DENSITY OF IDENTIFYING CODES OF HEXAGONAL GRIDS WITH FINITE NUMBER OF ROWS

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Abstract. In a graph $G$, a set $C \subseteq V(G)$ is an identifying code if, for all vertices $v$ in $G$, the sets $N[v] \cap C$ are all nonempty and pairwise distinct, where $N[v]$ denotes the closed neighbourhood of $v$. We focus on the minimum density of identifying codes of infinite hexagonal grids $H_k$ with $k$ rows, denoted by $d^*(H_k)$, and present optimal solutions for $k \leq 5$. Using the discharging method, we also prove a lower bound in terms of maximum degree for the minimum-density identifying codes of well-behaved infinite graphs. We prove that $d^*(H_2) = 9/20$, $d^*(H_3) = 6/13 \approx 0.4615$, $d^*(H_4) = 7/16 = 0.4375$ and $d^*(H_5) = 11/25 = 0.44$. We also prove that $H_2$ has a unique periodic identifying code with minimum density.

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1. Introduction

The concept of identifying code (idcode, for short), was introduced in 1998 by Karpovsky et al. [27] to identify a faulty processor in a multiprocessor system. The vertices of an idcode correspond to special processors (the monitors) that are able to check themselves and their neighbours to identify a faulty processor.

Problems on idcodes have been studied on finite and infinite graphs, being of great interest both from theoretical as well as practical viewpoint. Particular interest has been dedicated to grids as many processor networks have a grid topology (see [34, 35]). Among these, we mention the square grid $G_S$, the triangular grid $G_T$ and the king grid $G_K$, shown in Figure 1.

One fundamental problem on idcodes is that of finding idcodes of minimum density. The density captures the proportion of vertices in the code with respect to the whole graph. For finite graphs, Cohen et al. [7] proved that deciding the existence of an idcode of size at most $k$ in a graph is an NP-complete problem. On infinite graphs, studies on minimum-density idcodes have considered grids with infinite or with a finite number of rows (see [1–6, 9, 10, 12–14, 16–21, 24, 25, 27, 28]). For an updated bibliography covering this topic and related ones, the reader is referred to Jean [22].

We denote by $d^*(G)$ the minimum density of an idcode of a graph $G$. For the infinite grids mentioned previously, it is known that $d^*(G_S) = 7/20$ [1], $d^*(G_T) = 1/4$ [27] and $d^*(G_K) = 2/9$ [4]. When these grids have

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a finite number \( k \) of rows, idcodes of minimum density are known for \( k \leq 6 \), and for larger \( k \) only lower and upper bounds have been found.

In this work we focus on infinite graphs, specially the hexagonal grids (see Fig. 2). We denote these grids by \( \mathcal{G}_H \) when the number of rows is infinite, and by \( H_k \) when the number of rows is a positive integer \( k \). For \( \mathcal{G}_H \), new lower and upper bounds have been proved in the last years. Just to mention the more recent ones: in 2009, Cranston and Yu [9] proved a lower bound of \( 12/29 \approx 0.4138 \), and in 2013, Cuckierman and Yu [10] improved the lower bound to \( 5/12 \approx 0.4166 \). In 2014, Stolee [33] presented a computer-assisted framework showing that \( d^*(\mathcal{G}_H) \geq 23/55 \approx 0.4181 \). As for upper bounds, in 2000, Cohen et al. [6] constructed two idcodes of \( \mathcal{G}_H \) with density \( 3/7 \approx 0.4285 \). Other idcodes with the same density have also been reported in the literature. Recently, breaking the long-standing bound of \( 3/7 \), Salo and Törnä [29] showed that \( d^*(\mathcal{G}_H) \leq 53/126 \approx 0.4206 \). They found a periodic idcode using a computer-assisted proof that uses automata theory and Karp’s minimum mean cycle algorithm. No results on lower or upper bounds have appeared in the literature for \( d^*(H_k) \).

We prove that idcodes of well-behaved infinite graphs with maximum degree \( \Delta \) have density at least \( 2/\Delta + 2 \). This result and another one on infinite graphs with maximum degree 3 imply that \( d^*(H_k) \geq 2/5 \) for all \( k \geq 2 \), and that idcodes of \( H_k \) that do not induce trivial components have density at least \( 3/7 \). We prove that \( d^*(H_2) = 9/20 \), and exhibit an idcode with this minimum density, which we show to be unique. We also mention how we proved that \( d^*(H_3) = 6/13 \), \( d^*(H_4) = 7/16 \) and \( d^*(H_5) = 11/25 \), using computer-assisted tools.

In Section 2 we define the concepts used in this paper and establish the notation. We also present a density result on the infinite 3-regular tree, to show that this graph is not so well-behaved as the hexagonal grids, a fact (to be made precise) that has caused an erroneous proof in the literature on a related concept called locating-dominating set (and perhaps on other closely related concepts as well). These preliminary comments help understanding the property (named SG) that we require from the infinite graphs to guarantee that some density proof techniques work. In Sections 3 and 4, we define SG-property and prove results on the discharging method and the mentioned lower bound. In Section 5 we show a minimum-density periodic idcode for \( H_2 \), and prove that it is unique. Section 6 contains results on minimum-density idcodes for \( H_k \), \( k \in \{3, 4, 5\} \).

A preliminary version of this work (an extended abstract) appeared in [30]. This work contains additional novel results and a simplified and complete proof of Theorem 5.6.

2. Definitions, notation, and the infinite 3-regular tree

The hexagonal grid, denoted by \( \mathcal{G}_H \), is an infinite graph with vertex set \( V = \mathbb{Z} \times \mathbb{Z} \) and edge set \( E = \{\{u, v\} : u = (i, j), \, u - v \in \{(\pm 1, 0), (0, (1)^{i+j+1})\}\} \). See Figure 2. The hexagonal grid with \( k \) rows, \( k \geq 2 \), denoted by \( H_k \), is a graph isomorphic to the subgraph of \( \mathcal{G}_H \) induced by the vertex set \( \mathbb{Z} \times \{1, \ldots, k\} \).

Let \( G \) be a connected graph. If \( v \) is a vertex of \( G \), and \( r \) is a natural number, then \( N_r(v) \) denotes the set of vertices of \( v \) at distance at most \( r \) from \( v \), and \( N_r[v] = N_r(v) \cup \{v\} \) denotes the closed neighbourhood of \( v \). When \( r = 1 \), we omit the subscript \( r \) and simply write \( N(v) \) and \( N[v] \). Given \( C \subseteq V(G) \), let \( C[v] = N[v] \cap C \).

Figure 1. Partial representation of infinite square, triangular and king grids, and the corresponding minimum-density idcodes. (A) Square grid \( \mathcal{G}_S \). (B) Triangular grid \( \mathcal{T} \). (C) King grid \( \mathcal{G}_K \).
An idcode of $G$ is a set $C \subseteq V(G)$ such that $C[v] \neq \emptyset$ for every vertex $v$ of $G$, and $C[v] \neq C[w]$ for any pair of distinct vertices $v, w$ of $G$. Thus, if a graph $G$ has two distinct vertices $v$ and $w$ such that $N[v] = N[w]$, then $G$ has no idcode. Such vertices are called twins. Clearly, a graph has an idcode if and only if it is twin-free. If $C$ is an idcode, we say that $C[v]$ is the identifier of $v$.

We are interested in minimum-density idcodes of countably infinite connected graphs of bounded degree. For such a graph $G$, the density of a subset $C \subseteq V(G)$, denoted by $d(C,G)$, is defined as follows.

$$d(C,G) = \inf \{ d_w(C,G) : w \in V(G) \},$$

where

$$d_w(C,G) = \limsup_{r \to \infty} \frac{|C \cap N_r[w]|}{|N_r[w]|}.$$ 

The minimum density of an idcode of a graph $G$, denoted by $d^*(G)$, is defined as

$$d^*(G) = \inf \{ d(C,G) : C \text{ is an idcode of } G \}.$$ 

Notice that we use inf (infimum) in the definition of $d(C,G)$, instead of min (minimum), since the greatest lower bound does not always belong to the set. This definition (with inf) is also given by Jiang [24] to study densities of idcodes of $S_k$ (a topic to be mentioned in Sect. 6). Slater [31] defines density of locating-dominating sets (a notion similar to idcode) with min, but the definition of density $d(C,G)$ makes sense for any set $C$. In the proof of Lemma 2.1 we show an example of an infinite graph $G$ for which $d_w(C,G) > 0$ for all $w \in V(G)$, but $d(C,G) = 0$.

This definition of subset density given above has not always been used. In some papers, such as [10–13, 23], the density $d(C,G)$ was simply defined as $d_w(C,G)$ where $w$ is an “arbitrary vertex”. This contains an implicit assumption that $d_w(C,G) = d_v(C,G)$ for any two vertices $w, v$ of $G$, which is not always true as we show in Lemma 2.1. In most of these papers, this problem in the density definition did not lead to erroneous results, since the graphs considered were well-behaved grids, all of them satisfy an important condition (named SG-property in the next section) which guarantees that $d_w(C,G) = d_v(C,G)$ for any two vertices $w, v$ of $G$ (see Lem. 3.2). However, some papers contain erroneous statements, as we will see in Theorem 2.2.

