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ITALIAN, 2-RAINBOW AND ROMAN DOMINATION NUMBERS IN MIDDLE GRAPHS

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Abstract. Given a graph G, we consider the Italian domination number $\gamma_I(G)$, the 2-rainbow domination number $\gamma_{r2}(G)$ and the Roman domination number $\gamma_R(G)$. It is known that $\gamma_I(G) \leq \gamma_{r2}(G) \leq \gamma_R(G)$ holds for any graph G. In this paper, we prove that $\gamma_I(M(G)) = \gamma_{r2}(M(G)) = \gamma_R(M(G)) = n$ for the middle graph $M(G)$ of a graph G of order n, which gives an answer for an open problem posed by Chellali *et al.* [*Discrete Appl. Math.* **204** (2016) 22–28]. Moreover, we give a complete characterization of Roman domination stable middle graphs, 2-rainbow domination stable middle graphs and Italian domination stable middle graphs.

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1. Introduction

Let $G = (V, E)$ be an undirected graph with the vertex set $V = V(G)$ and edge set $E = E(G)$. The *order* of G is defined as $|V|$. The *open neighborhood* of $u \in V(G)$ is the set $N(u) = \{w \in V(G) \mid uw \in E(G)\}$ and the *closed neighborhood* of $u \in V(G)$ is the set $N[u] := N(u) \cup \{u\}$. For a subset $U \subseteq V$, we denote $\bigcup_{u \in U} N(u)$ by $N(U)$. The *degree* of $u \in V(G)$ is defined as $|N(u)|$, denoted by $deg_G(u)$. For a subset $S \subseteq V$, the subgraph obtained from G by deleting all vertices in S and all edges incident with S is denoted by $G - S$. We denote a complete graph and a complete bipartite graph by K_n and $K_{m,n}$, respectively. For terminology and notation on graph theory not given here, the reader is referred to [4].

In [12], Hamada and Yoshimura introduced the concept of middle graph. The *middle graph* $M(G)$ of a graph G is as follows. The vertex set $V(M(G))$ is $V(G) \cup E(G)$. Two vertices $v, w \in V(M(G))$ are adjacent in $M(G)$ if (1) $v, w \in E(G)$ and v, w are adjacent in G or (2) $v \in V(G)$, $w \in E(G)$ and v, w are incident in G. Recently, the authors of [14–19] studied domination-related parameters on middle graphs.

The study of Roman domination was motivated by the defense strategies used to defend the Roman Empire during the reign of Emperor Constantine the Great, 274–337 AD. The concept of Roman domination was introduced in [9, 22, 24]. A function $f : V(G) \to \{0, 1, 2\}$ is called a *Roman dominating function* on G if every vertex $v \in V(G)$ for which $f(v) = 0$ is adjacent to at least one vertex $u \in V(G)$ for which $f(u) = 2$. The weight of a Roman dominating function is the value $\sum_{v \in V(G)} f(v)$. The *Roman domination number* $\gamma_R(G)$ is the minimum

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weight of a Roman dominating function on G. We say that a function f is a $\gamma_R(G)$-function if it is a Roman dominating function with the weight $\gamma_R(G)$. A function $f : V(G) \to \mathcal{P}(\{1, 2\})$ is called a 2-*rainbow dominating function* on G if for each vertex $v \in V(G)$ for which $f(v) = \emptyset$, it holds that $\bigcup_{u\in N(v)} f(u) = \{1, 2\}$. The weight of a 2-rainbow dominating function is the value $\sum_{v\in V(G)} |f(v)|$. The 2-*rainbow domination number* $\gamma_{r2}(G)$ is the minimum weight of a 2-rainbow dominating function on G. A function $f : V(G) \to \{0, 1, 2\}$ is called an *Italian dominating function* on G if for each vertex $v \in V(G)$ for which $f(v) = 0$, it holds that $\sum_{u\in N(v)} f(u) \geq 2$. The weight of an Italian dominating function is the value $\sum_{v\in V(G)} f(v)$. The *Italian domination number* $\gamma_I(G)$ is the minimum weight of an Italian dominating function on G.

An Italian dominating function is a generalization of a Roman dominating function. A Roman dominating function is a monochromatic version of a 2-rainbow dominating function. Motivated by this similarity, several works have been done in [1, 2, 7, 25]. In [25], it was shown that $\gamma_{r2}(G) \leq \frac{3}{4}n$ for every connected graph of order $n \geq 3$. The authors of [7] proved that $\gamma_R(G) \leq \frac{3}{2}\gamma_{r2}(G)$ for every graph G and that $\gamma_R(T) \leq \frac{4}{3}\gamma_{r2}(T)$ for every tree T. The following result provides a relationship between the three parameters mentioned above. The proof follows from [25] and definitions of the three parameters.

