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MERGING DECISION-MAKING UNITS IN THE SIMULTANEOUS PRESENCE OF DESIRABLE AND UNDESIRABLE FACTORS

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Abstract. This paper is devoted to applying the inverse Data Envelopment Analysis (InvDEA) in the simultaneous presence of desirable and undesirable factors. One of the most common ways to improve units' performance in the business environment is through activity synergies called units' merging. The present study models how to identify the inherited input/output from the units participating in the merger process to achieve the desired efficiency goal. The proposed models are established based on the InvDEA approach and multiple-objective programming tools. Sufficient conditions to estimate desirable and undesirable data are obtained using Pareto solutions to multi-objective programming problems. The theory extended in the study is explained by an application in the banking sector.

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1. Introduction

The conventional data envelopment analysis (DEA) models are usually applied to evaluate the performance of decision making units (DMUs) based on linear programming (LP) problems. The input level in DEA models refers to the various resources or factors that DMUs utilize in their production or operational processes. DEA aims to assess how efficiently these inputs are being used to generate outputs. The output level involves identifying and quantifying the products, services, or outcomes (outputs) that decision-making units produce from utilizing their inputs. In these problems, input levels are minimized while the output levels are maximized. The traditional DEA approach is based on identifying the most desirable set of weights to maximize the efficiency of each unit based on an optimistic approach [6].

Merging units in DEA involves using efficiency analysis to identify and combine DMUs to create a more efficient and effective organizational structure through an inverse DEA (InvDEA) approach. The process requires a thorough understanding of the individual units, their inputs and outputs, and the potential synergies that can be obtained through the merging action. The motivation behind merging is deeply rooted in the pursuit of

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operational excellence and the desire to overcome inefficiencies that may persist at the particular DMU level. By combining forces, organizations aim to eliminate redundant processes and harness collective capabilities, achieving economies of scale and scope. In this area, our focus extends beyond the conventional DEA to address merging DMUs with desirable data. We will examine methodologies for handling undesirable outputs or inputs in the merger process. By navigating the complexities posed by undesirable data, organizations can ensure a more accurate evaluation of efficiency and a better understanding of the consequences and benefits of merging decision-making units.

This study primarily focuses on the merging of units containing undesirable data. It delves into three key concepts: the merging process, DEA applied to undesirable data, and the utilization of Inverse DEA as the primary tool for unit merging. The next section provides a literature review organized around these three themes. At the end of Section 2, we outline the specific contributions made by our research.

2. State of the art

This section presents a concise literature review on the merging of units within the context of DEA. The review is structured as follows: firstly, an examination of DEA with undesirable data is provided; secondly, an explanation of research related to inverse DEA is presented; and finally, some research on merging within the DEA framework is discussed.

2.1. DEA model with undesirable data

In ordinary DEA models, it is assumed that inputs are the resources or factors consumed or utilized by the decision-making units. These can include tangible resources like labor, capital, raw materials, energy, and machinery, and intangible resources like skills, knowledge, and technology. Outputs are the desirable results, products, or services the decision-making units produce by using inputs. However, this assumption may not be realized in some realistic situations *i.e.*, some output can be undesirable. In the environmental applications of DEA, pollutants or emissions are typical examples of undesirable outputs. For instance, a manufacturing plant seeks to minimize the amount of pollution it releases into the environment. In the context of the banking sector, an undesirable output could be the level of non-performing loans (NPLs). Non-performing loans are loans that are in default or close to default, representing a risk and financial loss for the bank. In this scenario, the bank want to minimize the level of non-performing loans. According to [28], the techniques to deal with undesirable data in the DEA can be divided into direct and indirect approaches. The main idea of the mentioned approaches is based on the fact that to improve the performance of an inefficient DMU, desirable and undesirable inputs should decrease and increase, respectively; besides, the desirable and undesirable outputs should increase and decrease, respectively. In a recent paper, Omrani *et al.* [41] used two existing computationally intensive DEA models (three-step and modified three-step methods) alongside two newly proposed models (common weight goal programming DEA and common weight DEA) to evaluate efficiencies under undesirable outputs. The paper utilizes the TOPSIS approach to integrate the results from the four models. It provides an overall ranking for Iranian airlines based on technical, social, environmental, and sustainable efficiencies, suggesting a managerial focus on improving identified weaknesses.

The direct approaches refer to treating the undesirable data in its original form, such as parametric data and directional distance functions; see, *e.g.*, [8, 15, 16, 28, 32, 45, 47, 48, 53, 61, 62] for some reviews. The indirect approaches refer to approaches that treat the undesirable output/input as a classical input/output. In contrast, the undesirable output/input is moved to the input/output side of the model after some transformation and treated as one of the inputs/outputs, as both inputs/outputs and undesirable outputs/inputs are the values that need to be minimized/maximized. Therefore, it is acceptable to treat both in the same manner [4, 34, 36, 43, 46, 52, 54, 55, 57]. Some of the latest research on undesirable data has been listed in Table 1 with a short description. According to the second approach, Liu *et al.* [36] proposed considering undesirable inputs as desirable outputs and undesirable outputs as desirable inputs. This method is currently used as an attractive technique in studying

TABLE 1. Some of the latest undesirable data research.

Method type	Reference	Short description
Direct	Toloo and Hanclova [53]	Proposed two directional distance models and developed multiplier- and envelopment-based selection approaches for combining multi-valued measures while considering undesirable outputs.
	Kao and Hwang [32]	Introduced a concept to assess the minimum level of undesirable outputs permissible for a DMU based on the principle of weak disposability.
	Shi *et al.* [47]	Proposed a new SBM-NDEA model that combines undesirable outputs to evaluate the performance of complex production processes with series and parallel components.
Indirect	Taher *et al.* [52]	Presented a method for interval data and used the inverse DEA method through a MOLP to improve the performance of the evaluated unit.
	Parashkouh *et al.* [43]	An alternative definition of the weak possibility assumption was introduced to address the problem of undesirable outputs.
	Liu and Xu [35]	The SBM-Undesirable model for strongly efficient projections was proposed considering undesirable outputs.

operational efficiency because of its simplicity and elegance. Since this approach is employed in our study, it will be explained in Section 3.1.

2.2. Inverse DEA

An important issue in the DEA field is Inverse DEA (InvDEA). This issue has been extensively studied from both theoretical and practical aspects in the last two decades. In InvDEA, a certain input/output level and a pre-defined efficiency goal are considered strategic targets. The underlying aim of this issue is to calculate the required output/input levels to attain this pre-defined efficiency goal. In comparison, the main aim of the conventional DEA models is to estimate the efficiency score of a unit with known input and output levels. In [60], under the preserving efficiency score, the input level increments of a unit are measured for its given output level increments. According to the DEA literature, the following basic problem is considered in InvDEA: Among a set of units, if the efficiency score of a unit remains unchanged but the output/input levels change, to what extent should the unit input/output levels change? This problem has been studied in many theoretical and applied publications, some of them given in Table 2 with brief descriptions.

2.3. Merging units in DEA

In the business environment, merging is a financial transaction in which at least two companies join each other and continue operations as one legal entity. Generally, merging can be divided into five different categories [31]: i) Horizontal merger, ii) Vertical merger, iii) Market-extension merger, iv) Product-extension merger, and v) Conglomerate merger. In this regard, the type of merger selected by a company primarily depends on the motives and objectives of the companies participating in a deal. In [17], the inherited input/output levels from merging units are estimated based on InvDEA to achieve the pre-determined goal level. Amin *et al.* [3] suggested a new method to anticipate whether a merger in a market is generating a major or a minor consolidation. According to the DEA literature, the InvDEA has been adopted to solve the problem of merging units in the

Table 2. Some research in the field of InvDEA.

Reference	Short description
Jahanshahloo *et al.* [30]	Extending the InvDEA problems for input/output estimation under inter-temporal dependence data addressed in [12].
Hadi-vencheh *et al.* [27]	InvDEA models are extended under interval data.
Dong Joon [10]	presented an InvDEA model to estimate the outputs based on decision weights according to the expected changes to the production frontier.
Ghiyasi [18]	The InvDEA problems extended in which the technical and cost (revenue) efficiency of all DMUs are maintained simultaneously.
Xiao *et al.* [58]	A two-stage cost efficiency model provided by minimizing the cost of the new unit while it maintains the sum of the output levels of the merging units.
Eyni *et al.* [14]	The InvDEA problems extended in the presence of undesirable and desirable data.
Chen *et al.* [7]	A novel InvDEA model provided for analyzing the investment problem in the presence of undesirable outputs.
Ghobadi [21]	The problems of the InvDEA solved using the non-radial enhanced Russell[44] under fuzzy data.
Emrouznejad *et al.* [13]	A new InvDEA model is developed to achieve pre-defined goals to allocate CO2 emission quota in different Chinese regions.
Ghiyasi [19]	The problems of InvDEA extended in the presence of controllable and non-controllable inputs. Also, some trade-off indexes were proposed for tracking the relationship between needed input, producible output and efficiency improvement percentage.
Ghiyasi [20]	the InvDEA problems were solved using novel criterion models. This leads to a reduction of computational complexity. Moreover, the proposed models solved some problematical fails of the InvDEA models.
Ghobadi and Jahangiri [24]	A new InvDEA model provided for optimal allocation of resources based on the ideal-solutions.
Wegener and Amin [56]	A novel InvDEA model presented to minimizing greenhouse gas emissions generated by a set of units for producing a certain level of outputs, provided that the units preserve at least their existing efficiency status.
Soleimani-Chamkhorami *et al.* [49]	responding to the following questions: To what extent outputs/inputs should be incremented in order to preserve the unit under evaluation revenue/cost-efficient, when the inputs/output of a unit increment?
Soleimani-Chamkhorami *et al.* [50]	The idea of InvDEA utilized for ranking extreme efficient units based on the growth potential.
Ghobadi [22]	A novel InvDEA model provided for resource allocation in the dynamic DEA framework.
Hu *et al.* [29]	presented a revised model where the investigated unit has no slack variable because radial InvDEA models may fail due to the neglect of slack variables.
Daryani *et al.* [9]	A four-stage method proposed to estimate of the inputs and outputs with a two-stage network structure method.

presence of negative data [2], interval data [23], fuzzy data [25], and under inter-temporal dependence data [59]. Furthermore, Guijarro *et al.* [26] proposed a new method for units' restructuring. In this method, the identification of merging units and estimation of inputs is done by genetic algorithm and InvDEA, respectively. Recently, Chang *et al.* [5] focuses on aiding acquirer companies in effectively selecting target companies in mergers and acquisitions (M&As) to enhance their performance through synergy, specifically in horizontal and vertical integration settings.The study introduces novel DEA-based Nash bargaining models to assist acquirer companies in achieving their desired outcomes when engaging in M&A activities. Moazeni *et al.* [40] used the SCOR model, which incorporates sustainability, resilience, and agility criteria and selects relevant criteria in the supply chain network. The merger optimization was facilitated through an inverse network DEA mathematical model, aiming to identify the optimal combination of merged units. Oukil *et al.* [42] proposed a hybrid DEA methodology consisting of two stages to identify optimal matches among hotels to enhance sector performance. The initial stage employs an inverse Data Envelopment Analysis (IDEA) model to evaluate the maximum gains that could potentially be generated from pairwise consolidations among hotels.

