COMPLEXITY RESULTS ON $k$-INDEPENDENCE IN SOME GRAPH PRODUCTS

MARCIA CAPPELLE*, ERIKA COELHO, OTAVIO MORTOSA AND JULIANO NASCIMENTO

Abstract. For a positive integer $k$, a subset $S$ of vertices of a graph $G$ is $k$-independent if each vertex in $S$ has at most $k - 1$ neighbors in $S$. We consider $k$-independent sets in two graph products: Cartesian and complementary prism. We show that the problem of determining $k$-independence remains NP-complete even for Cartesian products and complementary prisms. Furthermore, we present results on $k$-independence in the Cartesian product of two paths, known as grid graphs.

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1. Introduction

We consider finite, simple, and undirected graphs. For a graph $G$, the vertex set and the edge set are denoted $V(G)$ and $E(G)$, respectively. We use standard notation and terminology (see Bondy and Murty [5] for graph-theoretic terms not defined here).

A dominating set of a graph $G$ is a subset $D \subseteq V(G)$ such that every vertex in $G$ that is not in $D$ is adjacent to at least one member of $D$. An independent set of $G$ is a subset $S \subseteq V(G)$ such that its vertices are pairwise non-adjacent in $G$. An independent set $S$ is maximal if it is a dominating set, and maximum if it has the largest possible cardinality. The independence number of a graph $G$, denoted by $\alpha(G)$, is the maximum cardinality of an independent set of $G$.

Fink and Jacobson [16] generalized the concepts of independent and domination sets as follows. Let $k$ be a positive integer and $S$ be a subset of vertices of a graph $G$. We say that $S$ is $k$-independent if the maximum degree of the subgraph induced by $S$ is at most $k - 1$; and we say that $S$ is $k$-dominating if every vertex of $V(G) - S$ has at least $k$ neighbors in $S$. A $k$-dominating set $S$ is minimal if, for every vertex $v \in S$, $S - \{v\}$ is not $k$-dominating in $G$. The $k$-domination number $\gamma_k(G)$ is the minimum cardinality of a $k$-dominating set of $G$. A $k$-independent set $S$ is maximal if for every vertex $v \in V(G) - S$, $S \cup \{v\}$ is not $k$-independent. The minimum cardinality of a maximal $k$-independent set of a graph $G$ is denoted by $i_k(G)$ and its maximum cardinality is denoted by $\alpha_k(G)$, called the $k$-independence number. A $k$-independent set of $G$ with maximum cardinality is called an $\alpha_k(G)$-set. Thus, for $k = 1$, the 1-independent and 1-dominating sets are the classical independent and dominating sets, respectively. Hence $i_1(G) = \iota(G)$, $\alpha_1(G) = \alpha(G)$, and $\gamma_1(G) = \gamma(G)$. A $j$-independent set

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is also a $k$-independent set for $k \geq j$. Moreover, every set with $k$ vertices is $k$-independent; so $i_k(G) \geq k$ when $|V(G)| \geq k$.

Favaron et al. [15] called $k$-dependent domination a set that is $(k+1)$-independent and dominating simultaneously. They established relationships between $k$-dependent domination and concepts of classical domination. Blidia et al. [4] gave some relations between $\alpha_k(G)$ and $\alpha_j(G)$ and between $i_k(G)$ and $i_j(G)$ for $j \neq k$. They studied two families of extremal graphs for the inequality $i_2(G) \leq i(G) + \alpha(G)$, gave an upper bound on $i_2(G)$, and a lower bound when $G$ is a cactus. A cactus is a connected graph in which any two simple cycles have at most one vertex in common. Chellali et al. [9] studied graphs $G$ for which the removal of any edge $e$ yields a graph with the same $k$-independence number, that is, $\alpha_k(G-e) = \alpha_k(G)$. For a graph $G$ on $n$ vertices and average degree $d$, Caro and Hansberg [8] proved that $\alpha_k(G) \geq \frac{kn}{|V(G)|+k}$. This bound was improved by Kogan [26], who proved that $\alpha_k(G) \geq \frac{kn}{d+k}$ and, for $k = 2, 3$. Furthermore, he characterized the graphs for which the equality holds. Mao et al. [27] considered $k$-dependence on the lexicographic, strong, Cartesian, and direct products and presented several upper and lower bounds for these products of graphs. Aram et al. [12] studied 2-independence on trees. Bounds on $k$-independence were studied by Wang, Liu, and Liu [35] in the context of Nordhaus-Gaddum-Type results. For a given fixed integer $k \geq 1$, the $k$-INDEPENDENT SET decision problem is defined as follows.

**k-INDEPENDENT SET**

**Instance:** A graph $G$ and a positive integer $\ell$, where $\Delta(G) \geq k$ and $\ell \geq k$.

**Question:** Does $G$ have a $k$-independent set of size at least $\ell$?

For short, we write INDEPENDENT SET, when $k = 1$. The INDEPENDENT SET is a well-known NP-complete problem [17], Jacobson and Peters [24] established the NP-completeness of $k$-INDEPENDENT SET, for any fixed $k \geq 2$, and also provided linear-time algorithms for solving $k$-INDEPENDENT SET in trees and series-parallel graphs for all $k$. For more information on $k$-independence and $k$-domination, see a survey by Chellali et al. [10].