Theorem 1.1. *For every graph G, $\gamma_I(G) \leq \gamma_{r2}(G) \leq \gamma_R(G)$.*

The next theorem follows from [3].

Theorem 1.2. *If G is a graph of order n, then $\gamma_R(M(G)) = n$.*

An interesting problem is investigating the class of graphs for which the equalities in $\gamma_I(G) \leq \gamma_{r2}(G) \leq \gamma_R(G)$ are satisfied. As partial results, the authors of [8] proved that $\gamma_I(T) = \gamma_{r2}(T)$ for every tree T. In [5], it was given a constructive characterization of the class of trees T satisfying that $\gamma_I(T) = \gamma_R(T)$ and that $\gamma_{r2}(T) = \gamma_R(T)$. As related results, the authors of [6] described the trees T for which $\gamma_r(T) = \gamma_I(T)$, $\gamma_r(T) = \gamma_{r2}(T)$ and $\gamma_r(T) = \gamma_R(T)$, where $\gamma_r(T)$ is the weak Roman domination number of T. The authors of [10] presented a characterization of the connected graphs G for which $\gamma_R(G) - \gamma_{r2}(G) = k$ for some non-negative integer k at most $\frac{1}{2}\gamma_{r2}(G)$. In [2], it was proved that for every fixed non-negative integer k, the recognition of the connected K_4-free graphs with $\gamma_R(G) - \gamma_{r2}(G) = k$ is NP-hard. In [1], it was proved that if G is a connected graph of minimum degree at least 2 that is distinct from C_5, then $\gamma_{r2}(G) + \gamma_R(G) \leq \frac{4}{3}|V(G)|$. In [8], it was proved that $\gamma_{r2}(G) = \gamma_I(G)$ for trees and cactus graphs with no even cycle. The authors of [8] asked a question about the existence of other classes of graphs with $\gamma_{r2}(G) = \gamma_I(G)$. In this paper, we prove the following theorem, which is an answer for the above question.

Theorem 1.3. *Let G be a graph of order n. Then $\gamma_R(M(G)) = \gamma_{r2}(M(G)) = \gamma_I(M(G)) = n$.*

The concept of Roman domination stable graphs was introduced in [13]. A graph G is *Roman domination stable* if $\gamma_R(G) = \gamma_R(G - \{v\})$ for any vertex v of G. Properties and upper bounds for the Roman domination number in the class of Roman domination stable graphs were investigated in [11, 23]. This paper studies the above concept in the class of middle graphs. We find all Roman domination stable middle graphs. Since $\gamma_R(G) = \sum_{i=1}^{t} \gamma_R(G_i)$ for a graph G with components $G_1, \ldots, G_t$, we only consider connected graphs in the following theorem.

Theorem 1.4. *Let G be a connected graph. Then $M(G)$ is Roman domination stable if and only if $G \in \{K_{2m}, K_{m,m} \mid m \geq 1\}$.*

The concept of 2-rainbow domination stable graphs was introduced in [21]. A graph G is 2-*rainbow domination stable* if $\gamma_{r2}(G) = \gamma_{r2}(G - \{v\})$ for any vertex v of G. In this paper, we give the concept of Italian domination stable graphs. We say that a graph G is *Italian domination stable* if $\gamma_I(G) = \gamma_I(G - \{v\})$ for any vertex v of G. Since $\gamma_{r2}(G) = \sum_{i=1}^{t} \gamma_{r2}(G_i)$ and $\gamma_I(G) = \sum_{i=1}^{t} \gamma_I(G_i)$ for a graph G with components $G_1, \ldots, G_t$, we only consider connected graphs. Based on Theorems 1.3 and 1.4, we obtain the following results.

Theorem 1.5. *Let G be a connected graph. Then $M(G)$ is Italian domination stable if and only if $G \in \{K_{m,m} \mid m \geq 1\}$.*

Theorem 1.6. *Let G be a connected graph. Then $M(G)$ is 2-rainbow domination stable if and only if $G \in \{K_{2m}, K_{m,m} \mid m \geq 1\}$.*

This paper is organized as follows. In Section 2, we prepare basic concepts and results. In Sections 3 and 4, we give proof of the main theorems.

