RAIRO-Oper. Res. 58 (2024) 2379–2391
https://doi.org/10.1051/ro/2024110

RAIRO Operations Research
www.rairo-ro.org

DISTINCT SIZES OF MAXIMAL INDEPENDENT SETS ON GRAPHS WITH RESTRICTED GIRTH

Márcia Cappelle, Julliano Nascimento* and Vinícius Santos

Abstract. Let G be a graph. If G has exactly r distinct sizes of maximal independent sets, G belongs to a collection called $\mathcal{M}_r$. If $G \in \mathcal{M}_r$ and the distinct values of its maximal independent sets are consecutive, then G belongs to $\mathcal{I}_r$. The independence gap of G is the difference between the maximum and the minimum sizes of a maximal independent set in G. In this paper, we show that recognizing graphs in $\mathcal{I}_r$ is NP-complete, for every integer $r \geq 3$. On the other hand, we show that recognizing trees in $\mathcal{M}_r$ can be done in polynomial time, for every $r \geq 1$. Furthermore, we present characterizations of some graphs with girth at least 6 with independence gap at least 1, including graphs with independence gap $r-1$, for $r \geq 2$, belonging to $\mathcal{I}_r$. Moreover, we present a characterization of some trees in $\mathcal{I}_3$.

Mathematics Subject Classification. 05C69, 05C75, 05C85.

Received January 9, 2024. Accepted May 10, 2024.

1. Introduction

We consider finite, simple, and undirected graphs. For a graph G, the vertex set and the edge set are denoted by $V(G)$ and $E(G)$, respectively. We use standard notation and terminology. See [3] for graph-theoretic terms not defined here.

A subset $D \subseteq V(G)$ *dominates* a subset $S \subseteq V(G)$ if every vertex in S has a neighbor in D. If D dominates $V(G)$, then D is a *dominating set* of G. A subset $I \subseteq V(G)$ is *independent* if no two vertices in I are adjacent. An independent set I of G is *maximal* if it is a dominating set and it is *maximum* if G has no independent set J with $|J| > |I|$. For a graph G we denote by $\alpha(G)$ the maximum size of an independent set in G and by $i(G)$ the minimum size of a maximal independent set in G.

A graph is *well-covered* if all its maximal independent sets have the same size. These graphs were introduced by Plummer [16] and have been widely studied in the literature [13, 17]. Recognizing well-covered graphs is a co-NP-Complete problem [7, 19]. Such a decision problem is called Well-Coveredness.

On the other hand, if a graph is not well-covered, it is natural to know how far it is from being well-covered. Several new concepts have arisen, generalizing the concept of well-covered graphs. Some of them find applications, for instance, in commutative algebra [8, 15]. We are focused on a generalization by Finbow *et al.* [12], wherein they define the set $\mathcal{M}_r$, for every positive integer r, to be the set of graphs that have maximal independent sets of exactly r different sizes. With this notation, $\mathcal{M}_1$ is exactly the set of all well-covered graphs.

Keywords. Maximal independent sets, independent dominating sets, well-covered graphs, girth.

Instituto de Informática – Universidade Federal de Goiás, Goiânia, Brazil.
*Corresponding author: jullianonascimento@ufg.br

© The authors. Published by EDP Sciences, ROADEF, SMAI 2024

This is an Open Access article distributed under the terms of the Creative Commons Attribution License (https://creativecommons.org/licenses/by/4.0), which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

It was proved by Caro *et al.* [6] that recognizing graphs in $\mathcal{M}_r$ is a co-NP-complete problem. Such a decision problem is called $\mathcal{M}_r$-Membership. If a graph is a member of $\mathcal{M}_r$, where the distinct values of its maximal independent sets are consecutive, then it is also a member of $\mathcal{I}_r$ [1]. To the best of our knowledge, the recognition of graphs in $\mathcal{I}_r$ has not completely settled. For $r = 1, 2$, the problem is, respectively, co-NP-complete and NP-complete, directly by the reductions in [7, 19]. In this paper, we show that the recognition of graphs in $\mathcal{I}_r$ is NP-Complete, for every fixed $r \geq 2$.

