3}}{2} \ge \frac{11}{6}$ charge from $\mathcal{C}^a$ or $\frac{ch(\mathcal{C}^b) - \frac{4}{3}}{2} + \frac{ch(\mathcal{C}^c) - \frac{4}{3}}{2} \ge \frac{6 - \frac{8}{3}}{2} = \frac{5}{3}$ charge from $\mathcal{C}^b$ and $\mathcal{C}^c$. Therefore, the new charge of $\mathcal{C}$ is $ch'(\mathcal{C}) \ge \frac{5}{3} > \frac{4}{3}$.

From the above discussion, we know that the new charge of each column of $C_n \Box C_3$ is at least $\frac{4}{3}$. Since the discharging procedure preserves the total charge of $C_n \Box C_3$, we have $\sum_{v \in V(C_n \Box C_3)} f(v) = \sum_{i=1}^{n} ch(\mathcal{C}^i) = \sum_{i=1}^{n} ch'(\mathcal{C}^i) \ge \lceil \frac{4n}{3} \rceil$.

Next, we present a TR2DF of $C_n \Box C_3$ with weight $\lceil \frac{4n}{3} \rceil$. Denote $n = 3k + t$, where $k \ge 1$ and $0 \le t \le 2$. Let

$$A_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \quad A_5 = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

$$A_8 = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

- If $t = 0$, then the array A_3^k provides a pattern of TR2DF of $C_n \Box C_3$ with weight $4k = \lceil \frac{4n}{3} \rceil$.
- If $t = 1$, then the array $A_3^{k-1} A_4$ provides a pattern of TR2DF of $C_n \Box C_3$ with weight $4(k-1) + 6 = \lceil \frac{4n}{3} \rceil$.
- If $t = 2$, when $n = 5$, the array A_5 provides a pattern of TR2DF of $C_5 \Box C_3$; when $n \ge 8$, the array $A_3^{k-2} A_8$ provides a pattern of TR2DF of $C_n \Box C_3$ with weight $4(k-2) + 11 = \lceil \frac{4n}{3} \rceil$.

By the above discussion, all the upper bounds are established. A TR2DF of $C_7 \Box C_3$ and $C_8 \Box C_3$, respectively, is shown in Figure 2. □

We provide next the total Roman {2}-domination number of $C_n \Box C_4$ for $n \ge 4$.

Lemma 5.2. *Let $n \ge 4$. Then $\gamma_{\rm tR2}(C_n \Box C_4) \ge \lceil \frac{7n}{4} \rceil$.*

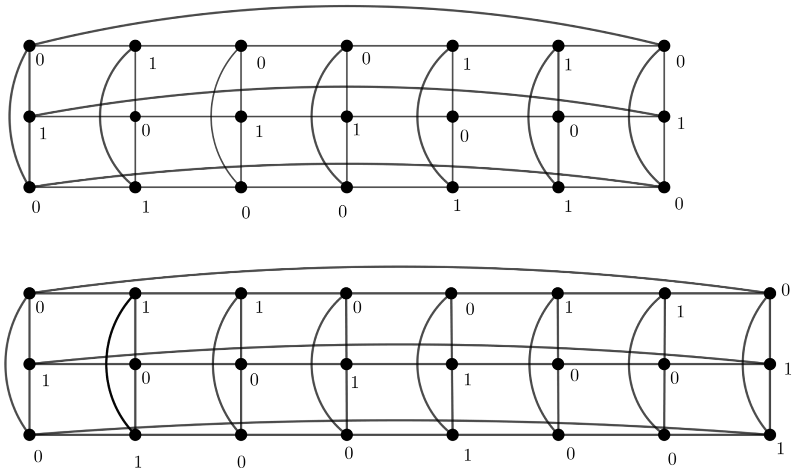


FIGURE 2. A TR2DF of $C_7\Box C_3$ and $C_8\Box C_3$, respectively.

Proof. Let f be a TR2DF of $C_n\Box C_4$. First, we assign an initial charge $ch(\mathcal{C}) = w_f(\mathcal{C})$ for each column $\mathcal{C}$ of $C_n\Box C_4$. Next, we discharge the initial charge of columns with weight at least 2 to their adjacent columns according to the following rule.

R4. Every column $\mathcal{C}$ with $ch(\mathcal{C}) \geq 2$ sends $\frac{ch(\mathcal{C})-\frac{7}{4}}{2}$ charge to each adjacent column with initial charge at most 1.

Let $ch'(\mathcal{C})$ be the new charge after the discharging process is finished according to **R4**. Now we show that $ch'(\mathcal{C}) \geq \frac{7}{4}$ for all columns of $C_n\Box C_4$.

