OPTIMAL CONFIDENCE REGIONS FOR THE PARAMETERS OF A GENERAL EXPONENTIAL CLASS UNDER TYPE-II PROGRESSIVE CENSORING

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Abstract. Under Type-II progressively censored data, joint confidence regions are proposed for the parameters of a general class of exponential distributions. The constrained optimization problem based on such censoring data can be adopted to obtain confidence regions for the unknown parameters of this general class with minimized size and a predetermined confidence level. The area of confidence sets are minimized by solving simultaneous non-linear equations. Two real data sets representing the duration of remission of leukemia patients and water level exceedances by River Nidd at Hunsingore located in New York, are analyzed by fitting appropriate well-known models. Further, numerical simulation study is performed to explain our procedures and findings here.

Mathematics Subject Classification. 62G15, 62N01.

Received September 2, 2023. Accepted January 25, 2024.

1. Introduction

Let \( \Lambda(x; \alpha, \beta) \) be the cumulative hazard function of a random variable (r.v.) \( X \) with CDF \( F(x; \alpha, \beta) \). Al-Hussaini [2] proposed a general exponential class (GEC) form of the underlying distribution to tackle with a general procedure for the Bayesian inference. The survival function (SF) of \( X \) can be written as

\[
F(x; \alpha, \beta) = e^{-\psi(x; \alpha, \beta)}, \beta > 0, \tag{1}
\]

where \( \psi(x; \alpha, \beta) = \beta \Lambda(x; \alpha) \) with \( \Lambda(x; \alpha) = -\ln F(x; \alpha) \) being a continuous increasing and differentiable function having \( \Lambda(-\infty; \alpha) = 0 \) and \( \Lambda(\infty; \alpha) = \infty \). Its probability density function (PDF) is obtained to be

\[
f(x; \alpha, \beta) = \beta \lambda(x; \alpha) e^{-\beta \Lambda(x; \alpha)}, \beta > 0, \tag{2}
\]

where \( \lambda(x; \alpha) = \frac{d}{dx} \Lambda(x; \alpha) \) is the hazard rate of the r.v. \( X \). The Weibull, Lomax and extreme value (Type-I) distributions are members of GEC family.

The GEC of distributions with the CDF and PDF given in (1) and (2) is widely used in reliability and survival studies. It worth mentioning that the expression in (1) shows a relationship between the cumulative hazard function, \( \psi(x; \alpha, \beta) \) and the CDF \( F(x; \alpha, \beta) \). The monotonicity behavior of \( \lambda(x; \alpha) \) determines whether the
parent distribution is increasing failure rate (IFR), decreasing failure rate (DFR), constant failure rate (CFR) or bathtub rate (BTR). Let us consider some examples. When $\alpha = 1$, $\Lambda(x; \alpha) = x$, then the exponential distribution $\text{EXP}(\beta)$ is characterized. For $\Lambda(x; \alpha) = x^\alpha$, the Weibull distribution $\text{WE}(\alpha, \beta)$ is obtained (IFR if $\alpha > 1$, DFR if $\alpha < 1$, CFR if $\alpha = 1$). The Burr type XII $(\alpha, \beta)$ follows if we choose $\Lambda(x; \alpha) = \ln(1 + x^\alpha)$. In this case, when $\alpha \leq 1$, $X$ is DFR and for $\alpha > 1$, $X$ is IFR on $(0, (\alpha - 1)^{1/\alpha})$ and DFR on $((\alpha - 1)^{1/\alpha}, \infty)$. Furthermore, this class of distributions includes the two-parameter bathtub-shaped (BS) distribution with parameters $\alpha$ and $\beta$, BS$(\alpha, \beta)$, where $\Lambda(x; \alpha) = -1 + e^{x^\alpha}$. The BS distribution has IFR for $\alpha \geq 1$ and it is bathtub-shaped otherwise. These distributional properties allow a flexibility to the experimenter to fit more practical situations such as electronic and mechanical systems as well as the lifetime of human beings. The GEC model is a rich flexible model since it provides different shapes and its hazard function can be increasing, decreasing, constant and bathtub-shaped. For further details, one may refer to Lawless [17] and Nadarajah and Kotz [22]. These special distributions and others deduced from the general exponential family are discussed heavily in the literature, see for example, Al-Hussaini and Al-Awadhi [3], Mohie El-Din et al. [19–21] and Balakrishnan and Shafay [8].

The topic of Type-II progressive censoring scheme (PCS) received great attention in the past few years. Due to the time availability and cost or money constraints, the experimenter allows for removing the items at various stages during the experiment process. The progressive censoring scheme can be outlined as follows. First, we start our experiment with $M$ units and only $m$ units of them are observed completely. Removal of the units will be occurred in $m$ stages. At the time of observing the first failure, $r_1$ of the $M - 1$ surviving units are removed intentionally from the experimentation. When we observe the second failure (second stage), $r_2$ of the $M - 2 - r_1$ surviving units are removed. We continue the process until reaching the $m$-th stage and $r_m = M - m - r_1 - \cdots - r_{m-1}$ surviving units are removed. The data extracted based on this process is called Type-II progressive censoring data with scheme $r = (r_1, r_2, \ldots, r_m)$. The complete sampling and Type-II censoring schemes can be obtained as special cases when $M = m$ and $r_1 = r_2 = \cdots = r_m = 0$ and $r_1 = r_2 = \cdots = r_{m-1} = 0$ and $r_m = M - m$, respectively. The lifetime data under Type-II PCS arises naturally in survival, reliability and medical studies. For a complete views, applications and references on progressively censoring and relevant references, see Cohen [11, 12], Valiollahi et al. [23] and Balakrishnan and Aggarwala [7].

Although the research work on the statistical inference based the joint confidence regions of the model parameters is a hot topic in recent years, there is no considerable works have been developed in this regard. Wu [24] established exact confidence intervals and exact joint confidence regions for the parameters of Weibull distribution. The constrained optimization procedures are formulated by Fernández [14, 15] to determine confidence regions of the Pareto parameters. Constrained optimization problems for the Rayleigh and exponential distributions, are also discussed by Asgharzadeh et al. [5, 6], respectively, to find the minimum-size confidence sets of location and scale parameters. In this context, our main aim of this paper is consider the problem of estimating the parameters of the GEC by joint confidence regions (JCRs) based on Type-II progressive censored data. Many well-known models can be included as special cases of the GEC. The optimal JCRs of the parameters are derived by minimizing the areas of the confidence sets under a pre-specified confident coefficient level, $1 - \gamma$. The main aim of this work is to consider a generalized class of distributions that cover various types of distributions used in real life experiments. Therefore, our confidence intervals and regions are quite general in nature and can be used widely in different applications.