However, the proposed models in all of the mentioned studies fail to solve the problem of merging units in the simultaneous presence of desirable and undesirable data. This shortcoming could be due to the restrictions of the fundamental InvDEA models utilized in their modeling. As a result, a large gap still exists in DEA research on the problem of merging units concerning undesirable factors, although the use or disposal of undesirable factors is of growing importance in the economy. The main contribution of our study is dedicated to fixing this gap, which addresses the following subjects:

– Novel InvDEA models are provided to achieve a pre-defined efficiency target of a merger in the simultaneous presence of desirable and undesirable data.
– This paper also provides models to determine the minimum amount of efficiency that can be achieved through a given merger. These models are multi-objective programming (MOP) that provides various scenarios through the Pareto solution, and the manager has diverse options for selection.
– Sufficient conditions to estimate desirable and undesirable data are obtained using Pareto solutions to MOP problems.

The rest of this article is organized as follows: Section 2 reviews some of the basic concepts and models used and generalized in our study. Section 3 is devoted to the main results of this study. This section provides the new InvDEA models for calculating the inherited input/output levels from merging units to achieve the pre-determined goal level, where desirable and undesirable factors are present. An application of the banking sector is provided to clarify the proposed approach in Section 4. Finally, a summary of our findings and further research work aspects is presented in Section 5.

3. Prerequisites

This section presents some applied models as prerequisites, such as initial DEA models under undesirable data and multiple objective programming.

3.1. Basic models for undesirable data

In this section, a mathematical programming model is reviewed to estimate the efficiency score of DMU in the presence of desirable and undesirable data. Suppose that there exists a set of n observations of DMUs, $\{\text{DMU}_j : \ j \in J = \{1, 2, \ldots, n\}\}$ which the related symbols have been shown in Table 3.

As can be seen in Table 3, there exist two kinds of inputs: desirable inputs (denoted by x^D) and undesirable inputs (represented by x^U), to produce two types of outputs: desirable outputs (represented by y^D)and undesirable outputs (denoted by y^U). There are a lot of models to evaluate DMUs with undesirable data. In this

Table 3. Main symbols.

Class	Description	Symbol
Indices sets	Index set of Units	$J = \{1, 2, \ldots, n\}$,
	Index set of Inputs	$I = \{1, 2, \ldots, m\}$,
	Index set of Outputs	$O = \{1, 2, \ldots, s\}$,
	Index set of Desirable inputs	$I_1 \subset I$,
	Index set of Undesirable inputs	$I_2 \subset I$
	Index set of Desirable outputs	$O_1 \subset O$,
	Index set of Undesirable outputs	$O_2 \subset O$,
Parameters	Desirable inputs	$x_j^D = (x_{ij}^D : \ \forall i \in I_1)$,
	Undesirable inputs	$x_j^U = (x_{ij}^U : \ \forall i \in I_2)$,
	Desirable outputs	$y_j^D = (y_{rj}^D : \ \forall r \in O_1)$,
	Undesirable outputs	$y_j^U = (y_{rj}^U : \ \forall r \in O_2)$,
	Parameters with $0-1$ values	$\delta_j \ , \ j \in \{1, 2, 3\}$
Variables	Decision Variables	$\lambda_j, \ j \in J$
	Efficiency score (Object function)	$\varphi \ , \ \theta$

Notes. $I_1 \cup I_2 = I, \ O_1 \cup O_2 = O, \ I_1 \cap I_2 = \phi$ and $O_1 \cap O_2 = \phi$.

study, the following input-oriented DEA model is considered to estimate the relative efficiency of the unit under assessment, DMU_o, $o \in J$ [36]:

$$
\begin{aligned}
\theta_o = \min \ & \theta \\
\text{s.t.} \ & \sum_{j=1}^{n} \lambda_j x_{ij}^D \leq \theta x_{io}^D, && \forall i \in I_1, \\
& \sum_{j=1}^{n} \lambda_j y_{rj}^U \leq \theta y_{ro}^U, && \forall r \in O_2, \\
& \sum_{j=1}^{n} \lambda_j x_{ij}^U \geq x_{io}^U, && \forall i \in I_2, \\
& \sum_{j=1}^{n} \lambda_j y_{rj}^D \geq y_{ro}^D, && \forall r \in O_1, \\
& \lambda = (\lambda_1, \ldots, \lambda_n) \in \Omega,
\end{aligned} \tag{1}
$$

where

$$
\begin{aligned}
\Omega = \{\lambda | \lambda = (\lambda_1, \ldots, \lambda_n), \ \delta_1 (\sum_{j=1}^{n} \lambda_j + \delta_2 (-1)^{\delta_3} \nu) = \delta_1, \\
\nu \geq 0, \ \lambda_j \geq 0, \ j = 1, 2, \ldots, n\}.
\end{aligned} \tag{2}
$$

where (λ, θ) is the variable vector and θ_o as an objective function is called the input-oriented efficiency score of DMU$_o$. In addition, δ_1, δ_2, and δ_3 are parameters with $0-1$ values. It is not difficult to see that:

(i) If $\delta_1 = 0$, then model (1) is under a constant returns to scale (CRS) assumption;

(ii) If $\delta_1 = 1$ and $\delta_2 = 0$, then model (1) is under a variable returns to scale (VRS) assumption;
(iii) If $\delta_1 = \delta_2 = 1$ and $\delta_3 = 0$, then model (1) is under a non-increasing returns to scale (NIRS) assumption;
(iv) If $\delta_1 = \delta_2 = \delta_3 = 1$, then model (1) is under a non-decreasing returns to scale (NDRS) assumption of the production technology.

The output-oriented version of model (1) is as follows:

$$\begin{aligned}
\varphi_o = \max\ & \varphi \\
\text{s.t.}\ & \sum_{j=1}^{n} \lambda_j x_{ij}^{D} \leq x_{io}^{D}, && \forall i \in I_1, \\
& \sum_{j=1}^{n} \lambda_j y_{rj}^{U} \leq y_{ro}^{U}, && \forall r \in O_2, \\
& \sum_{j=1}^{n} \lambda_j x_{ij}^{U} \geq \varphi x_{io}^{U}, && \forall i \in I_2, \\
& \sum_{j=1}^{n} \lambda_j y_{rj}^{D} \geq \varphi y_{ro}^{D}, && \forall r \in O_1, \\
& \lambda = (\lambda_1, \ldots, \lambda_n) \in \Omega.
\end{aligned} \tag{3}$$

where (λ, φ) is variables vector and φ_o is called the output-oriented efficiency score of DMU_o.

3.2. Multi-objective programming

A multi-objective programming (MOP) is an optimization problem that involves multiple objective functions and some constraints. A mathematical formulation of MOP is the following [11]:

$$\begin{aligned}
& Min f(x) \\
\text{s.t.}\quad & x \in X
\end{aligned} \tag{4}$$

where $X = \{x \in R^m : g_k(x) \leq 0,\ \ k = 1, 2, \ldots, p\}$ is the set of feasible solutions , $f : R^m \to R^q$ and $g : R^m \to R^p$ are the two vector functions, (*i.e.* $f(x) = (f_1(x), f_2(x), \ldots, f_q(x))$, and $g(x) = (g_1(x), g_2(x), \ldots, g_p(x))$). If $f(.)$ and $g(.)$ are linear functions, model (4) is renamed multiple-objective linear programming (MOLP). Single-objective programming aims to find the optimal value of a single objective function. In contrast, MOP seeks to find the optimal values of several objective functions, each representing a different goal. These objectives can conflict, meaning optimizing one objective may worsen the others. It means that sometimes there is no solution $x \in X$ to minimize all objective functions; therefore, the Pareto solutions should be considered instead of optimal solutions. Assume $\bar{x} \in X$ is a feasible solution of MOP (4), $\bar{x}$ is called a Pareto solution when there does not exist $x \in X$ in a sense that $f_i(x) \leq f_i(\bar{x})$ for all $i = 1, 2, \ldots, q$ or $f_i(x) < f_i(\bar{x})$ for some $i = 1, 2, \ldots, q$.

Inverse DEA suggests specific adjustments to inputs and outputs such that the efficiency score remains unchanged or would make those units more efficient. For this purpose, the MOP models are used which one kind of them is the following:

$$\begin{aligned}
& Min(\alpha_{1o}, \alpha_{2o}, \ldots, \alpha_{mo}) \\
\text{s.t.}\ & \sum_{j=1}^{n} x_{ij}\lambda_j \leq \theta^* \alpha_{io}, && i = 1, 2, \ldots, m, \\
& \sum_{j=1}^{n} y_{rj}\lambda_j \geq \beta_{ro}, && r = 1, 2, \ldots, s, \\
& \alpha_{io} \geq x_{io}, && i = 1, 2, \ldots, m, \\
& \lambda \in \Omega.
\end{aligned} \tag{5}$$

where θ^* is the efficiency score of DMU under evaluation (DMU_o). A lot of research in InvDEA applies MOP and multi-criteria decision-making algorithms (MCDM). MOP and MCDM are essential tools that were utilized in different areas, for example, [1, 37, 38] just as, in this paper, we use MOP to develop the new model for merging units.