2. Preliminaries

In this section, we give some necessary terminology and notation. First, we give the concept of a middle Roman dominating function to study the Roman domination number of the middle graph. A *middle Roman dominating function* (MRDF) on a graph G is a function $f : V \cup E \to \{0, 1, 2\}$ satisfying the following conditions: (1) every element $x \in V$ for which $f(x) = 0$ is incident to at least one element $y \in E$ for which $f(y) = 2$, (2) every element $x \in E$ for which $f(x) = 0$ is adjacent or incident to at least one element $y \in V \cup E$ for which $f(y) = 2$. An MRDF f gives an ordered partition $(V_0 \cup E_0, V_1 \cup E_1, V_2 \cup E_2)$ (or $(V_0^f \cup E_0^f, V_1^f \cup E_1^f, V_2^f \cup E_2^f)$ to refer to f) of $V \cup E$, where $V_i := \{x \in V \mid f(x) = i\}$ and $E_i := \{x \in E \mid f(x) = i\}$. The weight of a middle Roman dominating function f is $\sum_{x \in V \cup E} f(x)$. The *middle Roman domination number* $\gamma_R^\star(G)$ of G is the minimum weight of a middle Roman dominating function of G. A $\gamma_R^\star(G)$*-function* is an MRDF on G with weight $\gamma_R^\star(G)$. Similarly, we can define a middle 2-rainbow dominating function (M2RDF) and a middle Italian dominating function (MIDF), respectively. The *middle 2-rainbow domination number* $\gamma_{r2}^\star(G)$ of G is the minimum weight of a middle 2-rainbow dominating function of G. A $\gamma_{r2}^\star(G)$*-function* is an M2RDF on G with weight $\gamma_{r2}^\star(G)$. The *middle Italian domination number* $\gamma_I^\star(G)$ of G is the minimum weight of a middle Italian dominating function of G. A $\gamma_I^\star(G)$*-function* is an MIDF on G with weight $\gamma_I^\star(G)$. By the definition of middle graphs, we state the following remark.

Remark 2.1. For any graph G, $\gamma_R^\star(G) = \gamma_R(M(G))$, $\gamma_{r2}^\star(G) = \gamma_{r2}(M(G))$ and $\gamma_I^\star(G) = \gamma_I(M(G))$.

For $v \in V(G)$, we denote $\{e \in E(G) \mid e$ is incident with $v\}$ by $N_M(v)$. For $e \in E(G)$, we denote $\{x \in V(G) \cup E(G) \mid x$ is either adjacent or incident with $e\}$ by $N_M(e)$. We write $N_M[x] = N_M(x) \cup \{x\}$. For a subset $S \subseteq V(G) \cup E(G)$, we denote $\bigcup_{x \in S} N_M(x)$ by $N_M(S)$.

Observation 2.1. Let $f = (V_0 \cup E_0, V_1 \cup E_1, V_2 \cup E_2)$ be a $\gamma_R^\star(G)$-function. Then the following assertions hold:

(i) $V_2 = \emptyset$;
(ii) for $e_1, e_2 \in E_2$, e_1 is not adjacent to e_2;
(iii) for $v \in V_1, e \in E_2$, v is not incident to e;
(iv) $E_1 = \emptyset$.

Proof. (i) See Proof of Theorem 2.1 from [19].

(ii) Suppose that e_1 is adjacent to e_2 for some $e_1, e_2 \in E_2$. Let $e_1 = vu$ and $e_2 = uw$. We assume that there exist edges incident to v or w. Otherwise, by reassigning e_1 as an edge of E_0 and v as a vertex of V_1, we can reduce the value of $\gamma_R^\star(G)$. This is a contradiction.
Now we show that there exists no edge of $E_2 \setminus \{e_1, e_2\}$ incident to v or w. Without loss of generality, assume that there exists an edge $e_3 \in E_2$ incident to v. By reassigning e_1 as an edge of E_0, we can reduce the value of $\gamma_R^\star(G)$. This is a contradiction.
Now we consider $G - \{v, u, w\}$ and define $g : V(G - \{v, u, w\}) \cup E(G - \{v, u, w\}) \to \{0, 1, 2\}$ by $g(x) = f(x)$. Then g is an MRDF with $\omega(g) = n - 4$. Since $G - \{v, u, w\}$ has order $n - 3$, by Theorem 1.3 $\omega(g) \geq \gamma_R^\star(G - \{v, u, w\}) = n - 3$, a contradiction.

(iii) Clear.

(iv) See Proof of Theorem 2.1 from [19].

□

In general, Roman domination number can change by the removal of a vertex. In the class of middle graphs, the following results say that the Roman domination number of $M(G)$ cannot be increased by the removal of any vertex.

Observation 2.2. If $v \in V_1^f$ for any $\gamma_R(M(G))$-function f, then $\gamma_R(M(G)-\{v\}) = \gamma_R(M(G)) - 1$. If $v \in V_0^f$ for every $\gamma_R(M(G))$-function f, then $\gamma_R(M(G)-\{v\}) = \gamma_R(M(G))$.

Proof. If $v \in V_1$, then clearly $\gamma_R(M(G)-\{v\}) = \gamma_R(M(G)) - 1$.

