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SEMITOTAL DOMINATION *VERSUS* DOMINATION AND TOTAL DOMINATION IN TREES

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Abstract. A set S of vertices in G is a semitotal dominating set of G if it is a dominating set of G and every vertex in S is within distance 2 of another vertex of S. The *semitotal domination number*, $\gamma_{t2}(G)$, is the minimum cardinality of a semitotal dominating set of G. Clearly, $\gamma(G) \leq \gamma_{t2}(G) \leq \gamma_t(G)$. In this paper, for any nontrivial tree T that is not a star, we investigate the ratios $\gamma_{t2}(T)/\gamma(T)$ and $\gamma_t(T)/\gamma_{t2}(T)$, and provide constructive characterizations of trees achieving the upper bounds.

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1. Introduction

Let $G = (V, E)$ be a simple graph without isolated vertices, and let v be a vertex in G. The *open neighborhood* of v is $N(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood* of v is $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex v is $d(v) = |N(v)|$. For two vertices u and v in a connected graph G, the *distance* $d(u, v)$ between u and v is the length of a shortest (u, v)-path in G. The maximum distance among all pairs of vertices of G is the *diameter* of G which is denoted by $\operatorname{diam}(G)$. A *leaf* of G is a vertex of degree 1, and a *support vertex* of G is a vertex adjacent to a leaf. A support vertex that is adjacent to at least two leaves is called a *strong support vertex*. A *subdivided star*, denoted $K^*_{1,t}$, is the star $K_{1,t}$ $(t \geq 2)$ with each edge subdivided exactly once. For a vertex v in a rooted tree T, we denote by $C(v)$ and $D(v)$ the set of children and descendants, respectively, of v. The *maximal subtree* at v is the subtree of T induced by $D(v) \cup \{v\}$, and is denoted by T_v.

A set S of vertices of a graph G is called a *dominating set* (respectively, *total dominating set*) of G if every vertex in $V(G) \setminus S$ (respectively, $V(G)$) is adjacent to at least one vertex in S. The *domination number* (respectively, *total domination number*) of G, denoted by $\gamma(G)$ (respectively, $\gamma_t(G)$), is the minimum cardinality of a dominating set (respectively, total dominating set) of G.

The concept of semitotal domination in graphs was introduced by Goddard *et al.* [5], and studied further in [2, 4, 6, 8, 9, 11, 12] and elsewhere. A set S of vertices in G is a *semitotal dominating set* of G if it is a dominating set of G and every vertex in S is within distance 2 of another vertex of S. The *semitotal domination number*, $\gamma_{t2}(G)$, is the minimum cardinality of a semitotal dominating set of G. A semitotal dominating set (respectively, dominating set, total dominating set) of G with cardinality $\gamma_{t2}(G)$ (respectively, $\gamma(G)$, $\gamma_t(G)$) is called a $\gamma_{t2}(G)$*-set* (respectively, $\gamma(G)$*-set*, $\gamma_t(G)$*-set*).

Keywords. Domination number, total domination number, semitotal domination number.

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Since every total dominating set is a semitotal dominating set, and every semitotal dominating set is a dominating set, we have the following result first observed in [5].

Observation 1.1 ([5])**.** For every graph G with no isolated vertex, $\gamma(G) \le \gamma_{t2}(G) \le \gamma_t(G)$.

Hence, the semitotal domination number is squeezed between arguably the two most important domination parameters; namely, the domination number and the total domination numbers.

In 1979, B. Bollobás and Cockayne [1] studied the relationship between the domination and total domination numbers of a graph.

Observation 1.2 ([1])**.** For every graph G with no isolated vertex, $\gamma_t(G) \le 2\gamma(G)$.

A constructive characterization of trees T satisfying $\gamma_t(T)/\gamma(T) = 2$ is given in [7]. Motivated by the above results, we consider the ratios $\gamma_{t2}(T)/\gamma(T)$ and $\gamma_t(T)/\gamma_{t2}(T)$. More precisely, we prove the following results.

Theorem 1.3. *For every nontrivial tree T that is not a star, we have $\gamma_{t2}(T)/\gamma(T) < 2$, and the bound is asymptotically tight.*

Theorem 1.4. *For every nontrivial tree T, we have $\gamma_t(T)/\gamma_{t2}(T) < 2$, and the bound is asymptotically tight.*

2. Main results

From the definitions of domination number, semitotal domination number and total domination number, we have the following observations.

