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SOME EXISTENCE THEOREMS ON PATH-FACTOR CRITICAL AVOIDABLE GRAPHS

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Abstract. A spanning subgraph F of G is called a path factor if every component of F is a path of order at least 2. Let $k \geq 2$ be an integer. A $P_{\geq k}$-factor of G means a path factor in which every component has at least k vertices. A graph G is called a $P_{\geq k}$-factor avoidable graph if for any $e \in E(G)$, G has a $P_{\geq k}$-factor avoiding e. A graph G is called a $(P_{\geq k}, n)$-factor critical avoidable graph if for any $W \subseteq V(G)$ with $|W| = n$, $G - W$ is a $P_{\geq k}$-factor avoidable graph. In other words, G is $(P_{\geq k}, n)$-factor critical avoidable if for any $W \subseteq V(G)$ with $|W| = n$ and any $e \in E(G - W)$, $G - W - e$ admits a $P_{\geq k}$-factor. In this article, we verify that (i) an $(n + r + 2)$-connected graph G is $(P_{\geq 2}, n)$-factor critical avoidable if $I(G) > \frac{n+r+3}{2(r+2)}$; (ii) an $(n + r + 2)$-connected graph G is $(P_{\geq 3}, n)$-factor critical avoidable if $t(G) > \frac{n+r+2}{2(r+2)}$; (iii) an $(n + r + 2)$-connected graph G is $(P_{\geq 3}, n)$-factor critical avoidable if $I(G) > \frac{n+3(r+2)}{2(r+2)}$; where n and r are two nonnegative integers.

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1. Introduction

In this work, we discuss only finite, undirected and simple graphs. We denote by $G = (V(G), E(G))$ a graph, where $V(G)$ denotes the vertex set of G and $E(G)$ denotes the edge set of G. For a vertex x of G, the degree of x in G, denoted by $d_G(x)$, is the number of vertices adjacent to x in G. For a vertex subset X of G, $G[X]$ denotes the subgraph of G induced by X, and $G - X$ denotes the subgraph derived from G by removing all vertices in X. For an edge subset E' of G, $G - E'$ denotes the subgraph acquired from G by deleting all edges in E'. For a vertex (or an edge) subset Q, we denote $G - Q$ by $G - u$ for convenience if $Q = \{u\}$. Let $i(G)$, $\omega(G)$ and $\kappa(G)$ denote the number of isolated vertices, the number of connected components and the vertex connectivity of G, respectively. We use K_n and P_n to denote the complete graph and the path with n vertices, respectively. Let G_1 and G_2 be two graphs. Then the join $G_1 + G_2$ denotes the graph with vertex set $V(G_1 + G_2) = V(G_1) \cup V(G_2)$ and edge set

$$E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1), v \in V(G_2)\}.$$

Keywords. Graph, toughness, isolated toughness, connectivity, $(P_{\geq k}, n)$-factor critical avoidable graph.

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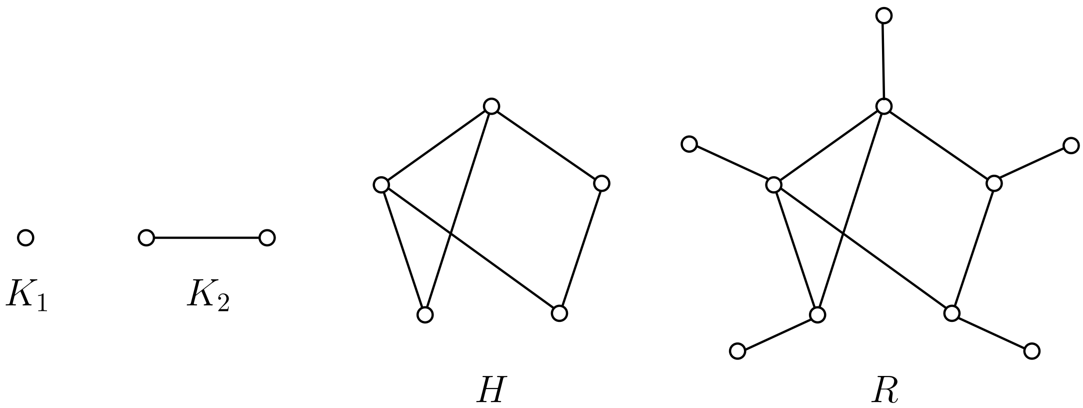

FIGURE 1. A factor-critical graph H and the sun R obtained from H.

The toughness of a graph G, denoted by $t(G)$, was first introduced by Chvátal [3]. If G is not complete, then

$$t(G) = \min\left\{ \frac{|X|}{\omega(G-X)} : X \subseteq V(G), \omega(G-X) \geq 2 \right\};$$

otherwise, $t(G) = +\infty$.