Assume that $v \in V_0^f$ for every $\gamma_R(M(G))$-function f. Suppose to the contrary that $\gamma_R(M(G)-\{v\}) < \gamma_R(M(G))$. Let g be a $\gamma_R(M(G)-\{v\})$-function. Then $g(N_M(v)) \le 1$. Otherwise, g can be extended to an RDF with weight $\gamma_R(M(G)-\{v\})$, a contradiction.

Let e_1 be an element of E_2^g incident to v, and let u be the other vertex incident to e_1. To dominate u, there must exist $e_2 \in E_2^g$ adjacent to u. Since e_1 is arbitrary, $g(N_M(v)) = 0$. By assigning the value 1 to v, it is possible to extend g to an RDF on $M(G)$ with weight $\gamma_R(M(G))$. This is a contradiction to our assumption. □

Observation 2.3. For any edge $e \in E(G)$, $\gamma_R(M(G)-\{e\}) = \gamma_R(M(G))$.

Proof. Since $M(G)-\{e\} \cong M(G')$ for some graph G' of order $|V(G)|$, by Theorem 1.3 we have $\gamma_R(M(G)-\{e\}) = |V(G)|$. □

3. Proof of Theorem 1.3

In this section, we show that for any graph G of order n, $\gamma_I^\star(G) = \gamma_{r2}^\star(G) = \gamma_R^\star(G) = n$.

Lemma 3.1. *Let G be a graph and $f = (V_0 \cup E_0, V_1 \cup E_1, V_2 \cup E_2)$ an MIDF of G. Then there exists an MIDF $f' = (V_0' \cup E_0', V_1' \cup E_1', V_2' \cup E_2')$ of G such that $N(V_1') \subseteq V_0'$, $N_M(V_1') \subseteq E_0'$, $V_2' = \emptyset$ and $\omega(f') \le \omega(f)$.*

Proof. If $N(V_1) \subseteq V_0$, $N_M(V_1) \subseteq E_0$ and $V_2 = \emptyset$, then we are done. Suppose that there exists a vertex $u \in V_2$. Then define $g : V \cup E \to \{0,1,2\}$ by $g(u) = 0, g(e) = 2$ for some $e \in N_M(u)$, $g(x) = f(x)$ otherwise. Clearly, g is an MIDF of G with weight at most $\omega(f)$. Since V_2 is finite, by repeating this process we can assume that V_2 is empty.

Let $u \in V_1$, and suppose there exists $e \in E_1 \cup E_2$ incident to u. If $e \in E_1$, then define $h : V \cup E \to \{0,1,2\}$ by $h(u) = 0, h(e) = 2$ and $h(x) = f(x)$ otherwise. If $e \in E_2$, then define $h : V \cup E \to \{0,1,2\}$ by $h(u) = 0$ and $h(x) = f(x)$ otherwise. Clearly, h is an MIDF of G with weight at most $\omega(f)$. Since V_1 is finite, this process terminates after a finite number of steps. From now on, we assume that $N_M(V_1) \subseteq E_0$ and $V_2 = \emptyset$.

Let $u \in V_1$, and suppose that there exists $w \in V_1 \cap N(u)$. Then let $e = uw$ and define $g : V \cup E \to \{0,1,2\}$ by $g(u) = g(w) = 0$, $g(e) = 2$ and $g(x) = f(x)$ otherwise. Clearly, g is an MIDF of G with weight at most $\omega(f)$. Since V_1 is finite, this process terminates after a finite number of steps. This completes the proof. □

Lemma 3.2. *Let G be a graph. Then there exists a $\gamma_I^\star(G)$-function $f = (V_0 \cup E_0, V_1 \cup E_1, V_2 \cup E_2)$ such that $N(V_1) \subseteq V_0$, $N_M(V_1) \subseteq E_0$, $V_2 = \emptyset$ and any two edges in E_2 are not adjacent.*

Proof. By Lemma 3.1, we can assume that $f = (V_0 \cup E_0, V_1 \cup E_1, V_2 \cup E_2)$ such that $N(V_1) \subseteq V_0$, $N_M(V_1) \subseteq E_0$, $V_2 = \emptyset$. It suffices to prove that any two edges in E_2 are not adjacent.

Suppose to the contrary that there exist two edges $e_1 = uv, e_2 = vw \in E_2$. Without loss of generality, fix the edge e_1. If $N_M(e_2) \setminus N_M[e_1] = \{w\}$, then define the function $g : V \cup E \to \{0,1,2\}$ by $g(w) = 1$, $g(e_2) = 0$ and $g(x) = f(x)$ otherwise. Clearly, g is an MIDF with weight less than $\omega(f)$, a contradiction. Thus, $E(G) \cap (N_M(e_2) \setminus N_M[e_1]) \ne \emptyset$.