4. Merging DMUs in the presence of desirable and undesirable data

Merging occurs when at least two companies combine their activities to create a new company with better performance. There are different motives for mergers in the business environment. Two firms may merge to increase the wealth of their shareholders. Generally, synergy means that the value of new generated firm exceeds the sum of the values of two individual firms. In this respect, revenue synergies and cost synergies are two common types of synergies. A merger can be motivated for diversification reasons, such as a market extension or product extension. Mergers can acquire certain assets that cannot be attained using other methods, *e.g.*, access to new technologies. Mergers are often done due to a lack of sufficient financial capacity. In fact, a merged company will secure a better financial capacity that can be used in further business development processes. A merger can be motivated by decreasing tax liability reasons. In other words, if a firm generates notable taxable revenue, it can merge with a firm with portable tax losses so that the total tax liability of the merged firm is much less than the tax debt of the independent firm. A merger can be motivated due to the personal interests and goals of a firm's top management. For example, the compensation of managers, base salary, and performance bonuses are correlated with the size of a firm. Such a motive can also be reinforced by the managers' ego and their intention to build the biggest company in the industry. According to the above discussion, the most common motivations for the firms merging could be divided into the following six groups: i) Value creation, ii) Diversification, iii) Acquisition of assets, iv) Increase in financial capacity, v) Tax purposes, and vi) Incentives for managers.

In this section, according to the attained approach to the most benefit, a method is proposed for treating the simultaneous presence of desirable and undesirable data in the problem of merging units. Let us to assume that there is a set of selected entities to merge, such as $\{DMU_j,\ j \in \Lambda \subset J\}$. In other words, Λ is a set of entities selected to create a synergy by merging and creating a new entity (denoted by DMU_q) with a specific efficiency objective. It is clear that the set of units participating in the merger process will be removed after the merger. Let $\{DMU_j,\ j \in \Pi = J - \Lambda\}$ be the set of units that have not participated in the merger process. With regard to the attain approach to the most benefit in the process of merger, it is assumed that DMU_q keeps the amount of undesirable/desirable inputs and desirable/undesirable outputs of a set of the merging entities and looks for the most/least amount of the inherited undesirable/desirable outputs and desirable/undesirable inputs to attain given efficiency target.

In this section, a method is proposed to treat the problem of merging units in the simultaneous presence of desirable and undesirable data.

4.1. Merging units under desirable inputs and undesirable outputs

Suppose that DMU_q keeps the amount of undesirable inputs and desirable outputs of a set of the merging DMUs ($\{DMU_j,\ j \in \Lambda \subset J\}$) and looks for the minimum amount of desirable inputs and undesirable outputs of these DMUs in order to reach the pre-defined target level, $\bar{\theta}_q$. Therefore, the undesirable inputs and desirable outputs of DMU_q are $x^U_{iq} = \sum_{j\in\Lambda} x^U_{ij}$ for each $i \in I_2$ and $y^D_{rq} = \sum_{j\in\Lambda} y^D_{rj}$ for each $r \in O_2$. We need to estimate the desirable input vector x^D_q and the undesirable output vector y^U_q in order to reach the desired given efficiency target $\bar{\theta}_q$. After estimating them, the following model measures the efficiency score of DMU_q:

$$\begin{aligned}
\theta_o = \min\ & \theta \\
\text{s.t. } & \sum_{j\in\Pi} \lambda_j x_{ij}^D + \lambda_q x_{iq}^D \le \theta x_{iq}^D, && \forall i \in I_1, \\
& \sum_{j\in\Pi} \lambda_j y_{rj}^U + \lambda_q y_{rq}^U \le \theta y_{rq}^U, && \forall r \in O_2, \\
& \sum_{j\in\Pi} \lambda_j x_{ij}^U + \lambda_q x_{iq}^U \ge x_{io}^U, && \forall i \in I_2, \\
& \sum_{j\in\Pi} \lambda_j y_{rj}^D + \lambda_q y_{rq}^D \ge y_{ro}^D, && \forall r \in O_1, \\
& \lambda = (\lambda_j;\ j \in \Pi \cup \{q\}) \in \Omega_q,
\end{aligned} \tag{6}$$

where

$$\Omega_q = \{\lambda | \lambda = (\lambda_j;\ j \in \Pi \cup \{q\}),\ \delta_1(\textstyle\sum_{j\in\Pi} \lambda_j + \lambda_q + \delta_2(-1)^{\delta_3}\nu) = \delta_1,\ \nu \ge 0,\ \lambda_j \ge 0,\ \forall j \in \Pi \cup \{q\}\}.$$

where $(\lambda,\ \theta)$ is variables vector. If the optimal value of Model (6) is equal $\bar{\theta}_q$, we say that the expected efficiency score of the generated unit has been achieved.

To estimate the desirable input vector x_q^D and the undesirable output vector y_q^U in order to reach the desired given efficiency target $\bar{\theta}_q$, along the lines of [17], we proposed the following multiple-objective non-linear programming (MONLP) model:

$$\begin{aligned}
\min\ & (\alpha_{ij}^D:\ \forall i \in I_1\ \ \&\ \ \beta_{rj}^U:\ \forall r \in O_2) \quad \forall j \in \Lambda, \\
\text{s.t. } & \sum_{j\in\Pi} \lambda_j x_{ij}^D + \lambda_q \sum_{j\in\Lambda} \alpha_{ij}^D \le \bar{\theta}_q \sum_{j\in\Lambda} \alpha_{ij}^D,\ \forall i \in I_1, && (5.1) \\
& \sum_{j\in\Pi} \lambda_j y_{rj}^U + \lambda_q \sum_{j\in\Lambda} \beta_{rj}^U \le \bar{\theta}_q \sum_{j\in\Lambda} \beta_{rj}^U,\ \forall r \in O_2, && (5.2) \\
& \sum_{j\in\Pi} \lambda_j x_{ij}^U + \lambda_q x_{iq}^U \ge x_{iq}^U, \quad \forall i \in I_2, && (5.3) \\
& \sum_{j\in\Pi} \lambda_j y_{rj}^D + \lambda_q y_{rq}^D \ge y_{rq}^D, \quad \forall r \in O_1, && (5.4) \\
& 0 \le \alpha_{ij}^D \le x_{ij}^D, \quad \forall i \in I_1, \forall j \in \Lambda, && (5.5) \\
& 0 \le \beta_{rj}^U \le y_{rj}^U, \quad \forall i \in O_2, \forall j \in \Lambda, && (5.6) \\
& \lambda = (\lambda_j;\ j \in \Pi \cup \{q\}) \in \Omega_q, && (5.7)
\end{aligned} \tag{7}$$

where $(\lambda,\ \alpha_{ij}^D,\ \beta_{rj}^U, \forall i \in I_1,\ \forall r \in O_2,\ \forall j \in \Lambda)$ is the variables vector. $\bar{\theta}_q$ is the expected efficiency score for DMU_q. α_{ij}^D and β_{rj}^U are indicates the amount of the ith desirable inputs and the rth undesirable outputs inherited by DMU_q from the jth unit of participating in the merger process, respectively. With regard to that the main motivation of M&A in our study is to achieve the maximum profit through saving resources of desirable and products of undesirable, so the goals of the inverse DEA model (7) are designed. In fact, this goals guarantees that the desirable inputs and undesirable outputs inherited by DMU_q from the participating units in the merger process are minimized to achieve efficiency aim, $\bar{\theta}_q$. equations (5.1)–(5.4), and (5.7) guarantees that DMU_q has the expected performance. Equations (5.5) and (5.6) ensures that the amount of the desirable resources and the undesirable products received by DMU_q does not exceed from the amount of desirable resources and undesirable products available to each of the units involved in the integration process.

In the real world, the most common consolidations happen between units to improve their respective performances and, in general, this naturally implies improving their technical efficiencies. This means that the merging do not change the pre-merging efficiency frontier. Clearly, if $\bar{\theta}_q < 1$ which is DMU_q is inefficient, or

$\bar{\theta}_q = 1$ and DMU_q is non-extreme efficient, then the corresponding λ_q will be zero in optimality ($\lambda_q^* = 0$), and this simplifies the non-linear InvDEA model (7) to multiple-objective linear programming model (8):

$$\begin{aligned}
&\min \ (\alpha_{ij}^D : \ \forall i \in I_1 \ \ \& \ \ \beta_{rj}^U : \ \forall r \in O_2) \quad \forall j \in \Lambda,\\
&\text{s.t.} \sum_{j\in\Pi} \lambda_j x_{ij}^D \le \bar{\theta}_q \sum_{j\in\Lambda} \alpha_{ij}^D, && \forall i \in I_1,\\
&\sum_{j\in\Pi} \lambda_j y_{rj}^U \le \bar{\theta}_q \sum_{j\in\Lambda} \beta_{rj}^U, && \forall r \in O_2,\\
&\sum_{j\in\Pi} \lambda_j x_{ij}^U \ge x_{iq}^U, && \forall i \in I_2,\\
&\sum_{j\in\Pi} \lambda_j y_{rj}^D \ge y_{rq}^D, && \forall r \in O_1,\\
&0 \le \alpha_{ij}^D \le x_{ij}^D, && \forall i \in I_1, \forall j \in \Lambda,\\
&0 \le \beta_{rj}^U \le y_{rj}^U, && \forall i \in O_2, \forall j \in \Lambda,\\
&\lambda = (\lambda_j ; \ j \in \Pi) \in \overline{\Omega}_q.
\end{aligned} \tag{8}$$

In this paper, we limit our development to the case of the consolidation where the merged unit is within the current PPS. Clearly, the merged unit will be inside of the current PPS, if and only if the virtual unit $(x_q^D, \ x_q^U, \ y_q^D, \ y_{rq}^U)$ where

$$\begin{cases}
x_{iq}^D = \sum_{j\in\Lambda} x_{ij}^D; \ \forall i \in I_1,\\
x_{iq}^U = \sum_{j\in\Lambda} x_{ij}^U; \ \forall i \in I_2,\\
y_{rq}^D = \sum_{j\in\Lambda} y_{rj}^D; \forall r \in O_1,\\
y_{rq}^U = \sum_{j\in\Lambda} y_{rj}^U; \forall r \in O_2,
\end{cases} \tag{9}$$

is within the PPS. This comes from the objective of the MONLP input-oriented InvDEA model (7) as well as the objective of the relaxed input-oriented InvDEA model (8), where it tries to keep the minimum level of the desirable inputs and undesirable outputs of the participating units in the merger process or equivalently the virtual unit. This assumption guarantees that the merged unit is inside the PPS when it is inefficient or on the frontier once it is efficient. Therefore, if we relax the MONLP InvDEA model (7) to the MOLP InvDEA model (8), we will not lose generality. Now, we show that the MOLP (8) can be used for the estimation of desirable inputs and undesirable outputs of the new unit.

