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ON SPECTRAL PROPERTIES OF DIGRAPHS ABOUT MAXIMUM DISTANCE

Leyou Xu[1], Shilin Wang[2] and Bo Zhou[1,*]

Abstract. The maximum distance matrix of a strongly connected digraph is a symmetric matrix whose rows and columns are indexed the vertices, the entries of which correspond to the maximum directed distance between the vertices. In this paper, we determine the digraphs that uniquely minimize the largest eigenvalue of the maximum distance matrix in some classes of strongly connected digraphs, and the n-vertex strongly connected digraphs for which the maximum distance matrices have an eigenvalue with multiplicity $n-1$.

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1. Introduction

Let D be a digraph with vertex $V(D)$ and arc set $E(D)$. Digraphs are understood to be finite loopless digraphs in which two vertices may be joined by at most one edge in each direction. A digraph D is strongly connected if there is a (directed) path from u to v in D for any $u, v \in V(D)$. A digraph D is symmetric if $(v, u) \in E(D)$ whenever $(u, v) \in E(D)$ for any $\{u, v\} \subseteq V(D)$. The length of a (directed) path of D is the number of arcs in the path.

Let D be a strongly connected digraph on n vertices. Let $u, v \in V(D)$. The length of a shortest (directed) path from u to v in D is called the directed distance from u to v in D, denoted by $d_D(u, v)$. Note that it is not possible that $d_D(u, v) = d_D(v, u)$ for all vertices u, v if D is not symmetric. So, in general, directed distance is not a metric though it is the standard distance in digraphs.

Chartrand and Tian [4,6] proposed the maximum distance (max-distance for short) as a metric on strongly connected digraphs. For $u, v \in V(D)$, the max-distance between u and v in D is defined as

$$md_D(u, v) = \max\{d_D(u, v), d_D(v, u)\}.$$

Definitely, this choice ignores the intrinsic asymmetry of digraphs. It was used to describe structural properties related to centers, medians and peripheries [4,6].

The maximum distance matrix (max-distance matrix for short) of D is the $n \times n$ matrix $\mathbf{D}^{\max}(D) = (md_D(u, v))_{u,v\in V(D)}$. As far as we know, there is no references on the spectral properties of the max-distance matrix of a digraph, which we propose to study. As $\mathbf{D}^{\max}(D)$ is symmetric, its eigenvalues are all real. Denote by

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[1] School of Mathematics, South China Normal University, Guangzhou 510631, P.R. China.

[2] Synchrony Bank, Chicago, IL 60606, USA.

*Corresponding author: zhoubo@scnu.edu.cn

$\rho_1(D) \geq \rho_2(D) \geq \cdots \geq \rho_n(D)$ the eigenvalues of $\mathbf{D}^{\max}(D)$, which are called the maximum distance eigenvalues (max-distance eigenvalues for short) of D, and the largest max-distance eigenvalue is said to be the max-distance spectral radius of D.

In this paper, we study the spectral properties of strongly connected digraphs, regarding the max-distance matrix. We characterize the digraphs that uniquely minimize the max-distance spectral radius in some families of strongly connected digraphs by giving some parameters such as order, dichromatic number and connectivity, and we show that an n-vertex strongly connected digraph has a max-distance eigenvalue with multiplicity $n-1$ if and only if it is a complete digraph or an all-kings tournament.

2. Preliminaries

For a digraph D with $\emptyset \neq S \subseteq V(D)$, denote by $D[S]$ the subdigraph of D induced by S. If $S \subset V(D)$, let $D - S := D[V(D) - S]$. For $S' \subseteq E(D)$, $D - S'$ denotes the subdigraph obtained from D by deleting the arcs in S'. Particularly, we write $D - v$ for $D - \{v\}$ if $S = \{v\}$ and $D - e$ for $D - \{e\}$ if $S' = \{e\}$. For $S^+ \subseteq V(D) \times V(D)$ with $S^+ \cap E(D) = \emptyset$, $D + S^+$ denotes the digraph obtained from D by adding the arcs in S^+, and we write $D + f$ for $D + \{f\}$ if $S^+ = \{f\}$.

A digraph is complete if both (u, v) and (v, u) are arcs for any pair u, v of distinct vertices. The complete digraph on n vertices is denoted by $\overleftrightarrow{K_n}$.

A digraph is acyclic if it has no (directed) cycle. For a digraph D, subset S of $V(D)$ is acyclic if its induced subdigraph $D[S]$ is acyclic. The dichromatic number of a digraph D is defined as the minimum number k that is required to partition $V(D)$ into k acyclic sets [13].

For a digraph D that is not complete, a separating set is a subset $S \subset V(D)$ such that $D - S$ is not strong, and the (vertex-strong) connectivity of D, denoted by $\kappa(D)$, is defined to be $\kappa(D) = \min\{|S| : S \text{ is a separating set of } D\}$. Define $\kappa(\overleftrightarrow{K_n}) = n - 1$. Similarly, an arc cut of a digraph is a subset $S' \subset E(D)$ such that $D - S'$ is not strong. The arc(-strong) connectivity of D, $\kappa'(D)$, is defined to be $\kappa'(D) = \min\{|S'| : S' \text{ is an arc cut of } D\}$. See, *e.g.* [2, 15].

A tournament is a digraph in which, for every pair of distinct vertices, there is exactly one arc connecting them.

A vertex u of a tournament T is a king if $d_T(u, v) = 1, 2$ for any $v \in V(T) \setminus \{u\}$. A tournament in which every vertex is a king is called an all-kings tournament. Maurer [10] showed that for all integers $n \geq 2$ there exists an all-kings tournament of order n if and only if $n \neq 2, 4$.