**Lemma 2.1.** There are infinite regular connected graphs $G$ with subsets $C \subseteq V(G)$ for which there are distinct vertices $w, v$ such that $d_w(C,G) \neq d_v(C,G)$.

**Proof.** Let us consider the infinite 3-regular tree $T$, obtained from two infinite binary trees $T_1$ and $T_2$ with roots $r_1$ and $r_2$, respectively, by adding the edge $r_1r_2$. We exhibit two examples of sets $C \subseteq V(T)$ and vertices $w, v$ of $V(T)$ for which $d_w(C,T) \neq d_v(C,T)$.

As a first example, consider $C = V(T_2)$. Let $w$ be a vertex of $T_1$ that is a neighbour of $r_1$. Then $d_w(C,T) = 1/6$. (More generally, if $w$ is at distance $d$ from $r_1$, we have that $d_w(C,T) = 2^{-d}/3$.) Let $v = r_2$. Then, $d_v(C,T) = 2/3$. (Note that here $d(C,T) = 0$.)

As a second example, let $C$ be the set consisting of all vertices of $T_2$ together with all vertices of $T_1$ whose distance to $r_1$ is even ($r_1$ included). In this case, $C$ is an idcode of $T$. Let $w$ (resp. $v$) be a vertex in $T_1$ (resp. $T_2$).
Lemma 2.1, which is obtained from two infinite binary trees with roots \( \mathcal{R} \). R.M. Sampaio

that is at distance \( d \) from \( r_1 \) (resp. \( r_2 \)). It is not difficult to check that \( d_w(C,T) \) converges to \( 2/3 \) and \( d_v(C,T) \) converges to 1 when \( d \) tends to \( \infty \). □

Even considering the correct definition of subset density \( d(C,G) \), when dealing with infinite graphs one has to be careful with arguments which sound plausible, but may be erroneous. One may consider the technique of covering the vertex set of \( G \) with a periodic pattern (subgraph) and assume that the density \( d(C,G) \) will be the density of \( C \) in the pattern. As an example, consider the infinite 3-regular tree \( \mathcal{T} \), used in the proof of Lemma 2.1, which is obtained from two infinite binary trees with roots \( r_1 \) and \( r_2 \) and the edge \( r_1r_2 \). Consider that \( \mathcal{T} \) is rooted at \( r_1 \). Let \( C \) be the set of vertices in \( \mathcal{T} \) whose distance to \( r_1 \) is even (\( r_1 \) included). Then, the vertices of \( \mathcal{T} \) can be covered by the pattern formed by an edge (a matching) consisting of a vertex and its leftmost child (being one in \( C \) and the other not in \( C \)), whose density is 1/2. Also, by ignoring \( r_2 \), the vertices of \( \mathcal{T} \) can be covered by the pattern formed by a vertex in \( C \) and its two children not in \( C \) (a cherry), whose density is 1/3. Finally, by ignoring \( r_1 \), the vertices of \( \mathcal{T} \) can also be covered by the pattern that is a cherry formed by a vertex not in \( C \) and its two children in \( C \), whose density is 2/3. Thus, considering three distinct periodic patterns, this technique gives three different values for the density of \( d(C,T) \), indicating that it may not be used in arbitrary graphs. For completeness, we mention that \( d(T,C) = 2/3 \) (the proof is not so short, but the reader may verify this).

Unfortunately, even more elaborate techniques to calculate density of sets on infinite graphs have led to some erroneous results in the literature. We will elaborate more on this in what follows, calling attention to a property that an infinite graph should satisfy for some techniques to work (see Lem. 3.2). The next theorem shows that one of the first results on the density of locating-dominating sets of regular graphs is not correct. We say that an infinite graph should satisfy for some techniques to work (see Lem. 3.2). The next theorem shows that

The minimum density of a locating-dominating set of the infinite 3-regular tree is at most 5/16 = 0.3125.

**Theorem 2.2.** The minimum density of a locating-dominating set of the infinite 3-regular tree is at most 5/16 = 0.3125.

**Proof.** Let \( \mathcal{T} \) be the infinite 3-regular tree with root \( \mathcal{R} \), and let layer \( L_i \) be the set of vertices of \( \mathcal{T} \) at a distance \( i \) from the root \( \mathcal{R} \). Thus, \( V(\mathcal{T}) = \bigcup_{i \geq 0} L_i \), \( L_0 = \{ \mathcal{R} \} \), and \( |L_i| = 3 \cdot 2^{i-1} \), for \( i \geq 1 \). Thus, for \( i \geq 5 \), \( |L_i| \) is a multiple of 16, and is composed of 3 groups with \( 2^{i-1} \) vertices.

To construct a set \( C \subseteq V(\mathcal{T}) \) which we shall prove to be an lds of \( \mathcal{T} \), we label first the vertices of \( \mathcal{T} \), and then we define which vertices belong to \( C \). The labelling procedure is the following.

(a) We assign label 1 to all vertices in \( L_0 \cup L_1 \cup \ldots \cup L_4 \).
(b) We label the vertices of \( L_5 \) as follows. We consider that \( L_5 \) is composed of 3 consecutive groups of 16 vertices (each of these groups consist of the leaves of the subtree of height 4 rooted at one of the children of root \( \mathcal{R} \)). We label identically these groups of 16 vertices, according to the following pattern:

<table>
<thead>
<tr>
<th>1 2 3 5</th>
<th>1 2 3 5</th>
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<tbody>
<tr>
<td>2 3 5 5</td>
<td>3 4 5 5</td>
</tr>
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</table>

(c) Once the vertices in \( L_i, i \geq 5 \), have been labelled, we label the vertices in \( L_{i+1} \). For that, we define for each vertex with label \( j \) (in \( L_i \)) which are the labels \( k, l \) of its children (in \( L_{i+1} \)), writing \( j \rightarrow \{k,l\} \). We let \( 1 \rightarrow \{3,4\}, 2 \rightarrow \{3,3\}, 3 \rightarrow \{1,5\}, 4 \rightarrow \{5,5\} \) and \( 5 \rightarrow \{2,5\} \). Representing this in a tree-like structure, we have:
Consider a group, say $H$, of 16 vertices in $L_5$, and let $x_j$ be the number of vertices in $H$ with label $j$. Then, $x_1 = 2$, $x_2 = 3$, $x_3 = 4$, $x_4 = 1$, $x_5 = 6$; or in a condensed form, $x(H) = (2, 3, 4, 1, 6)$.

Now, let $\text{chld}(H)$ be the group (in $L_6$) formed by the children of the vertices in $H$. Let now $x_j'$ be the number of vertices with label $j$ in $\text{chld}(H)$. Then, $x_1' = x_3 = 4 = 2x_1$, $x_2' = x_5 = 6 = 2x_2$, $x_3' = x_1 + 2x_2 = 8 = 2x_3$, $x_4' = x_1 = 2 = 2x_4$, and $x_5' = x_3 + 2x_4 + x_5 = 12 = 2x_5$. That is, $x_j' = 2x_j$ for $j \in \{1, 2, \ldots, 5\}$, and therefore, $x(\text{chld}(H)) = 2x(H)$. Since, at each layer $L_i$, $i \geq 5$, there are 3 groups with $2^{i-1}$ vertices, and each such group $G$ (by the labelling rule) gives rise to a (children) group with $x(\text{chld}(G)) = 2x(G)$, in each new layer the proportion of vertices with labels 1 or 2 (those in $C$) is exactly the proportion that holds in layer $L_5$. We have $|C \cap L_5| = 15$ and $|L_5| = 16$. Thus, $|C \cap L_5|/|L_5| = 15/16 = 5/16$. Since $|L_{i+1}| = 2|L_i|$ and $|C \cap L_{i+1}| = 2|C \cap L_i|$, for each layer $L_i$ the ratio $|C \cap L_i|/|L_i| = 5/16$ holds for all $i \geq 5$. Only for the initial layers $L_i$, $0 \leq i \leq 4$, we have $|C \cap L_i|/|L_i| = 1$. Thus, the density $d_R(C, T)$ is precisely

$$
:d_R(C, T) = \limsup_{r \to \infty} \frac{|C \cap N_r[R]|}{|N_r[R]|} = \limsup_{h \to \infty} \frac{|C \cap T_h(R)|}{|T_h(R)|} = \frac{5}{16},
$$

where $T_h(R)$ is the subtree of $T$ with height $h$ rooted at $R$. Since $d(C, T) = \inf\{d_w(C, T) : w \in V(T)\}$, we conclude that $d(C, T) \leq \frac{5}{16} = 0.3125$.