Many results concerning independence consider the *girth* of a graph, which is the length of its smallest cycle. If a graph has no cycle, its girth is, by definition, equal to infinity. A *tree* is a connected graph with no cycles. Well-covered graphs of girth at least 5 [11] and graphs of girth at least 8 belonging to $\mathcal{M}_2$ [12] were characterized. With those characterizations is possible to recognize trees in $\mathcal{M}_r$ in polynomial time, for $r = 1, 2$. In this context, we present a recursive algorithm able to recognize trees in $\mathcal{M}_r$, in polynomial time, for every positive integer r. In fact, the proposed algorithm returns a set of all sizes of maximal independent sets of the input tree.

Another concept that generalizes the well-covered graphs in a different way is called *independence gap*, which is the difference between the maximum and minimum sizes of a maximal independent set in G, that is, the difference $\alpha(G) - i(G)$. For a graph G, we denote its independence gap by $\mu_\alpha(G)$. Therefore, well-covered graphs have independence gap equal to zero. Graphs that have independence gap equal to one are also called *almost well-covered graphs* and are exactly the set of graphs in $\mathcal{I}_2$. An efficient characterization of a subclass of almost well-covered graphs of girth at least 6 was presented by Ekim *et al.* [9], as well as a polynomial-time algorithm for recognizing almost well-covered graphs that do not have cycles with sizes 3, 4, 5, and 7. A tight upper bound on the independence number of graphs of independence gap equal to k, where $k \geq 0$ was also presented [4].

When the restrictions on the graphs concern girth and minimum degree, for some values of r, there are only cycles belonging to $\mathcal{M}_r$. Finbow and Hartnell [10] proved that C_7 is the only well-covered connected graph of minimum degree at least 2 and girth at least 6. Similarly, Finbow *et al.* [12] proved that the cycles C_8, C_9, C_{10}, C_{11}, and C_{13} are the only graphs in $\mathcal{M}_2$ of minimum degree at least 2 and girth at least 8. For $r \geq 4$, Hartnell and Rall [14] proved that every graph in $\mathcal{M}_r$ of minimum degree at least 2 and girth at least $6r - 6$ is a cycle. Barbosa *et al.* [2] proved that, for integers r and d with $r \geq 2$ and $d \geq 3$, $\mathcal{M}_r$ contains only finitely many connected graphs of minimum degree at least 2, maximum degree at most d, and girth at least 7. When graphs may have leaves, $\mathcal{M}_r$ contains infinitely many such graphs.

A vertex is said to be of *type* r if it is adjacent to exactly r leaves. We consider $r \geq 2$. We show that graphs of girth at least 4 with $\mu_\alpha(G) = r - 1$ have at most 2 vertices of type r. We characterize some graphs of girth at least 6 with $\mu_\alpha(G) = r - 1$ and exactly two vertices of type r, generalizing results by Ekim *et al.* [9] for almost well-covered graphs. As a consequence of this result, we show that for girth at least 6, only graphs having girth exactly 6 in $\mathcal{I}_3$ have exactly two vertices of type r. We characterize trees in $\mathcal{I}_3$ that contain exactly one vertex of type 3 and exactly two vertices of type 2. A preliminary version of this work that focuses on independence gap was presented in [5].

The remainder of this paper is organized as follows. In Section 2, we present further definitions and some basic results. In Section 3, we address the complexity of recognizing graphs in $\mathcal{I}_r$. In Section 4, we present the results concerning the independence gap in graphs of girth at least 6, along with some consequences for the set $\mathcal{I}_r$. In Section 5, we show the results related to trees. The final remarks are presented in Section 6.

2. Definitions and preliminary results

For a vertex u of G, the neighborhood, the closed neighborhood, and the degree are denoted $N_G(u)$, $N_G[u]$, and $d_G(u)$, respectively. For a set U of vertices of G, $N_G[U] = \bigcup_{u \in U} N_G[u]$.

As usual, we denote by P_n, C_n, and K_n the path, the cycle, and the complete graph of order n, respectively. A *complete q-partite graph* is a graph that admits a partition of its vertex set into q independent sets $V_1, V_2, \ldots, V_q$, where every vertex in V_i is adjacent to all vertices in V_j, for every $j \neq i$. Given $|V_i| = n_i$, for $1 \leq i \leq q$, such a complete q-partite graph is denoted by $K_{n_1,n_2,\ldots,n_q}$.