The Cartesian product of two graphs $G$ and $H$ is a graph denoted as $G \square H$, whose vertex set is $V(G) \times V(H)$ and two vertices $(u,v)$ and $(u',v')$ are adjacent precisely if $u = u'$ and $vv' \in E(H)$, or $uu' \in E(G)$ and $v = v'$. The graphs $G$ and $H$ are called factors of $G \square H$. Cartesian product graphs are widely studied in the literature, having several properties that involve the structure of the factors in a meaningful way [20]. The disjoint union of two graphs is a graph constructed from the unions of their respective vertex sets and edge sets. The complementary prism of a graph $G$, denoted by $G\overline{G}$, is the graph formed from the disjoint union of $G$ and its complement $\overline{G}$ by adding the edges of the perfect matching between the corresponding vertices of $G$ and $\overline{G}$. Note that $V(G\overline{G}) = V(G) \cup V(\overline{G})$. The complementary prism of $C_5$, the graph $C_5 \overline{C_5}$, known as Petersen Graph, can be seen in Figure 1. The complementary prism is a particular case of a more general product called complementary product [22], which also generalizes Cartesian products.

Haynes et al. [21] investigated several graph theoretic properties of complementary prisms, such as independence, distance, and domination. Duarte et al. [13] proved that the clique, independent set and $k$-domination problems remain NP-complete for complementary prisms. Camargo, Souza, and Nascimento [7] obtained results regarding parameterized complexity on complementary prisms. In particular, they showed that when parameterized by the solution size, the clique and independent set problems are fixed-parameter tractable (FPT) but do
not admit polynomial kernel under some assumptions. Barbosa et al. [2] presented results for maximal independent sets in complementary prisms whose maximal independent sets are also maximum (known as well-covered graphs).

We consider $k$-independent sets in two graph products: Cartesian and complementary prism. We show that the problem of determining $k$-independence remains $\mathbf{NP}$-complete even for Cartesian products and complementary prisms. Furthermore, we present results on $k$-independence in the Cartesian product of two paths, known as grid graphs. Some results in this paper were previously presented in [30].

2. Preliminaries

For a positive integer $i$ we denote $[i]$ the set $\{1, 2, \ldots, i\}$. Two graphs $G = (V, E)$ and $G' = (V', E')$ are isomorphic if and only if there is a bijection, called isomorphism function, $\varphi : V(G) \rightarrow V'(G')$ such that $uv \in E(G)$ if and only if $\varphi(u)\varphi(v) \in E'(G')$, for every $u, v \in V(G)$. We denote by $G \cong G'$ if $G$ and $G'$ are isomorphic.

Let $G$ be a graph. For a vertex $v \in V(G)$, we denote its open neighborhood by $N_G(v)$, and its closed neighborhood by $N_G[v] : = N_G(v) \cup \{v\}$. For a set $U \subseteq V(G)$, let $N_G(U) = \bigcup_{v \in U} N_G(v) - U$, and $N_G[U] = N_G(U) \cup U$. The degree of a vertex $v \in V(G)$ on a set $U \subseteq V(G)$, denoted by $d_U(v)$, that is, $d_U(v) = |N_G(u) \cap U|$. If $U = V(G)$, we simply write $d_G(u)$, or $d(u)$ when $G$ is clear from the context. The maximum degree of $G$ is denoted by $\Delta(G)$.

We denote by $G \cup H$ the disjoint union of two graphs $G$ and $H$. For an integer $\ell \geq 2$, the graph $\ell G$ is obtained by the disjoint union of $\ell$ copies of $G$. The join of two graphs $G$ and $H$, denoted by $G + H$, is the graph with $V(G + H) = V(G) \cup V(H)$ and $E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G) \text{ and } v \in V(H)\}$.

Let $G \Box H$ be the Cartesian product of the graphs $G$ and $H$ with $V(G) = \{u_1, u_2, \ldots, u_n\}$ and $V(H) = \{v_1, v_2, \ldots, v_m\}$. For $i \in [n]$, we refer to row $\mathcal{L}_i$ as the subset of vertices $\{(u_i, v_1), (u_i, v_2), \ldots, (u_i, v_m)\}$ of $V(G \Box H)$, and for $j \in [m]$, we refer to column $\mathcal{C}_j$ as the subset of vertices $\{(u_1, v_j), (u_2, v_j), \ldots, (u_n, v_j)\}$ of $V(G \Box H)$. See Figure 2 for an illustration. The graph induced by $\mathcal{L}_i$, denoted by $G[\mathcal{L}_i]$, is isomorphic to $G$ and the graph induced by $\mathcal{L}_i$, denoted by $G[\mathcal{C}_j]$, is isomorphic to $H$. Let $S \subseteq V(G)$. The correspondence of $S$ to the column $\mathcal{C}_j$ is the set $S \mathcal{C}_j = \{(u_i, v_j) \in V(G \Box H) : u_i \in S\}$.

To simplify our discussion of complementary prisms, we say simply $G$ and $\overline{G}$ to refer to the subgraph copies of $G$ and $\overline{G}$, respectively, in $G \overline{G}$. Also, for a vertex $v$ of $G$, we let $\overline{v}$ be the corresponding vertex in $\overline{G}$, and for a set $X \subseteq V(G)$, let $\overline{X}$ be the corresponding set of vertices in $V(\overline{G})$.

![Figure 2. Graphs $G$, $H$, and the Cartesian product $G \Box H$. Rectangles $\mathcal{L}_2$ and $\mathcal{C}_3$ represent Row 2 and Column 3 of $G \Box H$, respectively.](image)
3. Cartesian Products

3.1. Complexity Results

A celebrated result stating bounds for the independence number of Cartesian product graphs appeared in 1963 by Vizing [34].