It remains to prove that $C$ is an lds of $T$. For that, it suffices to check that the vertices with labels 3, 4, 5 have distinct neighbourhood in $C$. The reader may check that a vertex with label 3 is identified by its parent and one child (with label 1); a vertex with label 4 is identified solely by its parent (which has label 1); and a vertex with label 5, if it belongs to $L_5$, is then identified by its parent and one child (with label 2), and if it belongs to layer $L_i$, $i \geq 6$, then it is identified solely by one child (the one with label 2). This concludes our proof that $C$ is an lds of $T$ with $d(C, T) \leq \frac{5}{16} = 0.3125$.

We understand that the erroneous proof of Theorem 2 stated in [31] happened because the infinite graph under consideration does not satisfy a property that would allow the application of the method that was used. The author used the share method, which is a simple application of the discharging method (to be discussed in the next section) to obtain a lower bound proof for the density of a set, say $C$.

Roughly speaking, the share method works as follows: each vertex of $C$ starts with charge $q > 0$ and each vertex outside $C$ starts with charge 0. For any vertex $c \in C$ and $u \in N[c]$, the vertex $c$ sends charge $1/|C[u]|$ to $u$ (this includes the case in which $c = u$). At the end of this procedure, all vertices outside $C$ will have charge exactly 1 and every vertex $c \in C$ will have charge $q + 1 - sh(c)$, where $sh(c) = \sum_{u \in N[c]} 1/|C[u]|$ is the total charge sent by $c$. The idea is that, if $sh(c) \leq q$ for every $c \in C$, all vertices in $G$ will have charge at least 1. Then, if $G$ is finite,

$$
1 \cdot |V(G)| \leq \sum_{c \in C} sh(c) \leq q \cdot |C|, \quad \text{and hence } \quad d(C, G) = \frac{|C|}{|V(G)|} \geq \frac{1}{q}.
$$

Now, let $G$ be an infinite connected graph and let $v$ be a vertex of $G$. To guarantee charge at least 1 at every vertex in $N_{r-1}[v]$, it suffices to consider the vertices in $C \cap N_r[v]$. Thus,

$$
1 \cdot |N_{r-1}[v]| \leq \sum_{c \in C \cap N_r[v]} sh(c) \leq q \cdot |C \cap N_r[v]|,
$$
which implies that
\[ d_v(C,G) = \limsup_{r \to \infty} \frac{|C \cap N_r[v]|}{|N_r[v]|} \geq \frac{1}{q} \cdot \limsup_{r \to \infty} \frac{|N_{r-1}[v]|}{|N_r[v]|}. \]

As we can see, the share method of [31] will work if \( \limsup_{r \to \infty} \frac{|N_{r-1}[v]|}{|N_r[v]|} = 1 \), which is a consequence (Lem. 3.2(a) with \( t = -1 \)) of our SG-property, defined in the next section.

3. THE USE OF DISCHARGING METHOD TO PROVE LOWER BOUNDS FOR THE DENSITY OF IDCODES

The discharging method is a proof technique in combinatorics, first used in graph theory, that has now been used in many different contexts, such as in graph colouring, decomposition, embedding, geometric and structural problems. For a guide on the use of this method to prove results on colouring and other structural properties of graphs see [8].

To prove results on a graph \( G \), this method involves two phases: charging and discharging. In the charging phase, we assign charges (a rational number) to certain structures of \( G \) using a charging rule, which describes the value of the charge and the structures of \( G \) which will receive the charge. These structures may be vertices, edges, faces (if \( G \) is planar), etc. In the discharging phase, we re-assign the charges using the discharging rules, which describe the structures that will send and/or receive charge from other vertices. The discharging must preserve the total charge that was assigned in the charging phase.

Both the charging and discharging rules are designed to guarantee that, after these phases some information on the charges of certain vertices/edges will help us prove some property of the graph. In some applications, the initial charges or the discharging rules may take into consideration the degree of the vertices.

The discharging method has been one of the main tools to prove lower bounds for density of idcodes. Theorem 3.3, proved in this section, tells how this method can be used to obtain density results in infinite graphs, once these graphs satisfy certain properties. Before that, we define SG-property and present a general result (Lem. 3.2) that is related to this property and is used in Theorem 3.3 and Lemma 3.4. (Here, the mnemonic SG stands for “slow growth”, the concept we want to emphasize.)

**Definition 3.1.** We say that a graph \( G \) satisfies the SG-property if \( G \) is connected and has a vertex \( s \) such that \( \lim_{r \to \infty} \frac{|N_{r+1}[s]|}{|N_r[s]|} = 1 \).

Notice that, since \( N_r[s] \subseteq N_{r+1}[s] \), then \( \lim_{r \to \infty} \frac{|N_{r+1}[s]|}{|N_r[s]|} = 1 \) if and only if \( \limsup_{r \to \infty} \frac{|N_{r+1}[s]|}{|N_r[s]|} = 1 \). Also notice that the integer \( t \) in the item (a) of the following lemma may be negative.

**Lemma 3.2.** Let \( G \) be an infinite connected graph satisfying the SG-property, and let \( s \in V(G) \) be such that \( \lim_{r \to \infty} \frac{|N_{r+1}[s]|}{|N_r[s]|} = 1 \). Then the following hold.

(a) For every vertex \( v \) and integer \( t \), we have \( \lim_{r \to \infty} \frac{|N_{r+t}[v]|}{|N_r[v]|} = 1 \).

(b) For every vertex \( v \) and subset \( C \subseteq V(G) \), we have \( d_v(C,G) = d_w(C,G) \). Thus the density of \( C \) is \( d(C,G) = d_w(C,G) \), where \( w \) is an arbitrary vertex of \( G \).

**Proof.** To simplify notation, let \( n_k[w] = |N_k[w]| \) for any positive integer \( k \) and vertex \( w \). For the vertex \( s \) stated in the lemma, and any integer \( t > 0 \), we have

\[
\lim_{r \to \infty} \frac{n_{r+t}[s]}{n_r[s]} = \lim_{r \to \infty} \left( \frac{n_{r+t}[s]}{n_{r+t-1}[s]} \cdot \frac{n_{r+t-1}[s]}{n_{r+t-2}[s]} \cdots \frac{n_{r+2}[s]}{n_{r+1}[s]} \cdot \frac{n_{r+1}[s]}{n_r[s]} \right) = 1. \quad (1)
\]

It is immediate that \( \lim_{r \to \infty} \frac{n_{r+t}[s]}{n_r[s]} = 1 \) also holds when \( t \) is negative (as long as \( r+t \geq 0 \)). Now, to prove (a), consider a vertex \( v \) and let \( d := \text{dist}(v,s) \). First, we prove that (for \( r \geq d \))

\[
N_{r-d}[s] \subseteq N_r[v] \subseteq N_{r+d}[s]. \quad (2)
\]
To prove the first inclusion, take a vertex \( y \) in \( N_{r-d}[s] \). Thus, \( \text{dist}(y, s) \leq r - d \). Since \( \text{dist}(y, v) \leq \text{dist}(y, s) + \text{dist}(s, v) \), it follows that \( \text{dist}(y, v) \leq r \), and therefore, \( y \in N_r[v] \). The proof of the second inclusion is analogous: take \( y \in N_r[v] \), which means that \( \text{dist}(y, v) \leq r \). Since \( \text{dist}(y, s) \leq \text{dist}(y, v) + \text{dist}(v, s) \), we have that \( \text{dist}(y, s) \leq r + d \), and therefore, \( y \in N_{r+d}[s] \). From (2), we have that

\[
N_{r+1-d}[s] \subseteq N_{r+1}[v] \subseteq N_{r+1+d}[s].
\]  

(3)

Combining (3) and (2), we have

\[
\frac{n_{r+1-d}[s]}{n_{r+d}[s]} \leq \frac{n_{r+1}[v]}{n_r[v]} \leq \frac{n_{r+1+d}[s]}{n_{r-d}[s]},
\]  

(4)

Since (1) holds for every integer \( t \) (see the observation in the paragraph following (1)), it follows that the limit of the fraction on the left (resp. right) side of (4) when \( r \) tends to \( \infty \) is 1, and therefore,

\[
\lim_{r \to \infty} \frac{n_{r+1}[v]}{n_r[v]} = 1.
\]  

(5)

From (5), we may conclude that (1) holds when \( s \) is replaced by \( v \), and this completes the proof of statement (a).

