

OPTIMALITY CONDITIONS FOR BILEVEL OPTIMAL CONTROL PROBLEMS WITH NON-CONVEX QUASI-VARIATIONAL INEQUALITIES

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Abstract. We establish Pontryagin optimality conditions for a generalized bilevel optimal control problem in which the leader is subject to a pure state inequality constraint, while the follower is governed by a non-convex quasi-variational inequality parameterized by the final state. To simplify the problem at hand, we convert it into a single-level optimal control problem by mapping the solution set of the quasi-variational inequality to a parametric optimization problem and employing the value function reformulation. Furthermore, we introduce certain regularity conditions to ensure that the derived maximum principle remains non-degenerate. Finally, we provide an illustrative example to elucidate our research findings.

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1. INTRODUCTION

In the realm of bilevel optimization, bilevel optimal control refers to scenarios where at least one of the decision-makers is tasked with solving optimal control problems that involve either ordinary or partial differential equations [7]. In this study, we take up the case when the optimal control mathematical program involves two levels of decision: the upper level, referred to as the ‘leader’, is an optimal control problem, which incorporates a dynamic system as one of its constraints. The lower level, denoted as the ‘follower’, is a non-convex quasi-variational inequality (QVI) parameterized by the liaison variable connecting the leader and the follower.

QVIs were originally introduced by Bensoussan and Lions in connection with impulse optimal control problems [4]. The rationale for incorporating them into the follower problems stems from their immense flexibility in modeling diverse real-world phenomena, from economics and continuum mechanics to biological interactions [16]. Building upon this foundation, we leverage this powerful tool in our bilevel optimal control problem, introducing a novel approach to handling its complexity induced by non-convexity in the lower level.

On the other hand, the characterization of optimal solutions through Pontryagin’s maximum has long been of significant interest, aiming to refine first-order optimality conditions for many classic bilevel optimal

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control problems. Numerous studies, such as [3, 5, 13, 14, 21], provide a comprehensive survey of the theoretical landscape on bilevel optimal control problems. Additionally, studies on practical applications can be found in [2, 9–11]. Most, however, involve mixed control-state constraints, in the sense that both control variables and state variables are involved in the constraints of the optimization problems at both levels.

For example, Bonnel and Morgan [5] investigated a bilevel problem with a scalar optimal control problem at the upper level and a multiobjective convex optimal control problem at the lower level. Through a specific scalarization, they showed that the optimal solutions of the follower coincide with the efficient set. Additionally, they derived sufficient conditions guaranteeing the existence of optimal solutions.

In [13], Mehlitz and Wachsmuth studied bilevel optimization problems in reflexive Banach spaces, where the follower deals with a convex problem. After deriving necessary optimality conditions, the authors applied their results to a class of bilevel optimal controls containing a state and its control for each level. The optimality conditions obtained were Pontryagin-weak and strong conditions. The same authors addressed in [14] an inverse optimal control problem. They initially established the existence of solutions, and subsequently, by employing the optimistic approach and the concept of relaxation, they established necessary optimality conditions for the given problem.

Ye focused in [21] on a bilevel optimization problem with a parametric control problem in the lower level. Using the optimal value function and the calmness property (some sufficient conditions of this property were given in this paper), the author derived necessary optimality conditions for the problem.

In [3], the authors investigated bilevel optimal control problems with pure state constraints and finite-dimensional lower levels, providing valuable insights into specific problem structures. While not directly addressing non-convex QVIs, their work demonstrated the importance of tailoring optimality conditions to the unique features of different bilevel control scenarios.

To our knowledge, prior research on bilevel optimal control has not specifically explored problems with non-convex QVI constraints. The presence of non-convexity in the lower level complicates matters, as the problem cannot be directly treated as a standard bilevel format. This work contributes to the field by addressing this gap, proposing a set of Pontryagin-type optimality conditions tailored to this specific scenario. These conditions aim to identify a subset of local optimal solutions, offering a valuable tool for analyzing and potentially solving such problems.

Within the time interval $[T_b, T_e] \subset \mathbb{R}^+$, we denote the begin point (initial state) $x(T_b)$ as x_b and the end point (final state) $x(T_e)$ as x_e for a trajectory $x : [T_b, T_e] \rightarrow \mathbb{R}^n$, then we proceed to examine the following generalized bilevel optimal control problem:

$$\left\{ \begin{array}{ll} \min_{x,u,y} f(x_e, y) & \\ \text{subject to } \dot{x}(t) = \phi(t, x(t), u(t)) & \text{a.e. } t \in [T_b, T_e] \\ x(T_b) = x_b & \\ g(t, x(t)) \leq 0 & \forall t \in [T_b, T_e] \\ u(t) \in \mathcal{U} & \text{a.e. } t \in [T_b, T_e] \\ y \in S(x_e). & \end{array} \right. \quad \text{(GBOCP)}$$

Here the functions $f : \mathbb{R}^n \times \mathbb{R}^d \rightarrow \mathbb{R}$ and $g : [T_b, T_e] \times \mathbb{R}^n \rightarrow \mathbb{R}$ are assumed to be continuously differentiable, $\phi : [T_b, T_e] \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ stands for the dynamic function, x_b is fixed, $\mathcal{U} \subset \mathbb{R}^m$ is a non-empty Borel measurable set, and $S(x_e)$ denotes the solution set of a quasi-variational inequality seeking vectors y in a feasible set $K(x_e, y)$ such that

$$\langle F(x_e, y), y - z \rangle \leq 0 \quad \text{for all } z \in K(x_e, y), \quad \text{(QVI}[x_e])$$

where $F : \mathbb{R}^n \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a continuously differentiable vector function, and $K : \mathbb{R}^n \times \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ denotes the feasibility set-valued map, explicitly given by:

$$K(x_e, y) := \{z \in \mathbb{R}^d : G(x_e, y, z) \leq 0\}, \quad (1)$$

with $G : \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^r$ is a continuously differentiable vector function.

Our goal in this study is to derive optimality conditions of the Pontryagin class for (GBOCP). When solving the quasi-variational inequality, a transformation of S together with the optimal value function reformulation are employed to convert (GBOCP) into a standard bilevel optimal control problem. Subsequently, we establish the optimality conditions for this problem using the initial data. A similar approach was recently employed in [8] but in the context of single-level optimization problems.

The basic structure of our paper is as follows: Section 2 introduces some basic tools and results from nonsmooth analysis. Section 3 presents the hypotheses utilized in our study, other formulations of the solution set-valued mapping S of (QVI $[x_e]$), and an equivalent single-level optimal control problem of (GBOCP). In Section 4, we employ an exact penalization to construct a locally equivalent problem of (GBOCP). In Section 5, we prove the main result of our work that provides Pontryagin-type necessary optimality conditions. We conclude this section with an illustrative example to demonstrate the applicability of our findings. Section 6 draws the conclusion and offers final remarks.

2. PRELIMINARIES

2.1. Basic tools

Let $(E, \|\cdot\|_E)$ be a Banach space, and $x \in E$. $\mathbb{B}_E^\varepsilon(x)$ refers to the open ball around x with a radius $\varepsilon > 0$. For a subset Δ of E and a vector x of E , we define the distance between Δ and x by $d_\Delta(x) = \inf \{\|x - y\|_E : y \in \Delta\}$.

The set of continuous functions from $[T_b, T_e]$ to \mathbb{R}^n is denoted as $\mathcal{C}([T_b, T_e], \mathbb{R}^n)$. Its dual space is represented by $\mathcal{C}^*([T_b, T_e], \mathbb{R}^n)$. We define $\mathcal{C}_p([T_b, T_e], \mathbb{R})$ as the set of elements belonging to $\mathcal{C}^*([T_b, T_e], \mathbb{R})$ that take nonnegative values on nonnegative functions in $\mathcal{C}([T_b, T_e], \mathbb{R})$. The support of a measure $\mu \in \mathcal{C}_p([T_b, T_e], \mathbb{R})$, denoted as $\text{supp}(\mu)$, is defined as the smallest closed set $X \subset [T_b, T_e]$ such that, for any relatively open subset $Y \subset [T_b, T_e] \setminus X$, we have $\mu(Y) = 0$. We use \mathcal{L} to represent the sigma-algebra generated by $[T_b, T_e]$, and \mathcal{B}^n to represent the Borel sigma-algebra on \mathbb{R}^n .

The sets $M([T_b, T_e], \mathbb{R}^n)$, $L^1([T_b, T_e], \mathbb{R}^n)$ and $L^\infty([T_b, T_e], \mathbb{R}^n)$ represent, respectively, the set of all functions $\varphi_1 : [T_b, T_e] \rightarrow \mathbb{R}^n$ that are measurable, the space of all functions $\varphi_2 : [T_b, T_e] \rightarrow \mathbb{R}^n$ that are Lebesgue-integrable, and the space of all measurable functions $\varphi_3 : [T_b, T_e] \rightarrow \mathbb{R}^n$ that are bounded almost everywhere on $[T_b, T_e]$. The Sobolev space $\mathcal{W}(n) := \mathcal{W}^{1,1}([T_b, T_e], \mathbb{R}^n)$, is defined as follows:

$$\mathcal{W}(n) = \{\varphi : [T_b, T_e] \rightarrow \mathbb{R}^n \mid \varphi, \dot{\varphi} \in L^1([T_b, T_e], \mathbb{R}^n)\},$$

where $\dot{\varphi}$ denotes the (weak) derivative of φ .