The isolated toughness of a graph G, denoted by $I(G)$, was first introduced by Yang *et al.* [20]. If G is not complete, then

$$I(G) = \min\left\{ \frac{|X|}{i(G-X)} : X \subseteq V(G), i(G-X) \geq 2 \right\};$$

otherwise, $I(G) = +\infty$.

A spanning subgraph F of G is called a path factor if every component of F is a path of order at least 2. Let $k \geq 2$ be an integer. A $P_{\geq k}$-factor of G means a path factor in which every component has at least k vertices.

Las Vergnas [14] showed a necessary and sufficient condition for graphs to possess $P_{\geq 2}$-factors.

Theorem 1 ([14]). *A graph G has a $P_{\geq 2}$-factor if and only if G satisfies*

$$i(G-X) \leq 2|X|$$

for every vertex subset X of G.

A graph H is factor-critical if any induced subgraph with $|V(H)| - 1$ vertices has a perfect matching. A graph R is called a sun if $R = K_1$, $R = K_2$ or R is the corona of a factor-critical graph H with at least three vertices, namely, R is acquired from H by adding a new vertex $z = z(y)$ together with a new edge yz for any $y \in V(H)$ to H (Fig. 1, which was shown by Kano *et al.* [12]). We easily see that $d_R(z) = 1$. In particular, a sun with at least six vertices is called a big sun. Let $sun(G)$ denote the number of sun components of G. In fact, $i(G) \leq sun(G) \leq \omega(G)$.

Kaneko [9] gave a necessary and sufficient condition for graphs admitting $P_{\geq 3}$-factors. Kano *et al.* [10] gave a simple proof.

Theorem 2 ([9, 10]). *A graph G contains a $P_{\geq 3}$-factor if and only if G satisfies*

$$sun(G-X) \leq 2|X|$$

for every vertex subset X of G.

In recent years, many results on path factors were derived. Kelmans [13] raised some results on the existence of path factors in claw-free graphs. Ando *et al.* [1] derived a minimum degree condition for a claw-free graph to have a path factor. Kano *et al.* [11] verified that every connected cubic bipartite graph with at least eight vertices admits a $P_{\geq 8}$-factor. Egawa and Furuya [4], Wang and Zhang [16] showed some sufficient conditions for graphs to have path factors. Zhou *et al.* [35] provided a sufficient condition for a graph with a $P_{\geq 2}$-factor. Kano *et al.* [12] presented a sufficient condition for the existence of $P_{\geq 3}$-factor. Wu [19] and Zhou *et al.* [25, 26, 31, 33] derived some sufficient conditions for graphs to possess $P_{\geq 3}$-factors with given properties. Gao *et al.* [8] posed some tight bounds for the existence of $P_{\geq 3}$-factors in graphs. Dauer *et al.* [2], Gao *et al.* [7], Liu and Zhang [15] established some relationships between toughness and graph factors. Gao and Wang [5], Gao *et al.* [6] established some relationships between isolated toughness and graph factors. More results on graph factors were acquired by Zhou [22–24, 28, 34], Zhou and Liu [27], Zhou *et al.* [30, 32], Wang and Zhang [17, 18], Yuan and Hao [21].

A graph G is called a $P_{\geq k}$-factor avoidable graph if for any $e \in E(G)$, G has a $P_{\geq k}$-factor avoiding e. A graph G is called a $(P_{\geq k}, n)$-factor critical avoidable graph if for any $W \subseteq V(G)$ with $|W| = n$, $G - W$ is a $P_{\geq k}$-factor avoidable graph. In other words, G is $(P_{\geq k}, n)$-factor critical avoidable if for any $W \subseteq V(G)$ with $|W| = n$ and any $e \in E(G - W)$, $G - W - e$ contains a $P_{\geq k}$-factor.

Zhou *et al.* [29] acquired some binding number conditions for graphs to be $(P_{\geq k}, n)$-factor critical avoidable graphs for $k = 2, 3$. Zhou *et al.* [31] provided a sun toughness condition for the existence of $(P_{\geq 3}, n)$-factor critical avoidable graphs. In this article, we proceed to investigate $(P_{\geq k}, n)$-factor critical avoidable graphs and derive three new results on $(P_{\geq k}, n)$-factor critical avoidable graphs depending on toughness and isolated toughness, which are shown in Sections 2 and 3.

2. $(P_{\geq 2}, n)$-FACTOR CRITICAL AVOIDABLE GRAPHS

In this section, we pose a sufficient condition using isolated toughness for graphs to be $(P_{\geq 2}, n)$-factor critical avoidable graphs.

Theorem 3. *Let n and r be two nonnegative integers, and let G be an $(n+r+2)$-connected graph. If its isolated toughness $I(G) > \frac{n+r+3}{2(r+2)}$, then G is $(P_{\geq 2}, n)$-factor critical avoidable.*

Proof. Obviously, Theorem 3 holds for a complete graph. Next, we assume that G is not a complete graph. Let $H = G - W - e$ for any $W \subseteq V(G)$ with $|W| = n$ and any $e \in E(G - W)$. It suffices to claim that H has a $P_{\geq 2}$-factor. Suppose that H has no $P_{\geq 2}$-factor. Then it follows from Theorem 1 that

$$i(H - X) \geq 2|X| + 1 \tag{1}$$

for some $X \subseteq V(H)$.