Let $\mathbf{M}$ be an irreducible nonnegative matrix. The spectral radius of $\mathbf{M}$ is the maximum modulus of all its eigenvalues, which is denoted by $\rho(\mathbf{M})$. By Perron–Frobenius Theorem (see [11], Thm. 4.1 in p. 11 and Thm. 4.3 in p. 14), $\rho(\mathbf{M})$ is a simple eigenvalue of $\mathbf{M}$ and there is precisely one positive unit eigenvector associated with $\rho(\mathbf{M})$, which is called the Perron vector of $\mathbf{M}$. Note that $\rho_1(D) = \rho(\mathbf{D}^{\max}(D))$ for any strongly connected digraph D. As $\mathbf{D}^{\max}(D)$ is an irreducible nonnegative matrix, we have $\rho_1(D) > \rho_2(D)$.

Denote by $\mathbf{I}_n$ and $\mathbf{J}_n$ the identity matrix and the all ones matrix of order n, respectively.

For nonnegative matrices $\mathbf{A}$ and $\mathbf{C}$ of the same size, if $\mathbf{A}$ is entrywise bigger than or equal to $\mathbf{C}$ and $\mathbf{A} \neq \mathbf{C}$, then we write $\mathbf{A} \succ \mathbf{C}$.

The following lemma is a restatement of Corollary 2.2 in p. 38 from [11].

Lemma 2.1. *For an irreducible nonnegative matrix $\mathbf{A}$ and a nonnegative matrix $\mathbf{C}$, $\mathbf{A} \succ \mathbf{C}$ implies that $\rho(\mathbf{A}) > \rho(\mathbf{C})$.*

Lemma 2.2. *Let u, v be two distinct vertices of a strongly connected digraph D. Let $\mathbf{x}$ be the Perron vector of $\mathbf{D}^{\max}(D)$. If there is an automorphism ϕ such that $\phi(u) = v$, then $x_u = x_v$.*

Proof. Let $\mathbf{P} = (p_{uv})_{u,v \in V(D)}$ be the permutation matrix of order n, where $p_{uv} = 1$ if and only if $\phi(v) = u$. Obviously, $\mathbf{P}^\top \mathbf{P} = \mathbf{I}_n$. Then $\mathbf{D}^{\max}(D) = \mathbf{P}^\top \mathbf{D}^{\max}(D) \mathbf{P}$. So

$$\rho_1(D) = \mathbf{x}^\top \mathbf{D}^{\max}(D) \mathbf{x} = \mathbf{x}^\top \mathbf{P}^\top \mathbf{D}^{\max}(D) \mathbf{P} \mathbf{x} = (\mathbf{P}\mathbf{x})^\top \mathbf{D}^{\max}(D)(\mathbf{P}\mathbf{x}),$$

implying that $\mathbf{Px}$ is also the Perron vector of $\mathbf{D}^{\max}(D)$, and by Perron–Frobenius Theorem, $\mathbf{Px} = \mathbf{x}$. $\square$

For a symmetric matrix $\mathbf{N}$ of order n, denote by $\rho_1(\mathbf{N}) \geq \cdots \geq \rho_n(\mathbf{N})$ the n eigenvalues of $\mathbf{N}$. Raylegh's principle (or relation) provides a tight lower bound for $\rho_1(\mathbf{N})$: $\rho_1(\mathbf{N}) \geq \mathbf{x}^\top \mathbf{N}\mathbf{x}$ for any n-dimensional unit vector $\mathbf{x}$, and equality holds if and only if $\mathbf{x}$ is an eigenvector of $\mathbf{N}$ associated with $\rho_1(\mathbf{N})$.

Lemma 2.3 ([8], Thm. 4.3.28)**.** *If* $\mathbf{N}$ *is a symmetric matrix of order* n *and* $\mathbf{B}$ *is its principal submatrix of order* p*, then* $\rho_{n-p+i}(\mathbf{N}) \leq \rho_i(\mathbf{B}) \leq \rho_i(\mathbf{N})$ *for* $i = 1, \ldots, p$.

For an arc $e = (u, v)$ of a digraph D, the reverse of e, denoted by e^-, is the arc (v, u).

3. Digraphs minimizing the max-distance spectral radius

In this section, we characterize the digraphs with minimum max-distance spectral radius over some classes of strongly connected digraphs.

Lemma 3.1. *Let* $V(\overleftrightarrow{K_n}) = \{v_1, \ldots, v_n\}$, $e = (v_1, v_2)$, $f = (v_3, v_4)$ *and* $h = (v_1, v_3)$*. Let* $D_1 = \overleftrightarrow{K_n} - e - f$ *and* $D_2 = \overleftrightarrow{K_n} - e - h$*. Then* $\rho_1(D_2) > \rho_1(D_1)$.