Now, let us prove (b). For that, we first note that, from (2) we have that

\[
\frac{n_{r-d}[s]}{n_r[s]} \leq \frac{n_r[v]}{n_r[s]} \leq \frac{n_{r+d}[s]}{n_r[s]},
\]  

(6)

Since the limit of the fraction on the left (resp. right) when \( r \) tends to \( \infty \) is 1, it follows that

\[
\lim_{r \to \infty} \frac{n_r[v]}{n_r[s]} = 1.
\]  

(7)

By definition, we have that

\[
d_v(C, G) = \limsup_{r \to \infty} \frac{|C \cap N_r[v]|}{n_r[v]}. \]  

(8)

From (2), we obtain

\[ C \cap N_{r-d}[s] \subseteq C \cap N_r[v] \subseteq C \cap N_{r+d}[s]. \]

Thus,

\[
\limsup_{r \to \infty} \frac{|C \cap N_{r-d}[s]|}{n_r[v]} \leq d_v(C, G) \leq \limsup_{r \to \infty} \frac{|C \cap N_{r+d}[s]|}{n_r[v]}.
\]  

(9)

The lower (resp. upper) bound of \( d_v(C, G) \) given by (9) is precisely \( d_s(C, G) \). Indeed, for the lower bound, using (8), (1) and (7), we have

\[
\limsup_{r \to \infty} \frac{|C \cap N_{r-d}[s]|}{n_r[v]} = \limsup_{r \to \infty} \left( \frac{|C \cap N_{r-d}[s]|}{n_{r-d}[s]} \cdot \frac{n_{r-d}[s]}{n_r[s]} \cdot \frac{n_r[s]}{n_r[v]} \right) = d_s(C, G).
\]

For the upper bound, the proof follows similarly. Thus, \( d_v(C, G) = d_s(C, G) \), and hence \( d(C, G) = d_w(C, G) \), where \( w \) is an arbitrary vertex in \( G \).

\( \square \)

The SG-property is very important for the forthcoming proofs on the minimum density based on the discharging method. Lemma 3.2 guarantees that if a connected graph \( G \) has this property, then the density of a vertex set \( C \) in \( G \) may be calculated by considering \( d_v(C, G) \) for an arbitrary vertex \( v \).

It is not difficult to see that the infinite hexagonal grids (\( G_H \) and \( H_k \)), as well as the grids mentioned in the introduction (square, triangular, king), and many others have the SG-property. In particular, for the grid \( G_H \),
it is known that \( n_{r+1}[s] = (3(r + 2)(r + 1))/2 + 1 \) for any vertex \( s \), from which we conclude that it has the SG-property. (For more information on \( n_r[s] \), see any reference on the \( r \)th centered triangular number.) For the grid \( H_k \), as \( k \) is fixed, it is easier to conclude that it has the SG-property. As an example of an infinite graph that does not have this property, consider an infinite complete binary tree. Recall that we have shown (see Lem. 2.1) that the infinite 3-regular tree does not have this property.

**Theorem 3.3 (Discharging Method).** Let \( G \) be an infinite graph with bounded maximum degree which satisfies the SG-property. Let \( C \) be a vertex set in \( G \). Suppose that the discharging method is applied to \( G \) in the following way. In the charging phase, charge 1 is assigned to each vertex in \( C \) and charge 0 is assigned to the remaining vertices. In the discharging phase, among other rules, the following one is respected: no vertex sends charge from it to a vertex at a distance greater than \( d \), for a fixed integer \( d \). If, at the end, every vertex \( v \) of \( G \) has final charge \( \text{chg}(v) \) such that \( q \leq \text{chg}(v) \leq q' \), where \( q \) and \( q' \) are rational numbers, then \( q \leq d(C,G) \leq q' \).

**Proof.** Given a set \( W \subseteq V(G) \), let \( \text{chg}(W) = \sum_{w \in W} \text{chg}(w) \). Let \( q, q' \) and \( d \) be as in the hypothesis of the lemma, and let \( s \) be an arbitrary vertex in \( G \). As in the proof of Lemma 3.2, to simplify notation, we let \( n_r[s] = |N_r[s]| \). Note that \( q \cdot n_r[s] \leq \text{chg}(N_r[s]) \leq q' \cdot n_r[s] \).

Moreover, notice that \( \text{chg}(N_r[s]) \) is at most \( |C \cap N_r[s]| \) plus the charge received from vertices outside \( N_r[s] \), which are contained in \( N_{r+d}[s] \). Then, \( q \cdot n_r[s] \leq \text{chg}(N_r[s]) \leq |C \cap N_r[s]| + n_{r+d}[s] - n_r[s] \). Therefore,

\[
d_s(C,G) = \limsup_{r \to \infty} \frac{|C \cap N_r[s]|}{n_r[s]} \geq q - \limsup_{r \to \infty} \frac{n_{r+d}[s] - n_r[s]}{n_r[s]} = q.
\]

The last equality holds because \( \lim_{r \to \infty} n_{r+d}[s]/n_r[s] = 1 \), by Lemma 3.2(a).

Moreover, for \( r > d \), \( \text{chg}(N_r[s]) \) is at least \( |C \cap N_r[s]| \) minus the charge sent to vertices outside \( N_r[s] \), which comes from vertices in \( N_r[s] \setminus N_{r-d}[s] \). Then, \( q' \cdot n_r[s] \geq \text{chg}(N_r[s]) \geq |C \cap N_r[s]| - (n_r[s] - n_{r-d}[s]) \). Therefore,

\[
d_s(C,G) = \limsup_{r \to \infty} \frac{|C \cap N_r[s]|}{n_r[s]} \leq q' + \limsup_{r \to \infty} \frac{n_r[s] - n_{r-d}[s]}{n_r[s]} = q'.
\]

Thus, from Lemma 3.2(b) we conclude that \( q \leq d(C,G) \leq q' \). \( \square \)

The next lemma shows that the method of determining the density of a set from periodic patterns, which we showed that may not work on arbitrary graphs, it works on graphs satisfying the SG-property.

**Lemma 3.4.** Let \( G \) be an infinite connected graph with bounded maximum degree that satisfies the SG-property. Let \( \ell, c, c', d \) be positive integers, and let \( C \) be a subset of \( V(G) \). Suppose that \( V(G) \) can be partitioned into subsets \( V_1, V_2, \ldots \) of size \( \ell \) such that \( c \leq |V_i \cap C| \leq c' \) for each \( i \geq 1 \), and the distance between any two vertices of \( V_i \) is at most \( d \). Then \( c/\ell \leq d(C,G) \leq c'/\ell \).

**Proof.** We use the discharging method as stated in Lemma 3.3 with \( q = c/\ell \) and \( q' = c'/\ell \). Recall that every vertex of \( C \) starts with charge 1 and the vertices outside \( C \) starts with charge 0. In the discharging phase, for every part \( V_i \) of \( V(G) \), the set of vertices in \( C \cap V_i \) can guarantee charge at least \( q = c/\ell \) and at most \( q' = c'/\ell \) for every vertex of \( V_i \). Since the distance between any two vertices of \( V_i \) is at most \( d \), no vertex sends charge to a vertex at a distance greater than \( d \). From Lemma 3.3, we conclude that \( c/\ell \leq d(C,G) \leq c'/\ell \). \( \square \)

In particular, for \( H_k \), the above result indicates that to prove a lower bound for the density of an idcode \( C \), one can show that if \( H_k \) can be covered with a periodic pattern \( H \), then \( H \) is a pattern (subgraph of \( H_k \) containing vertices of \( C \)) for which the ratio \( |C \cap V(H)|/|V(H)| \) is minimum possible (a result that might not be so easy to prove). This would lead us to the conclusion that this ratio gives a lower bound for \( d(C,H_k) \). In Section 4, we prove a lower bound for \( d'(H_2) \) using the discharging method, as stated in Theorem 3.3, and we also give another proof based on this idea of a pattern \( H \) with best possible ratio. The latter idea also yields a uniqueness proof of the minimum-density periodic idcode of \( H_2 \).
4. Lower bounds for the density of some idcodes of $H_k$

Karpovsky et al. [27] proved that for $d \geq 2$, every finite twin-free $d$-regular graph $G$ satisfies $d^*(G) \geq 2/(d+2)$. This was done using a double counting argument on the set of possible idcodes. The next theorem shows that the same bound holds for infinite connected graphs with maximum degree bounded by a constant $d$, if the graph has the SG-property. To prove this result, we use the discharging method, in a similar way that Cranston and Yu [9] proved the lower bound $2/5$ for the minimum density $d^*(G_H)$ of the hexagonal grid.

**Theorem 4.1.** Let $\Delta \geq 2$ be a fixed integer and $G$ be a connected infinite twin-free graph with maximum degree $\Delta$. If $G$ has the SG-property, then $d^*(G) \geq 2/(\Delta + 2)$. In particular, $d^*(H_k) \geq 2/5$ for every $k \geq 2$.

**Proof.** Let $C$ be an idcode of $G$, and let $q = 2/(\Delta + 2)$. We apply the discharging method with charging rules as stated in Lemma 3.3, and with the following discharging rule:

(R) If $v \notin C$ and $|C[v]| = p$, then $v$ receives a charge of $q/p$ from each vertex in $C[v]$.