The outline of the paper is organized as follows. In Section 2, we develop balanced confidence intervals (CIs) for the model parameters as well as respective JCRs. The JCRs of the unknown parameters have a general form and it is based on the functional behavior of $\Lambda(x; \alpha)$, viewed as a function of $\alpha$. In Section 3, we tackle the problem of obtaining optimal JCRs of the model parameters using smallest area criterion. In Section 4, we analyze two real data sets representing the duration of remission of leukemia patients and water level exceedances by River Nidd at Hunsingore located in New York. In Section 5, we conduct a numerical simulation analysis to evaluate the effectiveness of the proposed CIs and JCRs for the Weibull distribution parameters within the context of the GEC model. The outcomes and conclusions of our study are summarized in Section 6.
2. Balanced confidence regions

Based on the observed data under Type-II PCS, we consider here the problem of estimating $\alpha$ and $\beta$ from the GEC with CDF and PDF being in (1) and (2). The general pivotal quantities are obtained and then their respective confidence regions of the parameters can be established, accordingly under the set-up considered in the previous section.

Under the PCS, we set $Y_i(\alpha) = \Lambda(X_i; m; M, \alpha), i = 1, 2, \ldots, m$ and then $Y_1(\alpha) < Y_2(\alpha) < \ldots < Y_m(\alpha)$ are Type-II PCS ordered data from scaled exponential distribution, EXP($\beta$). The spacings can be described as follows:

$$E_1 = M\beta Y_1(\alpha)$$
$$E_2 = (M - r_1 - 1)\beta(Y_2(\alpha) - Y_1(\alpha))$$
$$\vdots$$
$$E_m = \left(M - \sum_{i=1}^{m-1} r_i - m + 1\right)\beta(Y_m(\alpha) - Y_{m-1}(\alpha)).$$

These spacings are independent and identically EXP(1) random variates (see, [7]). Then it follows that

$$Q_1 = 2\beta E_1 = 2M\beta Y_1(\alpha),$$

is a chi-square with 2 degrees of freedom, $\chi^2_2$ and

$$Q_2 = 2\beta \sum_{i=2}^{m} E_i = 2\beta \sum_{i=1}^{m} (r_i + 1)Y_i(\alpha) - 2M\beta Y_1(\alpha),$$

has a chi-square with $2(m - 1)$ degrees of freedom, $\chi^2_{2(m-1)}$. Since $Q_1$ and $Q_2$ are independent statistics, we have

$$W_1 = \frac{Q_2}{(m - 1)Q_1} = \frac{\sum_{i=1}^{m} (r_i + 1)Y_i(\alpha) - M Y_1(\alpha)}{(m - 1)M Y_1(\alpha)}$$

and

$$W_2 = Q_1 + Q_2 = 2\beta \sum_{i=1}^{m} (r_i + 1)Y_i(\alpha)$$

are $F(2(m - 1), 2)$ and $\chi^2_{2m}$. For known $\alpha$, Theorem 1 provides $(1 - \gamma)100\%$ CI for $\beta$.

Theorem 1. Let $X_1; m; M, X_2; m; M, \ldots, X_m; m; M$ be a Type-II PCS sample with censoring scheme $r = (r_1, \ldots, r_m)$ from the GEC. If for any $i \geq 1$, $\rho(\alpha) = Y_i(\alpha)/Y_1(\alpha)$ is increasing with $\rho(0^+) = 1$, then for $0 < \gamma < 1,$ $(1 - \gamma)100\%$ CI of $\alpha$ is

$$\left[ W_1^{-1}\left(F_{2m-2, 2}\left(\frac{\gamma}{2}\right)\right), W_1^{-1}\left(F_{2m-2, 2}\left(1 - \frac{\gamma}{2}\right)\right) \right],$$

and if $\rho(\alpha)$ is decreasing with $\rho(0^+) > 1,$ then $(1 - \gamma)100\%$ CI of $\alpha$ is

$$\left[ W_1^{-1}\left(F_{2m-2, 2}\left(1 - \frac{\gamma}{2}\right)\right), W_1^{-1}\left(F_{2m-2, 2}\left(\frac{\gamma}{2}\right)\right) \right],$$

where $W_1^{-1}(c)$ is the solution of $\alpha$ for the non-linear equation:

$$W_1(\alpha) = c,$$

with

$$W_1(\alpha) = W_1(\alpha; x, r) = \frac{\sum_{i=1}^{m} (r_i + 1)Y_i(\alpha)}{M(m - 1)Y_1(\alpha)} - \frac{1}{m - 1}.$$
Proof. Clearly, $W_1(\alpha)$ can be rewritten as

$$W_1(\alpha) = \frac{1}{M(m-1)} \sum_{i=1}^{m} (r_i + 1)\rho_i(\alpha) - \frac{1}{m-1}.$$  

This turns out that the monotonicity behavior of $W_1(\alpha)$ can be determined based on the behavior of $\rho_i(\alpha)$. It is evident that $\rho_i(\alpha) \geq 1$ and then $W_1(\alpha) \geq 0$. If $\rho(\alpha)$ is increasing with $\rho(0^+) = 1$, then $W_1(\alpha)$ is increasing with $W_1(0^+) = 0$ and $W_1(\infty) = \infty$. Further, if $\rho(\alpha)$ is decreasing with $\rho(0^+) > 1$, then $W_1(\alpha)$ is decreasing with $W_1(0^+) > 0$ and $W_1(\infty) = 0$. For both cases, $W_1(\alpha) = c$, for positive real constant $c$, has a unique solution for $\alpha$. Therefore,

$$P\left[ W_1^{-1} \left( F_{2m-2,2}^{-1} \left( \frac{\gamma}{2} \right) \right) < \alpha < W_1^{-1} \left( F_{2m-2,2}^{-1} \left( 1 - \frac{\gamma}{2} \right) \right) \right] = 1 - \gamma,$$

for increasing $W_1(\alpha)$, and

$$P\left[ W_1^{-1} \left( F_{2m-2,2}^{-1} \left( 1 - \frac{\gamma}{2} \right) \right) < \alpha < W_1^{-1} \left( F_{2m-2,2}^{-1} \left( \frac{\gamma}{2} \right) \right) \right] = 1 - \gamma,$$

for decreasing $W_1(\alpha)$. \hfill $\square$

Next, a $(1 - \gamma)100\%$ JCR for $(\alpha, \beta)$ is given in Theorem 2.