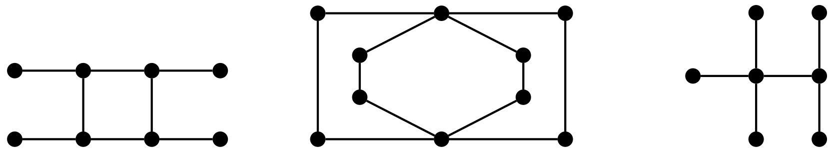

FIGURE 1. Graph G_1 is well-covered, and $\mu_\alpha(G_1) = 0$; $G_2 \in \mathcal{M}_3$, but $G_2 \notin \mathcal{I}_3$, and $\mu_\alpha(G_2) = 3$; $G_3 \in \mathcal{I}_3$, therefore $G_3 \in \mathcal{M}_3$, and $\mu_\alpha(G_3) = 2$. (a) miss(G1) = 4. (b) miss(G2) = 2, 4, 5. (c) miss(G3) = 3, 4, 5.

A vertex of degree 1 is a *leaf* and a *support vertex* is a vertex adjacent to a leaf. A vertex is called *internal* if it is not a leaf. A *pendant edge* is an edge between a support vertex and a leaf. We assign a non-negative integer, called the type of u, to some vertices u of G. Internal vertices adjacent to exactly k leaves will be of type k. Note that a leaf in a component other than K_2 is not assigned to any type.

A *perfect matching* in a graph is a set of pairwise disjoint edges such that every vertex of the graph is an endpoint of one of them. We abuse notation and write $X - a$, when X is a set and a is a possible member of X, instead of $X - \{a\}$. If $X \subseteq V(G)$, $G - X$ denotes the subgraph of G obtained by removing vertices in X and edges incident to them in G. We denote the set of positive integers between p and q, inclusive, for $[p, q]$. We abbreviate $[1, q]$ as $[q]$.

Let $A, B \subseteq \mathbb{N}$. The *Minkowski sum* of A and B is defined as the set $A \oplus B = \{a + b : a \in A \text{ and } b \in B\}$. If $B = \emptyset$ or $B = \{0\}$, we have $A \oplus B = A$. Since the standard sum of integers is associative, we have that Minkowski sum is also associative, that is, for $A, B, C \subseteq \mathbb{N}$, it holds that $(A \oplus B) \oplus C = A \oplus (B \oplus C)$. Let $A_1, A_2, \ldots, A_p \subseteq \mathbb{N}$, for an integer $p \geq 2$. When no parenthesis is given, we perform two Minkowski sums from left to right, that is, $A_1 \oplus A_2 \oplus A_3 \cdots \oplus A_p$ is read as $((A_1 \oplus A_2) \oplus A_3) \cdots \oplus A_p)$. If $|A_i| = n_i$, for $i \in [p]$, the Minkowski sum $A_1 \oplus A_2 \oplus \cdots \oplus A_p$ can be computed naively by nesting an iterator over every set, which runs in $O(n_1 \cdot n_2 \cdot \cdots \cdot n_p)$ time.

A *marked set* is a set $\mathcal{A} \subseteq A \times \{0, 1\}$. We say that $a \in A$ is *marked* in $\mathcal{A}$ if $(a, 1) \in \mathcal{A}$ and *unmarked* if $(a, 0) \in \mathcal{A}$. For short, to indicate that an element of A is marked in $\mathcal{A}$, we use the notation $\dot{a} = (a, 1)$ and by abuse of notation, we make no distinction between A and $\mathcal{A}$. The *mark all operation* on set A is defined as the set $\dot{A}$, obtained by a copy of A marking every unmarked element of A. If $\mathcal{A}$ and $\mathcal{B}$ are two marked sets, we define the *(marked) Minkowski sum* $\mathcal{A} \oplus \mathcal{B}$ of $\mathcal{A}$ and $\mathcal{B}$ as the Minkowski sum of the corresponding sets, where the marked elements of the marked Minkowski sum are precisely those that are the sum of a marked element of A and a marked element of B, that is, $\mathcal{A} \oplus \mathcal{B} = \{(a + b, i \cdot j) : (a, i) \in \mathcal{A} \text{ and } (b, j) \in \mathcal{B}\}$.