Lemma 2.1. [1] *There is a constant $C_{\text{emb}} > 0$, such that for any $\varphi \in \mathcal{W}(n)$, and for any $t \in [T_b, T_e]$, we have*

$$\|\varphi(t)\|_\infty \leq \|\varphi\|_{\mathcal{C}([T_b, T_e], \mathbb{R}^n)} \leq C_{\text{emb}} \|\varphi\|_{\mathcal{W}(n)}.$$

We employ classical nonsmooth analytical approaches. For a given locally closed subset Ω of \mathbb{R}^n , the Fréchet normal cone to Ω at x , denoted as $\hat{N}(\Omega, x)$, and the normal cone to Ω at $x \in \Omega$, denoted as $N(\Omega, x)$, can be defined as follows:

$$\hat{N}(\Omega, x) := \left\{ \lambda \in \mathbb{R}^n : \limsup_{y \rightarrow x, y \in \Omega} \frac{\lambda^\top (y - x)}{\|y - x\|_\infty} \leq 0 \right\},$$

$$N(\Omega, x) := \limsup_{y \rightarrow x, y \in \Omega} \hat{N}(\Omega, y).$$

It should be noted that in the definition of $N(\Omega, x)$, the superior limit must be considered in the Painlevé and Kuratowski sense [15]. Also, if Ω is a closed convex set, $N(\Omega, x)$ and $\hat{N}(\Omega, x)$ coincide with the normal cone of convex analysis.

Consider $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$ as a Lipschitz continuous function near a vector $x \in \mathbb{R}^n$. The directional derivative in the sense of Clarke of φ at x in a direction $\varrho \in \mathbb{R}^n$ is defined as

$$\varphi^\circ(x, \varrho) = \limsup_{y \rightarrow x, t \searrow 0} \frac{\varphi(x + t\varrho) - \varphi(x)}{t}.$$

Using $\varphi^\circ(x, \varrho)$, we define the Clarke subdifferential of φ at x as

$$\partial^c \varphi(x) = \{ \alpha \in \mathbb{R}^n : \langle \alpha, \varrho \rangle \leq \varphi^\circ(x, \varrho) \quad \forall \varrho \in \mathbb{R}^n \}.$$

The Clarke subdifferential at any reference point is a nonempty, convex, and compact set for every locally Lipschitz continuous function [6]. Additionally, the mapping $\varphi \mapsto \partial^c \varphi(x)$ for such functions is homogeneous. For convex functions, the Clarke subdifferential reduces to the classical subdifferential of convex analysis.

To end this section, the following results summarize the fundamental properties of locally Lipschitz continuous functions and associated Clarke subdifferentials.

Proposition 2.2. *Let $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$ be a Lipschitz continuous function at $x \in \mathbb{R}^n$.*

(i) *If φ attains a local optimal solution at x , then*

$$0 \in \partial^c \varphi(x), \quad \text{and} \quad 0 \leq \varphi^\circ(x, \varrho) \quad \forall \varrho \in \mathbb{R}^n.$$

(ii) *When φ is differentiable at x , $\nabla \varphi(x) \in \partial^c \varphi(x)$.*

(iii) *If φ is continuously differentiable at x , then $\partial^c \varphi(x) = \{ \nabla \varphi(x) \}$.*

Let $\Gamma : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a set-valued mapping. The graph of Γ is defined as follows:

$$gph(\Gamma) := \{ (x, y) \in \mathbb{R}^n \times \mathbb{R}^m : y \in \Gamma(x) \}.$$

We say that Γ is inner semicontinuous at a point $(\bar{x}, \bar{y}) \in gph(\Gamma)$ if for each sequence $x_i \rightarrow \bar{x}$ there is a sequence $y_i \in \Gamma(x_i)$ such that $y_i \rightarrow \bar{y}$. It is said to be calm at $(\bar{x}, \bar{y}) \in gph(\Gamma)$ if there exist neighborhoods U and V of \bar{x} and \bar{y} , respectively, and a scalar $\sigma > 0$ such that

$$d_{\Gamma(\bar{x})}(y) \leq \sigma \|\bar{x} - x\|, \quad \forall y \in V \cap \Gamma(x), \quad \forall x \in U.$$

2.2. The maximum principle for optimal control with pure state constraints

In this subsection, we will provide an overview of the maximum principle in the context of optimal control problems with pure state constraints. We follow the approach outlined in Theorem 9.3.1 from [19] and Theorem 4.1 from [12].

To proceed, let's consider a single-level optimal control problem with two state functions: $x : [T_b, T_e] \rightarrow \mathbb{R}^n$ and $y : [T_b, T_e] \rightarrow \mathbb{R}^d$. We assume that the trajectory x has a fixed starting point $x(T_b) := x_b$ in \mathbb{R}^n , is controlled by a control function $u \in M([T_b, T_e], \mathbb{R}^m)$, and influenced by a pure state constraint $h : [T_b, T_e] \times \mathbb{R}^n \rightarrow \mathbb{R}$. For the second trajectory, we assume that it is controlled by a control function $v \in M([T_b, T_e], \mathbb{R}^l)$. Mathematically, the considered optimal control problem takes the following form:

$$\left\{ \begin{array}{ll} \min_{(x,u),(y,v)} H(x_e, y_e) \\ \text{subject to } \dot{x}(t) = \phi_1(t, x(t), u(t)) & \text{a.e. } t \in [T_b, T_e] \\ \dot{y}(t) = \phi_2(t, y(t), v(t)) & \text{a.e. } t \in [T_b, T_e] \\ x(T_b) = x_b \\ h(t, x(t)) \leq 0 & \forall t \in [T_b, T_e] \\ u(t) \in \mathbf{U} & \text{a.e. } t \in [T_b, T_e] \\ v(t) \in \mathbf{V} & \text{a.e. } t \in [T_b, T_e] \\ (x_e, y_e) \in C, \end{array} \right. \tag{2}$$

where $H : \mathbb{R}^n \times \mathbb{R}^d \rightarrow \mathbb{R}$ is the objective function, $\phi_1 : [T_b, T_e] \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ and $\phi_2 : [T_b, T_e] \times \mathbb{R}^d \times \mathbb{R}^l \rightarrow \mathbb{R}^d$ represent the dynamic functions, \mathbf{U} and \mathbf{V} are nonempty and Borel measurable subsets of \mathbb{R}^n and \mathbb{R}^l , respectively, and $C \subset \mathbb{R}^n \times \mathbb{R}^d$.

A concept of local optimal solutions to this optimal control problem is stated below.

Definition 1. A feasible point $((\bar{x}, \bar{u}), (\bar{y}, \bar{v}))$ of (2) is said to be a $\mathcal{W}(n+d)$ -local optimal solution of (2) if there is a scalar $\varepsilon > 0$ such that

$$H(\bar{x}_e, \bar{y}_e) \leq H(x_e, y_e),$$

for each feasible point $((x, u), (y, v))$ which also verifies $(x, y) \in \mathbb{B}_{\mathcal{W}(n+d)}^\varepsilon(\bar{x}, \bar{y})$.

Now, to derive the maximum principle for problem (2), we need to make some hypotheses for its $\mathcal{W}(n+d)$ -local optimal solutions. For this purpose, let $((\bar{x}, \bar{u}), (\bar{y}, \bar{v}))$ be a $\mathcal{W}(n+d)$ -local optimal solution of (2).

(\mathcal{X}_1) H is Lipschitz continuous at (\bar{x}_e, \bar{y}_e) .

(\mathcal{X}_2) The functions ϕ_1 and ϕ_2 are continuous. Also, for each $(x, y) \in \mathbb{R}^n \times \mathbb{R}^d$, the mappings $\phi_1(\cdot, x, \cdot)$ and $\phi_2(\cdot, y, \cdot)$ are $\mathcal{L} \times \mathcal{B}^m$ and $\mathcal{L} \times \mathcal{B}^l$ -measurable, and for each $t \in [T_b, T_e]$, and $(u, v) \in \mathbf{U} \times \mathbf{V}$, the mappings $\phi_1(t, \cdot, u)$ and $\phi_2(t, \cdot, v)$ are continuously differentiable. Moreover, there are two measurable functions $\gamma_x : [T_b, T_e] \times \mathbb{R}^m \rightarrow \mathbb{R}$, $\gamma_y : [T_b, T_e] \times \mathbb{R}^l \rightarrow \mathbb{R}$, and scalars $\varepsilon_1, \varepsilon_2 > 0$ such that the mappings $t \mapsto \gamma_x(t, \bar{u}(t))$, $t \mapsto \gamma_y(t, \bar{v}(t))$ are integrable, and for almost every $t \in [T_b, T_e]$,

$$\begin{aligned} \|\phi_1(t, x_1, u) - \phi_1(t, x_2, u)\|_\infty &\leq \gamma_x(t, u) \|x_1 - x_2\|_\infty, \\ \|\phi_2(t, y_1, v) - \phi_2(t, y_2, v)\|_\infty &\leq \gamma_y(t, v) \|y_1 - y_2\|_\infty, \end{aligned}$$

for each $x_1, x_2 \in \mathbb{B}_{\mathbb{R}^n}^{\varepsilon_1}(\bar{x}(t))$, $u \in \mathbf{U}$, $y_1, y_2 \in \mathbb{B}_{\mathbb{R}^d}^{\varepsilon_2}(\bar{y}(t))$, $v \in \mathbf{V}$.