Proof. Let $\rho = \rho_1(D_1)$ and $\mathbf{x}$ be the Perron vector of $\mathbf{D}^{\max}(D_1)$. By Lemma 2.2, $x_{v_1} = x_{v_3}$, $x_{v_2} = x_{v_4}$ and $x_u = x_v$ for $u, v \in V(D_1) \setminus \{v_1, v_2, v_3, v_4\}$. Denote by x_1, x_2 and x_3 the entry of $\mathbf{x}$ corresponding to vertices in $\{v_1, v_3\}$, $\{v_2, v_4\}$ and $V(D_1) \setminus \{v_1, v_2, v_3, v_4\}$. From $\rho\mathbf{x} = \mathbf{D}^{\max}(D_1)\mathbf{x}$, we have

$$\rho x_1 = 3x_2 + x_1 + (n-4)x_3$$

and

$$\rho x_2 = 3x_1 + x_2 + (n-4)x_3.$$

So $(\rho + 2)(x_1 - x_2) = 0$, *i.e.*, $x_1 = x_2$. Thus

$$\rho x_1 = 4x_1 + (n-4)x_3. \tag{3.1}$$

Evidently, $D_2 = D_1 - (v_1, v_3) + (v_3, v_4)$. Note that $md_{D_1}(v_1, v_3) = md_{D_2}(v_1, v_3) - 1$, $md_{D_1}(v_3, v_4) = md_{D_2}(v_3, v_4) + 1$ and $md_{D_1}(u, v) = md_{D_2}(u, v)$ for every $(u, v) \neq (v_1, v_3), (v_3, v_4)$. As $\mathbf{D}^{\max}(D_1)$ is symmetric, we have

$$\rho_1(D_1) = \mathbf{x}^\top \mathbf{D}^{\max}(D_1)\mathbf{x} = 2 \sum_{\{u,v\} \subset V(D_1)} md_{D_1}(u, v) x_u x_v.$$

By Rayleigh's principle,

$$\rho_1(D_2) \geq 2 \sum_{\{u,v\} \subset V(D_2)} md_{D_2}(u, v) x_u x_v.$$

So we have

$$\begin{aligned}
\rho_1(D_2) - \rho_1(D_1) &\geq 2(md_{D_2}(v_1, v_3) - md_{D_1}(v_1, v_3)) x_{v_1} x_{v_3} \\
&\quad + 2(md_{D_2}(v_3, v_4) - md_{D_1}(v_3, v_4)) x_{v_3} x_{v_4} \\
&= 2x_{v_1} x_{v_3} - 2x_{v_3} x_{v_4} \\
&= 2x_1^2 - 2x_1^2 \\
&= 0.
\end{aligned}$$

Then $\rho_1(D_2) \geq \rho_1(D_1)$. Suppose that $\rho_1(D_2) = \rho_1(D_1)$. Then $\mathbf{x}$ is also the Perron vector of $\mathbf{D}^{\max}(D_2)$. From $\rho\mathbf{x} = \mathbf{D}^{\max}(D_2)\mathbf{x}$, we have

$$\rho x_1 = 5x_1 + (n-4)x_3. \tag{3.2}$$

Now it follows from (3.1) and (3.2) that $x_1 = 0$, which is a contradiction. So $\rho_1(D_2) > \rho_1(D_1)$. $\square$

Theorem 3.2. *Let D be a strongly connected digraph on n vertices, where $n \geq 2$.*

(i) $\rho_1(D) \geq n-1$ *with equality if and only if* $D \cong \overleftrightarrow{K_n}$.
(ii) *If* $n \geq 3$ *and* $D \ncong \overleftrightarrow{K_n}$, *then for some* $e \in E(\overleftrightarrow{K_n})$, $\rho_1(D) \geq \rho_1(\overleftrightarrow{K_n}-e)$ *with equality if and only if* $D \cong \overleftrightarrow{K_n}-e$ *or* $D \cong \overleftrightarrow{K_n}-e-e^-$.
(iii) *If* $n \geq 4$ *and* $D \ncong \overleftrightarrow{K_n}-e$, $\overleftrightarrow{K_n}-e-e^-$ *for some* $e \in E(\overleftrightarrow{K_n})$, *then* $\rho_1(D) \geq \rho_1(\overleftrightarrow{K_n}-e-f)$ *with equality if and only if* $D \cong \overleftrightarrow{K_n}-e-f$ *or* $D \cong \overleftrightarrow{K_n}-e-e^--f$ *or* $D \cong \overleftrightarrow{K_n}-e-e^--f-f^-$, *where* f *is an arc of* $\overleftrightarrow{K_n}-e$ *with no common vertex with* e.

Proof. Assume that $V(D) = V(\overleftrightarrow{K_n}) = \{v_1, \ldots, v_n\}$.

First, we prove (i). Note that $\mathbf{D}^{\max}(\overleftrightarrow{K_n}) = \mathbf{J}_n - \mathbf{I}_n$. Suppose that $D \ncong \overleftrightarrow{K_n}$. Then there exist $u, v \in V(D)$ with $u \neq v$ such that $(u,v) \notin E(D)$, so $md_D(u,v) \geq d_D(u,v) \geq 2$, implying that $\mathbf{D}^{\max}(D) \succ \mathbf{J}_n - \mathbf{I}_n = \mathbf{D}^{\max}(\overleftrightarrow{K_n})$. By Lemma 2.1, one gets $\rho_1(D) > \rho_1(\overleftrightarrow{K_n})$. Now (i) follows by noting that $\rho(\overleftrightarrow{K_n}) = \rho(\mathbf{J}_n - \mathbf{I}_n) = n-1$.

Next, we prove (ii). Suppose that D is a digraph with minimum max-distance spectral radius over all strongly connected digraphs on n vertices except $\overleftrightarrow{K_n}$. Then $n \geq 3$. As $D \ncong \overleftrightarrow{K_n}$, there are two vertices, say v_1 and v_2, such that $e = (u,v) \notin E(D)$. Note that $\mathbf{D}^{\max}(\overleftrightarrow{K_n}-e) = \mathbf{D}^{\max}(\overleftrightarrow{K_n}-e-e^-) = \mathbf{J}_n - \mathbf{I}_n + \mathbf{X}$, where $\mathbf{X}$ is an $n \times n$ matrix with precisely two nonzero entries in $(1,2)$ and $(2,1)$ positions, which are equal to 1.