Since G is $(n + r + 2)$-connected, H is $(r + 1)$-connected.

Claim 1. $|X| \geq r + 2$.

Proof. If $|X| = 0$, then it follows from (1) that $i(H) \geq 1$, which contradicts that H is $(r + 1)$-connected. Next, we discuss $1 \leq |X| \leq r + 1$.

According to (1), we derive $i(H - X) \geq 2|X| + 1 \geq 3$. Thus, we have

$$i(G - W - X) \geq i(G - W - X - e) - 2 = i(H - X) - 2 \geq 3 - 2 = 1,$$

which implies that there exists an isolated vertex v in $G - W - X$, namely, $d_{G-W-X}(v) = 0$. Combining this with $1 \leq |X| \leq r + 1$, we get

$$d_G(v) \leq d_{G-W-X}(v) + |W| + |X| = 0 + n + |X| \leq n + r + 1,$$

which contradicts that G is $(n + r + 2)$-connected. Hence, $|X| \geq r + 2$. Claim 1 is proved. □

In fact, $i(G-W-X-e) \geq i(G-W-X) \geq i(G-W-X-e)-2$. The following proof is divided into three cases.

Case 1. $i(G-W-X) = i(G-W-X-e)-2$.

In this case, it is obvious that there exists a K_2 component in $G-W-X$ with $e \in E(K_2)$. Let $u \in V(K_2)$. Then by (1) and Claim 1, we deduce

$$\begin{aligned} i(G-W-X-u) &= i(G-W-X)+1 = i(G-W-X-e)-2+1 \\ &= i(H-X)-1 \geq 2|X|+1-1 = 2|X| \\ &\geq 2(r+2) \geq 4, \end{aligned}$$

and so

$$I(G) \leq \frac{|W \cup X \cup \{u\}|}{i(G-W-X-u)} \leq \frac{n+|X|+1}{2|X|} = \frac{1}{2} + \frac{n+1}{2|X|} \leq \frac{1}{2} + \frac{n+1}{2(r+2)} = \frac{n+r+3}{2(r+2)},$$

which contradicts $I(G) > \frac{n+r+3}{2(r+2)}$.

Case 2. $i(G-W-X) = i(G-W-X-e)-1$.

In this case, there exists a vertex u such that $d_{G-W-X}(u) = 1$. Let v be a unique vertex adjacent to u in $G-W-X$, and $e = uv$. Similar to this discussion of Case 1, we easily deduce

$$i(G-W-X-v) \geq 2|X|+1 \geq 2(r+2)+1 \geq 5,$$

and so

$$\begin{aligned} I(G) &\leq \frac{|W \cup X \cup \{v\}|}{i(G-W-X-v)} \leq \frac{n+|X|+1}{2|X|+1} = \frac{1}{2} + \frac{n+\frac{1}{2}}{2|X|+1} \\ &< \frac{1}{2} + \frac{n+1}{2|X|} \leq \frac{1}{2} + \frac{n+1}{2(r+2)} = \frac{n+r+3}{2(r+2)}, \end{aligned}$$

which contradicts $I(G) > \frac{n+r+3}{2(r+2)}$.

Case 3. $i(G-W-X) = i(G-W-X-e)$.

According to (1), we obtain

$$I(G) \leq \frac{|W \cup X|}{i(G-W-X)} = \frac{n+|X|}{i(G-W-X-e)} = \frac{n+|X|}{i(H-X)} \leq \frac{n+|X|}{2|X|+1}. \tag{2}$$

If $n = 0$, then from (2) we have

$$I(G) \leq \frac{|X|}{2|X|+1} < \frac{1}{2},$$

which contradicts $I(G) > \frac{r+3}{2(r+2)} > \frac{1}{2}$.

If $n \geq 1$, then it follows from (2) and Claim 1 that

$$I(G) \leq \frac{n+|X|}{2|X|+1} = \frac{1}{2} + \frac{n-\frac{1}{2}}{2|X|+1} < \frac{1}{2} + \frac{n}{2|X|} \leq \frac{1}{2} + \frac{n}{2(r+2)} = \frac{n+r+2}{2(r+2)},$$

which contradicts $I(G) > \frac{n+r+3}{2(r+2)}$. This completes the proof of Theorem 3.

□

Remark 1. Next, we show that the condition on $I(G)$ in Theorem 3 is sharp.