(\mathcal{X}_3) The function h is upper semicontinuous, and for each $t \in [T_b, T_e]$, $h(t, \cdot)$ is continuously differentiable with $\nabla_x h(t, \bar{x}(t)) \neq 0$. Moreover, there are scalars $\varepsilon_h, L_h > 0$ such that $|h(t, x_1) - h(t, x_2)| \leq L_h \|x_1 - x_2\|_\infty$ for each $t \in [T_b, T_e]$, $x_1, x_2 \in \mathbb{B}_{\mathbb{R}^n}^{\varepsilon_h}(\bar{x}(t))$.

(\mathcal{X}_4) The set C is locally closed around (\bar{x}_e, \bar{y}_e) .

(\mathcal{X}_5) There is $\kappa > T_b$ such that the set $\{\phi_1(t, x, u) : u \in \mathbf{U}\}$ is convex for all $x \in \mathbb{B}_{\mathbb{R}^n}^\kappa(x_b)$, $t \in [T_b, \kappa]$. Furthermore, if $h(T_b, x_b) = 0$, then there are scalars $\nu_1, \nu_2, \nu_3, \nu_4 > 0$ and a control function $\hat{u} \in L^1([T_b, T_e], \mathbb{R}^m)$ with $\hat{u} \in \mathbf{U}$ a.e on $[T_b, T_e]$ such that

$$\begin{aligned} \|\phi_1(t, x_b, \bar{u}(t))\|_\infty &\leq \nu_3, \quad \|\phi_1(t, x_b, \hat{u}(t))\|_\infty \leq \nu_3, \\ \int_{T_b}^t \nabla_x h(a, x)^\top [\phi_1(\tau, x_b, \hat{u}(\tau)) - \phi_1(\tau, x_b, \bar{u}(\tau))] d\tau &\leq -\nu_4 t, \end{aligned}$$

for all $a, t \in [T_b, \nu_1]$ and $x \in \mathbb{B}_{\mathbb{R}^n}^{\nu_2}(x_b)$.

Remark 1. As is known, the primary objective of necessary optimality conditions is to reduce the cardinality of the feasible set. Unfortunately, sometimes we find that every feasible point satisfies the derived necessary optimality conditions, rendering them useless. To address this issue, as demonstrated in [12], we introduce the final hypothesis (\mathcal{X}_5) to ensure that the Pontryagin optimality conditions derived in Theorem 1 for the optimal control problem (2) do not lead to degeneracy.

The following result directly follows Theorem 4.1 from [12] and Theorem 9.3.1 from [19].

Theorem 1 (The Enhanced Maximum Principle for the Optimal Control Problem with Pure State Constraints). *Let $((\bar{x}, \bar{u}), (\bar{y}, \bar{v}))$ be a $\mathcal{W}(n+d)$ -local optimal solution of (2) under the assumptions (\mathcal{X}_1) – (\mathcal{X}_5). Then, there exist two functions $(p_1, p_2) \in \mathcal{W}(n) \times \mathcal{W}(d)$, a measure $\mu \in \mathcal{C}_p([T_b, T_e])$, and a constant $\tau \geq 0$, such that the following optimality conditions hold:*

The Enhanced Nontriviality Condition

$$\|R\|_{L^\infty([T_b, T_e], \mathbb{R}^n)} + \|p_2\|_{\mathcal{W}(d)} + \mu([T_b, T_e]) + \tau > 0.$$

The Adjoint Equations for almost every $t \in [T_b, T_e]$

$$\begin{aligned} -\dot{p}_1(t)^\top &= R(t)^\top \nabla_x \phi_1(t, \bar{x}(t), \bar{u}_1(t)), \\ -\dot{p}_2(t)^\top &= p_2(t)^\top \nabla_y \phi_2(t, \bar{y}(t), \bar{u}_2(t)). \end{aligned}$$

The Weierstrass-Pontryagin Conditions for almost every $t \in [T_b, T_e]$

$$\begin{aligned} R(t)^\top \phi_1(t, \bar{x}(t), \bar{u}(t)) &= \max_{w \in \bar{U}} R(t)^\top \phi_1(t, \bar{x}(t), w), \\ p_2(t)^\top \phi_2(t, \bar{y}(t), \bar{v}(t)) &= \max_{w \in \bar{V}} p_2(t)^\top \phi_2(t, \bar{y}(t), w). \end{aligned}$$

The Transversality Condition

$$\begin{aligned} p_2(T_b) &= 0, \\ (-R(T_e), -p_2(T_e)) &\in \tau \partial H(\bar{x}_e, \bar{y}_e) + N(C, (\bar{x}_e, \bar{y}_e)). \end{aligned}$$

The Support Condition

$$\text{supp}(\mu) \subseteq \{t \in [T_b, T_e] : h(t, \bar{x}(t)) = 0\}.$$

The function $R : [T_b, T_e] \rightarrow \mathbb{R}$ is given by

$$R(t) := \begin{cases} p_1(t) + \int_{[T_b, T_e[} \nabla_x h(a, \bar{x}(a)) \mu(da) & \text{if } t < T_e, \\ p_1(T_e) + \int_{[T_b, T_e]} \nabla_x h(a, \bar{x}(a)) \mu(da) & \text{if } t = T_e. \end{cases}$$

Remark 2. Without imposing the validity of hypothesis \mathcal{X}_5 , the enhanced nontriviality condition in Theorem 1 yields

$$(p_1, p_2, \mu, \tau) \neq (0, 0, 0, 0).$$

3. SINGLE-LEVEL REFORMULATION OF (GBOCP)

In this section, our focus is on transforming (GBOCP) into a single-level optimal control problem. We begin by examining a specific optimal solution of (GBOCP), as defined below. To this end, we define the feasible set Θ_L for the leader and the feasible set Θ for (GBOCP) as follows:

$$\begin{aligned} \Theta_L &:= \{(x, u) \in \mathcal{W}(n) \times M([T_b, T_e], \mathbb{R}^m) : \\ &\quad \dot{x}(t) = \phi(t, x(t), u(t)) \quad \text{a.e. } t \in [T_b, T_e] \\ &\quad x(T_b) = x_b \\ &\quad g(t, x(t)) \leq 0 \quad \forall t \in [T_b, T_e] \\ &\quad u(t) \in \mathcal{U} \quad \text{a.e. } t \in [T_b, T_e]\}, \\ \Theta &:= \{(x, u, y) \in \mathcal{W}(n) \times M([T_b, T_e], \mathbb{R}^m) \times \mathbb{R}^d : (x, u) \in \Theta_L, y \in S(x_e)\}. \end{aligned}$$

Now, we set $\mathcal{W} := \mathcal{W}(n) \times \mathbb{R}^d$, and consider the following concept of optimal solutions to (GBOCP).

Definition 2. Let $(\bar{x}, \bar{u}, \bar{y}) \in \Theta$ be a feasible process of (GBOCP). We say that $(\bar{x}, \bar{u}, \bar{y})$ is a \mathcal{W} -local optimal solution of (GBOCP) if a scalar $\varepsilon > 0$ exists such that $f(x_e, y) \geq f(\bar{x}_e, \bar{y})$ for all $(x, u, y) \in \Theta$ with $(x, y) \in \mathbb{B}_{\mathcal{W}}^\varepsilon(\bar{x}, \bar{y})$.

It is said to be a global optimal solution if, in addition, $\varepsilon = \infty$.

We point out that, as we are utilizing the method outlined by Vinter in [19], our analysis is mainly focused on the norm in the Banach space \mathcal{W} rather than the actions of the control. It is obvious that strong (and hence global) optimal solutions satisfy the same optimality conditions for our specific \mathcal{W} -local optimal solutions.

In our study, we consider that for any \mathcal{W} -local optimal solution $(\bar{x}, \bar{u}, \bar{y})$ of (GBOCP), the data validate the following hypotheses:

- (A₁) For the dynamic function: ϕ is continuous, and for a fixed x , it is measurable with respect to $\mathcal{L} \times \mathcal{B}^m$ for the first and third variables. Moreover, there is a measurable function $\gamma : [T_b, T_e] \times \mathbb{R}^m \rightarrow \mathbb{R}$ and some $\varepsilon > 0$ such that the function $t \mapsto \gamma(t, \bar{u}(t))$ is integrable, and for almost every $t \in [T_b, T_e]$:

$$\|\phi(t, x_1, u) - \phi(t, x_2, u)\|_\infty \leq \gamma(t, u) \|x_1 - x_2\|_\infty \quad \forall x_1, x_2 \in \mathbb{B}_{\mathbb{R}^n}^\varepsilon(\bar{x}(t)), \forall u \in \mathcal{U}.$$

- (A₂) For the pure state constraint: g is upper semicontinuous, and for any $t \in [T_b, T_e]$, $g(t, \cdot)$ is continuously differentiable with $\nabla_x g(t, \bar{x}(t)) \neq 0$. Moreover, there are scalars $\varepsilon_g, L_g > 0$ such that $|g(t, x_1) - g(t, x_2)| \leq L_g \|x_1 - x_2\|_\infty$ for each $t \in [T_b, T_e]$, $x_1, x_2 \in \mathbb{B}_{\mathbb{R}^n}^{\varepsilon_g}(\bar{x}(t))$.