**Theorem 3.1.** [34] For any graphs $G$ and $H$,
\[
\alpha(G \square H) \leq \min\{\alpha(G)|V(H)|, \alpha(H)|V(G)|\}, \\
\alpha(G \square H) \geq \alpha(G)\alpha(H) + \min\{|V(G)| - \alpha(G), |V(H)| - \alpha(H)|\}.
\]

This result has been improved by other researchers when $G$ or $H$ belong to specific graph classes such as bipartite [19], caterpillar [28], and odd cycle [25]. In particular, Jia and Slutzki [25] presented an algorithm that returns a maximal independent set of $G \square H$, which, for certain graphs, offers a superior lower bound compared to Vizing’s. Bounds for $\alpha(G \square H)$ have been established using the radius of its factors [1], with equalities noted when either $G$ or $H$ admit specific homomorphisms to independence graphs [6, 23]. In terms of complexity, Berge [3] and Nešetřil [31] observed an equivalence between the independence number of Cartesian products and the vertex coloring problem. A graph $G$ is $q$-colorable if the vertices of $G$ can be colored with at most $q$ colors such that no adjacent vertices receive the same color.

**Theorem 3.2.** [3, 31] A graph $G$ of order $n$ is $q$-colorable if and only if $\alpha(G \square K_q) = n$.

Since deciding whether a graph $G$ is $q$-colorable is NP-complete for $q \geq 3$ and polynomial time solvable otherwise [17], Theorem 3.2 implies that deciding whether $\alpha(G \square K_q) = n$ is NP-complete for $q \geq 3$. On the other hand, for $q = 2$ (resp. $q = 1$), it is clear that $\alpha(G \square K_q) = n$ if and only if $G$ is bipartite (resp. $G \simeq K_n$), which is polynomial-time solvable. In addition, structural relations between the independence number of Cartesian product graphs and vertex coloring have been studied by Pleammani et al. [32].

Regarding the $k$-independence number, Mao et al. [27] extended Vizing’s result, proving tight upper and lower bounds for $\alpha_k$ of $G \square H$. Moreover, they applied these bounds when analyzing Cartesian products of various graph classes, including $K_n \square K_m$, $P_n \square P_m$, and $C_n \square C_m$.

As far as we know, the complexity of determining $\alpha_k(G \square H)$ has not been settled, and there is no NP-completeness result for $\alpha(G \square H)$ when $H$ has two vertices. Thus, we fill this gap by showing that determining $\alpha_k$ of a graph, for every fixed $k \geq 1$, remains hard within the class of Cartesian product of graphs. Theorems 3.3 and 3.8 deal with the cases $k \geq 2$ and $k = 1$, respectively.

**Theorem 3.3.** The $k$-independent set problem is NP-complete when restricted to the Cartesian product of two nontrivial graphs, for any fixed $k \geq 2$.

**Proof.** Since $k$-INDEPENDENT SET is in NP [24], we just show its NP-hardness. We perform a reduction from $k$-INDEPENDENT SET on general graphs, which is NP-complete [24].

Let $(G, \ell)$ be an instance of $k$-INDEPENDENT SET, with $|V(G)| = n \geq 2$ and, since $k$ is fixed, we may consider $\ell \geq k$. We construct a graph $H \square K_k$ as follows. First, let $B$ be a graph arising from $kn$ disjoint copies of $K_2$. The construction of $H$ is obtained by a join of $G$ and $B$. Finally, we compute $H \square K_k$. An example of this construction follows in Figure 3. We denote each copy of $H$ in $H \square K_k$ by $H_i$, $i \in [k]$. Similarly, each copy of $G$ is denoted by $A_i$, while each copy of $B$ is denoted by $B_i$. Let $A = \bigcup_{i=1}^{k} V(A_i)$ and $B = \bigcup_{i=1}^{k} V(B_i)$.

We show that $G$ has a $k$-independent set of order at least $\ell$ if and only if $H \square K_k$ has a $k$-independent set of order at least $\omega + 2k^2n - 2kn$.

Given a $k$-independent set $S$ of $G$ with order at least $\ell$, let $X = S^{A_1} \cup (B - V(B_1))$. Since $A_1 \simeq G$ and $\Delta(K_2 \square K_{k-1}) = k - 1$, it follows that $X$ is a $k$-independent set of $H \square K_k$. Recall that $|X| = |S^{A_1}| + |B| - |V(B_1)| \geq \omega + 2k^2n - 2kn$.

For the converse, let $X$ be a $k$-independent set of $H \square K_k$ with $|X| \geq \omega + 2k^2n - 2kn$. We prove some claims first.
Claim 3.4. \( \ell + 2k^2n - 3kn \leq |X \cap B| \leq 2k^2n - 2kn. \)

Proof of Claim 3.4. Recall that, by construction, \(|A| = kn.\) For the lower bound, since \(|X| \geq \ell + 2k^2n - 2kn,\) we have that \(|X \cap B| \geq |X| - |A| = \ell + 2k^2n - 2kn - kn = \ell + 2k^2n - 3kn.\)