- For each column $\mathcal{C}$ with $ch(\mathcal{C}) \geq 2$, since it sends charges to at most two adjacent columns, the new charge $ch'(\mathcal{C}) \geq ch(\mathcal{C}) - 2 \times \frac{ch(\mathcal{C})-\frac{7}{4}}{2} = \frac{7}{4}$.
- For each column $\mathcal{C}$ with $ch(\mathcal{C}) = 1$, by Lemma 1.2(ii), $\mathcal{C}$ is either adjacent to a column $\mathcal{C}^a$ with $ch(\mathcal{C}^a) \geq 4$ or adjacent to two columns $\mathcal{C}^b$ and $\mathcal{C}^c$ with $ch(\mathcal{C}^b) \geq 2$, $ch(\mathcal{C}^c) \geq 2$ and $ch(\mathcal{C}^b) + ch(\mathcal{C}^c) \geq 5$. So $\mathcal{C}$ receives $\frac{ch(\mathcal{C}^a)-\frac{7}{4}}{2} \geq \frac{9}{8}$ charge from $\mathcal{C}^a$ or $\frac{ch(\mathcal{C}^b)-\frac{7}{4}}{2} + \frac{ch(\mathcal{C}^c)-\frac{7}{4}}{2} \geq \frac{5-\frac{7}{2}}{2} = \frac{3}{4}$ charge from $\mathcal{C}^b$ and $\mathcal{C}^c$. Therefore, the new charge of $\mathcal{C}$ is $ch'(\mathcal{C}) \geq 1 + \frac{3}{4} = \frac{7}{4}$.
- For each column $\mathcal{C}$ with $ch(\mathcal{C}) = 0$, by Lemma 1.2(i), $\mathcal{C}$ is either adjacent to a column $\mathcal{C}^a$ with $ch(\mathcal{C}^a) \geq 7$ or adjacent to two columns $\mathcal{C}^b$ and $\mathcal{C}^c$ with $ch(\mathcal{C}^b) \geq 2$, $ch(\mathcal{C}^c) \geq 2$ and $ch(\mathcal{C}^b) + ch(\mathcal{C}^c) \geq 8$. So $\mathcal{C}$ receives $\frac{ch(\mathcal{C}^a)-\frac{7}{4}}{2} \geq \frac{21}{8} > \frac{7}{4}$ charge from $\mathcal{C}^a$ or $\frac{ch(\mathcal{C}^b)-\frac{7}{4}}{2} + \frac{ch(\mathcal{C}^c)-\frac{7}{4}}{2} \geq \frac{9}{4} > \frac{7}{4}$ charge from $\mathcal{C}^b$ and $\mathcal{C}^c$. Therefore, the new charge of $\mathcal{C}$ is $ch'(\mathcal{C}) > \frac{7}{4}$.

In summary, $ch'(\mathcal{C}) \geq \frac{7}{4}$ holds for each column $\mathcal{C}$ of $C_n\Box C_4$. Since the discharging procedure preserves the total charge of $C_n\Box C_4$, we have $\sum_{v\in V(C_n\Box C_4)} f(v) = \sum_{i=1}^{n} ch(\mathcal{C}^i) = \sum_{i=1}^{n} ch'(\mathcal{C}^i) \geq \lceil \frac{7n}{4} \rceil$. □

We will show that the lower bound in Lemma 5.2 is tight except for $n = 4$.

Lemma 5.3. *For $n = 4$, $\gamma_{\rm tR2}(C_n\Box C_4) = 2n$.*

Proof. Since $P_n\Box P_4$ is a subgraph of $C_n\Box C_4$, then by Theorem 4.4, $\gamma_{\rm tR2}(C_4\Box C_4) \leq \gamma_{\rm tR2}(P_4\Box P_4) = 8$.

Next, we prove the lower bound. Assume to the contrary that f is a TR2DF of $C_4\Box C_4$ with $w(f) = 7$. Then by Lemma 1.2(i), we have $w_f(\mathcal{C}) \geq 1$ for each column $\mathcal{C}$ of $C_4\Box C_4$. Hence, there exists a column $\mathcal{C}^i$ with $w_f(\mathcal{C}^i) = 1$. By symmetry, let $w_f(\mathcal{C}^2) = 1$. Then by Lemma 1.2(ii), $w_f(\mathcal{C}^1) + w_f(\mathcal{C}^3) \geq 5$, and further, $w_f(\mathcal{C}^1) + w_f(\mathcal{C}^3) = 5$, $w_f(\mathcal{C}^2) = w_f(\mathcal{C}^4) = 1$ and $\{w_f(\mathcal{C}^1), w_f(\mathcal{C}^3)\} = \{2, 3\}$. Without loss of generality, assume

$f(x_{2,3}) = 1$. This implies that $f(x_{1,1}) + f(x_{3,1}) = 2$ and $f(x_{1,i}) + f(x_{3,i}) = 1$ for $i = 2, 3, 4$. So $V_1 \cap \mathcal{C}^4 = \{x_{4,3}\}$. Since f is total, one of $x_{1,3}$ and $x_{3,3}$ is assigned 1, say $f(x_{3,3}) = 1$. Then $f(x_{1,3}) = 0$.