Theorem 2. Let $X_{1:m:M}, X_{2:m:M}, \ldots, X_{m:m:M}$ be a Type-II PCS with censoring scheme $r = (r_1, \ldots, r_m)$ from the GEC. Then, a $(1 - \gamma)100\%$ equi-tailed JCR for $(\alpha, \beta)$ is given by

$$R_{1-\gamma} = \left\{ (\alpha, \beta) : W_1^{-1} \left( F_{2m-2,2}^{-1} \left( \frac{1 - \sqrt{1 - \gamma}}{2} \right) \right) < \alpha < W_1^{-1} \left( F_{2m-2,2}^{-1} \left( \frac{1 + \sqrt{1 - \gamma}}{2} \right) \right), \right. $$

$$\frac{\chi^2_{2m} \left( \frac{1 - \sqrt{1 - \gamma}}{2} \right)}{2 \sum_{i=1}^{m} (r_i + 1)Y_i(\alpha)} < \beta < \frac{\chi^2_{2m} \left( \frac{1 + \sqrt{1 - \gamma}}{2} \right)}{2 \sum_{i=1}^{m} (r_i + 1)Y_i(\alpha)},$$

for increasing $W_1(\alpha)$ in $\alpha$ and

$$R_{1-\gamma} = \left\{ (\alpha, \beta) : W_1^{-1} \left( F_{2m-2,2}^{-1} \left( \frac{1 + \sqrt{1 - \gamma}}{2} \right) \right) < \alpha < W_1^{-1} \left( F_{2m-2,2}^{-1} \left( 1 - \frac{\sqrt{1 - \gamma}}{2} \right) \right), \right. $$

$$\frac{\chi^2_{2m} \left( \frac{1 - \sqrt{1 - \gamma}}{2} \right)}{2 \sum_{i=1}^{m} (r_i + 1)Y_i(\alpha)} < \beta < \frac{\chi^2_{2m} \left( \frac{1 + \sqrt{1 - \gamma}}{2} \right)}{2 \sum_{i=1}^{m} (r_i + 1)Y_i(\alpha)},$$

for decreasing $W_1(\alpha)$ in $\alpha$.

Proof. Since $W_1$ and $W_2$ are independent statistics, we may write

$$P\left[ F_{2m-2,2} \left( \frac{1 - \sqrt{1 - \gamma}}{2} \right) \leq W_1(\alpha) \leq F_{2m-2,2} \left( \frac{1 + \sqrt{1 - \gamma}}{2} \right) \right],$$

$$P\left[ \chi^2_{2m} \left( \frac{1 - \sqrt{1 - \gamma}}{2} \right) \leq W_2(\beta) \leq \chi^2_{2m} \left( \frac{1 + \sqrt{1 - \gamma}}{2} \right) \right] = \sqrt{1 - \gamma} \sqrt{1 - \gamma},$$

which leads to the results of Theorem 2. \hfill $\square$
Example 1. Let $X_{1:m:M}, X_{2:m:M}, \ldots, X_{m:m:M}$ be a Type-II PCS sample with censoring scheme $r = (r_1, \ldots, r_m)$ from the Weibull distribution, $\text{WE}(\alpha, \beta)$ with parent CDF

$$F(x; \alpha, \beta) = 1 - e^{\beta x^{-\alpha}}, x \geq 0, \alpha > 0, \beta > 0.$$ 

Here, we have

$$\Lambda_i(\alpha) = \Lambda_i(x_{i:m:M}; \alpha) = x_{i:m:M}^\alpha,$$

and $\rho(\alpha) = Y_i(\alpha)/Y_1(\alpha)$ is increasing in $\alpha$ starting from 1 at $\alpha = 0$ and tends to $\infty$ as $\alpha$ moves to $\infty$. Further,

$$W_1(\alpha) = \frac{1}{M(m-1)} \sum_{i=1}^{m} (r_i + 1) \left( \frac{X_{i:m:M}}{X_{1:m:M}} \right)^\alpha - \frac{1}{m-1}.$$ 

It can be noticed that $W_1(\alpha)$ is increasing function with $W_1(0^+) = 0$ and $W_1(\alpha)$ tends to $\infty$ as $\alpha$ approaches $\infty$.

Example 2. Consider $X_{1:m:M}, X_{2:m:M}, \ldots, X_{m:m:M}$ is a Type-II PCS sample with censoring from the extreme value distribution, $\text{EV}(\alpha, \beta)$ with parent CDF

$$F(x; \alpha, \beta) = 1 - e^{-\beta e^{\frac{x}{\alpha}}}, -\infty < x < \infty, \alpha > 0, \beta > 0.$$ 

We readily have

$$\Lambda_i(\alpha) = \Lambda_i(x_{i:m:M}; \alpha) = e^{\frac{x_{i:m:M}}{\alpha}},$$

and

$$\rho(\alpha) = \frac{Y_i(\alpha)}{Y_1(\alpha)} = e^{\frac{1}{\alpha}(x_{i:n:M} - x_{1:n:M})},$$

is decreasing in $\alpha$ starting from $\infty$ at $\alpha = 0$ and tends to 1 as $\alpha$ moves to $\infty$. In addition,

$$W_1(\alpha) = \frac{1}{M(m-1)} \sum_{i=1}^{m} (r_i + 1) e^{(X_{i:n:M} - X_{1:n:M})/\alpha} - \frac{1}{m-1}.$$ 

Here, $W_1(\alpha)$ is decreasing in $\alpha$ with $W_1(0^+) = \infty$ and $W_1(\alpha)$ becomes 0 as $\alpha$ converges to $\infty$.

Example 3. Let us consider another member of the GEC, named, Lomax distribution. It is a special case of the second kind of Pareto distribution and it was proposed by Lomax [18]. It has been shifted from Pareto distribution so that its support begins at zero. It has been used in business, economics, insurance, queueing theory, and engineering. Its CDF is given by

$$F(x; \alpha, \beta) = 1 - (1 + \alpha x)^{-\beta}, x > 0, \alpha > 0, \beta > 0,$$

where $\alpha$ and $\beta$ are scale and shape parameters. Its hazard rate is DFR and therefore, the Lomax distribution may describe the lifetime of a decreasing failure rate items. Bryson [9] recommended Lomax distribution as an alternative to the exponential distribution when the data are heavy tailed. In our set-up, we have

$$\Lambda_i(x_{i:n:M}; \alpha) = \log(1 + \alpha x_{i:n:M}),$$

and

$$\rho(\alpha) = \frac{\log(1 + \alpha x_{n:n:M})}{\log(1 + \alpha x_{1:n:M})}.$$ 

In this case, it is not an easy task to show its monotonicity of $\rho(\alpha)$. Therefore, one may check the behavior of $W_1(\alpha)$

$$W_1(\alpha) = \frac{1}{M(m-1)} \sum_{i=1}^{m} (r_i + 1) \left( \frac{\log(1 + \alpha X_{i:n:M})}{\log(1 + \alpha X_{1:n:M})} \right)^\alpha - \frac{1}{m-1},$$

under the progressive censoring data to be used.
3. Confidence regions with smallest size

In this section, our aim is devoted to establishing optimal confidence sets of the parameters in the sense of smallest size. Mainly, we intend to use the constrained optimization problems in order to get these smallest sets. The Lagrangian method is implemented for solving the non-linear programming problems. For an ease notations, suppose the CDF and PDF of \( W_i \) sets. The Lagrangian method is implemented for solving the non-linear programming problems. For an ease notations, suppose the CDF and PDF of \( W_i \) sets.