Let $G = K_{n+r+2} + ((2r+3)K_1 \cup K_2)$, where n and r are two nonnegative integers with $n \geq r+1$. Clearly, G is $(n+r+2)$-connected and $I(G) = \frac{n+r+3}{2(r+2)}$. Let $W \subseteq V(K_{n+r+2}) \subseteq V(G)$ with $|W| = n$ and $e \in E(K_2)$. Then $G - W - e = K_{r+2} + ((2r+5)K_1)$. Let $X = V(K_{r+2}) \subseteq V(G - W - e)$. Then we possess

$$i(G - W - e - X) = 2r + 5 > 2(r+2) = 2|X|.$$

In terms of Theorem 1, $G - W - e$ has no $P_{\geq 2}$-factor. Hence, G is not $(P_{\geq 2}, n)$-factor critical avoidable.

Remark 2. Next, we explain that the condition on $(n+r+2)$-connected in Theorem 3 is best possible.

Let $G = K_{n+r+1} + ((2r+1)K_1 \cup K_2)$, where n and r are two nonnegative integers with $n \geq r$. It is obvious that G is $(n+r+1)$-connected and $I(G) = \frac{n+r+2}{2r+2} = \frac{1}{2} + \frac{n+1}{2(r+1)} > \frac{1}{2} + \frac{n+1}{2(r+2)} = \frac{n+r+3}{2(r+2)}$. Let $W \subseteq V(K_{n+r+1}) \subseteq V(G)$ with $|W| = n$ and $e \in E(K_2)$. Then $G - W - e = K_{r+1} + ((2r+3)K_1)$. Let $X = V(K_{r+1}) \subseteq V(G - W - e)$. Then we derive

$$i(G - W - e - X) = 2r + 3 > 2(r+1) = 2|X|.$$

In terms of Theorem 1, $G - W - e$ has no $P_{\geq 2}$-factor. So G is not $(P_{\geq 2}, n)$-factor critical avoidable.

3. $(P_{\geq 3}, n)$-factor critical avoidable graphs

We first verify the following lemma.

Lemma 1. *Let n and r be two nonnegative integers, let G be an $(n+r+2)$-connected graph, and let $H = G - W - e$ for any $W \subseteq V(G)$ with $|W| = n$ and any $e \in E(G - W)$. If $sun(H - X) \geq 2|X| + 1$ for $X \subseteq V(H)$, then $|X| \geq r + 2$.*

Proof. If $|X| = 0$, then $sun(H) \geq 1$.

On the other hand, since G is $(n+r+2)$-connected, H is $(r+1)$-connected. Thus, we have $sun(H) \leq \omega(H) = 1$. Hence, we obtain $sun(H) = 1$. Combining this with H being $(r+1)$-connected, H is a sun.

Note that G is $(n+r+2)$-connected, and so $|V(G)| \geq n+r+3$. Thus, $|V(H)| = |V(G)| - n \geq (n+r+3) - n = r + 3 \geq 3$, which implies that H is a big sun. Hence, there exist at least three vertices with degree 1 in H, and so there exists at least one vertex v with $d_{G-W}(v) = 1$. Thus, we acquire

$$d_G(v) \leq d_{G-W}(v) + |W| = n + 1 \leq n + r + 1,$$

which contradicts that G is $(n+r+2)$-connected. In what follows, we consider $1 \leq |X| \leq r + 1$.

According to $sun(H - X) \geq 2|X| + 1$, we admit

$$\omega(H - X) \geq sun(H - X) \geq 2|X| + 1 \geq 3.$$

Thus, we derive

$$\omega(G - W - X) \geq \omega(G - W - X - e) - 1 = \omega(H - X) - 1 \geq 3 - 1 = 2.$$

Combining this with $|W| = n$ and $1 \leq |X| \leq r + 1$, we know that G is at most $(n+r+1)$-connected, which contradicts that G is $(n+r+2)$-connected. Hence, $|X| \geq r + 2$. This completes the proof of Lemma 1. □

Next, we raise two sufficient conditions using toughness and isolated toughness for graphs being $(P_{\geq 3}, n)$-factor critical avoidable graphs.

Theorem 4. *Let n and r be two nonnegative integers, and let G be an $(n+r+2)$-connected graph. If its toughness $t(G) > \frac{n+r+2}{2(r+2)}$, then G is $(P_{\geq 3}, n)$-factor critical avoidable.*

Proof. For a complete graph G, Theorem 4 is true. In the following, we assume that G is not a complete graph. Let $H = G - W - e$ for any $W \subseteq V(G)$ with $|W| = n$ and any $e \in E(G-W)$. It suffices to prove that H admits a $P_{\geq 3}$-factor. On the contrary, we assume that H has no $P_{\geq 3}$-factor. In view of Theorem 2, we obtain

$$sun(H - X) \geq 2|X| + 1 \tag{3}$$

for some subset X of $V(H)$.