We note that only neighbouring vertices exchange charges (thus we may apply Lem. 3.3 with $d = 1$). We prove now that $\text{chg}(v) \geq q$ for every vertex $v$ in $G$. Clearly, if $v \notin C$, then $\text{chg}(v) = q$; so assume that $v \in C$. If $v$ has no neighbours in $C$, then for all $w \in N(v)$ we have $|C[w]| \geq 2$, otherwise $C[v] = C[w]$. Thus, vertex $v$ sends a charge of at most $q/2$ to each vertex in $N(v)$. As a vertex in $G$ has degree at most $\Delta$, it follows that $\text{chg}(v) \geq 1 - \Delta(q/2) = q$.

Suppose now that $v$ has a neighbour in $C$. Then for at most one vertex, say $w$, that is a neighbour of $v$ outside $C$, we have that $C[w] = \{v\}$; and for all the remaining neighbours $x$ of $v$ outside $C$, we have that $|C[x]| \geq 2$. Thus $v$ sends a charge of at most $q$ to $w$ and at most $q/2$ to the remaining neighbours $x$ in $N(v) \setminus C$. Since the degree of $v$ is at most $\Delta$, it follows that $\text{chg}(v) \geq 1 - q - (\Delta - 2)(q/2) = q$.

As $\text{chg}(v) \geq q$ for every vertex $v$ in $G$, by Lemma 3.3 we have that $d(C, G) \geq q$. As this holds for an arbitrary idcode $C$, it follows that $d^*(G) \geq q = 2/(\Delta + 2)$. When $G$ is the hexagonal grid $H_k$ with $k$ rows, the result we have shown implies that $d^*(H_k) \geq 2/5$ for every $k \geq 2$. □

If $C$ is an idcode of a graph $G$, then a component of $G[C]$, the subgraph induced by $C$, is called a cluster of $G$ (w.r.t. $C$). If a cluster has precisely (resp. at least) $t$ vertices, then it is called a $t$-cluster (resp. $t^+$-cluster). The unique vertex of a 1-cluster is also called a 1-cluster. Note that $G[C]$ has no 2-clusters, otherwise, the 2 vertices in such a cluster would have the same identifier. The idcodes shown in Figures 1B and 1C induce only 1-clusters.

In what follows, we show that if $C$ is an idcode of a graph $G$ such that $G[C]$ has no 1-clusters, and $G$ satisfies certain conditions, then $d(C, G) \geq 3/7$.

**Theorem 4.2.** Let $G$ be a connected infinite twin-free graph with maximum degree 3, and with the SG-property. If $C$ is an idcode of $G$ such that $G[C]$ has no 1-clusters, then $d(C, G) \geq 3/7$. In particular, $d(C, G_H) \geq 3/7$ and $d(C, H_k) \geq 3/7$ for every $k \geq 2$.

**Proof.** We use the discharging method with charging rules as stated in Lemma 3.3. We take $q = 3/7$, and consider the following discharging rules:

(R1) If $v \notin C$ and $|C[v]| = p$, then $v$ receives a charge of $3/(7p)$ from each vertex in $C[v]$.

(R2) If $c \in C$ and $|N[c] \cap C| \geq 2$, then $c$ sends a charge of $1/14$ to each neighbour in $N(c) \cap C$.

Let us prove now that $\text{chg}(v) \geq 3/7$ for every vertex $v$. Clearly, $\text{chg}(v) = 3/7$ if $v \notin C$. Consider now a vertex $c \in C$. By hypothesis, we have that $c$ has at least one neighbour in $C$. If $c$ has exactly one neighbour $c^\prime$ in $C$, then $c^\prime$ must have another neighbour in $C$. Since $c$ has at most 2 neighbours outside $C$, then $c$ sends a charge of at most $3/7$ to one of them, at most $3/14$ to the other, and receives $1/14$ from $c^\prime$. (Note that, if these two neighbours exist, then one of them must have another neighbour in $C$, distinct from $c$.) Hence, $\text{chg}(c) \geq 1 - 3/7 - 3/14 + 1/14 = 3/7$. If $c$ has exactly two neighbours in $C$, then $c$ sends a charge of at most $3/7$ to some neighbour $w \notin C$ and exactly $1/14$ to each one of the two neighbours in $C$. Thus, $\text{chg}(c) \geq 1 - 3/7 - 2(1/14) = 3/7$. If $c$ has exactly three neighbours in $C$, then $c$ sends exactly $1/14$ of charge to each of them. Hence, $\text{chg}(c) \geq 1 - 3(1/14) = 11/14 > 3/7$. The results follow from Lemma 3.3. □
5. An Identifying Code of $H_2$ with Minimum Density

In this section we prove that $d^*(H_2) = 9/20$. For that, we prove first the following result.

Lemma 5.1. The minimum density of an idcode of $H_2$ is at most 9/20.

Proof. Consider the subgraph, say $T$, indicated in Figure 3, which is a subgraph of $H_2$ induced by the vertices from columns 1 to 20. Let $C$ the set of 18 black vertices indicated in $T$.

Note that, the pattern defined by $C$ in the first 10 columns of $T$ is a reflected form of the pattern defined by $C$ in the next 10 columns. We claim that if we concatenate infinite copies of $T$ (side by side), the set of black vertices obtained is an idcode of $H_2$ (with period 20). We leave to the reader to check this fact (it is enough to check the first 11 columns, and the columns 20 and 21). By Lemma 3.4 we conclude that $d^*(H_2) \leq 9/20$. \qed

To show that $d^*(H_2) \geq 9/20$, we present two different proofs, which are closely related. Both are based on the patterns defined by an idcode $C$ in the graph $H_2$. To study these patterns, we consider that the graph $H_2$ is an infinite strip that can be “split” into “sequential” 4-vertex sets, defined formally in what follows.

For an integer $x$, we say that a vertex of column $x$ of $H_2$ is cubic if it has degree 3 in $H_2$. We adopt the convention that when $x$ is odd then the vertices in column $x$ are cubic. For an odd integer $x$, we denote by $Q_x$ the set of vertices $\{(x,1),(x+1,1),(x,2),(x+1,2)\}$, and call it a quartet.

Note that $H_2[Q_x]$ is a $\square$-shaped path in $H_2$ with 4 vertices, and $V(H_2)$ is the disjoint union of quartets $Q_x$ such that $x$ is an odd integer. Given a quartet $Q_x$, we also refer to $Q_{x-2}$ (resp. $Q_{x+2}$), its left (resp. right) quartet, as $Q^L_x$ (resp. $Q^R_x$), see Figure 4.

For a given idcode $C$, we say that $Q_x$ is type i (resp. type $i^+$) if $|Q_x \cap C| = i$ (resp. $|Q_x \cap C| \geq i$). Type 1 quartets $Q_x$ play an important role in the proofs. If the single vertex in the idcode that belongs to $Q_x$ is cubic (resp. not cubic) in $H_2$, we say that $Q_x$ is type 1-cubic (resp. type 1-noncubic). See Figure 6. All references to types assume that an idcode is clear from the context.

The next lemmas tell us, for each quartet $Q_x$ of type $i$ $(1 \leq i \leq 3)$, which are the possible (or forbidden) types of its neighbouring quartets $Q^L_x$ and/or $Q^R_x$. Once we have these results, we can either use the discharging method or an idea based on the average density of patterns defined by consecutive quartets.

We denote by $(H_2,C,x)$ a triple consisting of the grid $H_2$, an idcode $C$ of $H_2$, and an odd integer $x$. In the figures, vertices coloured black belong to $C$, vertices coloured gray may belong to $C$.

Lemma 5.2 ($Q_x$ is type 0). Consider a triple $(H_2,C,x)$. If $Q_x$ is type 0, then $Q^L_x$ is type 4; moreover, $Q^R_x$ is type 3$^+$ and $C \cap Q^R_x$ contains two cubic vertices.

Proof. If $Q_x$ is type 0, it is immediate that all vertices in columns $x-1$ and $x+2$ must be in $C$, since all vertices of $Q_x$ must have a nonempty identifier. As $C$ is an idcode, the vertices of column $x-2$ must belong to $C$; thus, $Q^L_x$ is type 4. See Figure 5. Since $H_2[C]$ has no 2-clusters, $Q^R_x$ is type 3$^+$. \qed
Lemma 5.3 \((Q_x \text{ is type 1})\). Consider a triple \((H_2, C, x)\). If \(Q_x \text{ is type 1}\), then the following holds.

(a) If \(Q_x \text{ is type 1-cubic}\), then \(Q^L_x \text{ is type } 2^+\) and \(Q^R_x \text{ is type } 3^+\).

(b) If \(Q_x \text{ is type 1-noncubic}\), then \(Q^L_x \text{ is type } 3^+\) and \(Q^R_x \text{ is type } 2^+\).

Proof. For simplicity, rename the vertices of \(Q^L_x \cup Q_x \cup Q^R_x\) as shown in Figure 6.

To prove (a), let \(Q_x \) be type 1-cubic, and assume without loss of generality that \(Q_x \cap C = \{x_1\}\). See Figure 6A.