- (A₃) For the quasi-variational inequality (QVI[x_e]): The Mangasarian-Fromovitz constraint qualification holds for the constraint describing K (1) at $(\bar{x}_e, \bar{y}, \bar{y})$, i.e.: there is a vector $d \in \mathbb{R}^d$ such that

$$\nabla_z G_j(\bar{x}_e, \bar{y}, \bar{y})^\top d < 0 \quad \forall j \in \{i : G_i(\bar{x}_e, \bar{y}, \bar{y}) = 0\}.$$

- (A₄) For the nondegeneracy of Pontryagin optimality conditions of (GBOCP): There is $\kappa > T_b$ such that the set $\{\phi(t, x, u) : u \in \mathcal{U}\}$ is convex for all $x \in \mathbb{B}_{\mathbb{R}^n}^\kappa(x_b)$ and $t \in [T_b, \kappa[$. Furthermore, if $g(T_b, x_b) = 0$, then there exist scalars $\nu_1, \nu_2, \nu_3, \nu_4 > 0$ and a control function $\hat{u} \in L^1([T_b, T_e], \mathbb{R}^m)$ verifying $\hat{u} \in \mathcal{U}$ a.e on $[T_b, T_e]$ such that

$$\begin{aligned} \|\phi(t, x_b, \bar{u}(t))\|_\infty &\leq \nu_3, \quad \|\phi(t, x_b, \hat{u}(t))\|_\infty \leq \nu_3 \\ \int_{T_b}^t \nabla_x g(a, x)^\top [\phi(\tau, x_b, \hat{u}(\tau)) - \phi(\tau, x_b, \bar{u}(\tau))] d\tau &\leq -\nu_4 t, \end{aligned}$$

for all $a, t \in [T_b, \nu_1[$ and $x \in \mathbb{B}_{\mathbb{R}^n}^{\nu_2}(x_b)$.

Remark 3. Similar to Remark 1, we introduce the hypothesis \mathcal{A}_4 to ensure that the maximum principle for (GBOCP) in Theorem 5.2 remains non-degenerate.

Now, according to [8], we present some alternative formulations of the solution set-valued map S . First, we can easily observe that the set $S(x_e)$ also represents the set of solutions to the following parametric problem:

$$\min_z z^\top F(x_e, y) \quad \text{subject to } z \in K(x_e, y). \quad (\text{P}[x_e, y])$$

The most practical method for reducing (GBOCP) to a single-level problem is by employing the optimal value function to replace an inequality constraint in the lower-level problem, specifically the quasi-variational inequality (QVI[x_e]). To proceed, we introduce the optimal value function $V : \mathbb{R}^n \times \mathbb{R}^d \rightarrow \mathbb{R}$ of the parametric problem (P[x_e, y]), defined as follows:

$$V(x_e, y) := \min_z \{z^\top F(x_e, y) : z \in K(x_e, y)\},$$

for each $(x_e, y) \in \mathbb{R}^n \times \mathbb{R}^d$. Therefore, the solution set $S(x_e)$ can be reformulated as:

$$S(x_e) = \{y \in K(x_e, y) : y^\top F(x_e, y) - V(x_e, y) \leq 0\}. \quad (3)$$

This set can also be seen as the fixed points set of the set-valued operator $\Lambda(x_e, \cdot)$ defined on \mathbb{R}^d by

$$\Lambda(x_e, y) := \arg \min_z \{z^\top F(x_e, y) : z \in K(x_e, y)\}, \quad \forall y \in \mathbb{R}^d.$$

Now, utilizing (3), we can express (GBOCP) as the following single-level optimal control problem:

$$\left\{ \begin{array}{ll} \min_{x,u,y} f(x_e, y) \\ \text{subject to } \dot{x}(t) = \phi(t, x(t), u(t)) & \text{a.e. } t \in [T_b, T_e] \\ x(T_b) = x_b \\ g(t, x(t)) \leq 0 & \forall t \in [T_b, T_e] \\ u(t) \in \mathcal{U} & \text{a.e. } t \in [T_b, T_e] \\ \tilde{G}(x_e, y) \leq 0 \\ \tilde{F}(x_e, y) - V(x_e, y) \leq 0, \end{array} \right. \tag{4}$$

with $\tilde{G}(x_e, y) := G(x_e, y, y)$, and $\tilde{F}(x_e, y) := y^\top F(x_e, y)$.

Note that the optimal value function V , which is characterized by the inner semicontinuity of the solution map Λ , has promising continuity properties, according to the unique structure of (QVI $[x_e]$). This offers crucial insights into how to handle the possible non-smoothness of V since we may utilize methods from variational analysis to distinguish this function.

However, examining (4) reveals the need to differentiate the function $(-V)$ as well. To address this issue, we employ Clarke’s concept of subdifferentiability throughout our subsequent analysis. This subdifferential is approximated in formulas provided by [17]. The following proposition, directly derived from [18, Corollary 5.3], outlines a method for calculating the Clarke subdifferential of the optimal value function V .

Proposition 1. *Let $\bar{x}_e \in \mathbb{R}^n$, $\bar{y} \in S(\bar{x}_e)$ and $\bar{z} \in \Lambda(\bar{x}_e, \bar{y})$. Assume that Λ is inner semicontinuous at $(\bar{x}_e, \bar{y}, \bar{z})$. Then, we have*

$$\partial^c V(\bar{x}_e, \bar{y}) \subset \bigcup_{\eta \in \mathcal{M}(\bar{x}_e, \bar{y}, \bar{z})} \left[\nabla F(\bar{x}_e, \bar{y})^\top \bar{z} + \nabla_{x_e, y} G(\bar{x}_e, \bar{y}, \bar{z})^\top \eta \right],$$

where

$$\mathcal{M}(\bar{x}_e, \bar{y}, \bar{z}) := \left\{ \eta \in \mathbb{R}^r : \begin{array}{l} F(\bar{x}_e, \bar{y}) + \nabla_z G(\bar{x}_e, \bar{y}, \bar{z})^\top \eta = 0 \\ \eta \geq 0 \\ \eta^\top G(\bar{x}_e, \bar{y}, \bar{z}) = 0 \end{array} \right\}.$$

Linear independent constraint qualification (LICQ) and MFCQ are typical constraint qualifiers that are too powerful to hold at the boundary conditions of (4). This failure mainly arises from the constraint $\tilde{F}(x_e, y) - V(x_e, y) \leq 0$. This specific constraint turns out to be a boundary condition. It’s worth noting that some maximum principles for classical optimal control problems only state the transversality condition, such as the normal cone to the constraint set of the boundary conditions, and make no mention of multipliers relating to the boundary conditions.

In this study, the key constraint involving the optimal value function is transferred to the leader’s objective through partial penalization. The resulting problem is likely to have an objective functional form of Mayer type that is not smooth. Since Vinter’s maximum principle can be applied to solve nonsmooth optimal control problems, we employ it in this method as well.

To close this section, we can use the distance function and the fact that the solution set $S(x_e)$ is closed for all $(x, u) \in \Theta_L$ to establish equivalence between our problem (GBOCP) and the following optimal control problem:

$$\left\{ \begin{array}{l} \min_{x,u,y} f(x_e, y) \\ \text{subject to } \dot{x}(t) = \phi(t, x(t), u(t)) \quad \text{a.e. } t \in [T_b, T_e] \\ \quad \quad \quad x(T_b) = x_b \\ \quad \quad \quad g(t, x(t)) \leq 0 \quad \quad \quad \forall t \in [T_b, T_e] \\ \quad \quad \quad u(t) \in \mathcal{U} \quad \quad \quad \text{a.e. } t \in [T_b, T_e] \\ \quad \quad \quad d_{S(x_e)}(y) = 0. \end{array} \right. \quad (5)$$

4. EXACT PENALIZATION

In this section, we adopt Ye and Zhu's theories [20], where the authors define a few constraints that must be fulfilled in order to reformulate an optimistic bilevel programming problem's optimal value. Then, we look into the conditions in which transferring the constraint to the leader's objective and weighting it with a penalty parameter yields a locally exact penalization.

Continuing with our discussion, we present the following definition, which takes into account our interest in \mathcal{W} -local optimal solutions.