Theorem 4.1. *Suppose that the merged unit (DMU_q) is within the PPS. Also, let $\Delta = (\lambda^*, \alpha_{ij}^{D*} : \ \forall i \in I_1, \ \beta_{rj}^{U*} : \ \forall r \in O_2)$ be a Pareto solution to Model (8). If $x_{iq}^D = \sum_{j\in\Lambda} \alpha_{ij}^{D*}$ ($\forall i \in I_1$) and $y_{rq}^U = \sum_{j\in\Lambda} \beta_{rj}^{U*}$ ($\forall r \in O_2$), then the optimal value of Model (6) is equal to $\bar{\theta}_q$.*

Proof. We need to prove that, $\theta_q = \bar{\theta}_q \le 1$. Feasibility of Δ for Model (8), implies:

$$\sum_{j\in\Pi} \lambda_j^* x_{ij}^D \le \bar{\theta}_q \sum_{j\in\Lambda} \alpha_{ij}^{D*} = \bar{\theta}_q x_{iq}^D \le x_{iq}^D, \qquad \forall i \in I_1, \tag{10}$$

$$\sum_{j\in\Pi} \lambda_j^* y_{rj}^U \le \bar{\theta}_q \sum_{j\in\Lambda} \beta_{rj}^{U*} = \bar{\theta}_q y_{rq}^U \le y_{rq}^U \qquad \forall r \in O_2, \tag{11}$$

$$\sum_{j\in\Lambda} \lambda_j^* x_{ij}^U \ge x_{iq}^U, \qquad \forall i \in I_2, \tag{12}$$

$$\sum_{j\in\Pi} \lambda_j^* y_{rj}^D \ge y_{rq}^D, \qquad \forall r \in O_1, \tag{13}$$

$$0 \leq \alpha_{ij}^{D*} \leq x_{ij}^{D}, \quad \forall i \in I_1, \ \forall j \in \Lambda, \tag{14}$$

$$0 \leq \beta_{rj}^{U*} \leq y_{rj}^{U}, \quad \forall r \in O_2, \ \forall j \in \Lambda, \tag{15}$$

$$\lambda^* = (\lambda_j^*; \ \forall j \in \Pi) \in \overline{\Omega}_q. \tag{16}$$

By equations (10)–(36) and (16), $(\lambda_j = \lambda_j^*; \forall j \in \Pi, \ \lambda_q = 0, \ \theta = \bar{\theta}_q)$ is obviously a feasible solution to problem (6). Therefore, $\theta_q \leq \bar{\theta}_q$.

Considering $(\tilde{\lambda}_j; \forall j \in \Pi, \tilde{\lambda}_q, \ \tilde{\theta})$ as an optimal solution to LP (6) and equation (10), we have

$$\tilde{\theta} x_{iq}^{D} \geq \sum_{j\in\Pi} \tilde{\lambda}_j x_{ij}^{D} + \tilde{\lambda}_q x_{iq}^{D} \geq \sum_{j\in\Pi} \tilde{\lambda}_j x_{ij}^{D} + \tilde{\lambda}_q (\sum_{j\in\Pi} \lambda_j^* x_{ij}^{D}) = \sum_{j\in\Pi} (\tilde{\lambda}_j + \tilde{\lambda}_q \lambda_j^*) x_{ij}^{D}, \ \forall i \in I_1$$

$$\tilde{\theta} y_{rq}^{U} \geq \sum_{j\in\Pi} \tilde{\lambda}_j y_{rj}^{U} + \tilde{\lambda}_q y_{rq}^{U} \geq \sum_{j\in\Pi} \tilde{\lambda}_j y_{rj}^{U} + \tilde{\lambda}_q (\sum_{j\in\Pi} \lambda_j^* y_{rj}^{U}) = \sum_{j\in\Pi} (\tilde{\lambda}_j + \tilde{\lambda}_q \lambda_j^*) y_{rj}^{U}, \ \forall r \in O_2.$$

Set $\bar{\lambda}_j := \tilde{\lambda}_j + \tilde{\lambda}_q \lambda_j^*$ for each $j \in \Pi$, then

$$\sum_{j\in\Pi} \bar{\lambda}_j x_{ij}^{D} \leq \tilde{\theta} x_{iq}^{D}, \quad \forall i \in I_1, \tag{17}$$

$$\sum_{j\in\Pi} \bar{\lambda}_j y_{rj}^{U} \leq \tilde{\theta} y_{rq}^{U}, \quad \forall r \in O_2. \tag{18}$$

Using the same procedure, the inequalities (12) and (36) will be used in LP (6), the following results are obtained:

$$\sum_{j\in\Pi} \bar{\lambda}_j x_{ij}^{U} \geq x_{iq}^{U}, \quad \forall i \in I_2, \tag{19}$$

$$\sum_{j\in\Pi} \bar{\lambda}_j y_{rj}^{D} \geq y_{rq}^{D}, \quad \forall r \in O_1, \tag{20}$$

It is easily seen that

$$\bar{\lambda} = (\bar{\lambda}_j; \forall j \in \Pi) \in \overline{\Omega}_q. \tag{21}$$

By contradiction assume that $\theta_q = \tilde{\theta} < \bar{\theta}_q$. There exists at least one $l \in I_1$ and $k \in \Lambda$ such that $\alpha_{lk}^{D*} > 0$ and also there exists at least one $h \in O_2$ and $f \in \Lambda$ such that $\beta_{hf}^{U*} > 0$, because $x_{iq}^{D} = \sum_{j\in\Lambda} \alpha_{ij}^{D*} \neq 0$ and $y_{rq}^{U} = \sum_{j\in\Lambda} \beta_{rj}^{U*} \neq 0$. By equations (17) and (18), we have

$$\sum_{j\in\Pi} \bar{\lambda}_j x_{lj}^{D} \leq \tilde{\theta} x_{lq}^{D} = \tilde{\theta} \sum_{j\in\Lambda} \alpha_{lj}^{D*} < \bar{\theta}_q \sum_{j\in\Lambda} \alpha_{lj}^{D*}, \tag{22}$$

$$\sum_{j\in\Pi} \bar{\lambda}_j y_{hj}^{U} \leq \tilde{\theta} y_{hq}^{U} = \tilde{\theta} \sum_{j\in\Lambda} \beta_{hj}^{U*} < \bar{\theta}_q \sum_{j\in\Lambda} \beta_{hj}^{U*}. \tag{23}$$

Clearly, there exists $\mu > 0$ such that

$$\alpha_{lj}^{D*} - \mu \geq 0. \tag{24}$$

Therefore, by equation (22), we have

$$\sum_{j\in\Pi}\bar{\lambda}_j x_{lj}^{D} \leq \bar{\theta}_q((\alpha_{lk}^{D*}-\mu)+\sum_{j\in\Lambda-\{k\}}\alpha_{lj}^{D*}). \tag{25}$$

Also, there exists $\kappa > 0$ such that

$$\beta_{rj}^{U*}-\kappa \geq 0. \tag{26}$$

Therefore, by equation (23), we get

$$\sum_{j\in\Pi}\bar{\lambda}_j y_{hj}^{U} \leq \bar{\theta}_q((\beta_{hf}^{D*}-\kappa)+\sum_{j\in\Lambda-\{f\}}\beta_{hj}^{U*}). \tag{27}$$

Now, define

$$\bar{\alpha}_{ij}^{U}=\begin{cases}\alpha_{ij}^{U*}-\mu & if \quad i=l, j=k,\\ \alpha_{ij}^{U*} & otherwise,\end{cases} \tag{28}$$

and

$$\bar{\beta}_{rj}^{U}=\begin{cases}\beta_{rj}^{D*}-\kappa & if \quad r=h, j=f,\\ \beta_{rj}^{D*} & otherwise.\end{cases} \tag{29}$$

By equations (17), (18), (24)–(27), and definitions of (28) and (29), we obtain

$$\sum_{j\in\Pi}\bar{\lambda}_j^{*} x_{ij}^{D} \leq \bar{\theta}_q\sum_{j\in\Lambda}\bar{\alpha}_{ij}^{D}, \qquad \forall i\in I_1, \tag{30}$$

$$\sum_{j\in\Pi}\bar{\lambda}_j^{*} y_{rj}^{U} \leq \bar{\theta}_q\sum_{j\in\Lambda}\bar{\beta}_{rj}^{U}, \qquad \forall r\in O_2, \tag{31}$$

$$0\leq \bar{\alpha}_{ij}^{D*}\leq x_{ij}^{D}, \quad \forall i\in I_1,\ \forall j\in\Lambda, \tag{32}$$

$$0\leq \bar{\beta}_{rj}^{U*}\leq y_{rj}^{U}, \quad \forall r\in O_2,\ \forall j\in\Lambda, \tag{33}$$

According to equations (19)–(21) and (30)–(33), $(\bar{\lambda},\bar{\alpha}_{ij}^{D*},\bar{\beta}_{rj}^{U*}:\forall i\in I_1,\forall r\in O_2,\ \forall j\in\Lambda)$ is a feasible solution to problem (8), such that $(\bar{\alpha}_{ij}^{D*},\bar{\beta}_{rj}^{U*}:\forall i\in I_1,\forall r\in O_2,\ \forall j\in\Lambda)\leq(\alpha_{ij}^{D*},\ \beta_{rj}^{U*}:\forall i\in I_1,\forall r\in O_2,\ \forall j\in\Lambda)$ and $(\bar{\alpha}_{ij}^{D*},\bar{\beta}_{rj}^{U*}:\forall i\in I_1,\forall r\in O_2,\ \forall j\in\Lambda)\neq(\alpha_{ij}^{D*},\ \beta_{rj}^{U*}:\forall i\in I_1,\forall r\in O_2,\ \forall j\in\Lambda)$. But this is impossible because Δ is a Pareto solution to MOLP (8). $\square$