Observation 2.1. Let G be a connected graph that is not a star. Then,

(i) there is a γ-set of G that contains no leaf, and
(ii) [8] there is a γ_{t2}-set of G that contains no leaf;
(ii) [10] there is a γ_t-set of G that contains no leaf.

For our purposes, we define a *labeling* of a tree T as a partition $S = (S_A, S_B, S_C, S_D, S_E)$ of $V(T)$ (this idea of labeling the vertices is introduced in [3]). We will refer to the pair (T, S) as a *labeled tree*. The label or *status* of a vertex v, denoted sta(v), is the letter $x \in \{A, B, C, D, E\}$ such that $v \in S_x$.

In what follows, we give three operations as follows:

Operation $\mathscr{O}_1$. Let v be a vertex with sta$(v) = A$ or B. Add a vertex u and the edge uv. Let sta$(u) = C$.

Operation $\mathscr{O}_2$. Let v be a vertex with sta$(v) = A$. Add a path $u_1u_2u_3u_4$ and the edge u_1v. Let sta$(u_1) = D$, sta$(u_2) = E$, sta$(u_3) = B$ and sta$(u_4) = C$.

Operation $\mathscr{O}_3$. Let v be a vertex with sta$(v) = A$. Add a path $u_1u_2u_3$ and the edge u_1v. Let sta$(u_1) = D$, sta$(u_2) = B$, sta$(u_3) = C$.

The three operations $\mathscr{O}_1$, $\mathscr{O}_2$ and $\mathscr{O}_3$ are illustrated in Figures 1c–1e.

Next, we are ready to give two families $\mathscr{T}$ and $\mathscr{T}_1$.

Let $\mathscr{T}$ be the family of labeled trees that: (i) contains (P_6, S_0) where S_0 is the labeling that assigns to the two leaves of the path P_6 status C, to the two support vertices status A and B respectively, and to the two center vertices status D and E respectively (see Fig. 1a); and (ii) is closed under the two operations $\mathscr{O}_1$ and $\mathscr{O}_2$ that are listed above, which extend the tree T' to a tree T by attaching a path to the vertex $v \in V(T')$.

Let $\mathscr{T}_1$ be the family of labeled trees that: (i) contains (P_5, S_0') where S_0' is the labeling that assigns to the two leaves of the path P_5 status C, to the two support vertices status B and A respectively, and to the central vertex status D (see Fig. 1b); and (ii) is closed under the two operations $\mathscr{O}_1$ and $\mathscr{O}_3$ that are listed above, which extend the tree T' to a tree T by attaching a path to the vertex $v \in V(T')$.

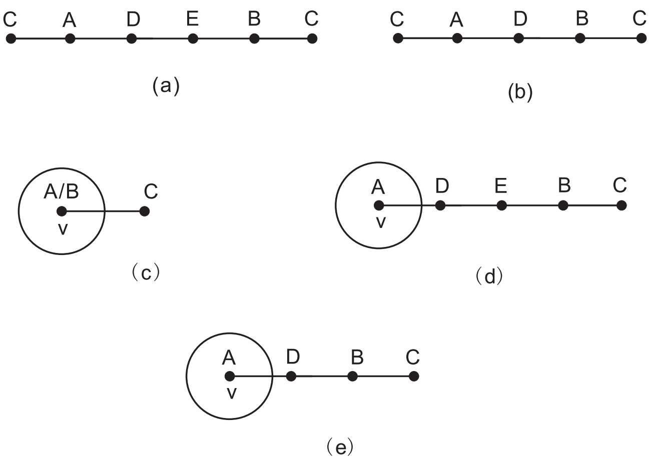

FIGURE 1. Labeled trees (P_6, S_0), (P_5, S_0') and three operations $\mathscr{O}_1$, $\mathscr{O}_2$, $\mathscr{O}_3$.

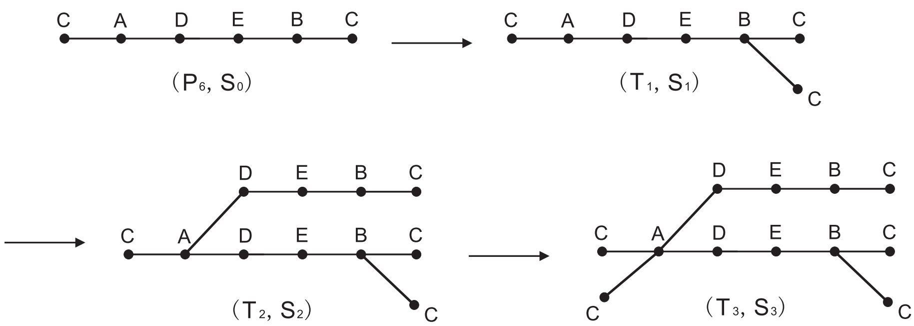

FIGURE 2. An example.