For the upper bound, recall that \(|E(B)| = kn, |V(B)| = 2kn\) and denote \(V(B) = \{b_1, b_2, \ldots, b_{2kn}\}.\) Let \(b_ib_j \in E(B),\) for \(i, j \in [2kn].\) We aim to determine an upper bound for \(|(L_i \cup L_j) \cap X|\). We consider whether every vertex of row \(L_i\) is in \(X.\) If \(L_i \cap X = L_i,\) then \(H \square K_k[L_i] \simeq K_k\) implies that \(d_X(v) = k - 1,\) for every \(v \in L_i.\) Since \(X\) is a \(k\)-independent set, we obtain that \(L_j \cap X = \emptyset\) and \(|(L_i \cup L_j) \cap X| = k.\) Otherwise, that is, \(L_i \cap X \neq L_i\) and \(L_j \cap X \neq L_j,\) we have that \(|L_i \cap X| \leq k - 1\) and \(|L_j \cap X| \leq k - 1.\) Consequently \(|(L_i \cup L_j) \cap X| \leq 2k - 2.\) Thus, \(|E(B)| = kn\) implies that \(|X \cap B| \leq kn(2k - 2) = 2k^2n - 2kn.\) \(\square\)

Claim 3.5. \( \ell \leq |X \cap A| \leq kn.\)

Proof of Claim 3.5. The upper bound is clear since \(|A| = kn.\) For the lower bound, by applying Claim 3.4, we have \(|X \cap A| = |X| - |X \cap B| \geq \ell + 2k^2n - 2kn - (2k^2n - 2kn) = \ell.\) \(\square\)

Claim 3.6. There exists \(j \in [k]\) such that \(X \cap V(A_j) \neq \emptyset\) and \(j\) is unique.

Proof of Claim 3.6. The existence of \(j \in [k]\) such that \(X \cap V(A_j) \neq \emptyset\) is implied by Claim 3.5. So, it remains to prove that \(j\) is unique. Suppose, by contradiction, that there exist distinct \(j, j' \in [k]\) such that \(X \cap V(A_j) \neq \emptyset\) and \(X \cap V(A_{j'}) \neq \emptyset.\)

Since \(|X \cap V(A_j)| \geq 1\) and \(|X \cap V(A_{j'})| \geq 1\) and \(X\) is a \(k\)-independent set, we have that \(|X \cap V(B_j)| \leq k - 1\) and \(|X \cap V(B_{j'})| \leq k - 1.\) Let \(F = A \cup (B - (V(B_j) \cup V(B_{j'}))).\) We have that

\[
|X \cap F| = |X| - |X \cap (V(B_j) \cup V(B_{j'})|)
\geq |X| - 2(k - 1)
= \ell + 2k^2n - 2kn - 2k + 2
> 2k^2n - 2kn - 2k
\geq 2k^2n - 3kn
\text{ (since } n \geq 2\)
= |F|.
\]

Since \(|X \cap F| > |F|,\) we reach a contradiction. \(\square\)

Claim 3.7. There exists \(j \in [k]\) such that \(X \cap V(B_j) = \emptyset\) and \(j\) is unique.

Proof of Claim 3.7. We first prove the existence. Suppose, by contradiction, that for every \(j \in [k], X \cap V(B_j) \neq \emptyset.\) The definition of \(k\)-independent set implies that \(|X \cap V(A_j)| \leq k - 1,\) for every \(j \in [k].\) By Claim 3.6, we obtain that \(|X \cap A| \leq k - 1.\) Thus,

\[
|X \cap B| = |X| - |X \cap A|
\geq \ell + 2k^2n - 2kn - k + 1
\geq 2k^2n - 2kn + 1
\text{ (since } \ell \geq k\)
> 2k^2n - 2kn,
\]

a contradiction to Claim 3.4.

Now, we show the uniqueness. Suppose, by contradiction, that there exist distinct \(j, j' \in [k]\) such that \(X \cap V(B_j) = \emptyset\) and \(X \cap V(B_{j'}) = \emptyset.\) This implies that \(|X \cap B| \leq 2kn(k - 2) = 2k^2n - 4kn.\) Hence,

\[
|X \cap A| = |X| - |X \cap B|
= \ell + 2k^2n - 2kn - (2k^2n - 4kn)
\geq \ell + 2kn
> n
\text{ (since } k \geq 2\).
\]
By Claim 3.6, we have that \(|X \cap \mathcal{A}| \leq k - 1\), if \(X \cap \mathcal{A} \subseteq V(A_j) \cup V(A_{j'})\), or \(|X \cap \mathcal{A}| \leq n\), otherwise. Then, we get a contradiction. \(\square\)

At this point, we know that \(\ell \leq |X \cap \mathcal{A}| \leq n\). Let \(j \in [k]\) such that \(X \cap V(A_j) \neq \emptyset\) (by Claim 3.6 we know that \(j\) is unique). Then \(\ell \leq |X \cap V(A_j)| \leq n\).

Since \(\ell \geq k\), \(|X \cap V(A_j)| \geq k\), and given that \(X\) is a \(k\)-independent set, we have that \(X \cap V(B_j) = \emptyset\).

Therefore, \(X \cap (\mathcal{B} - V(B_j))\) is a \(k\)-independent set in \(H \Box K_k\), as well as \(X \cap V(A_j)\). Finally, since \(A_j \simeq G\), \(X \cap V(A_j)\) gives to \(G\) a \(k\)-independent set of order at least \(\ell\). \(\square\)

Recall that the construction used for Theorem 3.8 when applied to \(k = 1\) produces a Cartesian product graph with a trivial factor. Since \(G\) is isomorphic to \(G \Box K_1\), this result is already known to be \(\text{NP}\)-complete [24]. Then, employing a specific construction, we demonstrate that \text{\(k\)-INDEPENDENT SET}, for \(k = 1\), is \(\text{NP}\)-complete on the Cartesian product of two nontrivial graphs.