Remark 4.2. It is worth noting that if model (8) is infeasible, it means that the process of merging units changed the pre-merging efficiency frontier. In fact, the merged unit falls outside the pre-merging frontier. In this case, there is an optimal solution such that $\lambda_q^* = 1$. Then, it can be assumed that in the model (7), we have $\lambda_q \in \{0,\ 1\}$. This gives the result that the model (7) can be linearized as follows:

$$
\begin{aligned}
\min\ & (\alpha_{ij}^D:\ \forall i \in I_1\ \ \&\ \ \beta_{rj}^U:\ \forall r \in O_2) \quad \forall j \in \Lambda,\\
\text{s.t.}\ & \sum_{j\in\Pi} \lambda_j x_{ij}^D + \sum_{j\in\Lambda} \overline{\alpha}_{ij}^D \le \bar{\theta}_q \sum_{j\in\Lambda} \alpha_{ij}^D,\ \forall i \in I_1,\\
& \sum_{j\in\Pi} \lambda_j y_{rj}^U + \sum_{j\in\Lambda} \overline{\beta}_{rj}^U \le \bar{\theta}_q \sum_{j\in\Lambda} \beta_{rj}^U,\ \forall r \in O_2,\\
& \sum_{j\in\Pi} \lambda_j x_{ij}^U + \lambda_q x_{iq}^U \ge x_{iq}^U, \qquad \forall i \in I_2,\\
& \sum_{j\in\Pi} \lambda_j y_{rj}^D + \lambda_q y_{rq}^D \ge y_{rq}^D, \qquad \forall r \in O_1,\\
& 0 \le \alpha_{ij}^D \le x_{ij}^D, \qquad \forall i \in I_1, \forall j \in \Lambda,\\
& 0 \le \beta_{rj}^U \le y_{rj}^U, \qquad \forall r \in O_2, \forall j \in \Lambda,\\
& \overline{\alpha}_{ij}^D \le x_{ij}^D \alpha_{ij}^D, \qquad \forall i \in I_1, \forall j \in \Lambda,\\
& \alpha_{ij}^D - (1-\lambda_q) x_{ij}^D \le \overline{\alpha}_{ij}^D \le \alpha_{ij}^D, \qquad \forall i \in I_1, \forall j \in \Lambda,\\
& \overline{\beta}_{rj}^U \le y_{rj}^U \beta_{rj}^U, \qquad \forall r \in O_2, \forall j \in \Lambda,\\
& \beta_{rj}^U - (1-\lambda_q) y_{rj}^U \le \overline{\beta}_{rj}^U \le \beta_{rj}^U, \qquad \forall r \in O_2, \forall j \in \Lambda,\\
& \overline{\alpha}_{ij}^D \ge 0, \qquad \forall i \in I_1, \forall j \in \Lambda,\\
& \overline{\beta}_{rj}^U \ge 0, \qquad \forall r \in O_2, \forall j \in \Lambda,\\
& \lambda_q \in \{0,\ 1\},\\
& \lambda = (\lambda_j;\ j \in \Pi \cup \{q\}) \in \Omega_q,
\end{aligned} \tag{34}
$$

where $\overline{\alpha}_{ij}^D = \lambda_q \alpha_{ij}^D$ for $i \in I_1,\ j \in \Lambda$, and $\overline{\beta}_{rj}^U = \lambda_q \beta_{rj}^U$ for $r \in O_2,\ j \in \Lambda$.

We close this subsection with a discussion on minimum achievable efficiency target of the merged DMU. It is worth noting that knowing minimum achievable efficiency target is necessary for the decision maker deliberating about engaging in the merging process. The lowest efficiency score realized through a merging could be determined *via* the following theorem.

Theorem 4.3. *Consider a merging with $\bar{\theta}_q$ as the efficiency target for the merged DMU.*

i) If model (8) is feasible, then it remains feasible for each efficiency target $\bar{\bar{\theta}}_q$, with $\bar{\theta}_q \le \bar{\bar{\theta}}_q \le 1$.

ii) Suppose that model (8) admits a Pareto solution. If

$$
\begin{aligned}
\min\ & \theta,\\
\text{s.t.}\ & \sum_{j\in\Pi} \lambda_j x_{ij}^D \le \sum_{j\in\Lambda} \alpha_{ij}^D,\ \forall i \in I_1,\\
& \sum_{j\in\Pi} \lambda_j y_{rj}^U \le \sum_{j\in\Lambda} \beta_{rj}^U,\ \forall r \in O_2,\\
& \sum_{j\in\Pi} \lambda_j x_{ij}^U \ge x_{iq}^U, \qquad \forall i \in I_2,\\
& \sum_{j\in\Pi} \lambda_j y_{rj}^D \ge y_{rq}^D, \qquad \forall r \in O_1,
\end{aligned}
$$

$$\begin{aligned}
&\sum_{j\in\Lambda}\alpha_{ij}^{D}\leq\theta\sum_{j\in\Lambda}x_{ij}^{D},\ \forall i\in I_1,\\
&\sum_{j\in\Lambda}\beta_{rj}^{U}\leq\theta\sum_{j\in\Lambda}y_{rj}^{U},\ \forall r\in O_2,\\
&0\leq\alpha_{ij}^{D}\leq x_{ij}^{D}, \qquad \forall i\in I_1,\forall j\in\Lambda,\\
&0\leq\beta_{rj}^{U}\leq y_{rj}^{U}, \qquad \forall r\in O_2,\forall j\in\Lambda,\\
&\lambda=(\lambda_j;\ j\in\Pi)\in\overline{\Omega}_q,
\end{aligned} \tag{35}$$

then $\bar{\theta}_q\geq\theta^*$ *(The minimum achievable efficiency target of the merged DMU is* θ^**).*

Proof. According to the model (8), the proof of part (i) is obvious. By contradiction assume that $\bar{\theta}_q<\theta^*$. Let $\Delta=(\lambda^*,\tilde{\alpha}_{ij}^{D*}:i\in I_1,\forall j\in\Lambda,\tilde{\beta}_{rj}^{U*}:r\in O_2,\forall j\in\Lambda)$ be a Pareto solution to model (8). Feasibility of Δ to model (8), implies:

$$\tilde{\alpha}_{ij}^{D*}\leq x_{ij}^{D},\quad i\in I_1,\ \forall j\in\Lambda, \tag{36}$$

$$\beta_{rj}^{\tilde{U}*}\leq y_{rj}^{U},\quad r\in O_2,\ \forall j\in\Lambda. \tag{37}$$

By (36), (37), and $0<\bar{\theta}_q\leq 1$, we get

$$\bar{\theta}_q\tilde{\alpha}_{ij}^{D*}\leq x_{ij}^{D},\quad i\in I_1,\ \forall j\in\Lambda, \tag{38}$$

$$\bar{\theta}_q\beta_{rj}^{\tilde{U}*}\leq y_{rj}^{U},\quad r\in O_2,\ \forall j\in\Lambda. \tag{39}$$

Now, define $\overline{\tilde{\alpha}}_{ij}^{D}:=\bar{\theta}_q\tilde{\alpha}_{ij}^{D*}$ and $\overline{\tilde{\beta}}_{rj}^{U}:=\bar{\theta}_q\tilde{\beta}_{rj}^{U*}$ for each $i\in I_1$, $r\in O_2$,and $j\in\Lambda$. Therefore,

$$\sum_{j\in\Lambda}\overline{\tilde{\alpha}}_{ij}^{D}\leq\theta_q\sum_{j\in\Lambda}x_{ij}^{D},\quad i\in I_1, \tag{40}$$

$$\sum_{j\in\Lambda}\overline{\tilde{\beta}}_{rj}^{U}\leq\theta_q\sum_{j\in\Lambda}y_{rj}^{U},\quad r\in O_2. \tag{41}$$

By (40), (41), and feasibility of Δ to model (8), it is obvious that

$\Psi=(\lambda^*,\ \bar{\theta}_q,\ \overline{\tilde{\alpha}}_{ij}^{D},\ \overline{\tilde{\beta}}_{rj}^{U}:\forall i\in I_1,\ \forall r\in O_2,\ \forall j\in\Lambda)$ is a feasible solution to model (35). Therefore, the optimal value of model (35) is less than or equal to $\bar{\theta}_q$. This contradicts the assumption that θ^* is the optimal value of model (35), and so the proof of part (ii) is completed. □

4.2. Merging units under undesirable inputs and desirable outputs

Assume that DMU_q holds the quantity of desirable inputs and undesirable outputs of a set of the merging DMUs and searches for the highest quantity of undesirable inputs and desirable outputs of these units to achieve of the pre-determined goal, $\bar{\theta}_q$. Then, the desirable inputs and undesirable outputs of DMU_q are $x_{iq}^{D}=\sum_{j\in\Lambda}x_{ij}^{D}$ for each $i\in I_1$ and $y_{rq}^{U}=\sum_{j\in\Lambda}y_{rj}^{U}$ for each $r\in O_1$. To achieve the desired efficiency score $\bar{\varphi}_q$, we must estimate the optimal desirable input vector x_q^U and the optimal undesirable output vector y_q^D. We use model (42) to estimate of the efficiency score of DMU_q after estimating all data:

$$\begin{aligned}
\varphi_o = \max\ & \varphi \\
\text{s.t.}\ & \sum_{j\in\Pi} \lambda_j x_{ij}^D + \lambda_q x_{iq}^D \le x_{iq}^D, && \forall i \in I_1, \\
& \sum_{j\in\Pi} \lambda_j y_{rj}^U + \lambda_q y_{rq}^U \le y_{rq}^U, && \forall r \in O_2, \\
& \sum_{j\in\Pi} \lambda_j x_{ij}^U + \lambda_q x_{iq}^U \ge \varphi x_{io}^U, && \forall i \in I_2, \\
& \sum_{j\in\Pi} \lambda_j y_{rj}^D + \lambda_q y_{rq}^D \ge \varphi y_{ro}^D, && \forall r \in O_1, \\
& \lambda = (\lambda_j;\ j \in \Pi \cup \{q\}) \in \Omega_q.
\end{aligned} \tag{42}$$

In model (42), $(\lambda,\ \varphi)$ is the variables vector. The expected efficiency goal of DMU_q has been attained when the optimal value of Model (6) is equal $\bar{\varphi}_q$. Along the lines of [17], we suggest MONLP problem (43) for calculating of the undesirable input and desirable output vectors of DMU_q:

$$\begin{aligned}
\max\ & (\alpha_{ij}^U:\ \forall i \in I_2\ \ \&\ \ \beta_{rj}^D:\ \forall r \in O_1) \quad \forall j \in \Lambda, \\
\text{s.t.}\ & \sum_{j\in\Pi} \lambda_j x_{ij}^D + \lambda_q x_{iq}^D \le x_{iq}^D, && \forall i \in I_1, \\
& \sum_{j\in\Pi} \lambda_j y_{rj}^U + \lambda_q y_{rq}^U \le y_{rq}^U, && \forall r \in O_2, \\
& \sum_{j\in\Pi} \lambda_j x_{ij}^U + \lambda_q \sum_{j\in\Lambda} \alpha_{ij}^U \ge \bar{\varphi}_q \sum_{j\in\Lambda} \alpha_{ij}^U, && \forall i \in I_2, \\
& \sum_{j\in\Pi} \lambda_j y_{rj}^D + \lambda_q \sum_{j\in\Lambda} \beta_{rj}^D \ge \bar{\varphi}_q \sum_{j\in\Lambda} \beta_{rj}^D, && \forall r \in O_1, \\
& \alpha_{ij}^U \ge x_{ij}^U, && \forall i \in I_2, \forall j \in \Lambda, \\
& \beta_{rj}^D \ge y_{rj}^D, && \forall i \in O_1, \forall j \in \Lambda, \\
& \lambda = (\lambda_j;\ j \in \Pi \cup \{q\}) \in \Omega_q.
\end{aligned} \tag{43}$$