If $f(x_{1,4}) = 0$, then $f(x_{3,4}) = 1$, $f(x_{1,1}) = 2$, $f(x_{3,1}) = 0$, $f(x_{3,2}) = 1$ and $f(x_{1,2}) = 0$. However, $x_{1,1}$ is an isolated vertex in $V_1 \cup V_2$, a contradiction.

If $f(x_{1,4}) = 1$, then $f(x_{3,4}) = 0$, $f(x_{3,1}) = 1$, $f(x_{1,1}) = 1$, $f(x_{1,2}) = 1$ and $f(x_{3,2}) = 0$. In this case, $x_{3,1}$ is an isolated vertex in $V_1 \cup V_2$, a contradiction.

Therefore, $\gamma_{\mathrm{tR2}}(C_4 \square C_4) \geq 8$ and then $\gamma_{\mathrm{tR2}}(C_4 \square C_4) = 8$. □

Theorem 5.4. *For $n \geq 5$, $\gamma_{\mathrm{tR2}}(C_n \square C_4) = \lceil \frac{7n}{4} \rceil$.*

Proof. To prove the upper bound, it is sufficient to provide a TR2DF of $C_n \square C_4$ with weight $\lceil \frac{7n}{4} \rceil$. Denote $n = 8k + t$, where $k \geq 0$ and $0 \leq t \leq 7$. Let

$$H_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \quad H_2 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad H_5 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$H_6 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad H_7 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad H_9 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$

$$H_{10} = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}, \quad H_{11} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix},$$

$$H_{12} = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

- If $t = 0$, then the array $(H_1H_2)^k$ provides a pattern of TR2DF of $C_n \square C_4$ with weight $14k = \lceil \frac{7n}{4} \rceil$.
- If $t = 1$, then the array $(H_1H_2)^{k-1}H_9$ provides a pattern of TR2DF of $C_n \square C_4$ with weight $14k + 2 = \lceil \frac{7n}{4} \rceil$.
- If $t = 2$, then the array $(H_1H_2)^{k-1}H_1H_{10}$ provides a pattern of TR2DF of $C_n \square C_4$ with weight $14k+4 = \lceil \frac{7n}{4} \rceil$.
- If $t = 3$, then the array $(H_1H_2)^{k-1}H_1H_{11}$ provides a pattern of TR2DF of $C_n \square C_4$ with weight $14k+6 = \lceil \frac{7n}{4} \rceil$.
- If $t = 4$, then the array $(H_1H_2)^{k-1}H_1H_{12}$ provides a pattern of TR2DF of $C_n \square C_4$ with weight $14k+7 = \lceil \frac{7n}{4} \rceil$.
- If $t = 5$, then the array $(H_1H_2)^kH_5$ provides a pattern of TR2DF of $C_n \square C_4$ with weight $14k + 9 = \lceil \frac{7n}{4} \rceil$.
- If $t = 6$, then the array $(H_1H_2)^kH_1H_6$ provides a pattern of TR2DF of $C_n \square C_4$ with weight $14k+11 = \lceil \frac{7n}{4} \rceil$.
- If $t = 7$, then the array $(H_1H_2)^kH_1H_7$ provides a pattern of TR2DF of $C_n \square C_4$ with weight $14k+13 = \lceil \frac{7n}{4} \rceil$.

This establish the upper bound. □

In what follows, we determine the total Roman $\{2\}$-domination number of $C_n \square C_5$ for $n \geq 5$.

Theorem 5.5. *For $n \geq 5$,* $\gamma_{\mathrm{tR2}}(C_n \square C_5) = \begin{cases} 2n, & n \equiv 0 \pmod 5; \\ 2n+1, & n \equiv 1, 2, 3 \pmod 5; \\ 2n+2, & n \equiv 4 \pmod 5. \end{cases}$

Proof. To show the lower bounds, we proceed with three claims.

Claim 1. If $n \geq 5$, then $\gamma_{\mathrm{tR2}}(C_n \square C_5) \geq 2n$.

Let f be a TR2DF of $C_n\Box C_5$. Set the initial charge $ch(\mathcal{C}) = w_f(\mathcal{C})$ to each column $\mathcal{C}$ of $C_n\Box C_5$. We then redistribute the initial charge of columns as follows.