In view of (3) and Lemma 1, we infer

$$\begin{aligned}\omega(G - W - X) &\geq \omega(G - W - X - e) - 1 = \omega(H - X) - 1\\ &\geq sun(H - X) - 1 \geq 2|X| + 1 - 1 = 2|X|\\ &\geq 2(r + 2) \geq 4.\end{aligned}$$

Combining this with the definition of $t(G)$, we get

$$t(G) \leq \frac{|W \cup X|}{\omega(G - W - X)} \leq \frac{n + |X|}{2|X|} = \frac{1}{2} + \frac{n}{2|X|} \leq \frac{1}{2} + \frac{n}{2(r+2)} = \frac{n + r + 2}{2(r+2)},$$

which contradicts $t(G) > \frac{n+r+2}{2(r+2)}$. We finish the proof of Theorem 4. □

Remark 3. Next, we show that the condition on $t(G)$ in Theorem 4 is sharp.

Let $G = K_{n+r+2} + ((2r+3)K_1 \cup K_2)$, where n and r are two nonnegative integers. Obviously, G is $(n+r+2)$-connected and $t(G) = \frac{n+r+2}{2(r+2)}$. Let $W \subseteq V(K_{n+r+2}) \subseteq V(G)$ with $|W| = n$ and $e \in E(K_2)$. Then $G - W - e = K_{r+2} + ((2r+5)K_1)$. Select $X = V(K_{r+2})$ in $G - W - e$. Thus, we derive

$$sun(G - W - e - X) = 2r + 5 > 2(r + 2) = 2|X|.$$

In terms of Theorem 2, $G - W - e$ has no $P_{\geq 3}$-factor. Hence, G is not $(P_{\geq 3}, n)$-factor critical avoidable.

Remark 4. Next, we explain that the condition on $(n + r + 2)$-connected in Theorem 4 cannot be replaced by $(n + r + 1)$-connected.

Let $G = K_{n+r+1} + ((2r+1)K_1 \cup K_2)$, where $n \geq 1$ and $r \geq 0$ are two integers. We see that G is $(n+r+1)$-connected and $t(G) = \frac{n+r+1}{2r+2} = \frac{1}{2} + \frac{n}{2(r+1)} > \frac{1}{2} + \frac{n}{2(r+2)} = \frac{n+r+2}{2(r+2)}$. Let $W \subseteq V(K_{n+r+1}) \subseteq V(G)$ with $|W| = n$ and $e \in E(K_2)$. Then $G - W - e = K_{r+1} + ((2r+3)K_1)$. Choose $X = V(K_{r+1})$ in $G - W - e$. Thus, we admit

$$sun(G - W - e - X) = 2r + 3 > 2(r + 1) = 2|X|.$$

In terms of Theorem 2, $G - W - e$ has no $P_{\geq 3}$-factor. Therefore, G is not $(P_{\geq 3}, n)$-factor critical avoidable.

Theorem 5. *Let n and r be two nonnegative integers, and let G be an $(n+r+2)$-connected graph. If its isolated toughness $I(G) > \frac{n+3(r+2)}{2(r+2)}$, then G is $(P_{\geq 3}, n)$-factor critical avoidable.*

Proof. It is obvious that Theorem 5 is true for a complete graph. In what follows, we assume that G is not complete. Let $H = G - W - e$ for any $W \subseteq V(G)$ with $|W| = n$ and any $e = uv \in E(G - W)$. It suffices to verify that H contains a $P_{\geq 3}$-factor. By means of contrary, we assume that H has no $P_{\geq 3}$-factor. Then by Theorem 2, we admit

$$sun(H - X) \geq 2|X| + 1 \tag{4}$$

for some vertex subset X of H.

Suppose that there exist a isolated vertices, b K_2's and c big sun components $H_1, H_2, \cdots, H_c$, where $|V(H_i)| \geq 6$, in $H - X$. Let R_i be the factor-critical subgraph of H_i. We select one vertex from every K_2 component of $H - X$, and denote the set of such vertices by Y. Thus, we admit

$$sun(H - X) = a + b + c. \tag{5}$$

According to (4), (5), Lemma 1 and $|V(R_i)| \geq 3$, we infer

$$a+b+\sum_{i=1}^{c}|V(R_i)| \geq a+b+3c \geq a+b+c = sun(H-X) \geq 2|X|+1 \geq 2(r+2)+1 \geq 5. \tag{6}$$

In fact, $sun(G-W-X-e)+1 \geq sun(G-W-X) \geq sun(G-W-X-e)-2$. The following proof is divided into four cases.

Case 1. $sun(G-W-X) = sun(G-W-X-e)-2$.

In this case, $u \in V(aK_1)$ and $v \in V(bK_2)$, or $u \in V(aK_1)$ and $v \in V(H_i)$, or u, v belong to two different K_2 components, or $u \in V(H_i)$ and $v \in V(bK_2)$, or $v \in V(H_i)$ and $u \in V(H_j)$ $(i \neq j)$.

Claim 1. $I(G) \leq \frac{n+|X|+b+\sum_{i=1}^{c}|V(R_i)|}{a+b+\sum_{i=1}^{c}|V(R_i)|}$.