For short, we write IS (resp. MIS) instead of (resp. maximal) independent set. For a graph G, let $\text{miss}(G) = \{|I| : I \text{ is a MIS of } G\}$, that is, the set with all sizes of MIS of G. In Figure 1, we provide examples of graphs and categorize them based on the sizes of their maximal independent sets (MIS). Note that if a graph G is a member of $\mathcal{I}_r$, then it belongs to $\mathcal{M}_r$ and $\mu_\alpha(G) = r - 1$. However, if $\mu_\alpha(G) = r - 1$, not necessarily it belongs to $\mathcal{M}_r$. Similarly, if G belongs to $\mathcal{M}_r$, not necessarily it belongs to $\mathcal{I}_r$.

Finbow *et al.* [12] showed that if G is a graph in $\mathcal{M}_r$, then for every independent set I of G, $G - N_G[I]$ belongs to $\mathcal{M}_s$ for some $s \leq r$. The following lemma is a straightforward implication of this result that will be used in some proofs.

Lemma 2.1. *([9]) For an independent set I of a graph G, $\mu_\alpha(G - N_G[I]) \leq \mu_\alpha(G)$.*

We consider connected graphs since the following lemma reduces the problem of determining the independence gap of a graph to its components.

Lemma 2.2. *([9]) If G is a graph with components $H_1, \ldots, H_k$, then $\mu_\alpha(G) = \sum_{j=1}^{k} \mu_\alpha(H_j)$.*

The following result, which is a characterization of well-covered graphs of girth at least 6, is used in some proofs.

Theorem 2.3. *([11]) Let G be a connected graph of girth at least 6 isomorphic to neither C_7 nor K_1. Then G is well-covered if and only if its pendant edges form a perfect matching.*

3. Complexity of $\mathcal{I}_r$-Membership

Given a graph G and a positive integer r, deciding whether $G \in \mathcal{M}_r$ is known to be co-NP-complete [6]. As far as we know, the complexity of the related problem, that restricts the sizes to be consecutive, is open. We define such a problem as follows.

> $\mathcal{I}_r$-Membership
> **Instance:** A graph G.
> **Question:** Does $G \in \mathcal{I}_r$, that is, does G have r consecutive distinct maximal independent set sizes, for some positive integer r?

If $r = 1$, we recall that $\mathcal{I}_r$-Membership coincides with Well-Coveredness, which is co-NP-complete [7,19]. We remark that the NP-completeness of $\mathcal{I}_2$-Membership follows directly from the reduction presented in the proofs in [7, 19]. Then, we investigate the case when $r \geq 3$.

Theorem 3.1. *For every $r \geq 3$, the $\mathcal{I}_r$-Membership problem is NP-complete.*

Proof. Let G be a graph. Given $i(G)$ and $\alpha(G)$ it is clear that $G \in \mathcal{I}_r$ if and only if $\alpha(G) - i(G) + 1 = r$. Since determining $i(G)$ and $\alpha(G)$ are in NP, we have that $\mathcal{I}_r$-Membership is in NP.

For the hardness, we perform a reduction from 3-SAT. Let C be an instance of 3-SAT, with clauses $C = \{c_1, \ldots, c_m\}$ and variables $U = \{u_1, \ldots, u_n\}$. We construct a graph H from the graph G constructed in [19]. The description of G is the following. Let $V_C = \{c_1, \ldots, c_m\}$, $V_L = \{u_1, \bar{u}_1, \ldots, u_n, \bar{u}_n\}$, and $V(G) = V_C \cup V_L$. For the edges, let $E(G) = \{c_i c_j : 1 \leq i, j \leq m, i \neq j\} \cup \{u_i \bar{u}_i : 1 \leq i \leq n\} \cup \{c_i u_j : u_j \text{ is a literal in clause } c_i\} \cup \{c_i \bar{u}_j : \bar{u}_j \text{ is a literal in clause } c_i\}$.