**Theorem 3.8.** The \textsc{independent set} problem is \(\text{NP}\)-complete when restricted to the Cartesian product of two nontrivial graphs.

**Proof.** We show the \(\text{NP}\)-hardness of the problem by performing a reduction from \textsc{independent set} on general graphs [17].

Let \((G, \ell)\) be an instance of \textsc{independent set}, with \(|V(G)| = n\). Let \(B = (n + 1)K_1\) and \(H\) be a graph arising from the join \(G + B\). We use \(H\) to compute \(H \Box K_2\). Each copy of \(G\) (resp. \(B\)) in \(H \Box K_2\) is denoted by \(G_i\) (resp. \(B_i\)), \(i \in [2]\). We show that \(G\) has an independent set of order at least \(\ell\) if and only if \(H \Box K_2\) has an independent set of order at least \(\ell + n + 1\).

Let \(S\) be an independent set of \(G\) with \(|S| \geq \ell\). It is easy to verify that \(X = S^{G_1} \cup V(B_2)\) is an independent set of \(H \Box K_2\), since \(S\) (resp. \(V(B_i)\), \(i \in [2]\)) is an independent set itself, and by construction, no vertex in \(G_1\) is adjacent to a vertex in \(B_2\). Recall that \(|X| \geq \ell + n + 1\).

Let \(X\) be an independent set of \(H \Box K_2\) with order at least \(\ell + n + 1\). Since \(X\) is an independent set, by construction, we have that \(|X \cap \mathcal{L}_i| \leq 1\), for every row \(\mathcal{L}_i\) of \(H \Box K_2\). Thus, \(\ell \leq |X \cap V(G_1 \cup G_2)| \leq n\) and
$\ell + 1 \leq |X \cap V(B_1 \cup B_2)| \leq n + 1$. Let $i \in [2]$. By construction, we have that $uv \in E(H \Box K_2)$, for every $u \in V(G_i)$ and for every $v \in V(B_i)$. This implies that $X \cap V(G_1) \neq \emptyset$ if and only if $X \cap V(B_1) = \emptyset$. Consequently, the cardinality of $X$ implies that either $X \cap V(G_1) \neq \emptyset$ or $X \cap V(G_2) \neq \emptyset$. Similarly, either $X \cap V(B_1) \neq \emptyset$ or $X \cap V(B_2) \neq \emptyset$. This implies that $X \cap V(G_1)$ gives an independent set to $G$ of order at least $\ell$. \hfill \Box

### 3.2. Results on grids

We denote $V(P_n) = \{u_1, u_2, \ldots, u_n\}$ and $V(P_m) = \{v_1, v_2, \ldots, v_m\}$, where two consecutive vertices in the sequence are adjacent. A Cartesian product of two paths is known as grid. Mao et al. [27] presented lower and upper bounds for the $k$-independence number in grid graphs.

**Proposition 3.9.** [27] Let $P_n \square P_m$ be a grid with $3 \leq n \leq m$.

1. If $k \geq 5$, then $\alpha_k(P_n \square P_m) = mn$.
2. If $k = 3, 4$, then $n \left\lceil \frac{n}{2} \right\rceil \leq \alpha_k(P_n \square P_m) \leq mn$.
3. If $k = 2$, then $n \left\lceil \frac{n}{2} \right\rceil + b \leq \alpha_k(P_n \square P_m) \leq (2 \left\lceil \frac{n}{2} \right\rceil + b)n$, where $b = m \mod 3$.
4. If $k = 1$, then $n \left\lceil \frac{n}{2} \right\rceil \leq \alpha_k(P_n \square P_m) \leq n \left\lceil \frac{n}{2} \right\rceil$.

The parameters $\alpha_k$ and $\gamma_k$ are related. For instance, if $S$ is a $q$-dominating set of a graph $G$ of maximum degree $\Delta \geq q$, then $V(G) - S$ is a $(\Delta - q + 1)$-independent set.

**Theorem 3.10.** [14] For every graph $G$ and positive integer $k \leq \Delta$, $\alpha_k(G) + \gamma_{\Delta-k+1}(G) \geq n$. Moreover, if $G$ is $d$-regular, then $\alpha_k(G) + \gamma_{d-k+1}(G) = n$.

Although the problem of determining the domination number of a proper subgraph of $P_n \square P_m$ is $\text{NP}$-complete [11], Gonalves et al. [18] completely solved the problem of determining the domination number of grids. For grids $P_n \square P_m$, with $16 \leq n \leq m$, they proved the result below.

**Theorem 3.11.** [18] Let $P_n \square P_m$ be a grid with $16 \leq n \leq m$. Then

$$\gamma(P_n \square P_m) = \left\lfloor \frac{(n+2)(m+2)}{5} \right\rfloor - 4.$$ 

Rao and Talon [33] gave a closed formula for the 2-domination number of grids when $9 \leq n \leq m$.

**Theorem 3.12.** [33] Let $P_n \square P_m$ be a grid with $9 \leq n \leq m$. Then

$$\gamma_2(P_n \square P_m) = \left\lfloor \frac{(n+2)(m+2)}{3} \right\rfloor - 6.$$ 

Grids have maximum degree four. Consequently, for $k \geq 5$, both the $k$-independent and $k$-dominating sets contain the whole set of vertices of these graphs. In view of Theorem 3.10, we use results on $q$-domination, $q = 1, 2$, for grids to give results on $k$-independence, $k = 3, 4$. For a grid $P_n \square P_m$, with $n, m \geq 3$, let $B = \{(u_i, v_j) : d((u_i, v_j)) < 4\}$. We call subgrid the subgraph $G^*$ induced by the vertex set $V(P_n \square P_m) - B$. Notice that, for every vertex $(u_i, v_j) \in V(G^*)$, $d_{P_n \square P_m}((u_i, v_j)) = 4$ and $G^*$ is isomorphic to $P_{n-2} \square P_{m-2}$.