Definition 4.1 (Partial calmness). We say that problem (4) is partially calm at a \mathcal{W} -local optimal solution of (GBOCP) $(\bar{x}, \bar{u}, \bar{y}) \in \Theta$ if there is a scalar $\beta > 0$ and a neighborhood U of $(\bar{x}, \bar{y}, 0)$ such that

$$f(x_e, y) - f(\bar{x}_e, \bar{y}) + \beta v \geq 0, \quad (6)$$

for all $(x, u, y) \in \Theta$, $v \geq 0$ satisfying also $(x, y, v) \in U$ and $\tilde{F}(x_e, y) - V(x_e, y) = v$.

As anticipated, the following argument demonstrates that the partial calmness property can be equivalently described by the partial penalization of problem (4).

Theorem 4.2. Let $(\bar{x}, \bar{u}, \bar{y})$ be a \mathcal{W} -local optimal solution of (GBOCP), and assume that the set-valued mapping Λ is inner semicontinuous at $(\bar{x}, \bar{y}, \bar{y})$. Then, problem (4) is partially calm at $(\bar{x}, \bar{u}, \bar{y})$ if and only if there is $\beta > 0$ such that $(\bar{x}, \bar{u}, \bar{y})$ is a \mathcal{W} -local optimal solution for the following optimal control problem

$$\left\{ \begin{array}{l} \min_{x,u,y} f(x_e, y) + \beta \left(\tilde{F}(x_e, y) - V(x_e, y) \right) \\ \text{subject to } \dot{x}(t) = \phi(t, x(t), u(t)) \quad \text{a.e. } t \in [T_b, T_e] \\ \quad \quad \quad x(T_b) = x_b \\ \quad \quad \quad g(t, x(t)) \leq 0 \quad \quad \quad \forall t \in [T_b, T_e] \\ \quad \quad \quad u(t) \in \mathcal{U} \quad \quad \quad \text{a.e. } t \in [T_b, T_e] \\ \quad \quad \quad \tilde{G}(x_e, y) \leq 0. \end{array} \right. \quad (7)$$

Proof. First, suppose that problem (4) is partially calm at $(\bar{x}, \bar{u}, \bar{y})$. On the basis of Definition 4.1, there are $\varepsilon, \beta > 0$ such that the inequality (6) holds for all $(x, u, y) \in \Theta$, $v \geq 0$ satisfying also $(x, y, v) \in U := \mathbb{B}_{\mathcal{W}(n) \times \mathbb{R}^d \times \mathbb{R}}^\varepsilon(\bar{x}, \bar{y}, 0)$ and $\tilde{F}(x_e, y) - V(x_e, y) = v$.

According to Proposition 1, there is a scalar $\varepsilon_1 > 0$ such that V is Lipschitz continuous function in $\mathbb{B}_{\mathbb{R}^n \times \mathbb{R}^d}^{\varepsilon_1}(\bar{x}_e, \bar{y})$ with modulus $L_V > 0$. In addition, since \tilde{F} is continuously differentiable function, then there is a scalar $\varepsilon_2 > 0$ such that \tilde{F} is a Lipschitz continuous function in $\mathbb{B}_{\mathbb{R}^n \times \mathbb{R}^d}^{\varepsilon_2}(\bar{x}_e, \bar{y})$ with modulus $L_{\tilde{F}} > 0$.

Let $(x, u, y) \in \Theta_L \times K(x_e, y)$ which verifies also $(x, y) \in \mathbb{B}_{\mathcal{W}(n) \times \mathbb{R}^d}^\varepsilon(\bar{x}, \bar{y})$ with

$$\bar{\varepsilon} := \min \left\{ \varepsilon, \frac{\varepsilon}{C_{\text{emb}}}, \frac{\varepsilon_1}{C_{\text{emb}}}, \frac{\varepsilon_2}{C_{\text{emb}}}, \frac{\varepsilon}{\max \{ (L_{\tilde{F}} + L_V) C_{\text{emb}}, L_{\tilde{F}} + L_V \}} \right\},$$

where $C_{\text{emb}} > 0$ is the constant of the Sobolev embedding of $\mathcal{W}(n)$ into the space $\mathcal{C}([T_b, T_e], \mathbb{R}^n)$ stated in Lemma 2.1.

Now, let $v := \tilde{F}(x_e, y) - V(x_e, y)$. One can check that

$$v = \tilde{F}(x_e, y) - V(x_e, y) = \tilde{F}(x_e, y) - V(x_e, y) - \left(\tilde{F}(\bar{x}_e, \bar{y}) - V(\bar{x}_e, \bar{y}) \right)$$

while taking account that $\bar{y} \in S(\bar{x}_e)$. Hence,

$$v \leq \left(\tilde{F}(x_e, y) - \tilde{F}(\bar{x}_e, \bar{y}) \right) + \left(V(\bar{x}_e, \bar{y}) - V(x_e, y) \right).$$

Exploiting the Lipschitz property of \tilde{F} and V , one gets

$$\begin{aligned} v &\leq L_{\tilde{F}} \|(x_e, y) - (\bar{x}_e, \bar{y})\|_{\infty} + L_V \|(x_e, y) - (\bar{x}_e, \bar{y})\|_{\infty} \\ &\leq (L_{\tilde{F}} + L_V) \max \{ \|x_e - \bar{x}_e\|_{\infty}, \|y - \bar{y}\|_{\infty} \}. \end{aligned}$$

From the embedding Lemma 2.1, one deduces

$$\begin{aligned} v &\leq \max \{ (L_{\tilde{F}} + L_V) C_{\text{emb}}, L_{\tilde{F}} + L_V \} \max \{ \|x - \bar{x}\|_{\mathcal{W}(n)}, \|y - \bar{y}\|_{\infty} \} \\ &< \max \{ (L_{\tilde{F}} + L_V) C_{\text{emb}}, L_{\tilde{F}} + L_V \} \bar{\varepsilon} \\ &\leq \varepsilon, \end{aligned}$$

which yields, $(x, y, v) \in U$. Henceforth, (6) writes

$$f(x_e, y) + \beta \left(\tilde{F}(x_e, y) - V(x_e, y) \right) \geq f(\bar{x}_e, \bar{y}).$$

Consequently, the following inequality

$$f(x_e, y) + \beta \left(\tilde{F}(x_e, y) - V(x_e, y) \right) \geq f(\bar{x}_e, \bar{y}) + \beta \left(\tilde{F}(\bar{x}_e, \bar{y}) - V(\bar{x}_e, \bar{y}) \right),$$

holds for all $(x, u, y) \in \Theta_L \times K(x_e, y)$ with $(x, y) \in \mathbb{B}_{\mathcal{W}}^{\bar{\varepsilon}}(\bar{x}, \bar{y})$. Therefore, $(\bar{x}, \bar{u}, \bar{y})$ is a \mathcal{W} -local optimal solution of (7).

Conversely, suppose that $(\bar{x}, \bar{u}, \bar{y})$ is a \mathcal{W} -local optimal solution of (7). Then, there is $\varepsilon > 0$ such that

$$f(\bar{x}_e, \bar{y}) \leq f(x_e, y) + \beta \left(\tilde{F}(x_e, y) - V(x_e, y) \right), \tag{8}$$

for all $(x, u, y) \in \Theta_L \times K(x_e, y)$ with $(x, y) \in \mathbb{B}_{\mathcal{W}}^{\varepsilon}(\bar{x}, \bar{y})$.

Let (x, u, y, v) in $\Theta_L \times K(x_e, y) \times \mathbb{R}_+$ with $\tilde{F}(x_e, y) - V(x_e, y) = v$ and $(x, y, v) \in \mathbb{B}_{\mathcal{W} \times \mathbb{R}}^{\varepsilon}(\bar{x}, \bar{y}, 0)$. Then, from (8), we have

$$f(x_e, y) - f(\bar{x}_e, \bar{y}) + \beta v \geq 0.$$

Consequently, problem (4) is partially calm at $(\bar{x}, \bar{u}, \bar{y})$. □

Next, we provide sufficient conditions to ensure that problem (4) is partially calm using uniform parametric error bound constraint qualification and the calmness property. First, we consider the following definition of error bounds on QVIs.

Definition 4.3. We say that $(\text{QVI}[x_e])$ possesses a uniform parametric error bound at $(\bar{x}_e, \bar{y}) \in \text{gph}(S)$ with modulus $\sigma > 0$ if there is a constant $\varepsilon > 0$ such that

$$\forall (x_e, y) \in \mathbb{B}_{\mathbb{R}^n \times \mathbb{R}^d}^{\varepsilon}(\bar{x}_e, \bar{y}) : y \in K(x_e, y) \Rightarrow d_{S(x_e)}(y) \leq \sigma \left(\tilde{F}(x_e, y) - V(x_e, y) \right).$$

Theorem 4.4. *Let $(\bar{x}, \bar{u}, \bar{y})$ be a \mathcal{W} -local optimal solution of (GBOCP) such that the quasi-variational inequality (QVI $[x_e]$) possesses a uniform parametric error bound at (\bar{x}_e, \bar{y}) with modulus $\sigma > 0$. Then, there is a constant $\bar{\beta} > 0$ such that $(\bar{x}, \bar{u}, \bar{y})$ is a \mathcal{W} -local optimal solution of (7) for each $\beta \geq \sigma \bar{\beta}$.*

Proof. Assume such $(\bar{x}, \bar{u}, \bar{y})$ exists. The proof consists of two steps.