R5. Every column $\mathcal{C}$ with $ch(\mathcal{C}) \geq 3$ sends $\frac{ch(\mathcal{C})-2}{2}$ charge to each adjacent column with initial charge at most 1.

Let $ch'(\mathcal{C})$ be the new charge after the discharging process is finished according to **R5**. Now we show that $ch'(\mathcal{C}) \geq 2$ for all columns of $C_n\Box C_5$.

- For each column $\mathcal{C}$ with $ch(\mathcal{C}) \geq 3$, since it sends charges to at most two adjacent columns, the new charge $ch'(\mathcal{C}) \geq ch(\mathcal{C}) - 2 \times \frac{ch(\mathcal{C})-2}{2} = 2$.
- For each column $\mathcal{C}$ with $ch(\mathcal{C}) = 1$, by Lemma 1.2(ii), $\mathcal{C}$ is either adjacent to a column $\mathcal{C}^a$ with $ch(\mathcal{C}^a) \geq 5$ or adjacent to two columns $\mathcal{C}^b$ and $\mathcal{C}^c$ with $ch(\mathcal{C}^b) \geq 3$, $ch(\mathcal{C}^c) \geq 3$ and $ch(\mathcal{C}^b) + ch(\mathcal{C}^c) \geq 7$. So $\mathcal{C}$ receives $\frac{ch(\mathcal{C}^a)-2}{2} \geq \frac{3}{2}$ charge from $\mathcal{C}^a$ or $\frac{ch(\mathcal{C}^b)-2}{2} + \frac{ch(\mathcal{C}^c)-2}{2} \geq \frac{7-4}{2} = \frac{3}{2}$ charge from $\mathcal{C}^b$ and $\mathcal{C}^c$. Therefore, the new charge of $\mathcal{C}$ is $ch'(\mathcal{C}) \geq 1 + \frac{3}{2} = \frac{5}{2}$.
- For each column $\mathcal{C}$ with $ch(\mathcal{C}) = 0$, by Lemma 1.2(i), $\mathcal{C}$ is either adjacent to a column $\mathcal{C}^a$ with $ch(\mathcal{C}^a) \geq 8$ or adjacent to two columns $\mathcal{C}^b$ and $\mathcal{C}^c$ with $ch(\mathcal{C}^b) \geq 3$, $ch(\mathcal{C}^c) \geq 3$ and $ch(\mathcal{C}^b) + ch(\mathcal{C}^c) \geq 10$. So $\mathcal{C}$ receives $\frac{ch(\mathcal{C}^a)-2}{2} \geq 3$ charge from $\mathcal{C}^a$ or $\frac{ch(\mathcal{C}^b)-2}{2} + \frac{ch(\mathcal{C}^c)-2}{2} \geq \frac{10-4}{2} = 3$ charge from $\mathcal{C}^b$ and $\mathcal{C}^c$. Therefore, the new charge of $\mathcal{C}$ is $ch'(\mathcal{C}) \geq 3$.

In summary, $ch'(\mathcal{C}) \geq 2$ holds for each column $\mathcal{C}$ of $C_n\Box C_5$. Since the discharging procedure preserves the total charge of $C_n\Box C_5$, we have $\sum_{v\in V(C_n\Box C_5)} f(v) = \sum_{i=1}^n ch(\mathcal{C}^i) = \sum_{i=1}^n ch'(\mathcal{C}^i) \geq 2n$. This completes the proof of Claim 1.

Claim 2. If $n \not\equiv 0 \pmod 5$ and $n \geq 5$, then $\gamma_{\mathrm{tR2}}(C_n\Box C_5) \geq 2n+1$.

Suppose, to the contrary that f is a TR2DF of $C_n\Box C_5$ with $\sum_{v\in V(C_n\Box C_5)} f(v) = 2n$ when $n \not\equiv 0 \pmod 5$. Then by the previous discharging result, $ch(\mathcal{C}) = w_f(\mathcal{C}) = 2$ for each column $\mathcal{C}$ of $C_n\Box C_5$. If there exists a vertex $x_{i,j} \in V(C_n\Box C_5)$ with $f(x_{i,j}) = 2$, then $V_1 \cap \mathcal{C}^i = \emptyset$, and therefore, $w_f(\mathcal{C}^{i-1}) + w_f(\mathcal{C}^{i+1}) \geq 5$, a contradiction. This leads to $f(x_{i,j}) = 0$ or 1 for each $x_{i,j} \in V(C_n\Box C_5)$.