In the above model, $(\lambda,\ \alpha_{ij}^U;\ \forall i \in I_2,\ \beta_{rj}^D;\ \forall r \in O_1)$ is variables vector. α_{ij}^U and β_{rj}^D are display the amount of the ith undesirable input and the rth desirable output inherited by DMU_q from the jth unit involved in the merger procedure, respectively. $\bar{\varphi}_q$ is the efficiency index indicator expected for DMU_q. The above model can be simplified to the following model based on the same argument for converting model (7) to model (8).

$$\begin{aligned}
\max\ & (\alpha_{ij}^U:\ \forall i \in I_2\ \ \&\ \ \beta_{rj}^D:\ \forall r \in O_1) \quad \forall j \in \Lambda, \\
\text{s.t.}\ & \sum_{j\in\Pi} \lambda_j x_{ij}^D \le x_{iq}^D, && \forall i \in I_1, \\
& \sum_{j\in\Pi} \lambda_j y_{rj}^U \le y_{rq}^U, && \forall r \in O_2, \\
& \sum_{j\in\Pi} \lambda_j x_{ij}^U \ge \bar{\varphi}_q \sum_{j\in\Lambda} \alpha_{ij}^U, && \forall i \in I_2,
\end{aligned}$$

$$\begin{aligned}
&\sum_{j\in\Pi}\lambda_j y^D_{rj} \geq \bar{\varphi}_q \sum_{j\in\Lambda}\beta^D_{rj}, && \forall r\in O_1,\\
&\alpha^U_{ij} \geq x^U_{ij}, && \forall i\in I_2, \forall j\in\Lambda,\\
&\beta^D_{rj} \geq y^D_{rj}, && \forall i\in O_1, \forall j\in\Lambda,\\
&\lambda = (\lambda_j;\ j\in\Pi)\in\Omega_q.
\end{aligned} \tag{44}$$

The following theorem indicates that model (44) can characterize the undesirable inputs and desirable outputs of DMU_q in order to reach the desired given efficiency target $\bar{\varphi}_q$.

Theorem 4.4. *Suppose that deleting units participating in the merger process does not change the PPS. Also, let* $\Delta = (\lambda^*, \alpha^{U*}_{ij}, \beta^{D*}_{rj}:\ \forall i\in I_2,\ \forall r\in O_1,\ \forall j\in\Lambda)$ *be a weak Pareto solution to Model (44). If* $x^U_{iq} = \sum_{j\in\Lambda}\alpha^{U*}_{ij}$ $(\forall i\in I_2)$ *and* $y^D_{rq} = \sum_{j\in\Lambda}\beta^{D*}_{rj}$ $(\forall r\in O_1)$*, then the optimal value of Model (42) is equal to* $\bar{\varphi}_q$.

Proof. We need to prove that, $\varphi_q = \bar{\varphi}_q \geq 1$. Feasibility of Δ for Model (44), implies:

$$\sum_{j\in\Pi}\lambda^*_j x^D_{ij} \leq x^D_{iq}, \qquad \forall i\in I_1, \tag{45}$$

$$\sum_{j\in\Pi}\lambda^*_j y^U_{rj} \leq y^U_{rq} \qquad \forall r\in O_2, \tag{46}$$

$$\sum_{j\in\Lambda}\lambda^*_j x^U_{ij} \geq \bar{\varphi}_q \sum_{j\in\Lambda}\alpha^{U*}_{ij} = \bar{\varphi}_q x^U_{iq} \geq x^U_{iq}, \qquad \forall i\in I_2, \tag{47}$$

$$\sum_{j\in\Pi}\lambda^*_j y^D_{rj} \geq \bar{\varphi}_q \sum_{j\in\Lambda}\beta^{D*}_{rj} = \bar{\varphi}_q y^D_{rq} \geq y^D_{rq}, \qquad \forall r\in O_1, \tag{48}$$

$$\alpha^{U*}_{ij} \geq x^U_{ij},\ \ \forall i\in I_2,\ \forall j\in\Lambda, \tag{49}$$

$$\beta^{D*}_{rj} \geq y^D_{rj},\ \ \forall r\in O_1,\ \forall j\in\Lambda, \tag{50}$$

$$\lambda^* = (\lambda^*_j;\ \forall j\in\Pi)\in\overline{\Omega}_q. \tag{51}$$

By equations (45)–(48) and (51), $(\lambda_j = \lambda^*_j; \forall j\in\Pi,\ \lambda_q = 0,\ \varphi = \bar{\varphi}_q)$ is obviously a feasible solution to problem (42). Therefore, $\varphi_q \geq \bar{\varphi}_q$. □

Considering $(\tilde{\lambda}_j; \forall j\in\Pi, \tilde{\lambda}_q,\ \tilde{\varphi})$ as an optimal solution to LP (42) and equations (47) and (47), we have

$$\tilde{\varphi} x^U_{iq} \leq \sum_{j\in\Pi}\tilde{\lambda}_j x^U_{ij} + \tilde{\lambda}_q x^U_{iq} \leq \sum_{j\in\Pi}\tilde{\lambda}_j x^U_{ij} + \tilde{\lambda}_q(\sum_{j\in\Pi}\lambda^*_j x^U_{ij}) = \sum_{j\in\Pi}(\tilde{\lambda}_j + \tilde{\lambda}_q\lambda^*_j)x^U_{ij},\ \forall i\in I_2$$

$$\tilde{\varphi} y^D_{rq} \leq \sum_{j\in\Pi}\tilde{\lambda}_j y^D_{rj} + \tilde{\lambda}_q y^D_{rq} \leq \sum_{j\in\Pi}\tilde{\lambda}_j y^D_{rj} + \tilde{\lambda}_q(\sum_{j\in\Pi}\lambda^*_j y^D_{rj}) = \sum_{j\in\Pi}(\tilde{\lambda}_j + \tilde{\lambda}_q\lambda^*_j)y^D_{rj},\ \forall r\in O_1.$$

Set $\bar{\lambda}_j := \tilde{\lambda}_j + \tilde{\lambda}_q \lambda_j^*$ for each $j \in \Pi$, then

$$\sum_{j\in\Pi} \bar{\lambda}_j x_{ij}^U \geq \tilde{\varphi} x_{iq}^U, \quad \forall i \in I_2, \tag{52}$$

$$\sum_{j\in\Pi} \bar{\lambda}_j y_{rj}^D \geq \tilde{\varphi} y_{rq}^D, \quad \forall r \in O_1. \tag{53}$$

Using the same procedure, the inequalities (45) and (46) will be used in LP (42), the following results are obtained:

$$\sum_{j\in\Pi} \bar{\lambda}_j x_{ij}^D \leq x_{iq}^D, \quad \forall i \in I_1, \tag{54}$$

$$\sum_{j\in\Pi} \bar{\lambda}_j y_{rj}^U \leq y_{rq}^U, \quad \forall r \in O_2, \tag{55}$$

It is easily seen that

$$\bar{\lambda} = (\bar{\lambda}_j; \forall j \in \Pi) \in \overline{\Omega}_q. \tag{56}$$

By contradiction, assume that $\varphi_q = \tilde{\varphi} > \bar{\varphi}_q$. By equations (17) and (18), we have

$$\sum_{j\in\Pi} \bar{\lambda}_j x_{ij}^U \geq \tilde{\varphi} x_{iq}^U = \tilde{\varphi} \sum_{j\in\Lambda} \alpha_{ij}^{U*} > \bar{\varphi}_q \sum_{j\in\Lambda} \alpha_{ij}^{U*}, \quad \forall i \in I_2, \tag{57}$$

$$\sum_{j\in\Pi} \bar{\lambda}_j y_{rj}^D \geq \tilde{\varphi} y_{rq}^D = \tilde{\varphi} \sum_{j\in\Lambda} \beta_{rj}^{D*} > \bar{\varphi}_q \sum_{j\in\Lambda} \beta_{rj}^{D*}, \quad \forall r \in O_1. \tag{58}$$

Therefore, there exists $\mu > 0$ such that

$$\sum_{j\in\Pi} \bar{\lambda}_j x_{ij}^U \geq \bar{\varphi}_q \sum_{j\in\Lambda} (\alpha_{ij}^{U*} + \mu), \quad \forall i \in I_2, \tag{59}$$

and

$$\sum_{j\in\Pi} \bar{\lambda}_j y_{rj}^D \geq \bar{\varphi}_q \sum_{j\in\Lambda} (\beta_{rj}^{D*} + \mu), \quad \forall r \in O_1. \tag{60}$$

Now, define

$$\bar{\alpha}_{ij}^U = \alpha_{ij}^{U*} + \mu, \qquad \forall i \in I_2, \tag{61}$$

and

$$\bar{\beta}_{rj}^D = \beta_{rj}^{D*} + \mu, \qquad \forall r \in O_1. \tag{62}$$

By equations (59)–(62), we obtain

$$\sum_{j\in\Pi}\bar{\lambda}_j x^U_{ij} \geq \bar{\varphi}_q \sum_{j\in\Lambda}\bar{\alpha}^U_{ij}, \quad \forall i\in I_2, \tag{63}$$

$$\sum_{j\in\Pi}\bar{\lambda}_j y^D_{rj} \geq \bar{\varphi}_q \sum_{j\in\Lambda}\bar{\beta}^D_{rj}, \quad \forall r\in O_1, \tag{64}$$

$$\bar{\alpha}^U_{ij} \geq x^U_{ij}, \quad \forall i\in I_2,\ \forall j\in\Lambda, \tag{65}$$

$$\bar{\beta}^D_{rj} \geq y^D_{rj}, \quad \forall r\in O_1,\ \forall j\in\Lambda. \tag{66}$$