For $r \geq 3$, let G' be a complete $(r-2)$-partite graph $K_{n+2,n+3,\ldots,n+r-1}$. The graph H arises from the disjoint union between G and G' by the addition of the edge set $E' = \{uu' : u \in V(G), u' \in V(G')\}$. Before we proceed, notice that G' is a complete multipartite graph. This implies that $\operatorname{miss}(G') = \{n+2, n+3, \ldots, n+r-1\}$.

We show that C is satisfiable if and only if $H \in \mathcal{I}_r$. From [7, 19] we know that $\operatorname{miss}(G) = \{n, n+1\}$, if C is satisfiable, and $\operatorname{miss}(G) = \{n\}$, otherwise. Then, by definition of E', $\operatorname{miss}(H) = \operatorname{miss}(G) \cup \operatorname{miss}(G')$. This implies that, if C is satisfiable, $\operatorname{miss}(H) = [n, n+r-1]$, hence $H \in \mathcal{I}_r$. Otherwise, $\operatorname{miss}(H) = \{n, n+2, n+3, \ldots, n+r-1\}$. Since $n + 1 \notin \operatorname{miss}(H)$, we conclude that $H \notin \mathcal{I}_r$. ☐

4. Independence gap and $\mathcal{I}_r$ class in graphs of girth at least 6

Ekim *et al.* [9] investigated graphs in $\mathcal{I}_2$, which are also called almost well-covered graphs. They considered graphs of girth at least 6. If G is almost well-covered, having girth at least 6, then G has at most two vertices of type 2. The authors did not present a complete characterization for these graphs; however, they provided characterizations for those with precisely two vertices of type 2 and those with only one vertex of type 2. We use some arguments presented in [9] to generalize results related to the independence gap in a given graph G, considering $\mu_\alpha(G) \leq r - 1$, for $r \geq 2$. We show the possible sizes of MIS for graphs of girth at least 6 with $\mu_\alpha(G) \leq r - 1$ having exactly two vertices of type r. We use this result to characterize graphs in $\mathcal{I}_3$ of girth at least 6 having exactly two vertices of type 3 and to show that graphs in $\mathcal{I}_r$ of girth at least 7 have at most one vertex of type r.

The following result bounds the number of leaves a vertex of a graph G can have considering that $\mu_\alpha(G) \leq r - 1$.

Lemma 4.1. *([9]) If G is a graph with $\mu_\alpha(G) \leq r-1$, then every internal vertex of G is adjacent to at most r leaves.*

In this section, we denote G_i the subgraph of G induced by internal vertices of G that are type i. For instance, G_0 is the subgraph of G that contains internal vertices with no leaves.

Lemma 4.2. *If G is a graph with $\mu_\alpha(G) = r-1$, for some $r \geq 2$, then G_r induces a complete graph.*

Proof. Let G be a graph such that $\mu_\alpha(G) = r-1$, for some $r \geq 2$. Toward a contradiction, suppose that G has two vertices of type r, say u_1 and u_2, with sets of leaves F_1 and F_2, respectively, and u_1 and u_2 are not adjacent. Hence there is an MIS I of G that contains both, u_1 and u_2. Let $I' = (I - \{u_1, u_2\}) \cup F_1 \cup F_2$. Extend I' to a maximal independent set of G. It follows that $|I'| - |I| \geq 2r-2$. Since $r \geq 2$, we conclude that $\mu_\alpha(G) > r-1$. □

Lemma 4.2 immediately implies that if $\mu_\alpha(G) = r-1$ for some $r \geq 2$ and G has girth at least 4, then G has at most two vertices of type r. From now on, we consider graphs of girth at least 6.