3.1. Shortest confidence intervals

Consider \( X = (X_{1:m:M}, \ldots, X_{m:m:M}) \), \( m \geq 2 \), be a Type-II PCS data from a GEC. Let

\[
W_1(\alpha) = \frac{1}{M(m-1)} \sum_{i=1}^{m} \rho_i(\alpha) - \frac{1}{m-1}.
\]

Given \( P(L_1 < W_1(\alpha) < U_1) = 1 - \gamma \) where \( 0 < \gamma < 1 \) and \( L_1 < U_1 \). Then, a \((1 - \gamma)100\%\) CI for \( \alpha \) is

\[
\begin{align*}
(W_1^{-1}(L_1), W_1^{-1}(U_1)), & \quad \text{for increasing } \rho(\alpha), \\
(W_1^{-1}(U_1), W_1^{-1}(L_1)), & \quad \text{for decreasing } \rho(\alpha).
\end{align*}
\]

Without loss of generality, assume that \( \rho(\alpha) \) is increasing in \( \alpha \) and then the minimization problem can be formulated as

\[
\begin{align*}
\text{Minimize} & \quad W_1^{-1}(U_1) - W_1^{-1}(L_1), \\
\text{Subject to} & \quad H_1(U_1) - H_1(L_1) = 1 - \gamma.
\end{align*}
\]

For obtaining the optimal solutions of \( L_1 \) and \( U_1 \) that minimize the above optimal function, we opt to apply the lagrangian method. The Lagrangian function to be minimized is

\[
D(L_1, U_1, \lambda) = W_1^{-1}(U_1) - W_1^{-1}(L_1) - \lambda[H_1(U_1) - H_1(L_1) - (1 - \gamma)],
\]

where \( \lambda \) is a Lagrange multiplier. By differentiating \( D(L_1, U_1, \lambda) \) with respect to \( L_1, U_1 \) and \( \lambda \) and equating the resulting equations to zero, we finally obtain the optimal values of \( L_1 \) and \( U_1 \), say \( L_1^* \) and \( U_1^* \), which can be found numerically by solving the non-linear equations:

\[
\frac{h_1(U_1)}{h_1(L_1)} = \frac{W_1'(W_1^{-1}(L_1))}{W_1'(W_1^{-1}(U_1))},
\]

and

\[
H_1(U_1) - H_1(L_1) = 1 - \gamma.
\]

The shortest \((1 - \gamma)100\%\) CI for \( \alpha \) is readily \((W_1^{-1}(L_1^*), W_1^{-1}(U_1^*))\) or \((W_1^{-1}(U_1^*), W_1^{-1}(L_1^*))\) according to increasing \( \rho(\alpha) \) or decreasing \( \rho(\alpha) \), respectively. For given \( \alpha \), \((1 - \gamma)100\%\) equal-tailed CI for \( \beta \) is

\[
\left( \frac{\chi_2^2 m (\gamma^2)}{2 \sum_{i=1}^{m} (r_i + 1) Y_i(\alpha)} - \frac{\chi_2^2 m (1 - \gamma^2)}{2 \sum_{i=1}^{m} (r_i + 1) Y_i(\alpha)} \right).
\]

Proceeding similarly, In this case, the \((1 - \gamma)100\%\) optimal CI for \( \beta \) can be obtained by solving the following minimization problem:

\[
\begin{align*}
\text{Minimize} & \quad U_2 - L_2, \\
\text{Subject to} & \quad H_2(U_2) - H_2(L_2) = 1 - \gamma.
\end{align*}
\]
Therefore, the corresponding the Lagrangian function to be minimized is
\[ D^*(L_2, U_2, \lambda) = U_2 - L_2 - \lambda[H_2(U_2) - H_2(L_2) - (1 - \gamma)]. \]
Consequently, for a given \( \alpha \), the shortest \( (1 - \gamma)100\% \) CI for \( \beta \) is determined to be
\[ \left( \frac{L_2}{2\sum_{i=1}^{m}(r_i + 1)Y_i(\alpha)}, \frac{U_2^*}{2\sum_{i=1}^{m}(r_i + 1)Y_i(\alpha)} \right), \]
where \( L_2^* \) and \( U_2^* \) are the solutions of
\[ h_2(L_2) = h_2(U_2), \]
and
\[ H_2(U_2) - H_2(L_2) = 1 - \gamma. \]

### 3.2. CRs with smallest area

In this subsection, we intend to obtain JCR for \((\alpha, \beta)\) of significance level \((1 - \gamma)\) based on minimizing its respective area. Assume that \( \rho(\alpha) \) is increasing and define the CR of \((\alpha, \beta)\) as
\[
\text{CR}_{1-\gamma} = \left\{ (\alpha, \beta) : W_1^{-1}(L_1) < \alpha < W_1^{-1}(U_1), \frac{L_2}{2\sum_{i=1}^{m}(r_i + 1)Y_i(\alpha)} < \beta < \frac{U_2}{2\sum_{i=1}^{m}(r_i + 1)Y_i(\alpha)} \right\},
\]
under the coverage probability
\[ 1 - \gamma = [H_1(U_1) - H_1(L_1)][H_2(U_2) - H_2(L_2)]. \]
The corresponding area of the \((1 - \gamma)100\%\) CR is computed to be
\[
A(\text{CR}_{1-\gamma}) = \int_{W_1^{-1}(L_1)}^{W_1^{-1}(U_1)} \int_{L_2}^{U_2} \frac{1}{2\sum_{i=1}^{m}(r_i + 1)Y_i(\alpha)} \, d\beta \, d\alpha = \int_{W_1^{-1}(L_1)}^{W_1^{-1}(U_1)} \frac{U_2 - L_2}{2\sum_{i=1}^{m}(r_i + 1)Y_i(\alpha)} \, d\alpha.
\]
The minimization problem needed for obtaining \((1 - \gamma)100\%\) optimal JCR for \((\alpha, \beta)\) can be formulated as follows:
\[
\begin{align*}
\text{Minimize} & \quad \int_{W_1^{-1}(L_1)}^{W_1^{-1}(U_1)} \frac{U_2 - L_2}{\eta(\alpha)} \, d\alpha, \\
\text{Subject to} & \quad [H_1(U_1) - H_1(L_1)][H_2(U_2) - H_2(L_1)] = 1 - \gamma,
\end{align*}
\]
where
\[ \eta(\alpha) = 2\sum_{i=1}^{m}(r_i + 1)Y_i(\alpha), \]
with \( 0 < L_1 < U_1 \), and \( 0 < L_2 < U_2 \). Now, we look for the optimal values of \( \varphi = (L_1, L_2, U_1, U_2) \) that minimize the following Lagrangian function:
\[
D(\varphi, \lambda) = \int_{W_1^{-1}(L_1)}^{W_1^{-1}(U_1)} \frac{U_2 - L_2}{\eta(\alpha)} \, d\alpha + \lambda([H_1(U_1) - H_1(L_1)][H_2(U_2) - H_2(L_2)] - (1 - \gamma)),
\]
where \( \lambda \) is a Lagrange multiplier. The Lagrangian partial derivatives of \( D(\varphi, \lambda) \) with respect to \( L_i \) and \( U_i(i = 1, 2) \). That is,
\[
\frac{\partial \varphi}{\partial L_1} = \frac{\partial \varphi}{\partial L_2} = \frac{\partial \varphi}{\partial U_1} = \frac{\partial \varphi}{\partial U_2} = 0,
\]
where

\[
\frac{\partial D(\varphi, \lambda)}{\partial L_1} = \frac{L_2 - U_2}{\eta(W_1^{-1}(L_1)) W_1'(W_1^{-1}(L_1))} - \lambda h_1(L_1)[H_2(U_2) - H_2(L_2)],
\]