According to equations (54)–(56) and (63)–(66), $(\bar{\lambda}, \bar{\alpha}^U_{ij}, \bar{\beta}^D_{rj} : \forall i\in I_2, \forall r\in O_1,\ \forall j\in\Lambda)$ is a feasible solution to problem (44), such that $(\bar{\alpha}^U_{ij}, \bar{\beta}^D_{rj} : \forall i\in I_2, \forall r\in O_1,\ \forall j\in\Lambda) < (\alpha^{U*}_{ij},\ \beta^{D*}_{rj} : \forall i\in I_2, \forall r\in O_1,\ \forall j\in\Lambda)$. But this is impossible because Δ is a weak Pareto solution to MOLP (44). □

Remark 4.5. According to the discussion made in Remark 4.2, if model (44) is infeasible, then model (44) can be converted to the following linear model by substitutions of $\overline{\alpha}^U_{ij} = \lambda_q \alpha^U_{ij}$ for $i\in I_2,\ j\in\Lambda$, and $\overline{\beta}^D_{rj} = \lambda_q \beta^D_{rj}$ for $r\in O_1,\ j\in\Lambda$.

$$\begin{aligned}
&\max\ \ (\alpha^U_{ij}:\ \forall i\in I_2\ \ \&\ \ \beta^D_{rj}:\ \forall r\in O_1) \qquad \forall j\in\Lambda,\\
&\text{s.t.}\ \sum_{j\in\Pi}\lambda_j x^D_{ij} + \lambda_q x^D_{iq} \leq x^D_{iq}, \qquad \forall i\in I_1,\\
&\quad \sum_{j\in\Pi}\lambda_j y^U_{rj} + \lambda_q y^U_{rq} \leq y^U_{rq}, \qquad \forall r\in O_2,\\
&\quad \sum_{j\in\Pi}\lambda_j x^U_{ij} + \sum_{j\in\Lambda}\overline{\alpha}^U_{ij} \geq \bar{\varphi}_q \sum_{j\in\Lambda}\alpha^U_{ij}, \quad \forall i\in I_2,\\
&\quad \sum_{j\in\Pi}\lambda_j y^D_{rj} + \sum_{j\in\Lambda}\overline{\beta}^D_{rj} \geq \bar{\varphi}_q \sum_{j\in\Lambda}\beta^D_{rj}, \quad \forall r\in O_1,\\
&\quad \alpha^U_{ij} \geq x^U_{ij}, \qquad \forall i\in I_2, \forall j\in\Lambda,\\
&\quad \beta^D_{rj} \geq y^D_{rj}, \qquad \forall i\in O_1, \forall j\in\Lambda,\\
&\quad \overline{\alpha}^U_{ij} \geq x^U_{ij}\alpha^U_{ij}, \qquad \forall i\in I_2, \forall j\in\Lambda,\\
&\quad \alpha^U_{ij} - (1-\lambda_q)x^U_{ij} \geq \overline{\alpha}^U_{ij} \geq \alpha^U_{ij}, \qquad \forall i\in I_2, \forall j\in\Lambda,\\
&\quad \overline{\beta}^D_{rj} \geq y^D_{rj}\beta^D_{rj}, \qquad \forall r\in O_1, \forall j\in\Lambda,\\
&\quad \beta^D_{rj} - (1-\lambda_q)y^D_{rj} \geq \overline{\beta}^D_{rj} \geq \beta^U_{rj}, \qquad \forall r\in O_1, \forall j\in\Lambda,\\
&\quad \overline{\alpha}^U_{ij} \geq 0, \qquad \forall i\in I_2, \forall j\in\Lambda,\\
&\quad \overline{\beta}^D_{rj} \geq 0, \qquad \forall r\in O_1, \forall j\in\Lambda,\\
&\quad \lambda_q \in \{0,\ 1\},\\
&\quad \lambda = (\lambda_j;\ j\in\Pi\cup\{q\}) \in \Omega_q.
\end{aligned} \tag{67}$$

We close this subsection with a discussion on the maximum achievable efficiency target of the merged DMU. It is worth noting that knowing maximum achievable efficiency target is necessary for the decision maker deliberating about engaging in the merging process. The highest efficiency score realized through a merging could be determined *via* the following theorem.

Theorem 4.6. *Consider a merging with $\bar{\varphi}_q$ as the efficiency target for the merged DMU.*

i) If model (44) is feasible, then it remains feasible for each efficiency target $\bar{\bar{\varphi}}_q$, with $\bar{\varphi}_q \geq \bar{\bar{\varphi}}_q \geq 1$.

ii) Suppose that model (44) admits a Pareto solution. If

$$
\begin{aligned}
\max \quad & \varphi, \\
\text{s.t.} \quad & \sum_{j\in\Pi} \lambda_j x_{ij}^D \leq x_{iq}^D, \ \forall i \in I_1, \\
& \sum_{j\in\Pi} \lambda_j y_{rj}^U \leq y_{rq}^U, \ \forall r \in O_2, \\
& \sum_{j\in\Pi} \lambda_j x_{ij}^U \geq \sum_{j\in\Lambda} \alpha_{ij}^U, \qquad \forall i \in I_2, \\
& \sum_{j\in\Pi} \lambda_j y_{rj}^D \geq \sum_{j\in\Lambda} \beta_{rj}^D, \qquad \forall r \in O_1, \\
& \sum_{j\in\Lambda} \alpha_{ij}^D \geq \varphi \sum_{j\in\Lambda} x_{ij}^D, \ \forall i \in I_1, \\
& \sum_{j\in\Lambda} \beta_{rj}^U \geq \varphi \sum_{j\in\Lambda} y_{rj}^U, \ \forall r \in O_2, \\
& \alpha_{ij}^U \geq x_{ij}^U, \qquad \forall i \in I_2, \forall j \in \Lambda, \\
& \beta_{rj}^D \geq y_{rj}^D, \qquad \forall r \in O_1, \forall j \in \Lambda, \\
& \lambda = (\lambda_j;\ j \in \Pi) \in \overline{\Omega}_q,
\end{aligned}
\tag{68}
$$

then $\bar{\varphi}_q \geq \varphi^*$ (The minimum achievable efficiency target of the merged DMU is θ^*).

Proof. The proof theorem is similar to the proof given in Theorem 4.3. □

5. A numerical illustration

In the current section, the InvDEA method provided so far is examined through a real-world data set. In fact, we considered the dataset of containing 10 branches of an Iranian commercial bank and presented in Table 4 [33]. Each branch uses two inputs to generate two outputs. Labor and deposits are inputs and performing loans and non-performing loans are outputs. Obviously, the amount of performing loans should be increased and amount of non-performing loans should be decreased to improve the efficiency score. Therefore, the performing loans is one desirable output factor, while the non- performing loans is one undesirable output factor. We used model (8) under the VRS assumption to estimate of the efficiency score of the branches as shown in Table 4.

According to Table 4, B02 and B10 branches are inefficient. Suppose market conditions require that these two banks be managed based on a synergistic scenario. In fact, it is decided to synergy through merging to improve their performance. Therefore, according to the approach proposed in this study, branches B02 and B10 are merged to produce a new unit (DMU_q) to achieve the pre-defined performance index ($\bar{\theta}_q$). According to the discussion in the Sub-section 4.1, new branch keeps the amount of desirable outputs of the branches B02 and B10 and looks for the minimum amount of desirable inputs and undesirable outputs of these branches in order to reach the pre-defined target level, $\bar{\theta}_q$. Then, the desirable output of DMU_q is $y_{1q}^D = y_{12}^D + y_{1,10}^D = 141183$. Now, with regard to the model (35), the minimum performance score that can be achieved by the new unit is 0.7131. It is obvious that the decision maker is encouraged to engage the merge process if this "pessimistic" level is judged satisfactory. Firstly, assume that the expected efficiency level of the DMU_q is equal to 0.7500. To estimate of the desirable inputs and undesirable output of the new branch, Model (8) corresponding to DMU_q

TABLE 4. The data of 10 bank branches.

	Inputs		Outputs		
DMUs	Labor	Deposits	Performing loans	Non-performing loans	Efficiency
B01	32	515578	1277833	446698	1.0000
B02	19	187679	102808	22585	0.6745
B03	14	150026	106734	12830	0.9612
B04	5	88358	14628	161	1.0000
B05	18	124349	75509	21035	0.7507
B06	18	127370	149860	39525	0.9325
B07	16	95288	55757	9632	0.9331
B08	17	89304	84631	13955	1.0000
B09	9	160138	102353	7153	1.0000
B10	13	148755	38375	7806	0.6240

TABLE 5. The desirable inputs and undesirable output of branch B_q based on the first scenario to achieve the efficiency goal $\bar{\theta}_q = 0.7500$.

	Desirable inputs				Undesirable output	
Factors	Labor		Deposits		Non-performing loans	
DMUs	$\alpha_{1,2}^{Dq}$	$\alpha_{1,10}^{Dq}$	$\alpha_{2,2}^{Dq}$	$\alpha_{2,10}^{Dq}$	$\beta_{2,2}^{Uq}$	$\beta_{2,10}^{Uq}$
OP_1	13.9417	0.0000	187679.0000	34208.3180	22585.0000	7806.0000
OP_2	0.0130	13.0000	187679.0000	41493.4820	22585.0000	6311.8374

is considered as follows:

$$\begin{aligned}
\min\ & (\alpha_{1,2}^{Dq},\ \alpha_{1,10}^{Dq},\ \alpha_{2,2}^{Dq},\ \alpha_{2,10}^{Dq},\ \beta_{2,2}^{Uq},\ \beta_{2,10}^{Uq}),\\
\text{s.t. } & \sum_{j\in\Pi} \lambda_j x_{ij}^D \le \bar{\theta}_q(\alpha_{i2}^{Dq} + \alpha_{i,10}^{Dq}), \qquad i = 1,2,\\
& \sum_{j\in\Pi} \lambda_j y_{1j}^U \le \bar{\theta}_q(\beta_{22}^{Uq} + \beta_{2,10}^{Uq}),\\
& \sum_{j\in\Pi} \lambda_j y_{1j}^D \ge y_{1q}^D = y_{12}^D + y_{1,10}^D = 141183,\\
& \sum_{j\in\Pi} \lambda_j = 1,\\
& 0 \le \alpha_{12}^{Dq} \le 19,\ 0 \le \alpha_{1,10}^{Dq} \le 13,\\
& 0 \le \alpha_{22}^{Dq} \le 187679,\ 0 \le \alpha_{2,10}^{Dq} \le 148755,\\
& 0 \le \beta_{22}^{Uq} \le 22585,\ 0 \le \beta_{2,10}^{Uq} \le 7806,\\
& \lambda_j \ge 0, \quad \forall j \in \Pi,
\end{aligned} \tag{69}$$

where $\Pi = \{1,3,4,\ldots,9\}$, $\Lambda = \{2,10\}$, and $\bar{\theta}_q = 0.7500$. According to the weight-sum method [11] by considering different weights, Model (69) is solved and two optimal solutions are obtained that the outcomes are presented in Table 5.