Lemma 4.3. *Let G be a graph of girth at least 6 such that $\mu_\alpha(G) = r-1$, for some $r \geq 1$. If G has a vertex u of type r, then u does not have a neighbor in G_0.*

Proof. Let G be a graph as described. Suppose that G has a vertex u of type r. For contradiction, suppose that u has a neighbor v in G_0. Let I be the set of vertices that are simultaneously at a distance of 3 from u and 2 from v. By the girth restriction, I is independent. Let $G' = G - N_G[I]$. Observe that in G' vertex v is a leaf of u that has $r+1$ leaves. By Lemma 4.1, $\mu_\alpha(G') > r-1$ and, by Lemma 2.1, $\mu_\alpha(G) > r-1$. □

Lemma 4.4. *Let G be a graph of girth at least 6 such that $\mu_\alpha(G) = r-1$, for some $r \geq 2$. If G has exactly two vertices of type r, then any other support vertex of G is of type 1.*

Proof. Let G be a graph as described. Suppose that G has two vertices of type r, say u_1 and u_2, with sets of leaves F_1 and F_2, respectively, such that $|F_1| = |F_2| = r$. By Lemma 4.2, u_1 and u_2 are adjacent. For a contradiction, suppose that G has another support vertex, say x, which has the leaves in the set F_3 with $|F_3| \geq 2$. Because of the girth restriction, x is not adjacent to at least one of u_1 and u_2. Without loss of generality, suppose that x is not adjacent to u_2. Let I be an MIS of G that contains both x and u_2. Hence $F_1 \subseteq I$. Let $I' = (I - \{x, u_2\}) \cup F_2 \cup F_3$. Extend I' to a maximal independent set of G. Since $|F_3| \geq 2$, it follows that $|I'| - |I| \geq |F_3| + r - 2 \geq r$, which contradicts that $\mu_\alpha(G) = r-1$. □

The following result is a generalization of Theorem 3.3 [9] for graphs G of girth at least 6 with $\mu_\alpha(G) = 1$. We adapt their proof considering $\mu_\alpha(G) \geq 1$. Additionally, we present the different sizes of MIS of G.

Theorem 4.5. *Let $r \geq 2$ and G be a connected graph of girth at least 6, with exactly two vertices u_1 and u_2 of type r, and with no type k vertices for $k \geq r+1$. Then $\mu_\alpha(G) = r-1$ if and only if u_1 and u_2 are adjacent, any other support vertex of G is type 1, and one of the following two conditions holds:*

(i) $V(G_0) = \emptyset$;
(ii) $G_0 \cong K_2$, *neither of u_1 and u_2 has a neighbor in G_0, and the two vertices of G_0 are of degree 2 in G and are contained in an induced 6-cycle containing u_1 and u_2.*

Moreover, if $V(G_0) = \emptyset$, then $\operatorname{miss}(G) = \{|V(G_1)| + r + 1, |V(G_1)| + 2r\}$ *otherwise* $\operatorname{miss}(G) = \{|V(G_1)| + r + 2, |V(G_1)| + 2r, |V(G_1)| + 2r + 1\}$.

Proof. Let G be a graph as described and let F_1 and F_2 be the sets of leaves, respectively, of vertices u_1 and u_2. Since both are type r, $|F_1| = |F_2| = r$. First suppose $\mu_\alpha(G) = r-1$. By Lemmas 4.3 and 4.4, the other neighbors of vertices u_1 and u_2 are vertices of type 1. By Lemma 4.2, u_1 and u_2 are adjacent. Suppose $V(G_0) \neq \emptyset$; Let $L_1 = N_G(u_1) - (F_1 \cup \{u_2\})$ and $L_2 = N_G(u_2) - (F_2 \cup \{u_1\})$. By the girth restriction, $L_1 \cup L_2$ is an independent set.

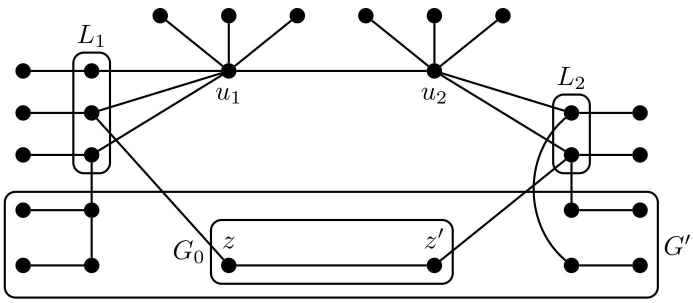

FIGURE 2. Graph G of girth 6 and two vertices of type 3. $G \in \mathcal{I}_3$ with $\text{miss}(G) = [14, 16]$.