(5)

\[
\frac{\partial D(\varphi, \lambda)}{\partial L_2} = \int_{W_1^{-1}(L_1)}^{W_1^{-1}(U_1)} \frac{-1}{\eta(\alpha)} d\alpha - \lambda h_2(L_2)[H_1(U_1) - H_1(L_1)],
\]

(6)

\[
\frac{\partial D(\varphi, \lambda)}{\partial U_1} = \frac{U_2 - L_2}{\eta(W_1^{-1}(U_1)) W_1'(W_1^{-1}(U_1))} + \lambda h_1(U_1)[H_2(U_2) - H_2(L_2)],
\]

(7)

\[
\frac{\partial D(\varphi, \lambda)}{\partial U_2} = \int_{W_1^{-1}(L_1)}^{W_1^{-1}(U_1)} \frac{1}{\eta(\alpha)} d\alpha + \lambda h_2(U_2)[H_1(U_1) - H_1(L_1)],
\]

(8)

and

\[
\frac{\partial D(\varphi, \lambda)}{\partial \lambda} = [H_1(U_1) - H_1(L_1)][H_2(U_2) - H_2(L_2)] - 1 + \gamma.
\]

(9)

By applying equations (5)–(9), simultaneously, we readily conclude that the optimal values \(L_1^*, L_2^*, U_1^*,\) and \(U_2^*\) are the numerical solutions to the following non-linear system:

\[
\frac{h_1(L_1)W_1'(W_1^{-1}(L_1))}{h_1(U_1)W_1'(W_1^{-1}(U_1))} = \frac{\eta(W_1^{-1}(U_1))}{\eta(W_1^{-1}(L_1))},
\]

(10)

\[
\int_{W_1^{-1}(L_1)}^{W_1^{-1}(U_1)} \frac{1}{\eta(\alpha)} d\alpha = \frac{[H_1(U_1) - H_1(L_1)][h_2(U_2)(U_2 - L_2)]}{\eta(W_1^{-1}(L_1)) h_1(L_1) W_1'(W_1^{-1}(U_1))[H_2(U_2) - H_2(L_2)]},
\]

(11)

\[
h_2(L_2) = h_2(U_2),
\]

(12)

and

\[
[H_1(U_1) - H_1(L_1)][H_2(U_2) - H_2(L_2)] = 1 - \gamma.
\]

(13)

Clearly, one of the numerical methods such as Newton–Raphson (NR) or bisection method should be used to find the solutions of equations (10)–(13). Further, the integral on the left hand side of (11) can’t be obtained explicitly and numerical integration method can be adopted to evaluate this type of integral.

### 4. Practical Applications

In this section, we discuss the analysis of the progressively type-II right censored sample produced from practical data sets with some well-known fitting distributions. The procedures discussed in Sections 2 and 3 are applied on a biometric and water level exceedances data sets.

**Example 4.** The data set to be analyzed represents the duration of remission of 20 leukemia patients considered by Wu and Wu [25] and analyzed by Lawless [17]. Recently Al-Mutairi and Raqab [4] used this data set to illustrate the estimation problem of population quantiles based on samples of random sizes. The durations of remission are recorded as follows:

\[
\begin{align*}
1.013 & \quad 1.034 \quad 1.109 \quad 1.169 \quad 1.266 \quad 1.509 \quad 1.533 \quad 1.563 \\
1.716 & \quad 1.929 \quad 1.965 \quad 2.061 \quad 2.344 \quad 2.546 \quad 2.626 \quad 2.778 \\
2.951 & \quad 3.413 \quad 4.118 \quad 5.136
\end{align*}
\]

Here, we consider the Weibull distribution \(\text{WE}(\alpha, \beta)\) as member of GEC with \(\psi(x; \alpha, \beta) = \beta \Lambda(x; \alpha)\) and \(\Lambda(x; \alpha) = x^\alpha\). Some basic descriptive statistics of the duration of remission data are summarized in Table 1. The coefficient skewness indicates that the data are positively skewed.
Table 1. Basic descriptive statistics of remission data.

<table>
<thead>
<tr>
<th>Mean</th>
<th>Median</th>
<th>Std.Dev</th>
<th>$Q_1$</th>
<th>$Q_3$</th>
<th>Skewness</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1890</td>
<td>1.947</td>
<td>1.090</td>
<td>1.327</td>
<td>2.740</td>
<td>1.269</td>
</tr>
</tbody>
</table>

Figure 1. Shapes of (a) $\rho(\alpha)$ for different $x$-values and shape of (b) $W_1(\alpha)$.

The MLEs of $\alpha$ and $\beta$ are computed numerically using NR method to be

$$\hat{\alpha} = 2.2011$$ and $$\hat{\beta} = 0.1350.$$  

The Kolmogorov–Smirnov (K-S) and Cramer–von Mises (CvM) distances between the fitted and the empirical distribution functions are 0.1152 and 0.0580, and the corresponding $p$-values are 0.9538 and 0.8269, respectively. It is evident that the WE is an appropriate fitting model for fitting the above data set.

Let us consider PCS with $n = 20, m = 12$ and $r_1 = 2, r_2 = 1, r_3 = \ldots = r_{10} = 0, r_{11} = 1, r_{12} = 4$. The extracted progressive data along with the censoring scheme are described as:

<table>
<thead>
<tr>
<th>$i$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_i$</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>$x_{i.m,M}$</td>
<td>1.013</td>
<td>1.169</td>
<td>1.509</td>
<td>1.533</td>
<td>1.563</td>
<td>1.716</td>
<td>1.929</td>
<td>1.965</td>
<td>2.061</td>
<td>2.344</td>
<td>2.546</td>
<td>2.778</td>
</tr>
</tbody>
</table>