If $(T, S) \in \mathscr{T}$ (or $\mathscr{T}_1$) is a labeled tree for some labeling S. Then there is a sequence of labeled trees (P_6, S_0) (or (P_5, S_0')), $(T_1, S_1), \ldots, (T_{k-1}, S_{k-1})$, (T_k, S_k) such that $(T_k, S_k) = (T, S)$. The labeled tree (T_i, S_i) can be obtained from (T_{i-1}, S_{i-1}) by one of the operations $\mathscr{O}_1$ and $\mathscr{O}_2$ (or $\mathscr{O}_1$ and $\mathscr{O}_3$), where $i \in \{1, 2, \ldots, k\}$. We remark that a sequence of labeled trees used to construct (T, S) is not necessarily unique.

We take an example to make it easier for reader to understand the family $\mathscr{T}$ (The graphs belonging to $\mathscr{T}_1$ can be constructed by a similar way). In Figure 2, $(P_6, S_0) \in \mathscr{T}$, (T_1, S_1) is obtained from (P_6, S_0) by operation $\mathscr{O}_1$, (T_2, S_2) is obtained from (T_1, S_1) by operation $\mathscr{O}_2$, and (T_3, S_3) is obtained from (T_2, S_2) by operation $\mathscr{O}_1$. Thus, (T_1, S_1), (T_2, S_2), $(T_3, S_3) \in \mathscr{T}$.

Two main conclusions of our paper are listed as follows.

Theorem 2.2. *If T is a nontrivial tree that is not a star, then $\gamma_{t2}(T) \leq 2\gamma(T) - 1$. Moreover, the trees T satisfying $\gamma_{t2}(T) = 2\gamma(T) - 1$ are precisely those trees T such that $(T, S) \in \mathscr{T}$ for some labeling S.*

Theorem 2.3. *If T is a nontrivial tree, then $\gamma_t(T) \leq 2\gamma_{t2}(T) - 1$. Moreover, the trees T satisfying $\gamma_t(T) = 2\gamma_{t2}(T) - 1$ are precisely those trees T such that $(T, S) \in \mathscr{T}_1$ for some labeling S.*

3. THE PROOF OF THEOREM 2.2

In what follows, we present a few preliminary results.

Observation 3.1. Let T be a tree of order at least 6 and S be a labeling of T such that $(T, S) \in \mathscr{T}$. Then, T has the following properties:

(a) A vertex is labeled A or B if and only if it is a support vertex.
(b) A vertex is labeled C if and only if it is a leaf.
(c) $|S_A| = 1$, $|S_B| = |S_D| = |S_E|$.
(d) The set $S_A \cup S_B$ is the unique γ-set of T.
(e) The set $S_A \cup S_B \cup S_D$ is a γ_{t2}-set of T.
(f) If a vertex has status A (respectively, B), then each of its non-leaf neighbors is labeled D (respectively, E).
(g) If a vertex has status D (respectively, E), then it has degree two and the two neighbors are labeled A and E (respectively, B and D).

From Observation 3.1(c), (d) and (e), the following corollary can be derived immediately.

Corollary 3.2. *Let T be a nontrivial tree that is not a star and S be a labeling of T such that $(T, S) \in \mathscr{T}$. Then, $\gamma_{t2}(T) = 2\gamma(T) - 1$.*

Lemma 3.3. *If T is a nontrivial tree that is not a star, then $\gamma_{t2}(T) \leq 2\gamma(T) - 1$. Moreover, if the equality holds, then $(T, S) \in \mathscr{T}$ for some labeling S.*

Proof. The proof is by induction on the order of T. If $|V(T)| \leq 6$, it is easy to verify that $\gamma_{t2}(T) \leq 2\gamma(T) - 1$, and $T = P_6$ when the equality holds. So we let $|V(T)| \geq 7$ and assume that for every non-star tree T' of order less than $|V(T)|$ we have $\gamma_{t2}(T') \leq 2\gamma(T') - 1$, if the equality holds, then $(T', S') \in \mathscr{T}$ for some labeling S'. By Observation 2.1(i), there exists a γ-set of T which contains no leaf, say D.