- If \(v_1 \in C\), then \(u_1 \in C\), otherwise \(\{v_1, x_1\}\) would induce a 2-cluster in \(H_2\), a contradiction. Thus, \(Q^L_x \cap C \supseteq \{v_1, u_1\}\) and therefore \(Q^L_x \) is type \(2^+\). If \(v_1 \notin C\), then \(v_2 \in C\), otherwise \(C[x_1] = C[x_2]\), a contradiction. Moreover, \(u_1 \in C\), otherwise \(C[v_1] = C[x_1]\). Thus, \(Q^L_x \cap C \supseteq \{v_2, u_1\}\) and therefore \(Q^L_x \) is type \(2^+\).

- Clearly, \(z_2 \in C\), otherwise \(C[y_2] = \emptyset\). If \(z_1 \in C\), then \(|Q^R_x \cap C| \geq 3\), otherwise \(\{z_1, z_2\}\) would induce a 2-cluster in \(H_2\). Hence, \(Q^R_x \) is type \(3^+\). If \(z_1 \notin C\), then \(w_1 \in C\), otherwise \(C[z_1] = C[y_2]\). Moreover, \(w_2 \in C\), otherwise \(C[y_2] = C[z_2]\). Thus, \(Q^R_x \cap C = \{z_2, w_1, w_2\}\), and \(Q^R_x \) is type \(3^+\).

To prove (b), let \(Q_x \) be type 1-noncubic, and assume without loss of generality that \(Q_x \cap C = \{y_2\}\). See Figure 6B.

- Clearly, \(v_1 \in C\), otherwise \(C[x_1] = \emptyset\). Moreover, \(u_1 \in C\), otherwise \(C[x_1] = C[v_1]\). If \(u_2 \notin C\), then \(v_2 \in C\) (because \(C[v_2] \neq \emptyset\)). Thus, \(Q^L_x \) is type \(3^+\).

- Clearly, \(z_1 \in C\) (because \(C[y_1] \neq \emptyset\)). If \(z_2 \in C\), then \(Q^R_x \) is type \(2^+\). If \(z_2 \notin C\), then \(w_1 \in C\), otherwise \(C[z_1] = C[y_1]\), a contradiction. Hence, \(Q^R_x \) is type \(2^+\).

\(\square\)

Lemma 5.4 \((Q_x \text{ is type 2})\). Consider a triple \((H_2, C, x)\). If \(Q_x \text{ is type 2}\), then \(Q^L_x \) and \(Q^R_x \) may not be both type 1.

Proof. Suppose, by contradiction, that both \(Q^L_x \) and \(Q^R_x \) are type 1. By Lemma 5.3, if \(Q^L_x \) (resp. \(Q^R_x \)) is type 1-cubic (resp. 1-noncubic), then \(Q_x \) is type \(3^+\). Thus, let us suppose now that \(Q^L_x \) is type 1-noncubic, \(Q^L_x \cap C = \{u\}\); and \(Q^R_x \) is type 1-cubic, \(Q^R_x \cap C = \{w\}\).

First, assume that \(u \) and \(w \) are in the same row, say 2. See Figure 7A. Then \((x_1, 2) \in C\), because \(C[(x_1 - 1, 1)] \neq \emptyset\). Note that one of the vertices \((x_1 + 1, 1)\) or \((x, 2)\) belongs to \(C\), because \(C[(x_1 - 1, 1)] \neq C[(x, 1)]\). If \((x_1 + 1, 1) \in C\), then \((x_1, 1)\) and \((x_1 + 1, 1)\) would induce a 2-cluster in \(H_2\), a contradiction. If \((x, 2) \in C\), then \(C[(x + 2, 2)] = \{w\} = C[(x + 2, 1)], a contradiction.
Figure 7. Quartet $Q_x$ is type 2. (A) Vertices $u$ and $w$ are in the same row. (B) Vertices $u$ and $w$ are in different rows.

If $u$ and $w$ are in different rows, assume without loss of generality that $u$ is in row 2 and $w$ is in row 1. See Figure 7B. Then $(x, 1) \in C$, because $C[(x - 1, 1)] \neq \emptyset$. If both $(x + 1, 1)$ and $(x + 1, 2)$ do not belong to $C$, then $C[(x + 2, 2)] = C[(x + 2, 1)]$, a contradiction. Thus, exactly one of them belongs to $C$. If $(x + 1, 1) \in C$, then $C[(x + 2, 2)] = C[(x + 2, 1)]$, a contradiction. Hence, $(x + 1, 2) \in C$. But in this case, $C[(x - 1, 1)] = C[(x, 1)]$, a contradiction. This concludes the proof of the lemma.

We state now a lemma that will be helpful to simplify the proof of the next theorem.

**Lemma 5.5.** The grid $H_2$ has idcodes of minimum density without type 0 quartets.

**Proof.** Let $C$ be an idcode of $H_2$, and $Q_x$ be a quartet of type 0. By Lemma 5.2, $Q^L_x$ is type 4. It is simple to verify that $C' = C \setminus \{(x - 1, 1)\} \cup \{(x, 1)\}$ is an idcode of $H_2$ such that $Q_x$ is type 1 and $Q^L_x$ is type 3. Thus, $d(C, H_2) = d(C', H_2)$. This means that If $C$ is an idcode of minimum density containing type 0 quartets, then $H_2$ has also an idcode of the same density without type 0 quartets.

**Remark.** The previous lemma does not guarantee anything about the elimination of type 4 quartets. We note that by doing a local change (more involved than the above one) we may also eliminate type 4 quartets and obtain an idcode of equal or possibly smaller density. We do not prove this statement as we do not use it here. Moreover, later we present arguments showing that type 4 quartets do not occur in minimum density idcodes of $H_2$.

Before going to the next proof, the reader may highlight in Figure 3 the 1-cubic and 1-noncubic quartets, and check the statements of Lemmas 5.3 and 5.4 with respect to the quartets of this figure. This will help the understanding of the discharging rule (resp. the idea based on the average density) used in the next two proofs.

**Theorem 5.6.** The minimum density of an idcode of $H_2$ is precisely $9/20$.

**Proof.** We use the discharging method to prove that $d^*(H_2) \geq 9/20$. For that, let $C$ be a minimum identifying code of $H_2$ that has no quartets of type 0 (cf. Lem. 5.5). In the charging phase, we proceed as stated in Lemma 3.3: we set $\text{chg}(v) = 1$ if $v \in C$, and $\text{chg}(v) = 0$, otherwise. We shall prove that after the discharging phase (to be defined), we have $\text{chg}(Q_x) \geq 9/5$ for each quartet $Q_x$. If this happens, then the total charge of each $Q_x$ can be distributed among its 4 vertices, and we get $\text{chg}(v) \geq 9/20$ for each vertex $v$ in $Q_x$. Thus, we say that a quartet $Q_x$ is satisfied if $\text{chg}(Q_x) \geq 9/5$, otherwise, it is unsatisfied.

After the charging phase, only type 1 quartets are unsatisfied. Apply the following discharging rule.

(R) As long as there are type 1 quartets $Q_x$ that are unsatisfied,

(a) if $Q_x$ is 1-cubic, then it receives $1/5$ from $Q^L_x$, and $3/5$ from $Q^R_x$;

(b) if $Q_x$ is 1-noncubic, then it receives $3/5$ from $Q^L_x$, and $1/5$ from $Q^R_x$.

We prove now that each quartet $Q_x$ is satisfied after the discharging phase.
Proof 2 of Theorem 5.6. Let idcode that we have found is unique.

Lemma 5.1. Thus, as those starting with a quartet of type 1, and by Lemma 5.3, we conclude that \( \text{chg}(Q_x) = 2 - 1/5 = 9/5 \).

Case 3. \( Q_x \) is type 3.

The only possibility for \( Q_x \) to decrease its initial charge is when it has type 1 neighbours. In the worst case, when both \( Q_x^L \) and \( Q_x^R \) are type 1, \( Q_x \) sends at most 3/5 to each of them. Thus, \( \text{chg}(Q_x) \geq 3 - 3/5 - 3/5 = 9/5 \).

Since every quartet \( Q_x \) is satisfied, by Lemma 3.3, we have that \( d(C, H_2) \geq 9/20 \). Using Lemma 5.1, we conclude that \( d^*(H_2) = 9/20 \).

From the previous result and the fact that the idcode shown in Lemma 5.1 has density at most 9/20, we conclude the following result.

**Corollary 5.7.** The idcode shown in Lemma 5.1 is a periodic idcode of \( H_2 \) with minimum density.

In what follows we present a second proof of Theorem 5.6 which is based on the idea of finding a periodic pattern that covers \( H_2 \) and has the minimum possible density. This proof also uses Lemmas 5.2–5.5, and it is based on the fact (mentioned in Sect. 6) that \( H_2 \) has a periodic idcode of minimum density. As we will see, the information provided by this proof, combined with further tests, will lead us to conclude that the periodic idcode that we have found is unique.