Let L'_i be the set of leaves adjacent to vertices of L_i, $i = 1, 2$. Note that $|L_i| = |L'_i|$. Now, let $I = F_1 \cup F_2 \cup L'_1 \cup L'_2$ and let $G' = G - N_G[I]$. The graph G' contains all vertices in G_0, and possibly other vertices in G_1, with their respective leaves. See Figure 2 for an illustration.

Observe that $\alpha(G) = \alpha(G') + |I| = \alpha(G') + |L_1| + |L_2| + 2r$ and $i(G) = i(G') + |L_1| + |L_2| + r + 1$, which implies that $i(G') = \alpha(G')$, *i.e.*, G' is well-covered. Furthermore, the independence gap of G is due to the two possibilities of selecting either $F_2 \cup \{u_1\}$ (symmetrically $F_1 \cup \{u_2\}$) or $F_1 \cup F_2$ to a MIS of G. The proof of the equivalent of (ii) in Theorem 3.3 [9] for graphs G of girth at least 6 with $\mu_\alpha(G) = 1$ shows that if this condition does not hold, G has a MIS of size $i(G) - 1$. They show that G' do not have a component isomorphic to either K_1 or C_7, which implies that G' has a perfect matching formed by its pendant edges (by Theorem 2.3), concluding that G_0 has exactly one component and it is isomorphic to K_2. So, by following the same steps in their proof, we can conclude that the same hold for G under the same conditions and $\mu_\alpha(G) = r - 1$, completing the proof of the forward direction of the implication.

Conversely, suppose $V(G_0) = \emptyset$. Hence $V(G)$ contains exactly the vertices u_1 and u_2, their leaves $(F_1 \cup F_2)$, the vertices in G_1, and their leaves. Note that any MIS of G contains exactly one vertex of every pendant edge with one of its endpoints in G_1. Let I be an MIS of G. If $I \cap \{u_1, u_2\} = \emptyset$, I contains the leaves in $F_1 \cup F_2$ and $|I| = 2r + |L_1| + |L_2| + |V(G')|/2 = |V(G_1)| + 2r$. If $I \cap \{u_1, u_2\} \neq \emptyset$, $|I| = r + 1 + |L_1| + |L_2| + |V(G')|/2 = |V(G_1)| + r + 1$. Hence $\text{miss}(G) = \{|V(G_1)| + r + 1, |V(G_1)| + 2r\}$ which implies $\mu_\alpha(G) = r - 1$.

Now, suppose that conditions of Case 4.5 hold. Let z and z' be the two vertices of G_0. Thus there exist $a \in N_G(u_1) \cap V(G_1)$ and $b \in N_G(u_2) \cap V(G_1)$ such that z is adjacent to a and z' is adjacent to b. Let I be an MIS of G. If $I \cap \{z, z'\} = \emptyset$, $\{a, b\} \subseteq I$, $I \cap \{u_1, u_2\} = \emptyset$, and $F_1 \cup F_2 \subseteq I$, which implies $|I| = |V(G_1)| + 2r$. If $I \cap \{z, z'\} \neq \emptyset$, there are two possibilities: $I \cap \{u_1, u_2\} = \emptyset$ and then $F_1 \cup F_2 \subseteq I$, which implies that $|I| = |V(G_1)| + 2r + 1$, and $I \cap \{u_1, u_2\} \neq \emptyset$, which implies that $|I| = |V(G_1)| + r + 2$. Hence $\text{miss}(G) = \{|V(G_1)| + r + 2, |V(G_1)| + 2r, |V(G_1)| + 2r + 1\}$ and we can conclude $\mu_\alpha(G) = r - 1$. □

Since the conditions of Theorem 4.5 can be tested in polynomial time, we have an efficient algorithm to recognize connected graphs G of girth at least 6, with exactly two type r vertices having $\mu_\alpha(G) = r - 1$.