Claim 1. If there exists a strong support vertex in T, then $\gamma_{t2}(T) \leq 2\gamma(T) - 1$. Moreover, if the equality holds, then $(T, S) \in \mathscr{T}$ for some labeling S.

Suppose that v is a support vertex which has at least two leaf-neighbors, say v_1, v_2. Let $T' = T - v_1$ and R be a γ_{t2}-set of T' containing no leaf. Then, R is also a semitotal dominating set of T. Hence, $\gamma_{t2}(T) \leq \gamma_{t2}(T')$. Combining the fact that $\gamma(T') \leq \gamma(T)$, we have that $\gamma_{t2}(T) \leq \gamma_{t2}(T') \leq 2\gamma(T') - 1 \leq 2\gamma(T) - 1$. If $\gamma_{t2}(T) = 2\gamma(T) - 1$, then we have that $\gamma_{t2}(T') = 2\gamma(T') - 1$. By the inductive hypothesis, $(T', S') \in \mathscr{T}$ for some labeling S'. It follows from Observation 3.1(a) that v has status A or B in S'. Let S be obtained from S' by labeling the vertex v_1 with label C. Then, (T, S) can be obtained from (T', S') by operation $\mathscr{O}_1$. Thus, $(T, S) \in \mathscr{T}$.

By Claim 1, we may assume that no strong support vertex in T.

Let $P = v_1 v_2 v_3 \dots v_t$ be a longest path in T such that $d(v_3)$ is as large as possible. Since T is a nontrivial tree that is not a star, $\operatorname{diam}(T) \geq 3$. Note that T is a double star when $\operatorname{diam}(T) = 3$, and each neighbor of v_3 outside P is a leaf or a support vertex of degree two when $\operatorname{diam}(T) = 4$. In either case, $\gamma_{t2}(T) = \gamma(T) = d(v_3) \geq 2$, and then $\gamma_{t2}(T) \leq 2\gamma(T) - 2$. If $\operatorname{diam}(T) = 5$, then for either of the vertices v_3 and v_4, there are two possibilities: (1) It has degree two; (2) Each of its neighbors outside P is a leaf or a support vertex of degree two. Then, we always have $2 \leq |D| \leq |R| \leq |D| + 1$, where R is a γ_{t2}-set of T containing no leaf. It means that $\gamma_{t2}(T) \leq 2\gamma(T) - 2$ unless $T = P_6$. Hence, we suppose that $\operatorname{diam}(T) \geq 6$.

Root the tree T at the vertex v_t.

Claim 2. If $d(v_3) \geq 3$, then $\gamma_{t2}(T) \leq 2\gamma(T) - 2$.

If $d(v_3) \geq 3$, let $T' = T - \{v_1, v_2\}$ and R' be a γ_{t2}-set of T' containing no leaf. Note that $D \setminus \{v_2\}$ is a dominating set of T'. On the other hand, $R' \cup \{v_2\}$ is a semitotal dominating set of T. Thus, $\gamma_{t2}(T) \leq \gamma_{t2}(T') + 1 \leq 2\gamma(T') - 1 + 1 \leq 2\gamma(T) - 2$.

By Claim 2, we may assume that $d(v_3) = 2$.

Claim 3. If $d(v_4) \geq 3$, then $\gamma_{t2}(T) \leq 2\gamma(T) - 2$.

Assume that $d(v_4) \geq 3$ and u_1 is a neighbor of v_4 outside P. From the assumption that no strong support vertex in T and the choice of P, we have that one of the three conditions as follows holds:

(1) u_1 is a leaf;
(2) u_1 has degree two and is adjacent to a support vertex of degree two, say u_2;
(3) u_1 is a support vertex of degree two.

In the first case, let $T' = T - \{v_1, v_2, v_3\}$ and R' be a γ_{t2}-set of T' containing no leaf. Note that v_4 belongs to D and R'. Because of $v_2 \in D$, $D \setminus \{v_2\}$ is a dominating set of T'. And then, $\gamma(T) - 1 \geq \gamma(T')$. On the other hand, $R' \cup \{v_2\}$ is a semitotal dominating set of T. That is, $\gamma_{t2}(T') + 1 \geq \gamma_{t2}(T)$. Then, $\gamma_{t2}(T) \leq \gamma_{t2}(T') + 1 \leq 2\gamma(T') - 1 + 1 \leq 2\gamma(T) - 2$.