Suppose that $D \ncong \overleftrightarrow{K_n}-e$ and $D \ncong \overleftrightarrow{K_n}-e-e^-$. Then $n \geq 4$ and $(w,z) \notin E(D)$ for some $\{w,z\} \subset V(D)$ with $\{w,z\} \neq \{v_1,v_2\}$. So $md_D(w,z) \geq d_D(w,z) \geq 2 > 1$. As $e \notin E(D)$, we have $md_D(v_1,v_2) \geq d_D(v_1,v_2) \geq 2$. Thus, $\mathbf{D}^{\max}(D) \succ \mathbf{J}_n - \mathbf{I}_n + \mathbf{X}$. By Lemma 2.1, $\rho_1(D) > \rho_1(\overleftrightarrow{K_n}-e)$, a contradiction. It thus follows that $D \cong \overleftrightarrow{K_n}-e$ or $D \cong \overleftrightarrow{K_n}-e-e^-$. Now (ii) follows.

In the following, we prove (iii). Suppose that D is a digraph with minimum max-distance spectral radius over all strongly connected digraphs on n vertices except $\overleftrightarrow{K_n}$, $\overleftrightarrow{K_n}-e$ and $\overleftrightarrow{K_n}-e-e^-$. Then $n \geq 4$. Assume that $e = (v_1,v_2)$. Note that $D \ncong \overleftrightarrow{K_n}$, $\overleftrightarrow{K_n}-e$, $\overleftrightarrow{K_n}-e-e^-$. So $(w,z) \notin E(D)$ for some $\{w,z\} \subset V(D)$ with $\{w,z\} \neq \{v_1,v_2\}$. There are two cases.

Suppose first that $\{w,z\} \subseteq V(D) \setminus \{v_1,v_2\}$. Assume that $w = v_3$, $z = v_4$ and $f = (v_3,v_4)$. Let

$$
\begin{aligned}
D_1 &= \overleftrightarrow{K_n}-e-f,\\
D_1' &= \overleftrightarrow{K_n}-e-e^--f,\\
D_1'' &= \overleftrightarrow{K_n}-e-e^--f-f^-.
\end{aligned}
$$

It is easy to see that $\mathbf{D}^{\max}(D_1) = \mathbf{D}^{\max}(D_1') = \mathbf{D}^{\max}(D_1'')$. If $D \ncong D_1, D_1', D_1''$, then $g = (x,y) \notin E(D)$ for some $x, y \in V(D)$ with $g \neq e, f, e^-, f^-$, so $\{x,y\} \neq \{v_1,v_2\}, \{v_3,v_4\}$ and $md_D(x,y) \geq 2 > 1 = md_{D_1}(x,y)$, implies that $\mathbf{D}^{\max}(D) \succ \mathbf{D}^{\max}(D_1)$, so we have by Lemma 2.1 that $\rho_1(D) > \rho_1(D_1)$. Thus $\rho_1(D) \geq \rho_1(D_1)$ with equality if and only if $D \cong D_1, D_1', D_1''$.

Suppose next that $|\{w,z\} \cap \{v_1,v_2\}| = 1$. Assume that $w = v_1$, $z = v_3$ and $h = (v_1,v_3)$. Let

$$
\begin{aligned}
D_2 &= \overleftrightarrow{K_n}-e-h,\\
D_2' &= \overleftrightarrow{K_n}-e-e^--h,\\
D_2'' &= \overleftrightarrow{K_n}-e-e^--h-h^-.
\end{aligned}
$$

It is easy to see that $\mathbf{D}^{\max}(D_2) = \mathbf{D}^{\max}(D_2') = \mathbf{D}^{\max}(D_2'')$. As above, if $D \ncong D_2, D_2', D_2''$, then we have by Lemma 2.1 that $\rho_1(D) > \rho_1(D_2)$. So, by Lemma 3.1, we have $\rho_1(D) \geq \rho_1(D_2) > \rho_1(D_1)$.

Combining the above two cases, we have $\rho_1(D) \geq \rho_1(D_1)$ with equality if and only if $D \cong D_1, D_1', D_1''$. This proves (iii). $\square$

The vertices of an acyclic digraph D of order n can be labeled as $v_1, \dots, v_n$ so that if (v_i, v_j) is an arc of D, then we have $1 \le i < j \le n$, see Proposition 2.1.3 in [2]. If an acyclic digraph is a tournament, then it is transitive.

Let D be a strongly connected digraph on n vertices with dichromatic number k with $2 \le k \le n$. Then $V(D)$ can be partitioned into k disjoint acyclic subsets $V_1, \dots, V_k$. If $(u,v), (v,u) \in E(D)$ for any $u \in V_i$ and $v \in V_j$ with $1 \le i < j \le k$, then we say D is proper. If $||V_i| - |V_j|| \le 1$ for all i and j with $1 \le i < j \le k$, then we say the acyclic subsets are balanced.

For a positive integer k, let $[k] = \{1, \dots, k\}$.

Theorem 3.3. *For integers k and n with $2 \le k \le n$ and $n \ge 3$, a digraph D minimizes the max-distance spectral radius over all strongly connected digraphs on n vertices with dichromatic number k if and only if D is proper and the acyclic sets of D are balanced.*

Proof. Suppose that D is a strongly connected digraph on n vertices with dichromatic number k that minimizes the max-distance spectral radius.

As the dichromatic number of D is k, $V(D)$ can be partitioned into k disjoint acyclic subsets $V_1, \dots, V_k$. Let $n_i = |V_i|$ for $i = 1, \dots, k$. For $i = 1, \dots, k$, as $D[V_i]$ is acyclic, the vertices in V_i can be labeled as $v_{i,1}, \dots, v_{i,n_i}$ so that each arc of $D[V_i]$ is of the form $(v_{i,p}, v_{i,q})$ for some p and q with $1 \le p < q \le n_i$.