Suppose that there are two adjacent vertices $x_{i,j}$ and $x_{i,j+1}$ with $f(x_{i,j}) = f(x_{i,j+1}) = 1$, by the symmetry of $C_n\Box C_5$, say $f(x_{1,3}) = f(x_{1,4}) = 1$. Then to Roman {2}-dominate $x_{1,1}$, we have $f(x_{2,1}) = f(x_{n,1}) = 1$, and to Roman {2}-dominate $x_{1,2}$, we have $f(x_{2,2}) = 1$ or $f(x_{n,2}) = 1$. Now the values of the vertices in $\bigcup_{i=1}^5 \mathcal{C}^i$ are determined. Specifically, If $f(x_{2,2}) = 1$, then $f(x_{2,j}) = 0$ for $j = 3,4,5$. It is easy to check that $V_1 \cap \mathcal{C}^3 = \{x_{3,4}, x_{3,5}\}$, $V_1 \cap \mathcal{C}^4 = \{x_{4,2}, x_{4,3}\}$ and $V_1 \cap \mathcal{C}^5 = \{x_{5,1}, x_{5,5}\}$. In this case, the TR2DF on the first five columns of $C_n\Box C_5$ is shown in Figure 3a. If $f(x_{n,2}) = 1$, then $f(x_{n,j}) = 0$ for $j = 3,4,5$. Hence, to Roman {2}-dominate $x_{1,5}$, we have $f(x_{2,5}) = 1$ and then $f(x_{2,j}) = 0$ for $j = 2,3,4$. It is easy to check that $V_1 \cap \mathcal{C}^3 = \{x_{3,2}, x_{3,3}\}$, $V_1 \cap \mathcal{C}^4 = \{x_{4,4}, x_{4,5}\}$ and $V_1 \cap \mathcal{C}^5 = \{x_{5,1}, x_{5,2}\}$. In this case, the TR2DF on the first five columns of $C_n\Box C_5$ is shown in Figure 3b. In both cases, we can find that the TR2DF on the remaining columns of $C_n\Box C_5$ is a repeat pattern of that on the first five columns. However, it is a TR2DF only when $n \equiv 0 \pmod 5$, a contradiction.

Hence, the two vertices assigned 1 in each column of $C_n\Box C_5$ are not adjacent. Without loss of generality, let $f(x_{1,1}) = f(x_{1,3}) = 1$. Since f is total, one of $f(x_{2,1})$ and $f(x_{n,1})$ is 1, say $f(x_{2,1}) = 1$. Hence, $f(x_{2,5}) = 0$ and $f(x_{n,5}) = 1$, which leads to $f(x_{n,4}) = 0$ and $f(x_{2,4}) = 1$. It is not hard to check that $V_1 \cap \mathcal{C}^3 = \{x_{3,2}, x_{3,4}\}$, $V_1 \cap \mathcal{C}^4 = \{x_{4,2}, x_{4,5}\}$ and $V_1 \cap \mathcal{C}^5 = \{x_{5,3}, x_{5,5}\}$. As before, we can find that the TR2DF on the remaining columns of $C_n\Box C_5$ is a repeat pattern of that on the first five columns. It is a TR2DF only when $n \equiv 0 \pmod 5$, a contradiction again. This completes the proof Claim 2.

Claim 3. If $n \equiv 4 \pmod 5$ and $n \geq 5$, then $\gamma_{\mathrm{tR2}}(C_n\Box C_5) \geq 2n+2$.

Suppose, to the contrary that f is a TR2DF of $C_n\Box C_5$ with $\sum_{v\in V(C_n\Box C_5)} f(v) = 2n+1$ when $n \equiv 4 \pmod 5$. Then by the discharging result in the proof of Claim 1, there exists either at most one column with weight 0 or at most two columns with weight 1. If $w_f(\mathcal{C}^i) = 0$ for some $i \in [n]$, then $w_f(\mathcal{C}^j) \geq 2$ for any $j \in [n] \setminus \{i\}$, and