In the second case, let $T' = T - \{v_1, v_2, v_3\}$ and R' be a γ_{t2}-set of T' containing no leaf. Note that $v_2 \in D$, then clearly $\gamma(T) - 1 \geq \gamma(T')$. On the other hand, since $u_2 \in R'$, $\{u_1, v_4\} \cap R' \neq \emptyset$. Without loss of generality, $v_4 \in R'$. (Otherwise, we replace the vertex u_1 in R' with v_4, and the resulting set is also a γ_{t2}-set of T'). Then $R' \cup \{v_2\}$ is a semitotal dominating set of T. And so, $\gamma_{t2}(T') + 1 \geq \gamma_{t2}(T)$. Hence, $\gamma_{t2}(T) \leq \gamma_{t2}(T') + 1 \leq 2\gamma(T') - 1 + 1 \leq 2\gamma(T) - 2$.

In the third case, we may assume that each neighbor of v_4 outside P (including u_1) is a support vertex of degree two. Let $T' = T - T_{v_4}$ and R' be a γ_{t2}-set of T' containing no leaf. Since $(N(v_4) \setminus \{v_3, v_5\}) \cup \{v_2\} \subseteq D$, $\gamma(T) - (d(v_4) - 1) \geq \gamma(T')$. On the other hand, note that $|T'| \geq 2$. If T' is a star, the tree T is determined. It is easy to check that $\gamma_{t2}(T) \leq 2\gamma(T) - 2$. So we assume that T' is not a star. Clearly, $R' \cup (N(v_4) \setminus \{v_5\}) \cup \{v_2\}$ is a semitotal dominating set of T, so $\gamma_{t2}(T') + d(v_4) \geq \gamma_{t2}(T)$. Hence, $\gamma_{t2}(T) \leq \gamma_{t2}(T') + d(v_4) \leq 2\gamma(T') - 1 + d(v_4) \leq 2\gamma(T) - d(v_4) + 1 \leq 2\gamma(T) - 2$.

By Claim 3, we may assume that $d(v_4) = 2$.

Now, we let $T' = T - \{v_1, v_2, v_3, v_4\}$ and R' be a γ_{t2}-set of T' (We may assume that T' is not a star, otherwise the tree T is determined, and it is easy to check that $\gamma_{t2}(T) \leq 2\gamma(T) - 2$). Clearly, $D' = (D \setminus \{v_3, v_4\}) \cup \{v_5\}$ is still a γ-set of T. It implies that $\gamma(T) - 1 \geq \gamma(T')$. On the other hand, $R' \cup \{v_2, v_3\}$ be a semitotal dominating set of T. Hence, $\gamma_{t2}(T) \leq \gamma_{t2}(T') + 2 \leq 2\gamma(T') - 1 + 2 \leq 2\gamma(T) - 1$. Suppose next that $\gamma_{t2}(T) = 2\gamma(T) - 1$. Then we have equality throughout the above inequality chain. In particular, $\gamma_{t2}(T') = 2\gamma(T') - 1$. By induction, $(T', S') \in \mathscr{T}$ for some labeling S'.

If v_5 has status A in S', let S be obtained from S' by labeling the vertices v_1, v_2, v_3, v_4 with label C, B, E, D, respectively. Then, (T, S) can be obtained from (T', S') by operation $\mathscr{O}_2$. Thus, $(T, S) \in \mathscr{T}$. If $\mathrm{sta}(v_5) \in \{C, D, E\}$ in S', or $\mathrm{sta}(v_5) = B$ in S' and $|T'| > 6$, we always have that $\gamma_{t2}(T) \leq 2\gamma(T) - 2$, a contradiction. If $\mathrm{sta}(v_5) = B$ in S' and $|T'| = 6$, then $T' = P_6$. Moreover, v_6, v_7, v_8 have status E, D, A in S', respectively. Let S'' be obtained from S' by relabeling the vertices v_5, v_6, v_7, v_8 with label A, D, E, B, respectively. And let S be obtained from S'' by labeling the vertices v_1, v_2, v_3, v_4 with label C, B, E, D, respectively. Thus, we can also obtain that $(T, S) \in \mathscr{T}$. □

Combining Corollary 3.2 and Lemma 3.3, we complete the proof of Theorem 2.2.