**Proof 2 of Theorem 5.6.** Let \( C \) be an idcode of minimum density in \( H_2 \) that has no quartets of type 0. If \( C \) has no quartets of type 1, then all quartets in \( C \) are of type \( 2^+ \), and in this case, \( d(C, H_2) \geq 1/2 \), contradicting Lemma 5.1. Thus, \( C \) has a quartet of type 1, and by Lemma 5.3 we conclude that \( C \) has a quartet of type \( 3^+ \).

Now let us consider that \( H_2 \) (seen as a concatenation of quartets) can be split into subgraphs corresponding to special sequences of consecutive quartets. We are interested in sequences, which we call \( S(3)\)-sequences, defined as those starting with a quartet of type \( 3^+ \) and containing exactly one quartet of type \( 3^+ \). The \( S(3)\)-sequences whose second quartet is of type 1 (resp. type 2) are called \( S(3, 1)\)-sequences (resp. \( S(3, 2)\)-sequences). (We remark that not allowing the presence of another quartet of type \( 3^+ \) is not a restriction to the size of the periods of the patterns we want to study. We may have different \( S(3)\)-sequences, and later we allow them to be concatenated, so that periods with many occurrences of quartets of type \( 3^+ \) are made possible.)

For an \( S(3)\)-sequence \( S \), let \( I(S) = (i_1, i_2, \ldots) \) be the sequence where each \( i_j \in \{1, 2, 3, 4\} \) indicates the type of each of the \( j \)th quartet in \( S \). In this proof, \( i_j = i^+ \) means that \( i_j \in \{i, i + 1\} \). A simplified notation such as \( I(S) = (3^+, 1, 2, 2, 1^+) \) stands for \( I(S) = \{(3, 1, 2, 2, 1), (3, 1, 2, 2, 2), (4, 1, 2, 2, 1), (4, 1, 2, 2, 2)\} \). We denote by \( H[S] \) the subgraph of \( H_2 \) induced by the quartets in \( S \), and denote by \( C(S) \) the restriction of \( C \) to \( H[S] \). We are interested in \( d(C(S), H[S]) \), the density of \( C(S) \) with respect to \( H[S] \).

Note that \( I(S) \) may not contain subsequences of the form \( (1, 2, 1), (2, 1, 2) \) or \( (1, 1) \) because of Lemmas 5.3 and 5.4. If \( S \) is an infinite \( S(3)\)-sequence, then \( I(S) = (3^+, 1, 2, 2, \ldots) \) or \( I(S) = (3^+, 2, 2, \ldots) \), and therefore \( d(C(S), H[S]) \geq 1/2 \). If \( S \) is a finite \( S(3, 1)\)-sequence, then \( I(S) \) contains at most two (non-consecutive) 1’s.

Let \( S_t \) be a finite \( S(3, 1)\)-sequence of length \( t \), let \( I_t = I(S_t) \), and let \( C_t \) be the restriction of \( C \) to \( S_t \). The possibilities for \( I_t \) are: \( I_1 = (3^+) \), \( I_2 = (3^+, 1) \), \( I_3 = (3^+, 1, 2) \), \( I_4 = (3^+, 1, 2, 2) \), \( I_5 = (3^+, 1, 2, 2, 1^+) \),
and \( I_t = (3^+, 1, 2, \ldots, 2, 1^+) \) if \( t > 5 \). Thus \( d(C_t, H[S_t]) \geq 1/2 \), for \( 1 \leq t \leq 4 \), \( d(C_5, H[S_5]) \geq 9/20 \) and \( d(C_t, H[S_t]) \geq (3 + 1 + 2(t - 3) + 1)/4t = (2t - 1)/4t > 9/20 \) if \( t > 5 \). Thus the minimum density 9/20 may possibly occur for \( S(3) \)-sequences of length 5 with sequence of types \((3, 1, 2, 2, 1)\).

It is easy to see that if \( S \) is a finite \( S(3, 2) \)-sequence, then \( d(C(S), H[S]) \geq 1/2 \) (because \( I(S) \) contains at most one 1). This ends the proof that all \( S(3) \)-sequences of \( H_2 \) have density at least 9/20. Thus, \( d(C, H_2) \geq 9/20 \) (as \( H_2 \) has a minimum-density periodic idcode). Combining this result with Lemma 5.1, we conclude that \( d^*(H_2) = 9/20 \).

**Remark on the uniqueness of a periodic minimum-density idcode for \( H_2 \)**. By Corollary 5.7, the idcode shown in Figure 3 is a periodic idcode of \( H_2 \) with minimum density. An interesting question is whether this idcode is unique, among the periodic ones. The meaning of uniqueness will be clear in what follows.

The second proof of Theorem 5.6 suggests that to construct a periodic minimum-density idcode for \( H_2 \) we should look for idcodes that define \( S(3, 1) \)-sequences of length 5 of type \((3, 1, 2, 2, 1)\), and try to concatenate them to see whether they yield a periodic idcode.

As the reader may check, the \( S(3, 1) \)-sequence, say \( S \), corresponding to the 5 initial quartets (first 10 columns) shown in Figure 3 is of type \((3, 1, 2, 2, 1)\). However, the concatenation \( SS \) does not define an idcode of \( H_2 \) restricted to these sequences. But, as one can see in Figure 3, after \( S \), the next sequence of 5 quartets, say \( S' \), which is a reflected form of \( S \) is also an \( S(3) \)-sequence of type \((3, 1, 2, 2, 1)\). As we mentioned before, this is an idcode of \( H_2 \) with period 20. This is not the way we obtained this idcode. In fact, this idcode was obtained by an ad hoc method, and we used it as an inspiration to derive the properties (Lems. 5.2–5.5) that we proved.

These lemmas, in turn, helped us in the lower bound proof. If a sequence such as \( S \) could not be found, one should look for \( S(3) \)-sequences of lengths \( t = 6, 7, \ldots \), as they would be the next candidates (if we did not know an idcode with density 9/20).

Let us now investigate whether the idcode shown in Figure 3 is the unique periodic idcode of \( H_2 \) with density 9/20. We note that \( S \) and \( S' \) are the unique \( S(3) \)-sequences of type \((3, 1, 2, 2, 1)\) (we have verified this by running a program). We also note that the concatenation \( S'S' \) does not define an idcode. So, for the moment we may say that the answer to this question is “yes”, if we consider minimum idcodes without type 0 quartets (as we proved).

The question now is whether there are minimum-density idcodes containing type 0 quartets. We will not go into details, but we can prove that carrying out analogous arguments as those we used for \( S(3, 1) \)- and \( S(3, 2) \)-sequences, the answer is “no”. By Lemma 5.2, a type 0 quartet is preceded by a type 4 quartet, and is succeeded by a type 3\(^+\) quartet. Using this fact, we can show that any \( S(3) \)-sequence that is of subtype \( S(4, 0) \) has density greater than 9/20. Thus, we conclude that the idcode shown in Figure 3 is the unique periodic idcode of \( H_2 \) with minimum density. This idcode was also obtained by running a computer program, about which we report in the next section.

We note that, the idea we mentioned after Lemma 3.4 to prove lower bound for the density of idcodes of \( H_k \) – based on periodic patterns with minimum density – is basically the idea behind the study we have carried out on the types of sequences of \( H_2 \). This study led us to conclude that the periodic pattern \( H \) defined by the concatenation \( SS' \) is the shortest periodic pattern that has the minimum density 9/20. Of course, we may say that \( S'S \) is also such a shortest periodic pattern, but here we consider that they are equivalent.

6. **Minimum-density identifying codes of \( H_3 \), \( H_4 \) and \( H_5 \)**

In this section we present minimum-density idcodes for \( H_3 \), \( H_4 \) and \( H_5 \) that we found with an algorithm implemented in C++. We describe briefly the algorithm, then exhibit some of these idcodes and the values \( d^*(H_3) \), \( d^*(H_4) \) and \( d^*(H_5) \).

The algorithm that we implemented searches for a periodic idcode for these grids, and uses an idea that was already proposed in 2018 by Jiang [24], to find minimum-density idcodes for square grids \( S_k \) with finite number \( k \) of rows. We were not aware of his algorithm, although we knew about his results on \( S_k \). Jiang [24] proved that such grids have idcodes with minimum density that are periodic, and described an algorithm to find them. His
work presents in detail an algorithm that constructs a weighted directed graph (associated with $S_k$) in which a minimum mean cycle corresponds to a periodic minimum-density idcode of $S_k$. Fortunately, the size of this graph is exponential in $k$. With his implementation in C, in 2018 Jiang was able to obtain optimum idcodes for $S_4$ and $S_5$. We used basically the same idea for $H_k$. For completeness, we describe briefly the construction of this graph, using the terminology introduced by Jiang.

We do not prove here that $H_k$ has finite periodic idcodes that have minimum density, but this result holds. A proof similar to the one presented by Jiang [24] for $S_k$ can be done for $H_k$, using the idea based on the concept of bars, which is central here, and is defined in what follows.