Claim 1. D is proper.

Suppose to the contrary that D is not proper. Then there exist vertices $u \in V_i$ and $v \in V_j$ with $1 \le i < j \le k$ so that $(u,v) \notin E(D)$ or $(v,u) \notin E(D)$. Adding all possible arcs between the acyclic subsets, we form a proper digraph D' on n vertices with dichromatic number k. For any two distinct vertices w and z of D, it is easy to see that $md_{D'}(w,z) = 2$ if they lie in the same acyclic subset and $md_{D'}(w,z) = 1$ otherwise. As $(u,v) \notin E(D)$ or $(v,u) \notin E(D)$, one gets $d_D(u,v) \ge 2$ or $d_D(v,u) \ge 2$, so $md_D(u,v) \ge 2 > 1 = md_{D'}(u,v)$. Thus $\mathbf{D}^{\max}(D) \succ \mathbf{D}^{\max}(D')$. By Lemma 2.1, $\rho_1(D) > \rho_1(D')$, contradicting the choice of D. This confirms Claim 1.

By Claim 1, D is proper. Assume that $n_1 \ge \cdots \ge n_k$.

Claim 2. $n_1 - n_k \le 1$.

Suppose that this is not true. Then $n_1 - n_k \ge 2$. Denote by D_0 the digraph obtained from D by deleting all possible arcs with both end vertices in the same acyclic subset. It is possible that $D = D_0$. Then $\mathbf{D}^{\max}(D) = \mathbf{D}^{\max}(D_0)$.

Let

$$D_0' = D_0 - \{(v_{1,n_1}, v_{k,t}), (v_{k,t}, v_{1,n_1}) : t \in [n_k]\} - \{(v_{1,n_1}, v_{1,t}), (v_{1,t}, v_{1,n_1}) : t \in [n_1 - 1]\}.$$

Then $V(D_0')$ can be partitioned into k disjoint acyclic subsets $V_1', \dots, V_k'$, where $V_t' = V_t$ if $t = 2, \dots, k-1$, $V_1' = V_1 \setminus \{v_{1,n_1}\}$ and $V_k' = V_k \cup \{v_{1,n_1}\}$. Rename v_{1,n_1} be $v_{k,n_k'}$. It is evident that D_0' is a strongly connected digraph on n vertices with dichromatic number k. Let $n_i' = |V_i'|$ for $i = 1, \dots, k$.

Let $\rho = \rho_1(D_0')$ and let $\mathbf{x}$ be the Perron vector of $\mathbf{D}^{\max}(D_0')$. Denote by $x_{i,j}$ the entry of $\mathbf{x}$ corresponding to $v_{i,j}$ for $1 \le i \le k$ and $1 \le j \le n_i'$. By Lemma 2.2, $x_{i,1} = \cdots = x_{i,n_i'}$ for $i \in [k]$. Hence, from now on, we denote by x_i the entry of $\mathbf{x}$ corresponding to the vertex in V_i' for $i \in [k]$. Note that

$$\rho x_1 = 2(n_1' - 1)x_1 + \sum_{j=2}^{k} n_j' x_j$$

and

$$\rho x_k = 2(n_k' - 1)x_k + \sum_{j=1}^{k-1} n_j' x_j.$$

So

$$\rho(x_1 - x_k) = (n'_k - 2)(x_1 - x_k) + (n'_1 - n'_k)x_1.$$

Let $\mathbf{1}_n$ be the all ones column vector. By Rayleigh's principle, we have

$$\begin{aligned}\rho &\geq \frac{1}{\mathbf{1}_n^\top \mathbf{1}_n}\mathbf{1}_n^\top \mathbf{D}^{\max}(D'_0)\mathbf{1}_n \\ &= \frac{1}{n}\sum_{j=1}^{k} n'_j(n + n'_j - 3) \\ &= n + \frac{1}{n}\sum_{j=1}^{k} n'^2_j - 3 \\ &> n'_k - 2.\end{aligned}$$

It thus follows that $x_1 \geq x_k$. So,

$$\begin{aligned}\rho_1(D_0) - \rho_1(D'_0) &\geq \mathbf{x}^\top \mathbf{D}^{\max}(D_0)\mathbf{x} - \mathbf{x}^\top \mathbf{D}^{\max}(D'_0)\mathbf{x} \\ &= 2(n_1 - 1)x_1x_k - 2n_kx_k^2 \\ &\geq 2(n_1 - 1 - n_k)x_k^2 \\ &> 0,\end{aligned}$$

implying that $\rho_1(D) = \rho_1(D_0) > \rho_1(D'_0)$, which is a contradiction. This completes the proof of Claim 2.

By Claim 2, $n_1 - n_k = 0, 1$. That is, the acyclic sets of D are balanced. □

The Turán graph $T_k(n)$ is the complete k-partite graph on n vertices such that the k partite sets are are as equal in cardinality as possible. Denote by $t_k(n)$ the number of edges of $T_k(n)$. It is known (by a direct calculation) that

$$t_k(n) = \binom{n}{2} - \frac{1}{2}\left\lfloor\frac{n}{k}\right\rfloor\left(2n - k\left\lfloor\frac{n}{k}\right\rfloor - k\right).$$

Let $\overleftrightarrow{T_k(n)}$ the digraph obtained from $T_k(n)$ by replacing edge uv with two arcs (u, v) and (v, u). Then $|E(\overleftrightarrow{T_k(n)})| = 2|E(T_k(n))| = 2t_k(n)$.