By applying Theorem 3.10, we get that $\alpha_4(G) \geq nm - \gamma(G)$. However, we show in Proposition 3.13 that $\alpha_4(G) = nm - \gamma(G^*)$.

**Proposition 3.13.** Let $G = P_n \square P_m$ be a grid with $3 \leq n \leq m$ and $G^*$ its subgrid. If $D \subseteq V(G^*)$ is a minimum dominating set of $G^*$, then $V(G) - D$ is a maximum $4$-independent set of $G$, that is, $\alpha_4(G) = nm - \gamma(G^*)$. 

There is a complexity result on $K$-independence in some graph products.
Proof. Suppose that \( D \subseteq V(G^*) \) is a minimum dominating set of \( G^* \). If \( v \in V(G^*) \), \( d(v) = 4 \), otherwise, \( d(v) < 4 \). Hence, every vertex in the subgraph induced by \( (V(G^*) - D) \) has degree at most 3. Then, \( V(G) - D \) is a 4-independent set of cardinality \( nm - \gamma(G^*) \). Therefore, \( \alpha_4(G) \geq nm - \gamma(G^*) \).

For a contradiction, suppose that there exists a set \( S \) which is maximum 4-independent of \( G \) such that \( |S| > nm - \gamma(G^*) \). Hence, there exists \( D \subseteq V(G) \) such that \( S = V(G) - D \) and \( |D| < \gamma(G^*) \). So, \( D \) does not dominate all vertices of \( V(G^*) \). Hence, \( S \) has at least a vertex \( v \) such that \( d_S(v) = 4 \), which is a contradiction. Therefore, \( \alpha_4(G) \leq nm - \gamma(G^*) \), which completes the proof.

\[ \square \]

Corollary 3.14. Let \( G = P_n \square P_m \) be a grid with \( 18 \leq n \leq m \). Then, \( \alpha_4(G) = 4\left(\lceil \frac{nm}{3} \rceil + 1 \right) + a \), where \( a = nm \text{ mod } 5 \).

Proof. By Proposition 3.13, we have \( \alpha_4(G) = nm - \gamma(G^*) \). Furthermore, by Theorem 3.11, \( \gamma(G^*) = \left\lfloor \frac{nm}{3} \right\rfloor - 4 \). Hence, \( \alpha_4(G) = nm - (\left\lfloor \frac{nm}{3} \right\rfloor - 4) = 4\left(\lceil \frac{nm}{3} \rceil + 1 \right) + (nm \text{ mod } 5) \). \[ \square \]

Corollary 3.15. If \( G = P_n \square P_m \) with \( 9 \leq n \leq m \), then \( \alpha_3(G) \geq nm - \gamma_2(G) \). By Theorem 3.12, \( \alpha_3(G) \geq nm - \left\lfloor \frac{(n+2)(m+2)}{3} \right\rfloor - 6 \).

Proof. By Theorem 3.10, we have \( \alpha_3(G) \geq nm - \gamma_2(G) \). By Theorem 3.12, \( \alpha_3(G) \geq nm - \left\lfloor \frac{(n+2)(m+2)}{3} \right\rfloor - 6 \). \[ \square \]

Next, we present a closed formula for \( \alpha(P_n \square P_m) \).

Proposition 3.16. Let \( G = P_n \square P_m \) such that \( 3 \leq n \leq m \). Then, \( \alpha(G) = n\left\lceil \frac{m}{2} \right\rceil + \left\lfloor \frac{n}{2} \right\rfloor a \), where \( a = m \text{ mod } 2 \).

Proof. For the lower bound, let \( S_1 = \{(u_i, v_j) : i \text{ mod } 2 = 1 \text{ and } j \text{ mod } 2 = 1\} \) and \( S_2 = \{(u_i, v_j) : i \text{ mod } 2 = 0 \text{ and } j \text{ mod } 2 = 0\} \). It is easy to verify that \( S_1 \cup S_2 \) is an independent set and \( |S_1 \cup S_2| = n\left\lceil \frac{m}{2} \right\rceil \), when \( m \) is even, and \( |S_1 \cup S_2| = n\left\lfloor \frac{m}{2} \right\rfloor + \left\lceil \frac{n}{2} \right\rceil \), when \( m \) is odd.

Since \( G \) has \( n \) copies of \( P_m \), a straightforward upper bound for \( \alpha(G) \) is \( n\left\lceil \frac{m}{2} \right\rceil \). This bound equals \( n\left\lceil \frac{m}{2} \right\rceil \) when \( m \) is even. However, in the case of odd \( m \), an additional consideration arises due to an extra copy of \( P_n \). This copy contributes with at most \( \left\lceil \frac{n}{2} \right\rceil \) vertices and therefore gives the bound \( n\left\lceil \frac{m}{2} \right\rceil + \left\lceil \frac{n}{2} \right\rceil \), completing the proof. \[ \square \]

Corollary 3.17. Let \( G = P_n \square P_m \) such that \( 3 \leq n \leq m \). Then, \( \alpha_2(G) \geq n\left\lceil \frac{m}{2} \right\rceil + \left\lfloor \frac{n}{2} \right\rfloor a \), where \( a = m \text{ mod } 2 \).