It is evident from Figure 1 that both $\rho(\alpha)$ and $W_1(\alpha)$ are increasing for all $\alpha > 0$. Based on the estimation methods used in Sections 2 and 3, we establish the balanced and smallest 95% CIs as well as CRs of the parameters involved in WE distribution. It follows from

$$F_{2m-2,2}\left(\frac{\gamma}{2}\right) = F_{2m-2,2}(0.025) = 0.2282,$$ and $$F_{2m-2,2}\left(1 - \frac{\gamma}{2}\right) = F_{2m-2,2}(0.975) = 39.4526.$$
that the balanced 95% CI of $\alpha$ is immediately obtained to be (1.7972, 7.1003). The shortest 95% CI for $\alpha$ is given by (1.5280, 6.5675). For given $\alpha$, the balanced and shortest 95% CIs for $\beta$ are computed to be, respectively,

$$\frac{3.5215}{\eta(\alpha)}, \frac{6.2741}{\eta(\alpha)}, \text{ and } \frac{3.4880}{\eta(\alpha)}, \frac{6.2370}{\eta(\alpha)}.$$  

For $\alpha = 0.25$ (exponential model), $\eta(\alpha) = \eta(1) = 46.705$, the 95% CIs for $\beta$ are reduced to (0.0754, 0.1343) and (0.075, 0.1335). Next, the $(1 - \gamma)100\%$ balanced and smallest CRs of $(\alpha, \beta)$ are

$$\text{CR}_{0.95} = \left\{ (\alpha, \beta) : 1.6181 \leq \alpha \leq 7.8079, \frac{3.3499}{\eta(\alpha)} \leq \beta \leq \frac{6.4865}{\eta(\alpha)} \right\},$$

and

$$\text{CR}_{0.95}^{\text{opt}} = \left\{ (\alpha, \beta) : 3.6828 \leq \alpha \leq 5.6215, \frac{3.5245}{\eta(\alpha)} \leq \beta \leq \frac{7.2481}{\eta(\alpha)} \right\},$$

respectively. Their corresponding areas of these regions are computed to be Area(CR$_{0.95}$) = 0.0844, and Area(CR$_{0.95}^{\text{opt}}$) = 0.0046.

**Example 5.** Here we discuss the analysis of real life data representing the water level exceedances over the level 65$m^3$ by the River Nidd at Hunsingore Weir which is located in North Yorkshire, England from 1934 to 1970 (35 years). This data set was reported in: Natural Environment Research Council, 1975. This data set was analyzed by Hosking and Wallis [16] and Davison and Smith [13]. Though the full set contains a series of values for each year, a valid assumption would be to assume that the data $x_1, x_2, \ldots, x_{21}$ are realisation from a sequence $X_1, X_2, \ldots, X_{21}$ of iid r.v.’s with common distribution. The ordered exceedance levels over the threshold $u = 80$ (in hundreds) are:

<table>
<thead>
<tr>
<th>$u$</th>
<th>$x$</th>
<th>$x$</th>
<th>$x$</th>
<th>$x$</th>
<th>$x$</th>
<th>$x$</th>
<th>$x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.75</td>
<td>1.4</td>
<td>1.4</td>
<td>2.54</td>
<td>3.56</td>
<td>3.69</td>
<td>11.8</td>
<td>11.8</td>
</tr>
<tr>
<td>14.08</td>
<td>17.24</td>
<td>17.87</td>
<td>22.92</td>
<td>35.52</td>
<td>36.77</td>
<td>39.28</td>
<td>51.82</td>
</tr>
<tr>
<td>63.06</td>
<td>69.3</td>
<td>73.04</td>
<td>82.99</td>
<td>109.02</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Let $T$ be the water flood levels of a river and let $X = T - T_0 > 0$, represents the exceedances over $T_0 = 80$. In this context, it is reasonable to assume the Lomax distribution having CDF

$$F(x; \alpha, \beta) = 1 - e^{-\beta \Lambda(x; \alpha)}, x > 0, \text{ with } \Lambda(x; \alpha) = \log(1 + \alpha x),$$

where $\alpha$ and $\beta$ are shape and scale parameters, respectively such that $\alpha > 0$ and $\beta > 0$. The Lomax distribution reduces to the exponential distribution. Its applications include environmental extreme events, ozone levels in the upper atmosphere, large insurance claims or large fluctuation in financial data, and reliability studies. Its areas of applications are successfully addressed in several books, such as those by Castillo et al. [10], and Ahsanullah [1]. The Lomax distribution is fitting the exceedances levels over a specific threshold $T_0$. However, we can easily check the fit of the data. Table 2 summarises some basic descriptive statistics of the water level exceedances. The MLEs of $\alpha$ and $\beta$ are computed numerically using NR method and founded to be $\hat{\alpha} = 0.0174$ and $\hat{\beta} = 4.9260$. The K-S distances between the fitted and the empirical distribution functions and the corresponding $p$-values are K-S = 0.1339 and $p$-value = 0.8458. The CvM distances between the fitted and the empirical distribution functions is 0.1299 and the corresponding $p$-value is 0.4569. This in turns out the Lomax distribution is an appropriate model for fitting the data set.

We have generated the following progressively censored samples using the sampling schemes $r = (0, 0, 2, 0, 0, 0, 3, 0, 0, 0, 5)$ from the full data set with $m = 11$. The generated data and corresponding censored scheme are given as follows.

It is worth mentioning that although the monotonicity of $\rho(\alpha)$ is not analytically clear, the plot of $W_1(\alpha)$ based on the observed progressive data shows that it is increasing function of $\alpha > 0$. Figure 2 shows that $\rho(\alpha)$ is decreasing for different choices of $X_{i:m:M}$ but $W_1(\alpha)$ is increasing function of $\alpha$. 

...
Table 2. Basic descriptive statistics of water levels data.

<table>
<thead>
<tr>
<th>Mean</th>
<th>Median</th>
<th>Std.Dev</th>
<th>$Q_1$</th>
<th>$Q_3$</th>
<th>Skewness</th>
</tr>
</thead>
<tbody>
<tr>
<td>31.898</td>
<td>17.870</td>
<td>31.637</td>
<td>3.625</td>
<td>57.440</td>
<td>0.989</td>
</tr>
</tbody>
</table>

Data set for Example 5.

<table>
<thead>
<tr>
<th>i</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_i$</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>$x_{i,m,M}$</td>
<td>0.75</td>
<td>1.4</td>
<td>1.4</td>
<td>3.69</td>
<td>11.8</td>
<td>11.8</td>
<td>14.08</td>
<td>35.52</td>
<td>36.77</td>
<td>39.28</td>
<td>51.82</td>
</tr>
</tbody>
</table>

Figure 2. Shapes of (a) $\rho(\alpha)$ for different $x$-values and shape of (b) $W_1(\alpha)$.