Step 1: We claim that problem (5) is partially calm at $(\bar{x}, \bar{u}, \bar{y})$. Indeed, from Definition 2 there is some $\varepsilon > 0$ such that

$$f(x_e, y) \geq f(\bar{x}_e, \bar{y}),$$

for all $(x, u, y) \in \Theta$ which satisfying also $(x, y) \in \mathbb{B}_{\mathcal{W}}^\varepsilon(\bar{x}, \bar{y})$. Let us fix $\varepsilon_1 = \frac{\varepsilon}{2}$ and $\varepsilon_2 \in [0, \varepsilon_1[$, and take $(x, u, y) \in \Theta_L \times \mathbb{R}^d$ such that $(x, y) \in \mathbb{B}_{\mathcal{W}(n) \times \mathbb{R}^d}^\varepsilon(\bar{x}, \bar{y})$ and $d_{S(x_e)}(y) = \varepsilon_2$. Since $S(x_e)$ is closed, there is $y^* \in S(x_e)$ such that $\|y^* - y\|_\infty = \varepsilon_2$. Hence,

$$\begin{aligned} \|(\bar{x}, \bar{y}) - (x, y^*)\|_{\mathcal{W}} &\leq \|(\bar{x}, \bar{y}) - (x, y)\|_{\mathcal{W}} + \|(x, y) - (x, y^*)\|_{\mathcal{W}} \\ &< \varepsilon_1 + \varepsilon_2 \\ &< \varepsilon. \end{aligned}$$

Consequently, $(x, y^*) \in \mathbb{B}_{\mathcal{W}(n) \times \mathbb{R}^d}^\varepsilon(\bar{x}, \bar{y})$, so we can write

$$f(\bar{x}_e, \bar{y}) \leq f(x_e, y^*) \Rightarrow f(x_e, y) - f(\bar{x}_e, \bar{y}) \geq f(x_e, y) - f(x_e, y^*).$$

Because f is continuously differentiable, then it is locally Lipschitz around (\bar{x}_e, \bar{y}) with modulus L_f , and one deduces that

$$f(\bar{x}_e, \bar{y}) \leq f(x_e, y^*) \Rightarrow f(x_e, y) - f(\bar{x}_e, \bar{y}) \geq -L_f \varepsilon_2.$$

Thus, $f(x_e, y) - f(\bar{x}_e, \bar{y}) + L_f \varepsilon_2 \geq 0$. Then, problem (5) is partially calm at $(\bar{x}, \bar{u}, \bar{y})$.

Step 2: Since the distance function is Lipschitz, from Theorem 4.2, we can find a constant $\bar{\beta} > 0$ such that for each $(x, u, y) \in \Theta$ with $(x, y) \in \mathbb{B}_{\mathcal{W}(n) \times \mathbb{R}^d}^\varepsilon(\bar{x}, \bar{y})$, we get

$$f(\bar{x}_e, \bar{y}) + \bar{\beta} d_{S(\bar{x}_e)}(\bar{y}) \leq f(x_e, y) + \bar{\beta} d_{S(x_e)}(y).$$

Then,

$$f(\bar{x}_e, \bar{y}) + \sigma \bar{\beta} \left(\tilde{F}(\bar{x}_e, \bar{y}) - V(\bar{x}_e, \bar{y}) \right) \leq f(x_e, y) + \sigma \bar{\beta} \left(\tilde{F}(x_e, y) - V(x_e, y) \right),$$

while using the fact that the quasi-variational inequality (QVI $[x_e]$) possesses a uniform parametric error bound at (\bar{x}_e, \bar{y}) with modulus σ , and that $\tilde{F}(\bar{x}_e, \bar{y}) - V(\bar{x}_e, \bar{y}) = 0$. Consequently, $(\bar{x}, \bar{u}, \bar{y})$ is a \mathcal{W} -local optimal solution of (7) for each $\beta \geq \sigma \bar{\beta}$. \square

Consider the set-valued mapping $\Pi : \mathbb{R} \rightrightarrows \mathcal{W}(n) \times M([T_b, T_e], \mathbb{R}^m) \times \mathbb{R}^d$ defined by

$$\Pi(\nu) := \left\{ (x, u, y) \in \mathcal{W}(n) \times M([T_b, T_e], \mathbb{R}^m) \times \mathbb{R}^d : \begin{array}{l} \tilde{F}(x_e, y) - V(x_e, y) \leq \nu \\ \tilde{G}(x_e, y) \leq 0 \end{array} \right\}$$

for each $\nu \in \mathbb{R}$.

Theorem 4.5. *Let $(\bar{x}, \bar{u}, \bar{y})$ be a \mathcal{W} -local optimal solution of (GBOCP) such that the quasi-variational inequality (QVI $[x_e]$) possesses a uniform parametric error bound at (\bar{x}_e, \bar{y}) with modulus $\sigma > 0$. Then, the set-valued mapping Π is calm at $(0, \bar{x}, \bar{u}, \bar{y})$.*

Proof. By assumption, we conclude that there is a constant $\varepsilon > 0$ such that

$$\forall (x_e, y) \in \mathbb{B}_{\mathbb{R}^n \times \mathbb{R}^d}^\varepsilon(\bar{x}_e, \bar{y}) : y \in K(x_e, y) \Rightarrow d_{S(x_e)}(y) \leq \sigma \left(\tilde{F}(x_e, y) - V(x_e, y) \right).$$

Set $\mathcal{N} := \mathbb{B}_{\mathbb{R} \times \mathcal{W}(n) \times M([T_b, T_e], \mathbb{R}^m) \times \mathbb{R}^d}^{\bar{\varepsilon}}(0, \bar{x}, \bar{u}, \bar{y})$, where $\bar{\varepsilon} := \frac{\varepsilon}{C_{\text{emb}}}$ and C_{emb} is the constant from Lemma 2.1. Let $(\nu, x, u, y) \in \mathcal{N} \cap \text{gph}(\Pi)$, one has

$$\begin{aligned} d_{\Pi(0)}(x, u, y) &= d_{\text{gph}(\bar{S})}(x, u, y) \\ &\leq d_{\bar{S}(x, u)}(y) = d_{S(x_e)}(y) \leq \sigma(\tilde{F}(x_e, y) - V(x_e, y)) \leq \sigma\nu, \end{aligned}$$

where $\bar{S}(x, u) := \{y \in \mathbb{R}^d : y \in S(x_e)\}$. Consequently, the set-valued mapping Π is calm at $(0, \bar{x}, \bar{u}, \bar{y})$. □

Remark 4.6. Theorem 4.5 remains true if we replace the set-valued mapping Π by the set-valued mapping $\Phi : \mathbb{R} \rightrightarrows \mathcal{W}(n) \times M([T_b, T_e], \mathbb{R}^m) \times \mathbb{R}^d$ defined for each $\nu \in \mathbb{R}$ by

$$\begin{aligned} \Phi(\nu) := \{ & (x, u, y) \in \mathcal{W}(n) \times M([T_b, T_e], \mathbb{R}^m) \times \mathbb{R}^d : \\ & \dot{x}(t) = \phi(t, x(t), u(t)) \quad \text{a.e. } t \in [T_b, T_e] \\ & x(T_b) = x_b \\ & g(t, x(t)) \leq 0 \quad \forall t \in [T_b, T_e] \\ & u(t) \in \mathcal{U} \quad \text{a.e. } t \in [T_b, T_e] \\ & \tilde{F}(x_e, y) - V(x_e, y) \leq \nu \\ & \tilde{G}(x_e, y) \leq 0 \quad \} \end{aligned}$$

Theorem 4.7. Let $(\bar{x}, \bar{u}, \bar{y})$ be a \mathcal{W} -local optimal solution of (GBOCP). Assume the set-valued mapping Φ is calm at $(0, \bar{x}, \bar{u}, \bar{y})$. Then, problem (5) is partially calm at $(\bar{x}, \bar{u}, \bar{y})$.