4. The proof of Theorem 2.3

Observation 4.1. Let T be a tree of order at least 5 and S be a labeling of T such that $(T, S) \in \mathscr{T}_1$. Then, T has the following properties:

(a) A vertex is labeled A or B if and only if it is a support vertex.
(b) A vertex is labeled C if and only if it is a leaf.
(c) $|S_A| = 1$, $|S_B| = |S_D|$.
(d) The set $S_A \cup S_B$ is the unique γ_{t2}-set of T.
(e) The set $S_A \cup S_B \cup S_D$ is a γ_t-set of T.
(f) If a vertex has status A (respectively, B), then each of its non-leaf neighbors is labeled D.
(g) If a vertex has status D, then it has degree two and the two neighbors are labeled A and B.

From Observation 4.1(c), (d) and (e), the following corollary can be derived immediately.

Corollary 4.2. *Let T be a nontrivial tree and S be a labeling of T such that $(T, S) \in \mathscr{T}_1$. Then, $\gamma_t(T) = 2\gamma_{t2}(T) - 1$.*

Lemma 4.3. *If T is a nontrivial tree, then $\gamma_t(T) \leq 2\gamma_{t2}(T) - 1$. Moreover, if the equality holds, then $(T, S) \in \mathscr{T}_1$ for some labeling S.*

Proof. The proof is by induction on the order of T. If $|V(T)| \leq 5$, it is easy to verify that $\gamma_t(T) \leq 2\gamma_{t2}(T) - 1$, and $T = P_5$ when the equality holds. So we let $|V(T)| \geq 6$ and assume that for every nontrivial tree T' of order less than $|V(T)|$ we have $\gamma_t(T') \leq 2\gamma_{t2}(T') - 1$, if the equality holds, then $(T', S') \in \mathscr{T}_1$ for some labeling S'. By Observation 2.1(ii), there exists a γ_{t2}-set of T which contains no leaf, say D.

If there is a strong support vertex in T, say v, then let $T' = T - v_1$, where v_1 is a leaf-neighbor of v. By a similar argument as in the proof of Lemma 3.3, we have that $\gamma_t(T) \leq \gamma_t(T') \leq 2\gamma_{t2}(T') - 1 \leq 2\gamma_{t2}(T) - 1$. If $\gamma_t(T) = 2\gamma_{t2}(T) - 1$, then we have that $\gamma_t(T') = 2\gamma_{t2}(T') - 1$. By the inductive hypothesis, $(T', S') \in \mathscr{T}_1$ for some labeling S'. It follows from Observation 4.1(a) that v has status A or B in S'. Let S be obtained from S' by labeling the vertex v_1 with label C. Then, (T, S) can be obtained from (T', S') by operation $\mathscr{O}_1$. That is, $(T, S) \in \mathscr{T}_1$. So we assume that no strong support vertex in T.

Let $P = v_1 v_2 v_3 \ldots v_t$ be a longest path in T such that

(i) $d(v_4)$ is as large as possible, and subject to this condition;
(ii) $d(v_3)$ is as large as possible.

If $\operatorname{diam}(T) \leq 4$, by a similar argument as in the proof of Lemma 3.3, we have that $\gamma_t(T) \leq 2\gamma_{t2}(T) - 2$. So, suppose that $\operatorname{diam}(T) \geq 5$.

Root the tree T at the vertex v_t.

Claim 4. If there exists a vertex u in T such that T_u is a subdivided star $K^*_{1,t}$ $(t \geq 2)$ and u is the central vertex of the subdivided star, then $\gamma_t(T) \leq 2\gamma_{t2}(T) - 2$.

Assume that such a vertex u exists. Note that each vertex of $C(u)$ is a support vertex of T, and belongs to D. If $u \in D$, then the parent of u (the vertex belonging to $N(u) \setminus C(u)$), say u', is not in D (Otherwise, $D \setminus \{u\}$ is still a semitotal dominating set of T, a contradiction). We let $D' = (D \setminus (C(u) \cup \{u\})) \cup \{u'\}$. If $u, u' \notin D$, let $D' = D \setminus C(u)$. In either case, D' is a semitotal dominating set of the tree T', where $T' = T - T_u$. Thus, $\gamma_{t2}(T) - t \geq \gamma_{t2}(T')$. On the other hand, let R' be a γ_t-set of T', and $R = R' \cup (N[u] \setminus \{u'\})$. Clearly, R is a total dominating set of T. So, $\gamma_t(T') + t + 1 \geq \gamma_t(T)$. And then, $\gamma_t(T) \leq \gamma_t(T') + t + 1 \leq 2\gamma_{t2}(T') - 1 + t + 1 \leq 2\gamma_{t2}(T) - t \leq 2\gamma_{t2}(T) - 2$.