Corollary 3.4. *For integers k and n with $2 \leq k \leq n$ and $n \geq 3$, let D be a strongly connected digraph on n vertices with dichromatic number k that minimizes the max-distance spectral radius. Then*

$$2t_k(n) \leq |E(D)| \leq t_k(n) + \binom{n}{2}$$

with left equality if and only if $D \cong \overleftrightarrow{T_k(n)}$ and with right equality if and only if D is proper, the acyclic sets of D are balanced and the subdigraph induced by any acyclic set is a transitive tournament.

Proof. By Theorem 3.3, $\overleftrightarrow{T_k(n)}$ is a spanning subdigraph of D. So

$$|E(D)| \geq |E(\overleftrightarrow{T_k(n)})| = 2t_k(n)$$

with equality if and only if $D \cong \overleftrightarrow{T_k(n)}$

Note that $V(D)$ can be partitioned into k disjoint acyclic subsets $V_1, \ldots, V_k$. By Theorem 3.3, adding an arc in $D[V_i]$ does not change the max-distance spectral radius for each $i = 1, \ldots, k$. So, if $|E(D)|$ is maximum, the vertices of V_i can be labeled as $v_{i,1}, \dots, v_{i,n_i}$ so that there are all the possible arcs $(v_{i,j}, v_{i,\ell})$ with $1 \le j < \ell \le n_i$. In this case, $D[V_i]$ is a transitive tournament. So we have by Theorem 3.3 that

$$|E(D)| \le |E(\overleftrightarrow{T_k(n)})| + \sum_{i=1}^{k} \binom{|V_i|}{2} = 2t_k(n) + \sum_{i=1}^{k} \binom{|V_i|}{2} = t_k(n) + \binom{n}{2}$$

with equality if and only if D is proper, the acyclic sets of D are balanced and the subdigraph induced by any acyclic set is a transitive tournament. $\square$

A strongly connected component of a digraph is a maximal induced subdigraph of the digraph that is strongly connected. If D is a digraph with t strongly connected components, then these strongly connected components can be labeled as $D_1, \dots, D_t$ such that there is no arc from D_j to D_i unless $j < i$, see p. 17 in [2]. Such an ordering is called an acyclic ordering of the strongly connected components of D. A strongly connected component of D that has no entering (leaving, respectively) arcs is called an initial (terminal, respectively) strongly connected component of D.

If D is a strongly connected digraph on n vertices with $\kappa(D) = n-1$, then $D \cong \overleftrightarrow{K_n}$.

For positive integers m, k and n with $1 \le m \le n-k-1$, let $\mathcal{D}_{n,k,m}$ be the set of digraphs for which the vertex set can be partitioned into $V_0 \cup V_1 \cup V_2$ so that $|V_0| = k$, $|V_1| = m$, $|V_2| = n-k-m$, $V_0 \cup V_1$ and $V_0 \cup V_2$ induce $\overleftrightarrow{K_{k+m}}$ and $\overleftrightarrow{K_{n-m}}$, respectively, and there is no arc (v, u) for any $u \in V_1$ and $v \in V_2$.

Theorem 3.5. *For $1 \le k \le n-2$, a digraph D minimizes the max-distance spectral radius over all strongly connected digraphs on n vertices with connectivity k if and only if $D \in \mathcal{D}_{n,k,m}$ with $m = 1, n-k-1$.*

Proof. Suppose that D is a strongly connected digraph on n vertices with connectivity k that minimizes the max-distance spectral radius.

Denote by V_0 a separating set of D of cardinality k. Then $D - V_0$ is not strongly connected. Let D_1 be the initial strongly connected component in an acyclic ordering of the strongly connected components of $D - V_0$. Let $V_1 = V(D_1)$ and $V_2 = V(D) \setminus (V_0 \cup V_1)$.

We claim that $D[V_0 \cup V_i]$ is complete for each $i = 1, 2$. Suppose that this is not true. Assume that $D[V_0 \cup V_1]$ is not complete. By adding all possible arcs between vertices in $V_0 \cup V_1$ and all possible arcs between vertices in $V_0 \cup V_2$, we form a digraph D' on n vertices with connectivity k. As $D[V_0 \cup V_1]$ is not complete, there are two vertices $u, v \in V_0 \cup V_1$ so that $(u, v) \notin E(D)$, so $md_D(u, v) \ge 2 > 1 = md_{D'}(u, v)$. Thus $\mathbf{D}^{\max}(D) \succ \mathbf{D}^{\max}(D')$. By Lemma 2.1, $\rho_1(D) > \rho_1(D')$, a contradiction. So $D[V_0 \cup V_i]$ is indeed complete for each $i = 1, 2$.

Let $m = |V_1|$. Then $D \in \mathcal{D}_{n,k,m}$. Note that all digraphs in $\mathcal{D}_{n,k,m}$ have equal max-distance matrix.

Case 1. $1 \le m \le \lfloor \frac{n-k}{2} \rfloor$.