Consider an edge $e_3 \in N_M(e_2) \setminus N_M[e_1]$. Since $N_M(e_2) \subseteq (N_M[e_1] \cup N_M[e_3])$, we have $f(e_3) = 0$, otherwise the assumption that $f(e_3) \geq 1$ induces an MIDF with weight less than $\omega(f)$.

Let $e_3 = wx$. Now it follows from our assumption that there exist the following three cases: (i) $N_M(e_3) \setminus N_M[e_2] = \{x\}$ and $f(x) = 1$, (ii) $f(x) = 1$, $N(x) \subseteq V_0$ and $N_M(x) \subseteq E_0$, (iii) $f(x) = 0$ and $\sum_{y \in N_M(x)} f(y) \geq 2$. For cases (i) and (ii), define the function $g : V \cup E \to \{0, 1, 2\}$ by $g(e_3) = 2$, $g(e_2) = 0$, $g(x) = 0$ and $g(z) = f(z)$ otherwise. Clearly, g is an MIDF with weight less than $\omega(f)$, a contradiction.

Now we can assume that for each $e \in N_M(e_2) \setminus N_M[e_1]$, e is incident to $x \in V_0 \setminus \{w\}$ and $\sum_{y \in N_M(x)} f(y) \geq 2$. Now we define $g(w) = 1$, $g(e_2) = 0$ and $g(x) = f(x)$ otherwise. Clearly g is an MIDF with weight less than $\omega(f)$, a contradiction. This completes the proof. □

Proof of Theorem 1.3. Without loss of generality, we may assume that G is connected. By Theorem 1.2, we have $\gamma_R^\star(G) = n$. By Theorem 1.1, we have $\gamma_I^\star(G) \leq \gamma_{r2}^\star(G) \leq \gamma_R^\star(G)$.

Now, we show that $\gamma_I^\star(G) \geq n$. We proceed by induction on $|V(G)|$. Obviously, $\gamma_I^\star(G) = n$ for a graph G of order $n \leq 3$, Let G be a graph of order $n \geq 4$. Suppose that every graph G' of order $n'(< n)$ has $\gamma_I^\star(G') = n'$. By Lemma 3.2, we can assume that there exists a $\gamma_I^\star(G)$-function $f = (V_0 \cup E_0, V_1 \cup E_1, V_2 \cup E_2)$ such that $N_M(V_1) \subseteq E_0$, $V_2 = \emptyset$ and any two edges in E_2 are not adjacent.

If $V_1 \neq \emptyset$, then take a vertex $v \in V_1$ and consider the subgraph $G - \{v\}$. Since the function $f|_{V(G-\{v\}) \cup E(G-\{v\})}$ is an MIDF of $G - \{v\}$, it follows that $\gamma_I^\star(G - \{v\}) \leq \gamma_I^\star(G) - 1$. Applying the induction hypothesis to $G - \{v\}$, we have $\gamma_I^\star(G - \{v\}) = n - 1$. Thus, $n \leq \gamma_I^\star(G)$.

Now assume that $V_1 = \emptyset$. Fix a vertex $v \in V_0$. Then there exists a unique edge $e = vw \in E_2$ incident to v. Note that if $deg(w) \neq 1$, then it follows from $V_1 = \emptyset$ that every edge in $N_M(w) \setminus \{e\}$ is adjacent to one edge of E_2. Since every edge in $(N_M(v) \cup N_M(w)) \setminus \{e\}$ is assigned 0 under f, the function g on $G - \{v\}$ defined by $g(w) = 1$ and $g(x) = f(x)$ otherwise is an MIDF of $G - \{v\}$ with weight $\gamma_I^\star(G) - 1 (\geq \gamma_I^\star(G - \{v\}))$. Applying the induction hypothesis to $G - \{v\}$, we have $n \leq \gamma_I^\star(G)$. This completes the proof. □

4. Proof of Theorems 1.4–1.6

In this section, we give a complete characterization of Roman domination stable middle graphs, 2-rainbow domination stable middle graphs and Italian domination stable middle graphs.

Lemma 4.1. *For a non-empty graph G, $M(G)$ is Roman domination stable if and only if every $\gamma_R^\star(G)$-function has $V_1 = \emptyset$.*

Proof. ($\Rightarrow$): suppose that there exists a $\gamma_R^\star(G)$-function f with $V_1^f \neq \emptyset$. Now we consider $M(G) - \{v\}$ for some $v \in V_1^f$. Define $g : V(M(G) - \{v\}) \to \{0, 1, 2\}$ by $g(x) = f(x)$. Then g is an RDF on $M(G) - \{v\}$ with $\omega(g) = |V(G)| - 1$. Thus, $M(G)$ is not Roman domination stable.
($\Leftarrow$): the result follows from Observations 2.2 and 2.3.