Proof. We consider the following two subcases.

Subcase 1.1. $u \in V(aK_1)$ and $v \in V(bK_2)$, or $u \in V(H_i)$ and $v \in V(bK_2)$, or u, v belong to two different K_2 components.

We choose such Y with $v \in Y$. Then we deduce

$$\begin{aligned} i\left(G-W-X-Y-\bigcup_{i=1}^{c}V(R_i)\right) &= i\left(G-W-X-e-Y-\bigcup_{i=1}^{c}V(R_i)\right) \\ &= i\left(H-X-Y-\bigcup_{i=1}^{c}V(R_i)\right) \\ &= a+b+\sum_{i=1}^{c}|V(R_i)|. \end{aligned}$$

In terms of (6) and the definition of $I(G)$, we get

$$I(G) \leq \frac{|W \cup X \cup Y \cup (\bigcup_{i=1}^{c}V(R_i))|}{i(G-W-X-Y-\bigcup_{i=1}^{c}V(R_i))} = \frac{n+|X|+b+\sum_{i=1}^{c}|V(R_i)|}{a+b+\sum_{i=1}^{c}|V(R_i)|}.$$

Subcase 1.2. $u \in V(aK_1)$ and $v \in V(H_i)$, or $v \in V(H_i)$ and $u \in V(H_j)$ $(i \neq j)$.

If $v \in V(R_i)$, then we write $Z = \bigcup_{i=1}^{c}V(R_i)$. If $v \in V(H_i) \setminus V(R_i)$, then $d_{H_i}(v) = 1$, which implies that there exists $w \in V(R_i)$ such that $vw \in E(H_i)$. Now we write $Z = ((\bigcup_{i=1}^{c}V(R_i)) \cup \{v\}) \setminus \{w\}$. Thus, we refer

$$\begin{aligned} i(G-W-X-Y-Z) &= i(G-W-X-e-Y-Z) \\ &= i(H-X-Y-Z) \\ &= a+b+\sum_{i=1}^{c}|V(R_i)|. \end{aligned}$$

Using (6) and the definition of $I(G)$, we derive

$$I(G) \leq \frac{|W \cup X \cup Y \cup Z|}{i(G-W-X-Y-Z)} = \frac{n+|X|+b+\sum_{i=1}^{c}|V(R_i)|}{a+b+\sum_{i=1}^{c}|V(R_i)|}.$$

This completes the proof of Claim 1.

□

According to (6), Claim 1 and Lemma 1, we obtain

$$\begin{aligned} I(G) &\leq \frac{n+|X|+b+\sum_{i=1}^{c}|V(R_i)|}{a+b+\sum_{i=1}^{c}|V(R_i)|} \\ &\leq \frac{n+|X|+a+b+\sum_{i=1}^{c}|V(R_i)|}{a+b+\sum_{i=1}^{c}|V(R_i)|} \\ &= 1+\frac{n+|X|}{a+b+\sum_{i=1}^{c}|V(R_i)|} \\ &\leq 1+\frac{n+|X|}{2|X|+1} \\ &< 1+\frac{n+|X|+\frac{1}{2}}{2|X|+1} \\ &= \frac{3}{2}+\frac{n}{2|X|+1} \\ &< \frac{3}{2}+\frac{n}{2|X|} \\ &\leq \frac{3}{2}+\frac{n}{2(r+2)} \\ &= \frac{n+3(r+2)}{2(r+2)}, \end{aligned}$$

which contradicts $I(G) > \frac{n+3(r+2)}{2(r+2)}$.

Case 2. $sun(G-W-X) = sun(G-W-X-e) - 1$.

In this case, $u, v \in V(aK_1)$, or $u \in V(M)$ and $v \in V(Q)$, where M is a sun component of $H-X$, Q is a non-sun component of $H-X$, and $M \cup Q \cup \{e\}$ is a non-sun component of $(H-X)\cup\{e\}$.

Subcase 2.1. $u, v \in V(aK_1)$.

In this subcase, $a \geq 2$. Thus, we admit

$$\begin{aligned} i\left(G-W-X-Y-\bigcup_{i=1}^{c}V(R_i)-v\right) &= i\left(G-W-X-e-Y-\bigcup_{i=1}^{c}V(R_i)-v\right) \\ &= i\left(H-X-Y-\bigcup_{i=1}^{c}V(R_i)-v\right) \\ &= a+b+\sum_{i=1}^{c}|V(R_i)|-1. \end{aligned}$$

Combining this with (6), $a \geq 2$, Lemma 1 and the definition of $I(G)$, we derive

$$\begin{aligned} I(G) &\leq \frac{|W\cup X\cup Y\cup(\bigcup_{i=1}^{c}V(R_i))\cup\{v\}|}{i(G-W-X-Y-\bigcup_{i=1}^{c}V(R_i)-v)} \\ &= \frac{n+|X|+b+\sum_{i=1}^{c}|V(R_i)|+1}{a+b+\sum_{i=1}^{c}|V(R_i)|-1} \\ &= 1+\frac{n+|X|+2-a}{a+b+\sum_{i=1}^{c}|V(R_i)|-1} \\ &\leq 1+\frac{n+|X|}{2|X|} \end{aligned}$$

$$= \frac{3}{2} + \frac{n}{2|X|}$$
$$\leq \frac{3}{2} + \frac{n}{2(r+2)}$$
$$= \frac{n+3(r+2)}{2(r+2)},$$

which contradicts $I(G) > \frac{n+3(r+2)}{2(r+2)}$.