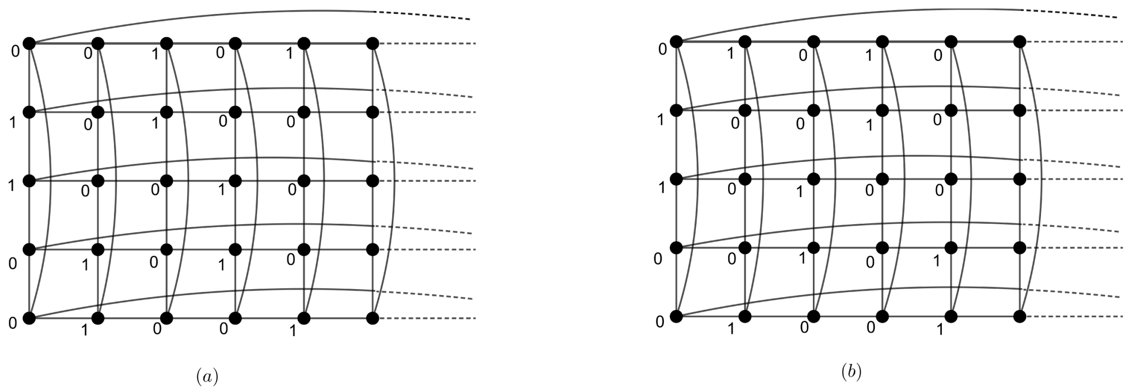


FIGURE 3. A TR2DF on the first five columns of $C_n\Box C_5$.

by Lemma 1.2(i), $w_f(\mathcal{C}^{i-1})+w_f(\mathcal{C}^{i+1})\geq 10$. So $\sum_{v\in V(C_n\Box C_5)}f(v)=\sum_{i=1}^{n}w_f(\mathcal{C}^i)\geq 10+2(n-3)=2n+4$, contradicting our assumption that $\sum_{v\in V(C_n\Box C_5)}f(v)=2n+1$. If there is only one column $\mathcal{C}^i$ with $w_f(\mathcal{C}^i)=1$ for some $i\in[n]$, then $w_f(\mathcal{C}^j)\geq 2$ for any $j\in[n]\setminus\{i\}$, and by Lemma 1.2(ii), $w_f(\mathcal{C}^{i-1})+w_f(\mathcal{C}^{i+1})\geq 7$. So $\sum_{v\in V(C_n\Box C_5)}f(v)=\sum_{i=1}^{n}w_f(\mathcal{C}^i)\geq 8+2(n-3)=2n+2$, contradicting our assumption again. If there are exactly two columns $\mathcal{C}^i$ and $\mathcal{C}^j$ $(i<j)$ with weight 1, then we should have $j=i+2$ according to the assumption $\sum_{v\in V(C_n\Box C_5)}f(v)=2n+1$. Furthermore, $w_f(\mathcal{C}^{i-1})+w_f(\mathcal{C}^{i+1})=7$, $w_f(\mathcal{C}^{i+1})+w_f(\mathcal{C}^{i+3})=7$ and $\sum_{k=i-1}^{i+3}w_f(\mathcal{C}^k)=11$. By inclusion-exclusion principle, we have $w_f(\mathcal{C}^{i+1})=5$, and hence, $w_f(\mathcal{C}^k)=2$ for any $k\in[n]\setminus\{i,i+1,i+2\}$. However, it is easy to check that $w_f(\mathcal{C}^{i-1})=2$ forces $w_f(\mathcal{C}^{i-2})+w_f(\mathcal{C}^i)\geq 4$, a contradiction.

Therefore, $w_f(\mathcal{C})\geq 2$ for any column $\mathcal{C}$ of $C_n\Box C_5$. Then except one column is weighted 3, all other columns are weighted 2. Without loss of generality, let $w_f(\mathcal{C}^{n-2})=3$. Observe that if $w_f(\mathcal{C}^i)=2$ and $V_2\cap\mathcal{C}^i\neq\emptyset$ for some $i\in[n]$, then $w_f(\mathcal{C}^{i-1})+w_f(\mathcal{C}^{i+1})\geq 5$. So $V_2\cap\mathcal{C}^i=\emptyset$ for $i=1,2,\ldots,n-4,n$. By a similar argument to the proof of Claim 2, we can see that no matter the two vertices assigned 1 in $\mathcal{C}^1$ are adjacent or not, the values of the vertices in the first five columns are determined according to $f(\mathcal{C}^1)$. And the TR2DF on the subsequent columns is a repeat pattern of that on the first five columns. Since $n-4\equiv 0 \pmod 5$, we conclude that $f(\mathcal{C}^5)$, $f(\mathcal{C}^{n-4})$ and $f(\mathcal{C}^n)$ share the same pattern, $f(\mathcal{C}^1)$ and $f(\mathcal{C}^{n-3})$ share the same pattern, and $f(\mathcal{C}^4)$ and $f(\mathcal{C}^{n-1})$ share the same pattern. However, it is straightforward to check that $w_f(\mathcal{C}^{n-2})\geq 4$, a contradiction. This completes the proof of Claim 3.