The lower bounds presented in Corollaries 3.15 and 3.17 are better than the ones shown in Proposition 3.9. Analogously to Proposition 3.13, we show that \( \gamma_4(G) = nm - \alpha(G^*) \).

Proposition 3.18. Let \( G = P_n \square P_m \) be a grid with \( 3 \leq n \leq m \) and \( G^* \) its subgrid. If \( D \subseteq V(G^*) \) is a maximum independent set of \( G^* \), then \( V(G) - D \) is a minimum 4-dominating set of \( G \), that is, \( \gamma_4(G) = nm - \alpha(G^*) \).

Proof. Suppose that \( D \subseteq V(G^*) \) is a maximum independent set of \( G^* \). If \( v \in V(G^*) \), \( d(v) = 4 \), otherwise, \( d(v) < 4 \). Since \( D \) is independent, every vertex in \( D \) has exactly four neighbors not in \( D \). Then, \( V(G) - D \) is a 4-dominating set of cardinality \( nm - \alpha(G^*) \). Therefore, \( \gamma_4(G) \geq nm - \alpha(G^*) \).

For a contradiction, let us suppose that a set \( S \) exists, which is minimum 4-dominating of \( G \), such as \( |S| < nm - \alpha(G^*) \). Hence, there exists \( D \subseteq V(G) \) such that \( S = V(G) - D \) and \( |D| > \alpha(G^*) \). Consequently, \( D \) is not independent in \( V(G^*) \). Hence, \( S \) has at least a vertex \( v \) such that \( v \) has at most three neighbors in \( S \), which is a contradiction. Therefore, \( \gamma_4(G) \leq nm - \alpha(G^*) \), which completes the proof. \[ \square \]
4. Complementary Prisms

Some preliminary results on $k$-independent sets in complementary prisms were presented in [29]. The authors showed sharp lower and upper bounds for the 2-independence number in these graphs and exact values for $\alpha_2$ for the complementary prism of some particular graph classes.

In Theorem 4.1 we present bounds on $\alpha_k(G\ov{G})$ for a general graph $G$.

**Theorem 4.1.** Let $k \geq 2$. For a graph $G$, $\alpha_{k-1}(G) + \alpha_k(G) \leq \alpha_k(G\ov{G}) \leq \alpha_k(G) + \alpha_k(G)$.

**Proof.** For the lower bound, let $S$ be an $\alpha_k(G\ov{G})$-set and $T$ an $\alpha_{k-1}(G\ov{G})$-set. For any vertices $v \in S$ and $\ov{v} \in T$, it follows that $d_S(v) \leq k - 2$ and $d_T(\ov{v}) \leq k - 2$. Hence, in $G\ov{G}$, a vertex $v \in S \cup T$ satisfies $d_{S\cup T}(v) \leq (k - 2) + 1 = k - 1$. Consequently, $S \cup T$ is a $k$-independent set of $G\ov{G}$. As $S$ and $T$ are disjoint sets in $G\ov{G}$, $|S \cup T| = |S| + |T| = \alpha_{k-1}(G) + \alpha_k(G\ov{G})$. Then, $\alpha_k(G\ov{G}) \geq \alpha_k(G) + \alpha_k(G\ov{G})$. For the upper bound, let $I$ be an $\alpha_k(G\ov{G})$-set, $S$ be the vertices of $I$ in $G$, and $T$ be the vertices of $I$ in $G\ov{G}$. Since $|S| \leq \alpha_k(G)$ and $|T| \leq \alpha_k(G\ov{G})$, it follows that $\alpha_k(G\ov{G}) = |S| + |T| \leq \alpha_k(G) + \alpha_k(G\ov{G})$.

The work by Duarte et al. [13] establishes the NP-completeness of INDEPENDENT SET on complementary prisms through a reduction from INDEPENDENT SET in general graphs. Inspired by that idea, we perform a reduction from $k$-INDEPENDENT SET, and through a straightforward adaptation, we generalized the construction by Duarte et al. [13] to obtain the result of Theorem 4.2. Since $k$-INDEPENDENT SET on complementary prisms for specifically $k = 1$ is due to Duarte et al. [13], we focus our attention on the case where $k \geq 2$.

**Theorem 4.2.** The $k$-INDEPENDENT SET problem remains NP-complete even when restricted to the complementary prisms.

**Proof.** Since $k$-INDEPENDENT SET is in NP [24], we just show its NP-hardness. We present a reduction from $k$-INDEPENDENT SET for this problem.