□

Lemma 4.2. *Let G be a connected graph of even order. For non-adjacent vertices $u, v \in V(G)$, if there exists a perfect matching in $G - \{u, v\}$, then $M(G)$ is not Roman domination stable.*

Proof. Let M be a perfect matching of $G - \{u, v\}$. Define $f : V(G) \cup E(G) \to \{0, 1, 2\}$ by $f(u) = f(v) = 1$, $f(x) = 2$ for $x \in M$ and $f(x) = 0$ otherwise. Then it is easy to see that f is a $\gamma_R^\star(G)$-function.

Now we consider $M(G) - \{u\}$. It follows from Observation 2.2 that $M(G)$ is not Roman domination stable. □

Example 4.1. For a complete graph K_{2m}, $M(K_{2m})$ is Roman domination stable.

Proof. Suppose that there exists a $\gamma_R^\star(K_{2m})$-function f with $V_1^f \neq \emptyset$. Then the order of K_{2m} implies that there exist at least two vertices $u, v \in V_1^f$ by Observation 2.1, and there must exist an edge $e \in E_2^f$ to dominate the edge uv. By Observation 2.1(iii), this is a contradiction. It follows from Lemma 4.1 that $M(K_{2m})$ is Roman domination stable. □

Theorem 4.1 (Tutte's theorem)**.** *A graph G has a perfect matching if and only if for every subset U of $V(G)$, the subgraph induced by $V(G) \setminus U$ has at most $|U|$ connected components with an odd number of vertices.*

Example 4.2. For a complete bipartite graph $K_{m,m}$, $M(K_{m,m})$ is Roman domination stable.

Proof. Suppose that there exists a $\gamma_R^\star(G)$-function f with $V_1^f \neq \emptyset$. Note that $|V_1^f| \geq 2$. Let $u, v \in V_1^f$. If two vertices $u, v \in V_1^f$ are adjacent, then there must exist an edge $e \in E_2^f$ to dominate the edge uv. By Observation 2.1(iii), this is a contradiction.

Assume that $u, v \in V_1^f$ are non-adjacent. To dominate all edges in $\{ux, vx \mid x \in N(u)\}$, there must exist a perfect matching in $K_{m,m} - \{u, v\}$. Note that $K_{m,m} - \{u, v\} \cong K_{m-2,m}$. Now we consider $U := V(K_{m,m}) \setminus (\{u, v\} \cup N(u))$ in $K_{m,m} - \{u, v\}$. The subgraph induced by $V(K_{m,m} - \{u, v\}) \setminus U$ has m singletons. By Theorem 4.1, there exists no perfect matching in $K_{m,m} - \{u, v\}$. This implies that such function f does not exist, a contradiction. It follows from Lemma 4.1 that $M(K_{m,m})$ is Roman domination stable. □

We make use of the notation $[m] := \{1, 2, \ldots, m\}$, and for two vertices u, v, we write $u \sim v$ to indicate that u is adjacent to v.

Proof of Theorem 1.4. (⇒): let $M(G)$ be a Roman domination stable graph. It follows from Observation 2.1, Theorem 1.3 and Lemma 4.1 that $|V(G)| = 2m$ and there exists a $\gamma_R^\star(G)$-function f with $|E_2^f| = m$. Without loss of generality, we may assume that $\{v_i w_i \in E_2^f \mid i \in [m]\}$, where $V(G) = \{v_i, w_i \mid i \in [m]\}$. We proceed by proving three claims.

Claim 1. If the subgraph I of G induced by $\{v_i, w_i, v_j, w_j\}$ is connected, then I is either a cycle $C_4 = v_i w_i v_j w_j v_i$ or a complete graph K_4.

Consider the number of edges between $\{v_i, w_i\}$ and $\{v_j, w_j\}$, which is denoted by $e(i, j) \in \{1, 2, 3, 4\}$. We show that $e(i, j)$ is neither 1 nor 3.

Suppose that $e(i, j) = 1$. In the induced subgraph I, denote by x, y two vertices with degree one. Since there exists a perfect matching in $G - \{x, y\}$, it follows from Lemma 4.2 that $M(G)$ is not Roman domination stable, a contradiction.

Suppose that $e(i, j) = 3$. In the induced subgraph I, denote by x, y two vertices with degree two. Since there exists a perfect matching in $G - \{x, y\}$, it follows from Lemma 4.2 that $M(G)$ is not Roman domination stable, a contradiction.

In the case of $e(i, j) = 2$, after relabeling v_j, w_j if necessary, the result follows. The case of $e(i, j) = 4$ is clear. This completes the proof of Claim 1.

Claim 2. G contains a complete bipartite graph $K_{m,m}$ as a subgraph.