The lower bounds of Theorem 5.5 follows from Claims 1–3. We further show the upper bounds of the theorem. Let

$$P=\begin{bmatrix}0&1&0&1&0\\1&0&0&1&0\\1&0&1&0&0\\0&0&1&0&1\\0&1&0&0&1\end{bmatrix},\qquad Q=\begin{bmatrix}0&0&0&1&1\\0&1&1&0&0\\1&0&0&0&1\\0&0&1&1&0\\1&1&0&0&0\end{bmatrix},$$

$$L_1=\begin{bmatrix}1\\0\\1\\0\\1\end{bmatrix},\quad L_2=\begin{bmatrix}1\\0\\0\\1\\1\end{bmatrix},\quad L_3=\begin{bmatrix}0\\0\\1\\1\\1\end{bmatrix},\quad L_4=\begin{bmatrix}1\\1\\0\\1\\1\end{bmatrix}.$$

Let $n=5k+t$, where $k\geq 1$ and $0\leq t\leq 4$. For $1\leq i\leq 5$, we denote by C^i the i-th column of the array P.

– If $t=0$, then the array P^k provides a pattern of TR2DF of $C_n\Box C_5$ with weight $10k=2n$.

- If $t=1$, then the array Q^kL_1 provides a pattern of TR2DF of $C_n\Box C_5$ with weight $10k+3=2n+1$.
- If $t=2$, then the array $P^kC^1L_2$ provides a pattern of TR2DF of $C_n\Box C_5$ with weight $10k+5=2n+1$.
- If $t=3$, then the array $P^kC^1C^2L_3$ provides a pattern of TR2DF of $C_n\Box C_5$ with weight $10k+7=2n+1$.
- If $t=4$, then the array $P^kC^1C^2C^3L_4$ provides a pattern of TR2DF of $C_n\Box C_5$ with weight $10k+10=2n+2$.

From the patterns above, all the upper bounds are established. This completes the proof of Theorem 5.5. □

6. Total Roman {2}-domination number of $P_n\Box C_3$

In previous sections, we determine the total Roman {2}-domination number for $P_n\Box P_3$ and $C_n\Box C_3$. Now, we investigate the exact value of $\gamma_{\mathrm{tR2}}(P_n\Box C_3)$. It can be seen that the difference between $\gamma_{\mathrm{tR2}}(P_n\Box C_3)$ and $\gamma_{\mathrm{tR2}}(C_n\Box C_3)$ is at most 1.

Theorem 6.1. *For $n\geq 2$,* $\gamma_{\mathrm{tR2}}(P_n\Box C_3)=\begin{cases}\lceil\frac{4n}{3}\rceil, & n\equiv 1 \pmod 3;\\ \lceil\frac{4n}{3}\rceil+1, & n\equiv 0,2 \pmod 3.\end{cases}$

Proof. Since $P_n\Box C_3$ is a spanning subgraph of $C_n\Box C_3$, Theorem 5.1 implies $\gamma_{\mathrm{tR2}}(P_n\Box C_3)\geq\gamma_{\mathrm{tR2}}(C_n\Box C_3)=\lceil\frac{4n}{3}\rceil$.

For $n\equiv 2 \pmod 3$, we first prove that $\gamma_{\mathrm{tR2}}(P_n\Box C_3)\geq\lceil\frac{4n}{3}\rceil+1$. Suppose to the contrary that f is a TR2DF of $P_n\Box C_3$ with $w(f)=\lceil\frac{4n}{3}\rceil=\frac{4n+1}{3}$. Let $ch(\mathcal{C})=w_f(\mathcal{C})$ be the initial charge of column $\mathcal{C}$ and $ch'(\mathcal{C})$ the new charge after the discharging process is finished according to **R3**. By a similar analysis as in the proof of Theorem 5.1, we have that there is at most one column with new charge $ch'(\mathcal{C})=\frac{5}{3}$, and the new charge of all the other columns is $\frac{4}{3}$. Specifically, there is at most one column $\mathcal{C}^t$ with $ch(\mathcal{C}^t)=0$, meanwhile it also satisfies $2\leq t\leq n-1$ and $ch(\mathcal{C}^{t-1})+ch(\mathcal{C}^{t+1})=6$. Hence, $ch('\mathcal{C}^t)=\frac{5}{3}$. In what follows, we show that such a column $\mathcal{C}^t$ does not exist. Since $ch(\mathcal{C}^{t-1})+ch(\mathcal{C}^{t+1})=6$, by symmetry we may assume $ch(\mathcal{C}^{t+1})\geq 3$. If $t+1=n$, then the new charge $ch'(\mathcal{C}^{t+1})\geq 3-\frac{3-\frac{4}{3}}{2}=\frac{13}{6}>\frac{4}{3}$. Assume $t+1<n$. Recall that $ch(\mathcal{C}^{t+2})\geq 1$. Thus the new charge $ch'(\mathcal{C}^{t+2})\geq\frac{11}{6}>\frac{4}{3}$, a contradiction. So the initial charge for each column of $P_n\Box C_3$ is positive. This further deduces that there is no column with initial charge $ch(\mathcal{C})\geq 3$.