With regard to Table 5, two scenarios are proposed to create B_q as follows Table 6. In other words, the inputs and outputs of B_q to achieve efficiency score 0.7500 must be as Table 6.

TABLE 6. The data of new bank branch.

	Inputs		Outputs	
DMUs	Labor	Deposits	Performing loans	Non-performing loans
B_q (Scenario 1)	13.9417	221887.3180	141183	30391.0000
B_q (Scenario 2)	13.0130	229172.4820	141183	28896.8374

TABLE 7. The desirable inputs and undesirable output of branch B_q to be fully efficient $\bar{\theta}_q = 1$.

	Desirable inputs				Undesirable output	
Factors	Labor		Deposits		Non-performing loans	
DMUs	$\alpha_{1,2}^{Dq}$	$\alpha_{1,10}^{Dq}$	$\alpha_{2,2}^{Dq}$	$\alpha_{2,10}^{Dq}$	$\beta_{2,2}^{Uq}$	$\beta_{2,10}^{Uq}$
OP_1	15.1787	0.0000	129370.7588	0.0000	22585.0000	7806.0000
OP_2	0.0000	9.7598	23124.3610	148755.0000	21672.6280	0.0000

TABLE 8. The data of new bank branch.

	Inputs		Outputs	
DMUs	Labor	Deposits	Performing loans	Non-performing loans
B_q (Scenario 1)	15.1787	129370.7588	141183	30391.0000
B_q (Scenario 2)	9.7598	171884.3610	141183	21672.6280

Table 6 shows the contribution of B02 and B10 branches to the supply of the desirable inputs and undesirable output of the B_q branch to achieve the efficiency goal. For example, the contribution of B02 and B10 branches to the supply of the deposits resource of the B_q is 84.58% and 15.42%, respectively while the share of these two branches to the supply of the undesirable output (Non-performing loans) of the B_q is 74.31% and 25.69%, respectively. Moreover, labor of B_q is fully supplied by B02 while B10 has no contribution.

Now, for the special case where the merged branch is to be fully efficient, namely $\theta_q = 1$, the model (8) related to this merging has the optimal solution listed in Table 7.

With regard to Table 7, two scenarios are proposed to create B_q as follows Table 8.

Table 8 shows the contribution of B02 and B10 branches to the supply of the desirable inputs and undesirable output of the B_q branch to be fully efficient.

In the real world, achieving fully efficient units depends on other parameters, *i.e.*, effective integration planning, cultural alignment, and strategic decision-making. The application of merging units in the banking sector, focusing on labor, deposits, performing loans, and non-performing loans, involves strategic considerations for optimizing operational efficiency, risk management, cost-effectiveness, and customer satisfaction. Economically, non-performing loans have secondary effects, including a negative impact on efficiency,affecting profitability operations, reducing the ability to produce new loans, and compromising banking [51]. The merging process can be strategically applied to achieve various objectives, including:

– Merging units can save costs by eliminating duplicate functions and resources. This includes streamlining administrative tasks, reducing the number of branches, and optimizing staff levels. As a result, the Cost-to-income ratio improves, making the bank more efficient [26].

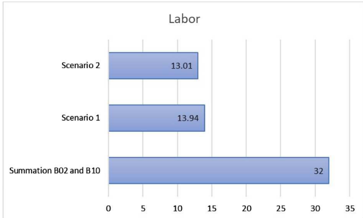
- A bank can diversify its risk exposure across different geographic regions and business segments by merging units. This diversification can help mitigate the impact of economic downturns or regional economic challenges, reducing the likelihood of a high concentration of non-performing loans in a specific area.
- This process enables the integration of systems, technologies, and procedures, fostering synergies that can improve operational efficiency. This can lead to faster loan processing, better customer service, and a more agile response to market changes.
- Merging units can increase economies of scale, allowing the bank to spread fixed costs over a larger customer base. This can contribute to increased profitability and a more competitive position in the market.

6. Discussion

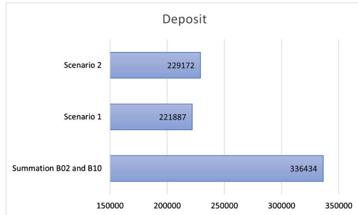
As discussed in Section 1, there are different approaches to dealing with undesirable data in the DEA literature. The most commonly used methods in the context of undesirable outputs are categorized into four groups [28]: 1) treating undesirable outputs as inputs, 2) data transformation, 3) impact assessment, and 4) necessary transformations. In the present study, our approach to dealing with undesirable data in unit performance evaluation is based on the approach proposed by Liu *et al.* [36] in Section 2.1 and on DEA-input (output) oriented models. In this approach, when using models with desirable inputs and undesirable outputs, undesirable outputs are treated as desirable inputs. In fact, to estimate efficiency, the proportional contraction of desirable inputs and undesirable outputs is considered. In this study, the integration of this approach with the inverse DEA concept leads to the establishment of a multi-objective nonlinear programming model (model (7)) and the expression of sufficient conditions for estimating data for the new unit (Thm. 4.1).

For clarification, consider two bank branches (B02 and B10 in the case study) as decision-making units, such that each branch uses two inputs (labor and deposits) to produce two outputs (performing loans and non-performing loans). It is clear that in order to improve the efficiency score of the units, the amount of performing loans should be increased and the amount of non-performing loans should be reduced. Therefore, performing loans are a desirable factor, and non-performing loans are undesirable. Suppose the decision-maker decides to merge these two units based on the first approach (model (7)) for efficiency estimation to improve their performance. Therefore, the proposed model for data estimation should focus on the proportional contraction of labor, deposits, and non-performing loans. If the decision-maker wants to merge B02 and B10 to achieve a new DMU, the summation of labor for these two units is 32. Similarly, the sums for deposits, Performing loans, and Non-performing loans are 336434, 141183, and 30391, respectively. It is worth noting that according to this approach, no expansion is expected in the output of performing loans. Therefore, in this state, we seek to maintain the total performing loans of B02 and B10 merging branches in the new branch. Thus, the proposed model should be designed in such a way that the new branch, while obtaining a pre-defined efficiency score, can use the sum of the performing loans of the two merged branches as production output and efforts to minimize the receipt of labor, deposits, and non-performing loans from the two merging branches as can be seen in Table 6. There are two scenarios for this purpose. In scenario 1, Labor and Deposits should be decreased, while Non-performing loans remain unchanged, relative to sums of these parameters. In scenario 2, the Non-performing loan should be decreased in addition to Labor and Deposits. The decrease in Deposits for scenario 1 is greater than in scenario 2 (see Fig. 1). Managers can select preferred scenarios according to company policy, facility, and other criteria.

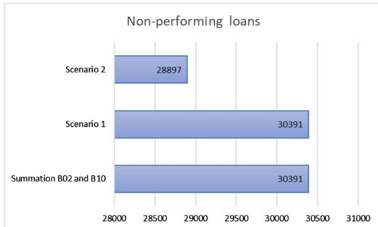
It is important to mention that in this approach, when using models with a performing output, undesirable inputs behave similarly to desirable outputs. In fact, to estimate efficiency, the proportional contraction of undesirable inputs and desirable outputs is considered. Therefore, combining this approach with the concept of inverse DEA leads to the establishment of a multi-objective nonlinear programming model (model (43)) and providing sufficient conditions for data estimation (Thm. 4.4).



(a)



(b)



(c)

Figure 1. Variation of merger parameters. (a) Labor. (b) Deposits. (c) Non-performing loans.

7. Conclusions

This study extends new links in the areas of firms merging and InvDEA. In this study, we extended the problem of the merging units for the inherited input/output estimation in the simultaneous presence of desirable and undesirable data. To achieve the pre-determined goal level:

- Multiple-objective programming tools were used to attain the sufficient conditions for desirable/undesirable inputs and desirable/undesirable output estimation.
- The proposed models utilize an achievement approach to maximize benefits through activity synergies.
- The approach recommends optimal sharing of inherited input and output levels from merging units to attain the desired efficiency goal.
- The study introduces models to identify the minimum efficiency level achievable through a specific merger.
- This minimum efficiency level can serve as a threshold for accepting a merger if it meets acceptable standards.

The results can be used for practical purposes, provided that relevant sectors have similar units and the possibility of merging. When merging units considering undesirable data, such as non-performing assets, high-risk portfolios, or other negative factors, managers must analyze the implications carefully and derive insights for effective decision-making. Managers must navigate the complexities associated with risk, culture, communication, and ongoing evaluation to ensure the merger's success despite the challenges posed by undesirable data. For example, in risk assessment, they assess the risks associated with the undesirable data and evaluate the potential impact on the financial health and stability of the merged entity. This is managerial insight. From a practical perspective, risk mitigation strategies should be developed to minimize the adverse effects of undesirable data on the merged organization. Also, managerial insights and practical implications are essential in the merger process. For instance, managers can assess how resources, including personnel, technology, and capital, can be optimized through the merger. From a practical perspective, the merging process enables the combined entity to leverage broader resources and improve productivity and competitiveness.

The limitation of work on merging units under undesirable outputs is a complex issue that can be approached from different perspectives. In the context of DEA, undesirable outputs can pose challenges to measuring efficiency. Several methods have been proposed to address this, such as treating undesirables as inputs to the

DEA model, data transformation, and impact assessment. However, a specific limitation of merging units under undesirable outputs depend on the particular application. In the banking sector, there are some challenges in merging units. The first limitation is the complexity of the process, which can involve integrating IT systems and core banking platforms, leading to data loss or customer service disruptions. The need for regulatory approvals, which can be time-consuming and may impose additional costs, is another limitation. Furthermore, merging units may involve combining different organizational and technical capabilities, which can lead to cultural clashes and operational inefficiencies. In this paper, the problem of merging entities is studied in the simultaneous presence of desirable and undesirable data. Obtaining similar models in dynamic and network structure sectors can be a helpful direction for further research.

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Conflict of Interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

Data availability statement

The authors confirm that all data generated or analysed during this study are included in this published article.

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