Under the progressive censored sample given above, the balanced and shortest CIs are computed to be $(0.3760, 1.6272)$ and $(0.3121, 1.4970)$, respectively. For given $\alpha$, we also obtain the balanced and shortest CIs for $\beta$ as

\[
\left( \frac{3.3140}{\eta(\alpha)}, \frac{6.0647}{\eta(\alpha)} \right), \text{ and } \left( \frac{3.2789}{\eta(\alpha)}, \frac{6.0258}{\eta(\alpha)} \right).
\]

Next, we establish 95% CRs of $(\alpha, \beta)$ based on the balanced and smallest area methods. The corresponding CRs are obtained and presented, respectively, as

\[
\text{CR}_{0.95} = \left\{ (\alpha, \beta) : 0.3354 \leq \alpha \leq 1.7960, \frac{3.1436}{\eta(\alpha)} \leq \beta \leq \frac{6.2777}{\eta(\alpha)} \right\},
\]

and

\[
\text{CR}^{\text{opt}}_{0.95} = \left\{ (\alpha, \beta) : 0.8154 \leq \alpha \leq 1.2749, \frac{3.5000}{\eta(\alpha)} \leq \beta \leq \frac{6.0562}{\eta(\alpha)} \right\},
\]
respectively. Their corresponding areas of these regions are computed to be

\[
\text{Area}(\text{CR}_{0.95}) = 0.0199, \quad \text{and} \quad \text{Area}(\text{CR}_{0.95}) = 0.0011.
\]

5. NUMERICAL EXPERIMENTS

A Monte Carlo simulation is executed to explore the effectiveness of the proposed CIs and JCRs established for the Weibull distribution parameters. This exploration involved employing the Monte Carlo (MS) simulation technique across various censoring schemes and for different sample sizes. All numerical computations are executed using Mathematica package. In evaluating the performance of the proposed CIs and JCRs, two significant optimality criteria are considered: the mean size (MS) and the coverage percentage (CP). The MS and CP of a JCR (say, \(\mathcal{R}\)) of a vector of parameters \(\theta\) are defined as

\[
\text{MS}(\mathcal{R}) = \frac{1}{It} \sum_{i=1}^{It} \text{Area}(\mathcal{C}_i), \quad \text{and} \quad \text{CP}(\mathcal{R}) = \frac{1}{It} \sum_{i=1}^{It} I_{\theta}(\mathcal{C}_i),
\]

where \(\mathcal{C}_i\), for \(i = 1, 2, \ldots, It\) are the simulated regions generated based on \(It = 1000\) iterations, \(I_{\theta}(\mathcal{C}_i) = 1\), if \(\theta \in \mathcal{C}_i\), 0, if \(\theta \notin \mathcal{C}_i\).

These criteria were applied to both the 95% balanced and smallest CIs, along with the balanced and optimal JCRs. By conducting this simulation analysis, the influence of distinct sample sizes and varying levels can be observed on the characteristics of the intervals and regions. It is worthy to note that the analysis highlighted the relationship between interval lengths and region areas under different cases of sample size and censoring degree.

In this context, we conduct a series of simulations involving 1000 generated samples from Weibull distribution characterized by parameters \((\alpha = 1, \beta = 1)\) and \((\alpha = 1, \beta = 2)\). These simulations are carried out using two distinct sample sizes: \(n = 25\) and \(n = 50\). To explore the impact of various censoring schemes, we consider a range of censoring levels changing from a minimal 20% up to a substantial 80%. Specifically, the following censoring schemes are considered:

Censoring Schemes with \(n=25\): \((19*0.5), (5,19*0), (4*0.20), (4*1,16)\);

Censoring Schemes with \(n=50\): \((39*0.10),(10,39*0),(9*0.40), (9*1,31)\).

We further employ the subsequent algorithm to simulate Weibull PCS data, followed by simulated MSs and CPs.

**Algorithm 1.** MC sampling for obtaining MSs and CPs.

**Step 1.** Generate Type-II PCS data following the WE distribution, using the specified shape parameter \(\alpha\) and inverted scale parameter \(\beta\):

(i) Generate a specific number (say \(m\)) independent uniform \(U(0,1)\) observations \(U_1, U_2, \ldots, U_m\);

(ii) Set \(V_i = U_i^{1/(\alpha + \sum_{k=m-i+1}^{m} r_k)}\) for \(i = 1, 2, \ldots, m\);

(iii) Set \(U_{i:m:n} = 1 - V_m V_{m-1} \cdots V_{m-i+1}\) for \(i = 1, 2, \ldots, m\);

(iv) Obtain \(X_{i:m:n} = \left[-\frac{1}{\beta} \log(1 - U_{i:m:n})\right]^{1/\alpha}, i = 1, 2, \ldots, m\). Then \(X_{1:m:n}, \ldots, X_{m:m:n}\), are the required Type-II PCS data from Weibull distribution.

**Step 2.** Given a censoring scheme with \(n\) and \(r\), compute the balanced and smallest confidence sets of \(\alpha, \beta\) and \((\alpha, \beta)\) discussed in Sections 2 and 3;

**Step 3.** Repeat steps 1 and 2, \(It = 1000\) iterations;

**Step 4.** Compute the MSs and CPs of the balanced and smallest confidence sets of \(\alpha, \beta\) and \((\alpha, \beta)\).
Table 3. MSs and CPs of the balanced and smallest confidence sets of \( \alpha, \beta \) and \( (\alpha, \beta) \) with \( n = 25 \) from Weibull distribution.

<table>
<thead>
<tr>
<th>( r = (19 \times 0, 5) )</th>
<th>( r = (5, 19 \times 0) )</th>
<th>( r = (4 \times 0, 20) )</th>
<th>( r = (4 \times 1, 16) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha ) Balanced</td>
<td>1.521</td>
<td>1.356</td>
<td>1.975</td>
</tr>
<tr>
<td>Balanced</td>
<td>1.521</td>
<td>1.356</td>
<td>1.975</td>
</tr>
<tr>
<td>Smallest</td>
<td>1.274</td>
<td>1.090</td>
<td>1.835</td>
</tr>
<tr>
<td>Smallest</td>
<td>1.274</td>
<td>1.090</td>
<td>1.835</td>
</tr>
<tr>
<td>( \beta ) Balanced</td>
<td>1.654</td>
<td>1.375</td>
<td>1.786</td>
</tr>
<tr>
<td>Balanced</td>
<td>1.654</td>
<td>1.375</td>
<td>1.786</td>
</tr>
<tr>
<td>Smallest</td>
<td>1.432</td>
<td>1.278</td>
<td>1.599</td>
</tr>
<tr>
<td>Smallest</td>
<td>1.432</td>
<td>1.278</td>
<td>1.599</td>
</tr>
<tr>
<td>( \alpha, \beta ) Balanced</td>
<td>0.0719</td>
<td>0.0505</td>
<td>0.3540</td>
</tr>
<tr>
<td>Balanced</td>
<td>0.0719</td>
<td>0.0505</td>
<td>0.3540</td>
</tr>
<tr>
<td>Smallest</td>
<td>0.0084</td>
<td>0.0031</td>
<td>0.0668</td>
</tr>
<tr>
<td>Smallest</td>
<td>0.0084</td>
<td>0.0031</td>
<td>0.0668</td>
</tr>
</tbody>
</table>