Consequently, the initial charge of each column of $P_n\Box C_3$ is either 1 or 2. If $ch(\mathcal{C}^1)=1$, then by Lemma 1.2 $ch(\mathcal{C}^2)\geq 3$, a contradiction. Hence, $ch(\mathcal{C}^1)=2$. If $ch(\mathcal{C}^2)=2$, then the new charge $ch'(\mathcal{C}^1)=2$, a contradiction. So $ch(\mathcal{C}^2)=1$. Now the new charge of $\mathcal{C}^1$ is $ch'(\mathcal{C}^1)=\frac{5}{3}$. So the new charge of each other column of $P_n\Box C_3$ is $\frac{4}{3}$. Based on the initial charge of the first two columns, we observe that for each $2\leq i\leq n$, $ch(\mathcal{C}^{i-1})+ch(\mathcal{C}^{i+1})=3$ if $ch(\mathcal{C}^i)=1$ and $ch(\mathcal{C}^{i-1})=ch(\mathcal{C}^{i+1})=1$ if $ch(\mathcal{C}^i)=2$. By this observation, the pattern of the initial charge of the columns is $21\underline{121}$, where $\underline{121}$ means repeat the three numbers as n becomes large. Therefore, $ch(\mathcal{C}^{n-1})=2$ and $ch(\mathcal{C}^n)=1$. By symmetry, let $f(x_{n,1})=1$, $f(x_{n,2})=f(x_{n,3})=0$. Then we have $f(x_{n-1,2})=f(x_{n-1,3})=1$ and $f(x_{n-1,1})=0$. However, $x_{n,1}$ has no neighbor with positive weight, a contradiction.

For $n\equiv 0 \pmod 3$, we first prove that $\gamma_{\mathrm{tR2}}(P_n\Box C_3)\geq\lceil\frac{4n}{3}\rceil+1$. Suppose to the contrary that f is a TR2DF of $P_n\Box C_3$ with $w(f)=\lceil\frac{4n}{3}\rceil=\frac{4n}{3}$. Let $ch(\mathcal{C})=w_f(\mathcal{C})$ be the initial charge of the column $\mathcal{C}$. By the analysis in the proof of Theorem 5.1, the new charge of each column of $P_n\Box C_3$ is $\frac{4}{3}$. Consequently, there is no column with initial charge $ch(\mathcal{C})=0$ or $ch(\mathcal{C})\geq 3$. As discussed in the previous case where $n\equiv 2 \pmod 3$, we get $ch(\mathcal{C}^1)=2$ and $ch(\mathcal{C}^2)=1$. Now the new charge of $\mathcal{C}^1$ is $ch'(\mathcal{C}^1)=\frac{5}{3}$, a contradiction.

Finally, we prove the upper bounds by establishing a TR2DF of $P_n\Box C_3$ as follows. Let $V_2=\emptyset$, $V_1\cap\mathcal{C}^i=\{x_{i,1},x_{i,2}\}$ if $i\equiv 1 \pmod 3$ and $V_1\cap\mathcal{C}^i=\{x_{i,3}\}$ if $i\equiv 0,2 \pmod 3$, $V_0=V(P_n\Box C_3)\setminus V_1$. If $n\equiv 0,2 \pmod 3$, redefine V_1 by setting $V_1\cap\mathcal{C}^n=\{x_{n,2},x_{n,3}\}$. It is easy to check that $w(f)=4\times\frac{n-1}{3}+2=\frac{4n+2}{3}=\lceil\frac{4n}{3}\rceil$ if $n\equiv 1 \pmod 3$, $w(f)=4\times\frac{n-2}{3}+4=\lceil\frac{4n}{3}\rceil+1$ if $n\equiv 2 \pmod 3$ and $w(f)=4\times\frac{n}{3}+1=\lceil\frac{4n}{3}\rceil+1$ if $n\equiv 0 \pmod 3$, proving the result. □

7. Conclusion

Notice that all the $\gamma_{tR2}(G)$-functions $f = (V_0, V_1, V_2)$ provided for the Cartesian product graphs under our discussion satisfy $V_2 = \emptyset$. As a consequence, we derive the following corollary.

Corollary 7.1. *If* $G \in \{P_n \Box P_m | m = 2, 3\} \cup \{C_n \Box C_m | m = 3, 4, 5\} \cup \{P_n \Box P_4 | n = 4, 5, 6\} \cup \{C_n \Box P_2, P_n \Box C_3\}$, *then* $\gamma_{\times 2}(G) = \gamma_{tR2}(G)$.

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