For $\ell \geq 1$ and $k \geq 2$, any subgraph of $H_k$ induced by $\{j_1, \ldots, j_\ell\} \times [k]$, where $j_1 \leq j_2 \leq \ldots \leq j_\ell$ are vertices of $H_k$, is called an $\ell$-bar (see Fig. 9). Let $R$ be any $\ell$-bar with $\ell \geq 3$ in $H_k$, and let $R'$ be the $(\ell - 2)$-bar consisting of the middle columns of $R$ (obtained by excluding the first and the last columns of $R$). We say that a subset $C$ of vertices of $R$ is a barcode of $R$ if $C[v] \neq \emptyset$ and $C[u] \neq C[v]$ for every distinct $u, v \in R'$. We adopt the convention that the first column of each 4-bar of $H_k$ is indexed by an odd number.

6.1. Construction of the arc-weighted directed graph $G_{k,4,j}$

For $k \geq 2$ and $5 \leq j \leq 8$, let $G_{k,4,j} = (V, A)$ denote the $j$-configuration graph of the idcodes of $H_k$ defined as follows. The vertex set $V$ of this graph consists of barcodes $C$ of any 4-bar of $H_k$. There is an arc from $C$ to $C'$ if there is a barcode $Q$ of a $j$-bar $B$ of $H_k$ such that $C$ (resp. $C'$) is the restriction of $Q$ to the first (resp. last) $4$ columns of $B$. In this case, the arc from $C$ to $C'$ gets weight $|Q| - |C|$. Note that, $|V| \leq 24k$ and $|A| \leq 24k$.

In our implementation, we used $j = 6$ and $j = 8$ (as in this case we have to deal only with 4-bars whose first column is indexed by an odd number).

Jiang [24] considered, for the grid $S_k$, the graph $G_{k,4,5}$, described above for $H_k$ (for $S_k$, the 4-bars correspond to subgraphs of $S_k$). He showed that in this graph, each 4-bar pattern of a periodic idcode for $H_k$ (as in this there is an overlap of 2 columns for each two adjacent barcodes).

In Figure 9 we show a minimum density periodic idcode (with period 8) for $H_4$ that was found in the 8-configuration graph $G_{4,4,8}$. The two curly braces indicate two consecutive 4-bars (corresponding to two barcodes, say $C$ and $C'$, which are adjacent vertices in this graph). In this case, $Q$ is the barcode of the 8-bar (formed by the indicated 4-bars), and the weight of the arc from $C$ to $C'$ is $|Q| - |C| = 14 - 7 = 7$. This solution corresponds to the weighted directed cycle $Z = (C, C')$ that has length $|Z| = 2$ and weight $w(Z) = 14$ (with mean weight $w(Z)/2 = 14/2 = 7$). In this case, the period is $p = 8$. Thus, the density of this solution is $w(Z)/(8 \cdot 4) = 14/32 = 7/16$. We observe that when $j = 8$ the period is $|Z| \cdot 4$ (but the period is $|Z| \cdot 2$ if $j = 6$, as in this there is an overlap of 2 columns for each two adjacent barcodes).

It is well known that the minimum mean cycle problem on a graph with $n$ vertices and $m$ arcs can be solved in $O(nm)$ time by Karp’s algorithm [26]. This is the algorithm that Jiang [24] used in his implementation for $S_k$. For $H_k$, we use Hartmann–Orlin’s algorithm [15], which is an improved version of Karp’s algorithm, to find a minimum mean cycle. We implemented a program in C++, using lemon library for graphs: it builds the graph $G_{k,4,j}$, finds a minimum mean cycle and outputs an idcode with minimum density for $H_k$. This implementation can be found in [32].

We run this program to find minimum-density idcodes for $H_3$, $H_4$ and $H_5$. This program constructed $G_{3,4,6}$, $G_{4,4,8}$, $G_{5,4,6}$, and obtained $d^*(H_3) = 6/13$, $d^*(H_4) = 7/16$ and $d^*(H_5) = 11/25$. The corresponding idcodes for these grids are depicted in Figures 8–10. In Table 1, we indicate the size of these configuration graphs and the total running time the program needed to find an optimal solution. The running times for $j = 8$ are included.

\[1\text{https://lemon.cs.elte.hu/trac/lemon}\]
to show the difference when compared with \( j = 6 \). The code was compiled with g++ 11.4.0 and option \(-O3\), and executed in a computer with Intel(R) Xeon(R) CPU E7-2870 @ 2.40 GHz processor with 512 GB of RAM.

**Theorem 6.1.** For \( k = 3, 4, 5 \), the idcodes for \( H_k \) shown in Figures 8–10 have minimum density. The corresponding densities of these idcodes are \( d^*(H_3) = 6/13 \), \( d^*(H_4) = 7/16 \) and \( d^*(H_5) = 11/25 \).

As a side remark, we observe that if instead of considering 4-bars, we consider 3-bars (to define the vertices of the graph), and define adjacency of vertices in an analogous way, the corresponding graphs \( G_{k,3,5} \) or \( G_{k,3,6} \) for \( S_k \) or \( H_k \) do not have the desired property (as some arcs would indicate a wrong adjacency). We leave to the reader finding examples to verify this statement. But such incorrect adjacencies occur rarely. Since it is much faster to work with 3-bars, one possibility is to work with 3-bars, and check whether the solution found does not have wrong adjacencies, as in this case, an optimum solution may be found more quickly.

We conclude this section mentioning that with our implementation we were not able to find a minimum-density idcode for \( H_6 \) using the computer resources available to us.

## 7. Concluding remarks

We note that for \( H_3 \) we have found only the minimum-density idcode shown in Figure 8. But we are not claiming that it is unique. For \( H_4 \) and \( H_5 \), we have found other minimum-density idcodes with different periods. For \( H_5 \) we note that the minimum-density idcode shown in Figure 11 is different from the idcode shown in...
Table 1. Sizes of the configuration graphs generated by our implementation and total running times.

<table>
<thead>
<tr>
<th>Configuration graph</th>
<th>#vertices</th>
<th>#edges</th>
<th>Total running time</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_{2,4,6} (H_2)$</td>
<td>144</td>
<td>1359</td>
<td>8 ms</td>
</tr>
<tr>
<td>$G_{3,4,6} (H_3)$</td>
<td>1896</td>
<td>57723</td>
<td>253 ms</td>
</tr>
<tr>
<td>$G_{4,4,6} (H_4)$</td>
<td>5870</td>
<td>63095</td>
<td>8 s</td>
</tr>
<tr>
<td>$G_{5,4,6} (H_5)$</td>
<td>63751</td>
<td>1650188</td>
<td>87 m</td>
</tr>
</tbody>
</table>

(B) Data for $j = 8$

<table>
<thead>
<tr>
<th>Configuration graph</th>
<th>#vertices</th>
<th>#edges</th>
<th>Total running time</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_{2,4,8} (H_2)$</td>
<td>144</td>
<td>12894</td>
<td>46 ms</td>
</tr>
<tr>
<td>$G_{3,4,8} (H_3)$</td>
<td>1896</td>
<td>1784401</td>
<td>9 s</td>
</tr>
<tr>
<td>$G_{4,4,8} (H_4)$</td>
<td>5870</td>
<td>3291346</td>
<td>8209 s</td>
</tr>
<tr>
<td>$G_{5,4,8} (H_5)$</td>
<td>63751</td>
<td>248161004</td>
<td>928 m</td>
</tr>
</tbody>
</table>

Figure 11. A minimum-density idcode of $H_5$ found in the graph $G_{5,4,8}$ (density $11/25 = 0.44$).

Figure 10, but both have period 10. By considering the graph $G_{5,4,8}$, the corresponding program output the solution of Figure 11 indicating that the period is 20. We noted that the columns from 1 to 10 of this idcode is equal to the columns from 11 to 20. Thus, we may say that the period of this idcode is 10. This does not indicate that the program is incorrect. Clearly, when $j = 8$, the program outputs a solution whose period is always a multiple of 4, while when $j = 6$ the program outputs a solution whose period is a multiple of 2.

With this respect, we note that if $H_k$ has a minimum-density idcode with period $p$, even when $p$ is odd, an idcode with the same density and possibly different period can be found in the graph $G_{k,4,6}$ and $G_{k,4,8}$. This is true because there is a (smallest) multiple of $p$ which is always a multiple of 2 or of 4, and therefore such a solution will be present in the corresponding graphs. We observe that our program finds one optimal solution (a minimum mean cycle) but not all optimal solutions.

Our implementation may possibly be improved if we can eliminate from the graph $G_{k,4,j}$ some vertices and arcs which we are sure will not occur in an optimal solution. For example, barcodes corresponding to the set of all vertices in a 4-bar, or possibly barcodes whose densities are much larger than some known upper bound for the minimum-density idcode. But to implement such steps safely, some proofs are needed. We also believe that a more substantial improvement is needed to be able to solve for larger $k$. We are working on this topic and hope that in a forthcoming paper we will be able to present good upper bounds for $d^*(H_k)$, for all $k \geq 6$.

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