Proof. First, since $(\bar{x}, \bar{u}, \bar{y})$ is a \mathcal{W} -local optimal solution of (GBOCP), then there exists a neighborhood \mathcal{N}_1 of $(\bar{x}, \bar{u}, \bar{y})$ such that

$$f(\bar{x}_e, \bar{y}) \leq f(x_e, y), \quad \forall (x, u, y) \in \mathcal{N}_1 \cap \Theta.$$

Second, since the set-valued mapping Φ is calm at $(0, \bar{x}, \bar{u}, \bar{y})$, then there exists a neighborhood \mathcal{N}_2 of $(0, \bar{x}, \bar{u}, \bar{y})$ and a scalar $\rho > 0$ such that

$$d_{\Phi(0)}(x, u, y) \leq \rho|\nu|, \quad \forall (\nu, x, u, y) \in \mathcal{N}_2 \cap \text{gph}(\Phi). \tag{9}$$

Now, set $\mathcal{N} := (\mathbb{R} \times \mathcal{N}_1) \cap \mathcal{N}_2$, and let $(\nu, x, u, y) \in \mathcal{N} \cap \text{gph}(\Phi)$. Since $(\bar{x}, \bar{u}, \bar{y}) \in \Phi(0)$, then $\Phi(0) \neq \emptyset$, then we can obtain $(x^*, u^*, y^*) \in \Phi(0)$ such that

$$d_{\Phi(0)}(x, u, y) = \|(x^*, u^*, y^*) - (x, u, y)\|_{\mathcal{W}(n) \times M([T_b, T_e], \mathbb{R}^m) \times \mathbb{R}^d}. \tag{10}$$

Then, we have

$$\begin{aligned}
f(\bar{x}_e, \bar{y}) - f(x_e, y) &\stackrel{(a)}{\leq} f(x_e^*, y^*) - f(x_e, y) \\
&\stackrel{(b)}{\leq} L_f \|(x_e^*, y^*) - (x_e, y)\|_\infty \\
&\leq L_f \max\{\|x_e^* - x_e\|_\infty; \|y^* - y\|_\infty\} \\
&\stackrel{(c)}{\leq} \max\{L_f C_{\text{emb}}; L_f\} \max\{\|x^* - x\|_{\mathcal{W}(n)}; \|y^* - y\|_\infty\} \\
&\leq \max\{L_f C_{\text{emb}}; L_f\} \max\{\|x^* - x\|_{\mathcal{W}(n)}; \|u^* - u\|_{M([T_b, T_e], \mathbb{R}^m)}; \|y^* - y\|_\infty\} \\
&\leq \max\{L_f C_{\text{emb}}; L_f\} \|(x^*, u^*, y^*) - (x, u, y)\|_{\mathcal{W}(n) \times M([T_b, T_e], \mathbb{R}^m) \times \mathbb{R}^d} \\
&\stackrel{(d)}{=} \max\{L_f C_{\text{emb}}; L_f\} d_{\Phi(0)}(x, u, y) \\
&\stackrel{(e)}{\leq} \max\{L_f C_{\text{emb}}; L_f\} \rho|\nu|.
\end{aligned}$$

We note here that (a) follows from $\Phi(0) = \Theta$ and the fact that we can choose \mathcal{N}_2 small enough to verify $f(\bar{x}_e, \bar{y}) \leq f(x_e^*, y^*)$, (b) corresponds to the fact that f is locally Lipschitz with the modulus $L_f > 0$, (c) follows from Lemma 2.1 with $C_{\text{emb}} > 0$, (d) and (e) follow from (10) and (9), respectively. Consequently,

$$f(x_e, y) - f(\bar{x}_e, \bar{y}) + \max\{L_f C_{\text{emb}}; L_f\} \rho|\nu| \geq 0,$$

which means that (5) is partially calm at $(\bar{x}, \bar{u}, \bar{y})$. \square

5. NECESSARY OPTIMALITY CONDITIONS

In our work, to derive Pontryagin optimality conditions for problem (GBOCP), we utilize the results from Theorem 1. To achieve this, we transform problem (7) into a minimization problem that optimizes its objective function with respect to the state function and the control only, similar to problem (2). To proceed, any vector $y \in \mathbb{R}^d$ can be treated as a constant function defined over the interval $[T_b, T_e]$, with $\dot{y}(t) = 0$ for all $t \in [T_b, T_e]$. Consequently, y can be viewed as a state function. Therefore, problem (7) can be rewritten as follows:

$$\left\{ \begin{array}{l} \min_{(x,y),u} f(x_e, y_e) + \beta \left(\tilde{F}(x_e, y_e) - V(x_e, y_e) \right) \\ \text{subject to } \dot{x}(t) = \phi(t, x(t), u(t)) \quad \text{a.e. } t \in [T_b, T_e] \\ \dot{y}(t) = 0 \quad \text{a.e. } t \in [T_b, T_e] \\ x(T_b) = x_b \\ g(t, x(t)) \leq 0 \quad \forall t \in [T_b, T_e] \\ u(t) \in \mathcal{U} \quad \text{a.e. } t \in [T_b, T_e] \\ (x_e, y_e) \in \tilde{K}, \end{array} \right. \quad (11)$$

where $\tilde{K} := \{(x_e, y) \in \mathbb{R}^n \times \mathbb{R}^d : \tilde{G}(x_e, y) \leq 0\}$.

Since for a \mathcal{W} -local optimal solution $(\bar{x}, \bar{u}, \bar{y})$ of (GBOCP), the MFCQ hold at $(\bar{x}_e, \bar{y}, \bar{y})$ (hypothesis (\mathcal{A}_3)), we get the next characterization of the normal cone of the set \tilde{K} at (\bar{x}_e, \bar{y}) as follows:

Lemma 5.1. *Let $(\bar{x}, \bar{u}, \bar{y})$ be a \mathcal{W} -local optimal solution of (GBOCP), then*

$$N\left(\tilde{K}, (\bar{x}_e, \bar{y})\right) \subset \left\{ \left(\nabla_{x_e} \tilde{G}(\bar{x}_e, \bar{y})^\top \xi, \nabla_y \tilde{G}(\bar{x}_e, \bar{y})^\top \xi \right) : \xi \geq 0, \tilde{G}(\bar{x}_e, \bar{y})^\top \xi = 0 \right\}.$$

Now, we are in a position to establish the maximum principle for (GBOCP).

Theorem 5.2. *Let $(\bar{x}, \bar{u}, \bar{y})$ be a \mathcal{W} -local solution of problem (GBOCP) such that the hypotheses $(\mathcal{A}_1) - (\mathcal{A}_4)$ hold. Assume that Λ is inner semicontinuous at $(\bar{x}_e, \bar{y}, \bar{y})$ and problem (4) is partially calm at $(\bar{x}, \bar{u}, \bar{y})$. Then, there is a function $p \in \mathcal{W}(n)$, two positives vectors $(\xi, \eta) \in \mathbb{R}^r \times \mathbb{R}^r$, a measure $\mu \in \mathcal{C}_p([T_b, T_e])$ and two constants $\tau \geq 0, \beta > 0$ such that:
the enhanced non-triviality condition:*

$$\|R\|_{L^\infty([T_b, T_e], \mathbb{R}^n)} + \mu([T_b, T_e]) + \tau > 0;$$

the adjoint equation: for almost every $t \in [T_b, T_e]$

$$-\dot{p}(t)^\top = R(t)^\top \nabla_x \phi(t, \bar{x}(t), \bar{u}(t));$$

the WeierstrassPontryagin condition: for almost every $t \in [T_b, T_e]$

$$R(t)^\top \phi(t, \bar{x}(t), \bar{u}(t)) = \max_{w \in \mathcal{U}} R(t)^\top \phi(t, \bar{x}(t), w);$$

the transversality condition:

$$-R(T_e) = \tau \nabla_{x_e} f(\bar{x}_e, \bar{y}) + \nabla_{x_e} G(\bar{x}_e, \bar{y}, \bar{y})^\top (\xi - \tau \beta \eta);$$

the support condition:

$$\text{supp}(\mu) \subseteq \{t \in [T_b, T_e] : g(t, \bar{x}(t)) = 0\};$$

the quasi-variational inequality condition:

$$F(\bar{x}_e, \bar{y}) + \nabla_z G(\bar{x}_e, \bar{y}, \bar{y})^\top \eta = 0;$$

the complementarity conditions:

$$\eta^\top G(\bar{x}_e, \bar{y}, \bar{y}) = 0, \quad \xi^\top G(\bar{x}_e, \bar{y}, \bar{y}) = 0;$$

the multiplier condition:

$$\tau \nabla_y f(\bar{x}_e, \bar{y}) + \tau \beta F(\bar{x}_e, \bar{y}) + \nabla_y G(\bar{x}_e, \bar{y}, \bar{y})^\top (\xi - \tau \beta \eta) = 0;$$

where the function $R : [T_b, T_e] \rightarrow \mathbb{R}$ is given by

$$R(t) := \begin{cases} p(t) + \int_{[T_b, T_e[} \nabla_x g(a, \bar{x}(a)) \mu(da) & \text{if } T_b \leq t < T_e, \\ p(T_e) + \int_{[T_b, T_e]} \nabla_x g(a, \bar{x}(a)) \mu(da) & \text{if } t = T_e. \end{cases}$$

Proof. Let $(\bar{x}, \bar{u}, \bar{y})$ be a \mathcal{W} -local solution of problem (GBOCP). Theorem 4.2 ensures that there is a constant $\beta > 0$ such that $(\bar{x}, \bar{u}, \bar{y})$ is a \mathcal{W} -local optimal solution of problem (7). Furthermore, it is clear that $(\bar{x}, \bar{u}, \bar{y})$ is a \mathcal{W} -local optimal solution of problem (11).