Let $D' \in \mathcal{D}_{n,k,1}$ so that $V_0' \cup V_1' \cup V_2'$ is a subset partition of $V(D')$ as above with $V_1' = \{w\}$ and there is an arc from w to any vertex in V_2'. Let $\rho = \rho_1(D')$ and $\mathbf{x}$ be the Perron vector of $\mathbf{D}^{\max}(D')$. By Lemma 2.2, $x_u = x_v$ for any $u, v \in V_i'$ for $i = 0, 2$. Denote by x_i the common entry of $\mathbf{x}$ at any vertex in V_i' for $i = 0, 1, 2$. From $\rho\mathbf{x} = \mathbf{D}^{\max}(D')\mathbf{x}$, we have

$$\rho x_1 = kx_0 + 2(n-k-1)x_2$$

and

$$\rho x_2 = kx_0 + 2x_1 + (n-k-2)x_2.$$

Then

$$(\rho + 2)(x_1 - x_2) = (n-k-2)x_2 > 0$$

and

$$2x_1 = \rho(x_2 - x_1) + (n-k)x_2 < (n-k)x_2,$$

so $x_1 > x_2$ and $2x_1 < (n-k)x_2$.

Suppose that $2 \le m \le \lfloor \frac{n-k}{2} \rfloor$. Let

$$D^* = D' - \{(v,u) : v \in V_2' \setminus U, u \in U\} + \{(u,w) : u \in U\},$$

where U is a subset of V_2' with $m-1$ vertices. Then $D^* \in \mathcal{D}_{n,k,m}$ and $\mathbf{D}^{\max}(D^*) = \mathbf{D}^{\max}(D)$. So by Rayleigh's principle, one gets

$$\begin{aligned}\rho_1(D^*) - \rho_1(D') &\ge \mathbf{x}^\top(\mathbf{D}^{\max}(D^*) - \mathbf{D}^{\max}(D'))\mathbf{x}\\ &= 2(m-1)(n-k-m)x_2^2 - 2(m-1)x_1x_2\\ &= 2(m-1)x_2((n-k-m)x_2 - x_1)\\ &\ge 2(m-1)x_2\left(\frac{n-k}{2}x_2 - x_1\right)\\ &> 0,\end{aligned}$$

implying that $\rho_1(D) = \rho_1(D^*) > \rho_1(D')$, a contradiction. It follows that $m = 1$.

Case 2. $\lfloor \frac{n-k}{2} \rfloor < m \le n-k-1$.

In this case, we write D as $D_{k,m}$. As $\mathbf{D}^{\max}(D_{k,m})$ and $\mathbf{D}^{\max}(D_{k,n-k-m})$ are permutation similar, $\rho_1(D) = \rho_1(D_{k,n-k-m})$. As $1 \le n-k-m \le \lfloor \frac{n-k}{2} \rfloor$, we have by the conclusion of Case 1 that $n-k-m = 1$. That is, $m = n-k-1$.

The result follows by combining Cases 1 and 2. □

Corollary 3.6. *For $1 \le r \le n-2$, a digraph D minimizes the max-distance spectral radius among strongly connected digraphs on n vertices with arc connectivity r if and only if $D \in \mathcal{D}_{n,r,m}$ with $m = 1, n-r-1$.*

Proof. Suppose that D is a strongly connected digraph on n vertices with arc connectivity r that minimizes the max-distance spectral radius. Let k be the connectivity of D. From Corollary 5.4.3 in [2], we have $k \le r$.

We claim that $k = r$. Suppose that $k < r$. Let D' be the digraph in $\mathcal{D}_{n,k,1}$ so that $V_0 \cup V_1 \cup V_2$ is a partition of $V(D')$ as in the proof of Theorem 3.5 with $V_1 = \{u\}$, and there is an arc from u to any vertex in V_2. By Theorem 3.5, $\rho_1(D) \ge \rho_1(D')$. Let $W \subset V_2$ with $|W| = r-k$. Let $D'' = D' + \{(w,u) : w \in W\}$. Then $D'' \in \mathcal{D}_{n,r,1}$, and the arc connectivity of D'' is r. By Lemma 2.1, we have $\rho_1(D') > \rho_1(D'')$. Therefore, we have $\rho_1(D) > \rho_1(D'')$, which is a contradiction. So $k = r$ and the result follows from Theorem 3.5. □

4. Properties of other max-distance eigenvalues

In this section, we consider the max-distance eigenvalues different from the max-distance spectral radius.

Lemma 4.1. *Let D be a strongly connected digraph on n vertices, where $n \ge 2$. Then*

$$\rho_2(D) \ge -\min\{md_D(u,v) : \{u,v\} \subseteq V(D)\}$$

and

$$\rho_n(D) \le -\max\{md_D(u,v) : \{u,v\} \subseteq V(D)\}.$$

Proof. Let $u,v \in V(D)$ with $u \ne v$. It is evident that $\mathbf{B} = \begin{pmatrix} 0 & md_D(u,v) \\ md_D(u,v) & 0 \end{pmatrix}$ is a principal submatrix of $\mathbf{D}^{\max}(D)$. By Lemma 2.3, we have

$$\rho_n(D) \le \rho_2(\mathbf{B}) = -md_D(u,v) \le \rho_2(D)$$

for any $u,v \in V(D)$ with $u \ne v$. □

Corollary 4.2. *Let D be a strongly connected digraph on n vertices. Then $\rho_n(D) \leq -1$ with equality if and only if $D \cong \overleftrightarrow{K_n}$.*

Proof. By Lemma 4.1, $\rho_n(D) \leq -\max\{md_D(u,v) : \{u,v\} \subseteq V(D)\} \leq -1$ with equality if and only if $md_D(u,v) = 1$ for any $\{u,v\} \subseteq V(D)$, that is, $D \cong \overleftrightarrow{K_n}$. □

For a digraph D with vertex $u \in V(D)$, let $N_D^{+i}(u) = \{v \in V(D) : d_D(u,v) = i\}$ and $N_D^{-i}(u) = \{v \in V(D) : d_D(v,u) = i\}$ for $i = 1, \dots, n-1$. It is evident that $N_D^{+i}(u) = \emptyset$ ($N_D^{-i}(u) = \emptyset$, respectively) if $i > \max\{d_D(u,v) : v \in V(D)\}$ ($i > \max\{d_D(v,u) : v \in V(D)\}$, respectively). It is easy to see that a tournament T of order n with $|N_T^{+1}(u)| = |N_T^{-1}(u)| = \frac{n-1}{2}$ for every $u \in V(T)$ is an all-kings tournament.