Now, we consider the case that $u \notin D$ and $u' \in D$. Let $T' = T - T_x$, where $x \in C(u)$. Clearly, $D \setminus \{x\}$ is a semitotal dominating set of T'. Let R' be a γ_t-set of T' containing no leaf, then $R' \cup \{x\}$ is a total dominating set of T. Thus, $\gamma_t(T) \leq \gamma_t(T') + 1 \leq 2\gamma_{t2}(T') - 1 + 1 \leq 2\gamma_{t2}(T) - 2$.

By Claim 4, we may assume that there is no such vertex u in T.

Claim 5. If $d(v_3) \geq 3$, then $\gamma_t(T) \leq 2\gamma_{t2}(T) - 2$.

Assume that $d(v_3) \geq 3$, and u is a neighbor of v_3 outside P.

We distinguish two cases.

Case 1. u is a leaf of T.

Clearly, $v_3 \in D$. If v_3 is within distance two from a vertex in $D \setminus \{v_2, v_3\}$, then let $T' = T - \{v_1, v_2\}$. We conclude that $D \setminus \{v_2\}$ is a semitotal dominating set of T'. That is, $\gamma_{t2}(T) - 1 \geq \gamma_{t2}(T')$. Let R' be a γ_t-set of T', then $v_3 \in R'$. It means that $R' \cup \{v_2\}$ is a total dominating set of T, and so $\gamma_t(T') + 1 \geq \gamma_t(T)$. We have that $\gamma_t(T) \leq \gamma_t(T') + 1 \leq 2\gamma_{t2}(T') - 1 + 1 \leq 2\gamma_{t2}(T) - 2$.

Now we consider the case that all vertices in $D \setminus \{v_2, v_3\}$ are distance at least three from v_3. Then, $v_4, v_5 \notin D$. Moreover, $d(v_3) = 3$.

Subcase 1.1. $d(v_4) = 2$.

Let $T' = T - T_{v_4}$. Note that $|T'| \geq 2$. Clearly, $D \setminus \{v_2, v_3\}$ is a semitotal dominating set of T'. That is, $\gamma_{t2}(T) - 2 \geq \gamma_{t2}(T')$. Let R' be a γ_t-set of T', then $R' \cup \{v_2, v_3\}$ is a total dominating set of T, and so $\gamma_t(T') + 2 \geq \gamma_t(T)$. We have that $\gamma_t(T) \leq \gamma_t(T') + 2 \leq 2\gamma_{t2}(T') - 1 + 2 \leq 2\gamma_{t2}(T) - 3$.

Subcase 1.2. $d(v_4) > 2$.

Let u_1 be a neighbor of v_4 outside P. Note that u_1 is neither a leaf nor a support vertex. Combining this with Claim 4, we have that T_{u_1} is a path P_3. Clearly, $|\{u_1, u_2, v_4\} \cap D| \geq 2$, where u_2 is the neighbor of u_1 rather than v_4. But it contradicts with the condition that all vertices in $D \setminus \{v_2, v_3\}$ are distance at least three from v_3.

Case 2. u is a support vertex of T, and v_3 has no leaf-neighbor.

In this case, $d(u) = 2$. Note that T_{v_3} is a subdivided star. By Claim 4, we are done.

By Claim 5, we may assume that $d(v_3) = 2$.

Since $v_2 \in D$, $\{v_3, v_4\} \cap D \neq \emptyset$. Without loss of generality, $v_4 \in D$. (Otherwise, we replace the vertex v_3 in D with v_4, and the resulting set is also a γ_{t2}-set of T).

Claim 6. If all vertices in $D \setminus \{v_2, v_4\}$ are distance at least three from v_4, then $\gamma_t(T) \leq 2\gamma_{t2}(T) - 2$.