| \( \alpha = 1, \beta = 2 \) |
|---------------------------|---------------------------|---------------------------|---------------------------|
| \( r = (39 \times 0, 10) \) | \( r = (10, 39 \times 0) \) | \( r = (9 \times 0, 40) \) | \( r = (9 \times 1, 31) \) |
| \( \alpha \) Balanced     | 1.477                     | 1.057                     | 1.762                     | 1.499                     |
| Balanced                  | 1.477                     | 1.057                     | 1.762                     | 1.499                     |
| Smallest                  | 1.287                     | 0.988                     | 1.591                     | 1.349                     |
| Smallest                  | 1.287                     | 0.988                     | 1.591                     | 1.349                     |
| \( \beta \) Balanced      | 1.298                     | 1.176                     | 1.705                     | 1.499                     |
| Balanced                  | 1.298                     | 1.176                     | 1.705                     | 1.499                     |
| Smallest                  | 1.209                     | 1.104                     | 1.473                     | 1.299                     |
| Smallest                  | 1.209                     | 1.104                     | 1.473                     | 1.299                     |
| \( \alpha, \beta \) Balanced | 0.0290                 | 0.0177                  | 0.1547                    | 0.0754                    |
| Balanced                  | 0.0290                 | 0.0177                  | 0.1547                    | 0.0754                    |
| Smallest                  | 0.0038                  | 0.0021                  | 0.0135                    | 0.0053                    |
| Smallest                  | 0.0038                  | 0.0021                  | 0.0135                    | 0.0053                    |

| \( \alpha = 1, \beta = 2 \) |
|---------------------------|---------------------------|---------------------------|---------------------------|
| \( r = (19 \times 0, 5) \) | \( r = (5, 19 \times 0) \) | \( r = (4 \times 0, 20) \) | \( r = (4 \times 1, 16) \) |
| \( \alpha \) Balanced     | 1.521                     | 1.356                     | 1.975                     | 1.864                     |
| Balanced                  | 1.521                     | 1.356                     | 1.975                     | 1.864                     |
| Smallest                  | 1.274                     | 1.090                     | 1.835                     | 1.634                     |
| Smallest                  | 1.274                     | 1.090                     | 1.835                     | 1.634                     |
| \( \beta \) Balanced      | 1.654                     | 1.375                     | 1.786                     | 1.564                     |
| Balanced                  | 1.654                     | 1.375                     | 1.786                     | 1.564                     |
| Smallest                  | 1.432                     | 1.278                     | 1.599                     | 1.433                     |
| Smallest                  | 1.432                     | 1.278                     | 1.599                     | 1.433                     |
| \( \alpha, \beta \) Balanced | 0.0719                 | 0.0505                  | 0.3540                    | 0.1722                    |
| Balanced                  | 0.0719                 | 0.0505                  | 0.3540                    | 0.1722                    |
| Smallest                  | 0.0084                  | 0.0031                  | 0.0668                    | 0.0375                    |
| Smallest                  | 0.0084                  | 0.0031                  | 0.0668                    | 0.0375                    |

Table 4. MSs and CPs of the balanced and smallest confidence sets of \( \alpha, \beta \) and \( (\alpha, \beta) \) with \( n = 50 \) from Weibull distribution.

<table>
<thead>
<tr>
<th>( r = (39 \times 0, 10) )</th>
<th>( r = (10, 39 \times 0) )</th>
<th>( r = (9 \times 0, 40) )</th>
<th>( r = (9 \times 1, 31) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha ) Balanced</td>
<td>1.477</td>
<td>1.057</td>
<td>1.762</td>
</tr>
<tr>
<td>Balanced</td>
<td>1.477</td>
<td>1.057</td>
<td>1.762</td>
</tr>
<tr>
<td>Smallest</td>
<td>1.287</td>
<td>0.988</td>
<td>1.591</td>
</tr>
<tr>
<td>Smallest</td>
<td>1.287</td>
<td>0.988</td>
<td>1.591</td>
</tr>
<tr>
<td>( \beta ) Balanced</td>
<td>1.298</td>
<td>1.176</td>
<td>1.705</td>
</tr>
<tr>
<td>Balanced</td>
<td>1.298</td>
<td>1.176</td>
<td>1.705</td>
</tr>
<tr>
<td>Smallest</td>
<td>1.209</td>
<td>1.104</td>
<td>1.473</td>
</tr>
<tr>
<td>Smallest</td>
<td>1.209</td>
<td>1.104</td>
<td>1.473</td>
</tr>
<tr>
<td>( \alpha, \beta ) Balanced</td>
<td>0.0290</td>
<td>0.0177</td>
<td>0.1547</td>
</tr>
<tr>
<td>Balanced</td>
<td>0.0290</td>
<td>0.0177</td>
<td>0.1547</td>
</tr>
<tr>
<td>Smallest</td>
<td>0.0038</td>
<td>0.0021</td>
<td>0.0135</td>
</tr>
<tr>
<td>Smallest</td>
<td>0.0038</td>
<td>0.0021</td>
<td>0.0135</td>
</tr>
</tbody>
</table>

Using the above mentioned censoring schemes and different sample sizes, the calculated MSs and the associated CPs for the estimated CIs of the model parameters are presented in Tables 3 and 4. Evidently, as sample size gets large, the mean sizes of the CIs consistently decrease. An interesting observation is that as the censoring level increases, the mean sizes of the CIs tend to expand. This highlights the relationship between censoring percentage and interval size. Furthermore, a clear trend is observed in the MS of the shortest length CIs with respect to the ones based on the balanced method. Remarkably, when the sample size decreases or the censoring level increases, these CIs based on the smallest size method exhibit a more substantial reduction in mean size compared to their corresponding balanced counterparts. This finding shows the sensitivity of these intervals to variations in both sample size and censoring degree, taking into accounts their nature in response to changing...
conditions. Furthermore, it is apparent that the CPs of the simulated CIs consistently tend to be around 95% which is the expected nominal level.

6. Conclusion

In this paper, we consider the problem of obtaining the joint confidence regions of the unknown parameters involved in the generalized exponential class which is very flexible model including families of distributions with increasing failure, decreasing failure and bathtub failure rates. The optimal regions of the parameters are obtained using the Lagrangian approach for minimizing the corresponding area under a specified coverage probability, $1 - \gamma$. The optimal confidence regions are compared with the alternative balanced confidence regions. Our results show that the optimal confidence regions achieve minimized area when compared with their counterparts obtained via balanced regions or the ones obtained marginally. Further, the so developed results here are quite general ones and they can be applied to any family member belonging to the generalized exponential class. Another problem may be arisen naturally is to extend the work to other censoring schemes such as hybrid and adaptive censorings. A follow-up study along these lines will take place in the near future.

Acknowledgements. The authors are also grateful to the Editor and referees for many constructive suggestions which have helped to improve the paper.

Conflict of interest. On behalf of all authors, the corresponding author states that there is no conflict of interest.

References

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