Subcase 2.2. $u \in V(M)$ and $v \in V(Q)$, where M is a sun component of $H - X$, Q is a non-sun component of $H - X$, and $M \cup Q \cup \{e\}$ is a non-sun component of $(H - X) \cup \{e\}$.

If $M = K_1$, then $a \geq 1$. Thus, we have

$$\begin{aligned} i\left(G - W - X - Y - \bigcup_{i=1}^{c} V(R_i) - v\right) &= i\left(G - W - X - e - Y - \bigcup_{i=1}^{c} V(R_i) - v\right) \\ &= i\left(H - X - Y - \bigcup_{i=1}^{c} V(R_i) - v\right) \\ &= a + b + \sum_{i=1}^{c} |V(R_i)|. \end{aligned}$$

Then using (6), $a \geq 1$, Lemma 1, the definition of $I(G)$ and $I(G) > \frac{n+3(r+2)}{2(r+2)}$, we infer

$$\begin{aligned} \frac{n+3(r+2)}{2(r+2)} < I(G) &\leq \frac{|W \cup X \cup Y \cup (\bigcup_{i=1}^{c} V(R_i)) \cup \{v\}|}{i(G - W - X - Y - \bigcup_{i=1}^{c} V(R_i) - v)} \\ &= \frac{n + |X| + b + \sum_{i=1}^{c} |V(R_i)| + 1}{a + b + \sum_{i=1}^{c} |V(R_i)|} \\ &= 1 + \frac{n + |X| + 1 - a}{a + b + \sum_{i=1}^{c} |V(R_i)|} \\ &\leq 1 + \frac{n + |X|}{2|X| + 1} \\ &< 1 + \frac{n + |X|}{2|X|} \\ &= \frac{3}{2} + \frac{n}{2|X|} \\ &\leq \frac{3}{2} + \frac{n}{2(r+2)} \\ &= \frac{n+3(r+2)}{2(r+2)}, \end{aligned}$$

which is a contradiction.

If $M = K_2$, then we choose such Y with $u \in Y$ and select $Z = \bigcup_{i=1}^{c} V(R_i)$. If $M = H_i$, then $u \in V(R_i)$ or $u \in V(H_i) \setminus V(R_i)$. If $u \in V(R_i)$, then we choose $Z = \bigcup_{i=1}^{c} V(R_i)$. If $u \in V(H_i) \setminus V(R_i)$, then there exists $w \in V(R_i)$ such that $uw \in E(H_i)$. Thus, we select $Z = ((\bigcup_{i=1}^{c} V(R_i)) \cup \{u\}) \setminus \{w\}$.

Hence, we deduce

$$\begin{aligned} i(G - W - X - Y - Z) &= i(G - W - X - e - Y - Z) \\ &= i(H - X - Y - Z) \end{aligned}$$

$$= a + b + \sum_{i=1}^{c} |V(R_i)|.$$

Combining this with (6), Lemma 1 and then definition of $I(G)$, we get

$$\begin{aligned} I(G) &\le \frac{|W \cup X \cup Y \cup Z|}{i(G - W - X - Y - Z)} \\ &= \frac{n + |X| + b + \sum_{i=1}^{c} |V(R_i)|}{a + b + \sum_{i=1}^{c} |V(R_i)|} \\ &\le \frac{n + |X| + a + b + \sum_{i=1}^{c} |V(R_i)|}{a + b + \sum_{i=1}^{c} |V(R_i)|} \\ &= 1 + \frac{n + |X|}{a + b + \sum_{i=1}^{c} |V(R_i)|} \\ &\le 1 + \frac{n + |X|}{2|X| + 1} \\ &< 1 + \frac{n + |X|}{2|X|} \\ &= \frac{3}{2} + \frac{n}{2|X|} \\ &\le \frac{3}{2} + \frac{n}{2(r + 2)} \\ &= \frac{n + 3(r + 2)}{2(r + 2)}, \end{aligned}$$

which contradicts $I(G) > \frac{n+3(r+2)}{2(r+2)}$.

Case 3. $sun(G - W - X) = sun(G - W - X - e)$.