Given a graph $G$ of order $n > k$, we construct a graph $H$, which is the disjoint union of $G$ and the complete multipartite graph $K_{m_1,m_2,...,m_{n+1}}$, where $m_1 = m_2 = ... = m_{n+1} = 2k - 1$. Let $H\ov{H}$ be the complementary prism of $H$. The construction of $H\ov{H}$ is depicted in Figure 4. Let $K = V(K_{m_1,m_2,...,m_{n+1}})$, that is, $K$ is a set of $(2k - 1)(n + 1)$ vertices of $H$. We show that, for integers $\ell'$ and $k$, $G$ has a $k$-independent set of order at least $\ell'$ if and only if the complementary prism $H\ov{H}$ has a $k$-independent set of order at least $\ell = kn + 3k - 2 + \ell'$. First, suppose that $I$ is a $k$-independent set of $G$ of order at least $\ell'$. Let $K^1, ..., K^{2k-1}$ be the set of vertices of each part of $H[K]$, and $\ov{K}^1, ..., \ov{K}^{2k-1}$ be the corresponding sets of vertices in $\ov{K}$. Let $D \subseteq K \cup \ov{K}$ containing $2k - 1$ vertices of $K^1$, $k - 1$ vertices of $\ov{K}^1$, and $k$ vertices of each one of the sets $\ov{K}^2, ..., \ov{K}^{2k-1}$. That is, $|D \cap K| = kn + k - 1$ and $|D \cap K| = 2k - 1$. Each vertex in $H[D]$ has degree at most $k - 1$. Hence, $I \cup D$ is a $k$-independent set of $H\ov{H}$ of order $kn + k - 1 + 2k - 1 + \ell' = kn + 3k - 2 + \ell'$.

Now, suppose that $H\ov{H}$ has a $k$-independent set $J$ of order at least $kn + 3k - 2 + \ell'$. We prove some claims first.

**Claim 4.3.** $|J \cap K| \leq 2k - 1$.

**Proof of Claim 4.3.** Suppose, for contradiction, that $|J \cap K| > 2k - 1$. Consequently, $J$ has vertices in at least two parts of $H[K]$. If $J$ contains all the $k$ vertices of a part of $H[K]$, say $K^i$, a vertex $u$ of another part of $H[K]$ which is a member of $J$ has $d_J(u) \geq k$, a contradiction, since $J$ is a $k$-independent set. If $J$ contains less than $k$ vertices of $K^i$, then $J$ contains at least $k$ vertices in other parts of $H[K]$. Hence, for a vertex $u \in K^i$, it follows that $d_J(u) \geq k$, which again leads to a contradiction. Therefore, $|J \cap K| \leq 2k - 1$.

**Claim 4.4.** $J \cap V(\ov{G}) = \emptyset$.
Proof of Claim 3.5. Suppose, for contradiction, that $J \cap V(G) \neq \emptyset$. If $J \cap K = \emptyset$, then $J$ can contain at most the $2n$ vertices of $V(G) \cup V(G)$, and at most $2k - 1$ vertices of $K$, by (i). Recall that $k \geq 2$, $n > k$ and $\ell' \geq k$. Therefore, we have

$$|J| \leq 2n + 2k - 1$$
$$\leq kn + 2k - 1$$
$$< kn + 2k + (k - 2) + \ell'$$
$$= kn + 3k - 2 + \ell'.$$

If $J \cap K \neq \emptyset$, it follows that $|J \cap (V(G) \cup K)| \leq 2k - 2$, since each vertex in $V(G)$ is adjacent to each vertex in $K$, which implies in $|V(G) \cap J| \leq k - 1$ and $|K \cap J| \leq k - 1$. Furthermore, $|J \cap V(G)| \leq n$ and, by (i), $|J \cap K| \leq 2k - 1$. Therefore, we have

$$|J| \leq 2k - 2 + 2k - 1 + n$$
$$= (n + k) + 3k - 3$$
$$< 2n + 3k - 2$$
$$\leq kn + 3k - 2$$
$$< kn + 3k - 2 + \ell'.$$

Hence, any combination of vertices of $J$ containing at least one vertex of $V(G)$ has cardinality less than $kn + 3k - 2 + \ell'$, which is a contradiction. So, we can conclude that $J \cap V(G) = \emptyset$. 

Claim 4.5. $|J \cap (K \cup K)| \leq kn + 3k - 2$.

Proof of Claim 3.6. Suppose, for contradiction, that $|J \cap (K \cup K)| \geq kn + 3k - 1$. As $|J \cap K| \leq 2k - 1$, it follows that $|J \cap K| \geq kn + k = k(n + 1)$. If $|J \cap K| = 2k - 1$, there exists $i$ with $K^i \subseteq J$ and $\nu \in K^i$ such that its corresponding vertex $v \in K^i$ also belonging to $J$, which implies that $d_J(v) = k$. If $|J \cap K| < 2k - 1$, we have $|J \cap K| \geq kn + k + 1 = k(n + 1) + 1$. Hence, there is at least a clique $K^i$ of $K$ such that $|K^i \cap J| \geq k + 1$, contradicting the fact that $J$ is $k$-independent. Therefore, $|J \cap (K \cup K)| \leq kn + 3k - 2$. 

By Claims 4.4 and 4.5, it can be concluded that $J - (K \cup K)$ is a $k$-independent set of $G$ of order at least $\ell'$. Therefore, the $k$-INDEPENDENT SET problem remains NP-complete even when restricted to the class of complementary prisms. 

Figure 4. Example of $H\bar{H}$, with $k = 3$. The circled black vertices represent a $k$-independent set. The edges of $G$, $\bar{G}$, and $K$ are omitted.
5. Concluding Remarks

We study $k$-independence and $k$-domination in Cartesian products and complementary prisms. We establish that the problem of determining a $k$-INDEPENDENT SET remains NP-complete when the input graph is one of these two products. Moreover, we present closed formulas and upper bounds for maximum $k$-independent sets in grid graphs, improving results presented in [27].

Furthermore, as a natural extension of our investigation, exploring the complexity of $k$-domination in these products is conceivable. Moreover, the quest for closed formulas for $\alpha_k$, when $k = 2, 3$, and $\gamma_q$, when $q = 3$, in grid graphs, could be a compelling avenue for future research.

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