Since G is connected, without loss of generality, we can assume that the induced subgraph by $\{v_1, w_1, v_2, w_2\}$ is connected. By Claim 1, the induced subgraph by $\{v_1, w_1, v_2, w_2\}$ is either a cycle $C_4 = v_1 w_1 v_2 w_2 v_1$ or a complete graph K_4. By the connectedness of G, we can assume that the induced subgraph by $\{v_2, w_2, v_3, w_3\}$ is connected. By Claim 1, the induced subgraph by $\{v_2, w_2, v_3, w_3\}$ is either a cycle $C_4 = v_2 w_2 v_3 w_3 v_2$ or a complete graph K_4.

We consider the adjacency relation between $\{v_1, w_1\}$ and $\{v_3, w_3\}$. If $v_1 \not\sim w_3$, then there exists a perfect matching $\{v_2 w_1, v_3 w_2, v_4 w_4, \ldots, v_m w_m\}$ in $G - \{v_1, w_3\}$. By Lemma 4.2, $M(G)$ is not Roman domination stable, a contradiction. Thus, $v_1 \sim w_3$ and by symmetry $w_1 \sim v_3$.

By the connectedness of G, we can assume that the induced subgraph by $\{v_3, w_3, v_4, w_4\}$ is connected. By Claim 1, the induced subgraph by $\{v_3, w_3, v_4, w_4\}$ is either a cycle $C_4 = v_3w_3v_4w_4v_3$ or a complete graph K_4. We consider the adjacency relation between $\{v_1, w_1\}$ and $\{v_4, w_4\}$. If $v_1 \not\sim w_4$, then there exists a perfect matching $\{v_2w_1, v_3w_2, v_4w_3, v_5w_5, \ldots, v_mw_m\}$ in $G - \{v_1, w_4\}$. By Lemma 4.2, $M(G)$ is not Roman domination stable, a contradiction. Thus, $v_1 \sim w_4$ and by symmetry $w_1 \sim v_4$. By applying the same argument for the adjacency relation between $\{v_2, w_2\}$ and $\{v_4, w_4\}$, we obtain $v_2 \sim w_4$ and $w_2 \sim v_4$.

By repeating this process, the proof of Claim 2 is completed.

Claim 3. $G \in \{K_{2m}, K_{m,m} \mid m \geq 1\}$.

Note that it follows from Examples 4.1 and 4.2 that K_{2m} and $K_{m,m}$ are Roman domination stable. Now suppose that G is neither a complete graph nor a complete bipartite graph. Then there exists a pair of vertices which are not adjacent, say v_1 and v_2. It follows from Claim 1 that $w_1 \not\sim w_2$. By the assumption that $G \ncong K_{2m}, K_{m,m}$, there exists at least one edge in the subgraph induced by $\{w_1, \ldots, w_m\}$. Without loss of generality, we can divide our consideration into two case, *i.e.*, $w_2 \sim w_3$ and $w_3 \sim w_4$. In the case of $w_2 \sim w_3$, $\{v_3w_1, w_2w_3, v_4w_4, \ldots, v_mw_m\}$ is a perfect matching in $G - \{v_1, v_2\}$. In the case of $w_3 \sim w_4$, $\{v_3w_1, v_4w_2, w_3w_4, v_5w_5, \ldots, v_mw_m\}$ is a perfect matching in $G - \{v_1, v_2\}$. In either case, it follows from Lemma 4.2 that $M(G)$ is not Roman domination stable. This completes the proof of Claim 3.

($\Leftarrow$): it follows from Examples 4.1 and 4.2.

□

Proof of Theorem 1.5. Theorems 1.1, 1.3 and Observations 2.2, 2.3 imply that $\gamma_I(M(G)) = \gamma_R(M(G)) \geq \gamma_R(M(G) - \{x\}) \geq \gamma_I(M(G) - \{x\})$ for any $x \in V(G) \cup E(G)$. So, if $M(G)$ is Italian domination stable, then $M(G)$ is Roman domination stable. By Theorem 1.4, it suffices to consider whether $M(K_{2m})$ and $M(K_{m,m})$ are Italian domination stable.

In the case of $G = K_{2m}$, let v be an arbitrary vertex of G. Consider $M(G) - \{v\}$. By assigning 1 to each edge in a Hamiltonian cycle of $G - \{v\}$, $M(G) - \{v\}$ has an Italian dominating function with weight $2m - 1$. Thus, $M(K_{2m})$ is not Italian domination stable.

In the case of $G \cong K_{m,m}$, let W and W' be the bipartition, let v be an arbitrary vertex of W', and let f be a $\gamma_I(M(G) - \{v\})$-function. In $M(G) - \{v\}$, we consider the sum of weight of edges incident to v, *i.e.*, $f(N_M(v))$. Suppose that $\gamma_I(M(G)) > \gamma_I(M(G) - \{v\}) = 2m - 1$.