In this case, $u \in V(R_i)$ and $v \in V(H_i) \setminus V(R_i)$, or $u, v \in V(M)$, or $u \in V(M)$ and $v \in V(Q)$, where M is a non-sun component of $H - X$, $M \cup \{e\}$ is a non-sun component of $(H - X) \cup \{e\}$, Q is a non-sun component of $H - X$, and $M \cup Q \cup \{e\}$ is a non-sun component of $(H - X) \cup \{e\}$. Then we infer

$$\begin{aligned} i\left(G - W - X - Y - \bigcup_{i=1}^{c} V(R_i)\right) &= i\left(G - W - X - e - Y - \bigcup_{i=1}^{c} V(R_i)\right) \\ &= i\left(H - X - Y - \bigcup_{i=1}^{c} V(R_i)\right) \\ &= a + b + \sum_{i=1}^{c} |V(R_i)|. \end{aligned}$$

It follows from (6), Lemma 1, the definition of $I(G)$ and $I(G) > \frac{n+3(r+2)}{2(r+2)}$ that

$$\begin{aligned} \frac{n + 3(r + 2)}{2(r + 2)} < I(G) &\le \frac{|W \cup X \cup Y \cup (\bigcup_{i=1}^{c} V(R_i))|}{i(G - W - X - Y - \bigcup_{i=1}^{c} V(R_i))} \\ &= \frac{n + |X| + b + \sum_{i=1}^{c} |V(R_i)|}{a + b + \sum_{i=1}^{c} |V(R_i)|} \\ &\le 1 + \frac{n + |X|}{a + b + \sum_{i=1}^{c} |V(R_i)|} \end{aligned}$$

$$\begin{aligned}&\leq 1+\frac{n+|X|}{2|X|+1}\\&< 1+\frac{n+|X|}{2|X|}\\&=\frac{3}{2}+\frac{n}{2|X|}\\&\leq\frac{3}{2}+\frac{n}{2(r+2)}\\&=\frac{n+3(r+2)}{2(r+2)},\end{aligned}$$

which is a contradiction.

Case 4. $sun(G-W-X) = sun(G-W-X-e)+1$.

In this case, there exists a non-sun component H_{c+1} in $H-X$ such that $H_{c+1}+e$ is a big sun component of $H-X+e = G-W-X$, where R_{c+1} be the factor-critical subgraph of $H_{c+1}+e$. Obviously, $u, v \in V(R_{c+1})$ and $R_{c+1}-e$ is not a factor-critical graph. The following proof is similar to that of Case 3, we easily deduce

$$I(G) \leq \frac{|W \cup X \cup Y \cup (\bigcup_{i=1}^{c} V(R_i))|}{i(G-W-X-Y-\bigcup_{i=1}^{c} V(R_i))} < \frac{n+3(r+2)}{2(r+2)},$$

which contradicts $I(G) > \frac{n+3(r+2)}{2(r+2)}$. We finish the proof of Theorem 5.

□

Remark 5. Now, we show that the condition $I(G) > \frac{n+3(r+2)}{2(r+2)}$ in Theorem 5 cannot be replaced by $I(G) \geq \frac{n+3(r+2)}{2(r+2)}$.

Let $G = K_{n+r+2} + ((2r+4)K_2)$, where n and r are two nonnegative integers. It is obvious that G is $(n+r+2)$-connected and $I(G) = \frac{n+3(r+2)}{2(r+2)}$. Set $W \subseteq V(K_{n+r+2}) \subseteq V(G)$ with $|W| = n$ and $e \in E(K_2)$. Then $G-W-e = K_{r+2} + ((2r+3)K_2 \cup (2K_1))$. Let $X = V(K_{r+2})$ in $G-W-e$. Thus, we acquire

$$sun(G-W-e-X) = 2r+5 > 2(r+2) = 2|X|.$$

According to Theorem 2, $G-W-e$ has no $P_{\geq 3}$-factor. So G is not $(P_{\geq 3}, n)$-factor critical avoidable.

Remark 6. Now, we claim that the condition that $(n+r+2)$-connected in Theorem 5 is sharp.

Let $G = K_{n+r+1} + ((2r+2)K_2)$, where $n \geq 1$ and $r \geq 0$ are two integers. Clearly, G is $(n+r+1)$-connected and $I(G) = \frac{n+3(r+1)}{2(r+1)} = \frac{3}{2} + \frac{n}{2(r+1)} > \frac{3}{2} + \frac{n}{2(r+2)} = \frac{n+3(r+2)}{2(r+2)}$. Let $W \subseteq V(K_{n+r+1}) \subseteq V(G)$ with $|W| = n$ and $e \in E(K_2)$. Then $G-W-e = K_{r+1} + ((2r+1)K_2 \cup (2K_1))$. Let $X = V(K_{r+1})$ in $G-W-e$. Thus, we refer

$$sun(G-W-e-X) = 2r+3 > 2(r+1) = 2|X|.$$

Using Theorem 2, $G-W-e$ has no $P_{\geq 3}$-factor. Hence, G is not $(P_{\geq 3}, n)$-factor critical avoidable.

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