If all vertices in $D \setminus \{v_2, v_4\}$ are distance at least three from v_4, it is easy to verify that $d(v_4) = 2$. If $d(v_5) = 2$, let $T' = T - T_{v_5}$ (We may assume that $|T'| \geq 2$, otherwise the tree T is determined, and it is easy to check that $\gamma_t(T) \leq 2\gamma_{t2}(T) - 2$). Clearly, $D \setminus \{v_2, v_4\}$ is a semitotal dominating set of T'. That is, $\gamma_{t2}(T) - 2 \geq \gamma_{t2}(T')$. Let R' be a γ_t-set of T', then $R' \cup \{v_2, v_3, v_4\}$ is a total dominating set of T, and so $\gamma_t(T') + 3 \geq \gamma_t(T)$. We have that $\gamma_t(T) \leq \gamma_t(T') + 3 \leq 2\gamma_{t2}(T') - 1 + 3 \leq 2\gamma_{t2}(T) - 2$.

Hence, suppose that $d(v_5) > 2$, and w_1 is the neighbor of v_5 outside P. There must be a leaf outside P, say x, and a path P', such that P' is the shortest path between w_1 and x, and $V(P') \cap V(P) = \emptyset$. Let $P' = w_1 w_2 \cdots w_k$, where $w_k = x$. By the choice of P, $k \leq 4$. If $1 \leq k \leq 3$, it follows from Claim 4 and the choice of D that there always exists a vertex $y \in D$ such that $d(y, v_4) \leq 2$. So we consider the case of $k = 4$. From the assumption that no strong vertex in T and the choice of P, we have that $d(w_1) = d(w_2) = d(w_3) = 2$, and $w_3 \in D$, $|\{w_1, w_2\} \cap D| \geq 1$. If $w_1 \in D$, then $d(v_4, w_1) = 2$, a contradiction. If $w_2 \in D$ and $w_1 \notin D$, let $T' = T - T_{w_1}$. Since $D \setminus \{w_2, w_3\}$ is a semitotal dominating set of T', $\gamma_{t2}(T) - 2 \geq \gamma_{t2}(T')$. It is easy to see that $R' \cup \{w_2, w_3\}$ is a total dominating set of T, where R' is a γ_t-set of T'. That is, $\gamma_t(T') + 2 \geq \gamma_t(T)$. We have that $\gamma_t(T) \leq \gamma_t(T') + 2 \leq 2\gamma_{t2}(T') - 1 + 2 \leq 2\gamma_{t2}(T) - 3$.

By Claim 6, we assume that v_4 is within distance two from some vertex in $D \setminus \{v_2, v_4\}$.

Next, let $T' = T - \{v_1, v_2, v_3\}$. By Claim 6, $D \setminus \{v_2\}$ is a semitotal dominating set of T'. Then, $\gamma_{t2}(T) - 1 \geq \gamma_{t2}(T')$. Let R' be a γ_t-set of T', $R' \cup \{v_2, v_3\}$ is a total dominating set of T. That is, $\gamma_t(T') + 2 \geq \gamma_t(T)$. We have that $\gamma_t(T) \leq \gamma_t(T') + 2 \leq 2\gamma_{t2}(T') - 1 + 2 \leq 2\gamma_{t2}(T) - 1$. Suppose next that $\gamma_t(T) = 2\gamma_{t2}(T) - 1$. Then we have equality throughout the above inequality chain. In particular, $\gamma_t(T') = 2\gamma_{t2}(T') - 1$. By induction, $(T', S') \in \mathscr{T}_1$ for some labeling S'.

If v_4 has status A in S', let S be obtained from S' by labeling the vertices v_1, v_2, v_3 with label C, B, D, respectively. Then, (T, S) can be obtained from (T', S') by operation $\mathscr{O}_3$. Thus, $(T, S) \in \mathscr{T}$. If $\mathrm{sta}(v_4) \in \{C, D\}$ in S', or $\mathrm{sta}(v_4) = B$ in S' and $|T'| > 5$, we always have that $\gamma_{t2}(T) \leq 2\gamma(T) - 2$, a contradiction. If $\mathrm{sta}(v_4) = B$ in S' and $|T'| = 5$, then $T' = P_5$. Moreover, v_5, v_6 have status D, A in S', respectively. Let S'' be obtained from S' by relabeling the vertices v_4, v_5, v_6 with label A, D, B, respectively. And let S be obtained from S'' by labeling the vertices v_1, v_2, v_3 with label C, B, D, respectively. Thus, we can also obtain that $(T, S) \in \mathscr{T}_1$. □

Combining Corollary 4.2 and Lemma 4.3, we complete the proof of Theorem 2.3.

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