When $f(N_M(v)) \geq 2$, by extending f to an Italian dominating function on $M(G)$, clearly $\gamma_I(M(G) - \{v\}) = \gamma_I(M(G)) = 2m - 1$, a contradiction.

When $f(N_M(v)) = 1$, let $uv \in E_1$. Consider $M(G - \{v\}) + uv$, where $V(M(G - \{v\}) + uv) = V(M(G - \{v\})) \cup \{uv\}$ and $E(M(G - \{v\}) + uv) = E(M(G - \{v\})) \cup \{uv \cdot x \mid x \in N_M[u] \setminus \{uv\}\}$. Then this graph has an Italian dominating function with weight $2m - 1$. By combining such two graphs and the vertex v, we can have a middle graph $M(H)$ with wight $4m - 2$, where H is a graph obtained by identifying a vertex of $K_{m,m-1}$ at each end vertex of a path P_3. But, $|V(H)|$ is $4m - 1$, a contradiction.

When $f(N_M(v)) = 0$, to dominate each edge $uv \in N_M(v)$, it must be $f(N_m[u]) \geq 2$. So, we have $\gamma_I(M(G) - \{v\}) \geq 2m$, a contradiction. This implies $\gamma_I(M(G)) = \gamma_I(M(G) - \{v\})$. □

Proof of Theorem 1.6. Theorems 1.1, 1.3 and Observations 2.2, 2.3 imply that $\gamma_{r2}(M(G)) = \gamma_R(M(G)) \geq \gamma_R(M(G) - \{x\}) \geq \gamma_{r2}(M(G) - \{x\})$ for any $x \in V(G) \cup E(G)$. So, if $M(G)$ is 2-rainbow domination stable, then $M(G)$ is Roman domination stable. By Theorem 1.4, it suffices to consider whether $M(K_{2m})$ and $M(K_{m,m})$ are 2-rainbow domination stable.

In the case of $G = K_{2m}$, let v be an arbitrary vertex of G. Consider $M(G) - \{v\}$. Let f be a $\gamma_{r2}(M(G)) - \{v\})$-function. For each $u \in V(G) \setminus \{v\}$, uv have to be dominated by $\{1, 2\}$. So, $\cup_{x \in N_M[u]} f(x) = \{1, 2\}$. On the other hand, every edge of $G - \{v\}$ is adjacent to exactly two edges in $\{uv \mid u \in V(G) \setminus \{v\}\}$. When $Y := \cup_{u \in V(G) \setminus \{v\}} \cup_{x \in N_M[u]} f(x)$ is considered as a multiset, Y have to contain at least m elements 1 and 2, respectively. Thus, the weight of f is at least $2m$ and so $M(K_{2m})$ is 2-rainbow domination stable.

In the case of $G \cong K_{m,m}$, let W and W' be the bipartition, let v be an arbitrary vertex of W', and let f be a $\gamma_I(M(G)-\{v\})$-function. In $M(G)-\{v\}$, we consider the sum of weight of edges incident to v, *i.e.*, $f(N_M(v))$. Suppose that $\gamma_I(M(G)) > \gamma_I(M(G)-\{v\}) = 2m-1$.

When $f(N_M(v)) \geq 2$, by extending f to a 2-rainbow dominating function on $M(G)$, clearly $\gamma_{r2}(M(G)-\{v\}) = \gamma_{r2}(M(G)) = 2m-1$, a contradiction.

When $f(N_M(v)) = 1$, without loss of generality, assume $uv \in E_{\{1\}}$. Consider $M(G-\{v\})+uv$, where $V(M(G-\{v\})+uv) = V(M(G-\{v\})) \cup \{uv\}$ and $E(M(G-\{v\})+uv) = E(M(G-\{v\})) \cup \{uv \cdot x \mid x \in N_M[u] \setminus \{uv\}\}$. Then this graph has a 2-rainbow dominating function h with weight $2m-1$. Also, this graph can have a 2-rainbow dominating function h' with weight $2m-1$ by switching the assignment of 1, 2 in h. By combining such two graphs and the vertex v, we can have a middle graph $M(H)$ with wight $4m-2$, where H is a graph obtained by identifying a vertex of $K_{m,m-1}$ at each end vertex of a path P_3. $M(H)$ has a 2-rainbow dominating function with weight $4m-2$. But, $|V(H)| = 4m-1$, a contradiction.

When $f(N_M(v)) = 0$, to dominate each edge $uv \in N_M(v)$, it must be $|f(N_m[u])| \geq 2$ for $u \in W$. So, we have $\gamma_{r2}(M(G)-\{v\}) \geq 2m$, a contradiction. This implies $\gamma_{r2}(M(G)) = \gamma_{r2}(M(G)-\{v\})$. □

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