Theorem 4.3. *Let D be a strongly connected digraph on n vertices, where $n \geq 2$. Then D has a max-distance eigenvalue with multiplicity $n-1$ if and only if $D \cong \overleftrightarrow{K_n}$ or $n \neq 2, 4$ and D is an all-kings tournament.*

Proof. As $\mathbf{D}^{\max}(\overleftrightarrow{K_n}) = \mathbf{J}_n - \mathbf{I}_n$, we have $\rho_1(\overleftrightarrow{K_n}) = n-1 > \rho_i(\overleftrightarrow{K_n}) = -1$ for $i = 2, \dots, n$. If D is an all-kings tournament with $n \neq 2, 4$, then for every $u, v \in V(D)$ with $u \neq v$, we have $d_D(u,v) = 1, 2$, so $md_D(u,v) = 2$, and $\mathbf{D}^{\max}(D) = 2\mathbf{D}^{\max}(\overleftrightarrow{K_n})$, implying that $\rho_1(D) > \rho_2(D) = \cdots = \rho_n(D)$. In either case, D has a max-distance eigenvalue of multiplicity $n-1$.

Conversely, suppose that D has a max-distance eigenvalue of multiplicity $n-1$. By Perron–Frobenius Theorem, we have $\rho_1(D) > \rho_2(D) = \cdots = \rho_n(D)$. By Lemma 4.1,

$$\begin{aligned}\max\{md_D(u,v) : \{u,v\} \subseteq V(D)\} &\leq -\rho_n(D)\\ &= -\rho_2(D)\\ &\leq \min\{md_D(u,v) : \{u,v\} \subseteq V(D)\}.\end{aligned}$$

So $\max\{md_D(u,v) : \{u,v\} \subseteq V(D)\} = \min\{md_D(u,v) : \{u,v\} \subseteq V(D)\}$. That is, there is some constant integer r, $md_D(u,v) = r$ for any $\{u,v\} \subseteq V(D)$.

Claim. $r \leq 2$.

Suppose that $r \geq 3$. Let $(v,u) \in E(D)$. Then $d_D(u,v) = md_D(u,v) = r$. Let $u \dots v'v$ be the (directed) path of length r from u to v. As $(v',v), (v,u) \in E(D)$, one gets $d_D(v',u) \leq 2 < r$ and $d_D(u,v') = r-1 < r$, implying that $md_D(u,v') < r$, a contradiction. It follows that $r \leq 2$.

By the above claim, $r = 1, 2$.

If $r = 1$, then $d_D(u,v) = 1$ for any $\{u,v\} \subseteq V(D)$, so $D \cong \overleftrightarrow{K_n}$.

Suppose that $r = 2$. Then for any $(u,v) \in E(D)$, $(v,u) \notin E(D)$. Let u be any vertex in $V(D)$. Then $V(D) = \{u\} \cup N_D^{+1}(u) \cup N_D^{+2}(u) = \{u\} \cup N_D^{-1}(u) \cup N_D^{-2}(u)$. So $N_D^{+2}(u) = N_D^{-1}(u)$ and $N_D^{-2}(u) = N_D^{+1}(u)$. It follows that D is a tournament. As $d_D(u,v) = 1, 2$ for any $v \in V(D) \setminus \{u\}$, u is a king of D. By the arbitrariness of u, D is an all-kings tournament. By Maurer's result [10], we have $n \neq 2, 4$. □

5. Concluding remarks

For strongly connected digraphs, the (directed) distance matrix, defined as $(d_D(u,v))_{u,v \in V(D)}$ for a strongly connected digraph D, is the standard distance matrix for digraphs. This matrix, however, is not symmetric unless the digraph considered is symmetric, in this case, it corresponds naturally to the distance matrix for undirected graphs. The spectral radius of the directed version received due attention [14], while the undirected version has been extensively studied, on both the spectral radius and the other eigenvalues [1].

In this paper, we investigate the spectral properties of strongly connected digraphs, regarding the maximum distance matrix. The entries of this matrix correspond to the maximum distance between the vertices, which constitutes a metric on the vertex set of the considered digraph. This choice makes the matrix symmetric, allowing the use of linear algebra tools.

In the literature, endeavours are made to associate symmetric (or Hermitian) matrices to digraphs. For example, the Hermitian matrix of a digraph was studied in [7, 9, 12]. Another is the sum-distance matrix [16] that is defined as $(d_D(u,v)+d_D(v,u))_{u,v\in V(D)}$ for a strongly connected digraph D, based on Chartrand and Tian's another suggestion [6]. The max-distance matrix is a new try. So the systematic study of this matrix is still in its early stages. We note that the complete digraph uniquely minimizes the spectral radius of both matrices among all strongly connected digraphs on n vertices. In other cases, the extremal digraphs are different. For example, the digraphs uniquely minimizing the spectral radius of sum-distance matrix are parts of the digraphs that minimize the spectral radius of max-distance matrix among strongly connected digraphs on n vertices with given (arc) connectivity k ($1 \leq k \leq n-2$). It was shown in [16] that the directed cycle uniquely maximizes the spectral radius of the sum-distance matrices among all strongly connected digraphs on n vertices. However, the corresponding problem for max-distance matrix remains unsolved.

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