In order to use the results in Subsection 2.2, we define the functions: $H : \mathbb{R}^n \times \mathbb{R}^d \rightarrow \mathbb{R}$, $\phi_1 : [T_b, T_e] \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$, $\phi_2 : [T_b, T_e] \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d$, and $h : [T_b, T_e] \times \mathbb{R}^n \rightarrow \mathbb{R}$ as follows

$$\begin{aligned} H(x_e, y_e) &:= f(x_e, y_e) + \beta \left(\tilde{F}(x_e, y_e) - V(x_e, y_e) \right), \\ \phi_1(t, x(t), u(t)) &:= \phi(t, x(t), u(t)), \\ \phi_2(t, y(t)) &:= 0, \\ h(t, x(t)) &:= g(t, x(t)), \end{aligned}$$

for all $(t, x, u, y) \in [T_b, T_e] \times \mathcal{W}(n) \times M([T_b, T_e], \mathbb{R}^m) \times \mathcal{W}(d)$. In addition, we set $\mathcal{U}_1 := \mathcal{U}$, $l := 0$, and $C := \tilde{K}$. According to Theorem 1, there are two functions $(p_1, p_2) \in \mathcal{W}(n) \times \mathcal{W}(d)$, a measure $\mu \in \mathcal{C}_p([T_b, T_e])$ and a scalar $\tau \geq 0$ such that

$$\|R\|_{L^\infty([T_b, T_e], \mathbb{R}^n)} + \|p_2\|_{\mathcal{W}(d)} + \mu([T_b, T_e]) + \tau > 0; \quad (12)$$

$$-\dot{p}_1(t)^\top = R(t)^\top \nabla_x \phi(t, \bar{x}(t), \bar{u}(t)); \quad (13)$$

$$-\dot{p}_2(t) = 0; \quad (14)$$

$$R(t)^\top \phi(t, \bar{x}(t), \bar{u}(t)) = \max_{w \in \mathcal{U}} R(t)^\top \phi(t, \bar{x}(t), w); \quad (15)$$

$$p_2(0) = 0; \quad (16)$$

$$\begin{aligned} (-R(T_e), -p_2(T_e)) \in \{ & \tau (\nabla_{x_e} f(\bar{x}_e, \bar{y}_e), \nabla_{y_e} f(\bar{x}_e, \bar{y}_e)) \\ & + \tau \beta (\nabla_{x_e} \tilde{F}(\bar{x}_e, \bar{y}_e, \bar{y}_e), \nabla_{y_e} \tilde{F}(\bar{x}_e, \bar{y}_e, \bar{y}_e)) \} \\ & - \tau \beta \partial^c V(\bar{x}_e, \bar{y}_e) + N(\tilde{K}, (\bar{x}_e, \bar{y}_e)); \end{aligned} \quad (17)$$

$$\text{supp}(\mu) \subset \{t \in [0, T] : g(t, \bar{x}(t)) = 0\}; \quad (18)$$

where the function $R : [T_b, T_e] \rightarrow \mathbb{R}$ is given by

$$R(t) := \begin{cases} p_1(t) + \int_{[T_b, T_e[} \nabla_x g(a, \bar{x}(a)) \mu(da) & \text{if } t < T_e, \\ p_1(T_e) + \int_{[T_b, T_e]} \nabla_x g(a, \bar{x}(a)) \mu(da) & \text{if } t = T_e. \end{cases}$$

In combining (14) and (16), we can see that $p_2 \equiv 0$. From here now, we take $p \equiv p_1$. Therefore, the enhanced non-triviality condition and the adjoint equation follow from (12) and (13), respectively. One can also check that the assertions (15) and (18) are, respectively, the WeierstrassPontryagin condition and the support condition.

From Proposition 1 and Lemma 5.1, there exist positive multipliers $\xi \geq 0$ and $\eta \in \mathcal{M}(\bar{x}_e, \bar{y}, \bar{y})$ satisfying the complementarity conditions, the quasi-variational inequality condition (for η), and (as per (17)) the following equations:

$$\begin{aligned} -R(T_e) &= \tau \nabla_{x_e} f(\bar{x}_e, \bar{y}_e) + \tau \beta \nabla_{x_e} F(\bar{x}_e, \bar{y})^\top \bar{y} - \tau \beta \nabla_{x_e} F(\bar{x}_e, \bar{y})^\top \bar{y} \\ &\quad - \tau \beta \nabla_{x_e} G(\bar{x}_e, \bar{y}, \bar{y})^\top \eta + \nabla_{x_e} G(\bar{x}_e, \bar{y}, \bar{y})^\top \xi, \\ 0 &= \tau \nabla_y f(\bar{x}_e, \bar{y}) + \tau \beta \nabla_y F(\bar{x}_e, \bar{y})^\top \bar{y} - \tau \beta \nabla_y F(\bar{x}_e, \bar{y})^\top \bar{y} + \tau \beta F(\bar{x}_e, \bar{y}) \\ &\quad - \tau \beta \nabla_y G(\bar{x}_e, \bar{y}, \bar{y})^\top \eta + \nabla_y G(\bar{x}_e, \bar{y}, \bar{y})^\top \xi. \end{aligned}$$

The transversality condition and the multiplier condition follow, respectively, from the last two equations. \square

Example 5.3. Consider the non-convex (GBOCP) with the following data:

$$\begin{aligned} T_b &:= 0, \quad T_e := 1, \quad n := 1, \quad m := 1, \quad d := 1, \quad r := 2, \\ f(x_e, y) &:= 1 - x_e + (x_e - y)^2, \\ \phi(t, x(t), u(t)) &:= (t^2 - x(t))^2 u(t) + 2t, \\ x_b &:= 0, \quad g(t, x(t)) := x(t) - t, \quad \mathcal{U} := [0, 1], \\ F(x_e, y) &:= x_e y^2, \quad G(x_e, y, z) := (x_e - z^2, x_e - z)^\top. \end{aligned}$$

One can check that

$$\Lambda(x_e, y) = \begin{cases} \emptyset & \text{if } x_e < 0 & \text{and } y \neq 0, \\ \mathbb{R} & \text{if } x_e = 0 & \text{or } y \neq 0, \\ \{\sqrt{x_e}\} & \text{if } 0 < x_e \leq 1 & \text{or } y \neq 0, \\ \{x_e\} & \text{if } 1 \leq x_e & \text{or } y \neq 0, \end{cases} \quad \text{and}$$

$$V(x_e, y) = \begin{cases} -\infty & \text{if } x_e < 0 & \text{and } y \neq 0, \\ 0 & \text{if } x_e = 0 & \text{or } y \neq 0, \\ x_e \sqrt{x_e} y^2 & \text{if } 0 < x_e \leq 1 & \text{or } y \neq 0, \\ (x_e)^2 y^2 & \text{if } 1 \leq x_e & \text{or } y \neq 0. \end{cases}$$

Hence,

$$S(x_e) = \begin{cases} \emptyset & \text{if } x_e < 0, \\ \mathbb{R} & \text{if } x_e = 0, \\ \{\sqrt{x_e}\} & \text{if } 0 < x_e \leq 1, \\ \{x_e\} & \text{if } 1 \leq x_e. \end{cases}$$

Therefore, the triplet $(\bar{x}, \bar{u}, \bar{y})$ defined by $\bar{x}_1(t) := t^2$, $\bar{u}(t) := t$, for all $t \in [0, 1]$, and $\bar{y} := 1$ is a global optimal solution of our (GBOCP).

First, we can see that Λ is inner semicontinuous at $(\bar{x}_e, \bar{y}, \bar{y})$. Then, problem (4) is partially calm at $(\bar{x}, \bar{u}, \bar{y})$ by letting $U := \mathbb{B}_{\mathcal{W}(1) \times \mathbb{R}^2}^3(\bar{x}, \bar{y}, 0)$ as we have

$$f(x_e, y) - f(\bar{x}_e, \bar{y}) + 2v = f(x_e, y) + 2v \geq 0,$$

for all $(x, u, y) \in \Theta$, $v \geq 0$ satisfying also $(x, y, v) \in U$ and $\tilde{F}(x_e, y) - V(x_e, y) = v$.

Moreover, all hypothesis $(\mathcal{A}_1) - (\mathcal{A}_4)$ are verified for $(\bar{x}, \bar{u}, \bar{y})$.

Let μ_{Leb} the Lebesgue measure, $A := \mathbb{R} \setminus [0, 1]$, and χ_A is the intersection function defined by

$$\chi_A(B) = A \cap B, \quad \forall B \subseteq \mathbb{R}.$$

The arguments $P \equiv R \equiv -4$, $\xi := (1, 3)^\top$, $\eta := \left(\frac{1}{4}, \frac{1}{2}\right)^\top$, $\mu := \mu_{Leb} \circ \chi_A$, $\tau := 0$, and $\beta := 2$ satisfy the Pontryagin optimality conditions stated in Theorem 5.2.

6. CONCLUSION

In this research, we have proposed Pontryagin optimality conditions for a class of bilevel optimal control problems where the leader imposes a pure state inequality constraint, and the follower’s variable is governed by a nonconvex quasi-variational inequality parameterized by the final state. To tackle this problem, we have represented the solution set of the quasi-variational inequality as a solution set of a parametric optimization problem, and used the value function approach. To address the potential non-smoothness of the value function, we have employed the Clarke subdifferential for accurate estimation. Furthermore, we have applied an exact penalization technique to transform the constraint resulting from the value function approach into the objective function. This transformation has allowed us to recast the bilevel optimal control problem into a more tractable form. Finally, we have provided the necessary optimality conditions for our generalized bilevel control problem, along with certain assumptions that ensure the nondegeneracy of the proposed optimality conditions.

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