Corollary 4.6. *Given a connected graph G of girth at least 6 and with exactly two type r vertices, it can be decided in polynomial time whether $\mu_\alpha(G) = r - 1$.*

Barbosa *et al.* [1] show sufficient conditions for a graph to belong to $\mathcal{M}_2$ or $\mathcal{I}_2$. They show that for a positive integer r, the set $\mathcal{I}_r$ is not empty, since given any graph G of order n, it is possible to obtain G' from G by adding two leaves to each vertex of G. If $\alpha(G) = t - 1$, then G' has MIS of sizes $2n - (t - 1), 2n - (t - 2), \ldots, 2n$, and, therefore, G' belongs to $\mathcal{I}_t$.

Theorem 4.5 has some consequences to $\mathcal{I}_r$. Observe that the graphs G considered have at least two and at most three distinct sizes of MIS and they are consecutive in the following conditions: if $r = 2$ and the girth of G is at least 6, as proved in [9]; and if $r = 3$, the girth of G is exactly 6, and $V(G_0) \neq \emptyset$. In the following cases, the sizes of MIS of G are not consecutive: if $r \geq 3$ and the girth of G is at least 7, and if $r \geq 4$ and the girth of G is at least 6. We summarize these conditions in Corollary 4.7.

Corollary 4.7. *Let $r \geq 3$ and let G be a graph of girth at least 6 with $\mu_\alpha(G) = r - 1$ such that G contains exactly two vertices of type r. Then, $G \in \mathcal{I}_r$ only if $r = 3$ and the girth of G is exactly 6.*

Proof. Suppose $G \in \mathcal{I}_r$. Toward a contradiction, suppose first $r \geq 4$. By Theorem 4.5, G has at most 3 distinct sizes of MIS, which contradicts that $G \in \mathcal{I}_r$. Now, suppose that the girth of G is at least 7. If $V(G_0) \neq \emptyset$, by Theorem 4.5, Case 4.5, G has girth exactly 6. If $V(G_0) = \emptyset$, by Case 4.5, G has at most 2 distinct sizes of MIS. However, since $r \geq 3$, this contradicts that $G \in \mathcal{I}_r$. □

Next, we consider $r \geq 3$ and G is a graph of girth at least 6 with exactly one vertex of type r such that $\mu_\alpha(G) = r - 1$. We bound the number of vertices of type $r - 1$ in G.

Lemma 4.8. *Let $r \geq 3$ and let G be a graph of girth at least 6 with exactly one vertex u_1 of type r. If $\mu_\alpha(G) = r - 1$, any vertex of G of type $r - 1$ is adjacent to u_1.*

Proof. Let G be a graph having $\mu_\alpha(G) = r - 1$ with a vertex u_1 that is a vertex of type r. Suppose that there is a vertex u_2 of type $r - 1$, which is not adjacent to u_1. Let F_i be the set of leaves of u_i, $i = 1, 2$ and let I be an MIS of G that contains $\{u_1, u_2\}$. Let $I' = (I - \{u_1, u_2\}) \cup F_1 \cup F_2$. Extend I' to a maximal independent set of G. It follows that $|I'| - |I| \geq |F_1| + |F_2| - 2 = 2r - 3 \geq r$, which contradicts that $\mu_\alpha(G) = r - 1$. Hence, u_1 and u_2 are adjacent. □

Proposition 4.9. *Let G be a graph of girth at least 6 with $\mu_\alpha(G) = r - 1$ and exactly one vertex u_1 of type r. If $r = 3$, then G has at most two vertices of type 2; and if $r \geq 4$, then G has at most one vertex of type $r - 1$.*

Proof. Let G be a graph having $\mu_\alpha(G) = r-1$ with a vertex u_1 that is a vertex of type r. First suppose $r = 3$ and there are three vertices in G, say u_2, u_3 and u_4, which are of type 2. By Lemma 4.8, each one of these vertices must be adjacent to u_1. By the girth restriction, vertices u_2, u_3, and u_4 are independent. Let F_i be the set of leaves of u_i, $i = 1, 2, 3, 4$ and let I be an MIS of G that contains $\{u_2, u_3, u_4\}$. Let $I' = (I - \{u_2, u_3, u_4\}) \cup F_2 \cup F_3 \cup F_4$. Extend I' to a maximal independent set of G. It is clear that $|I'| - |I| \geq |F_2| + |F_3| + |F_4| - 3 = 3$. This contradicts $\mu_\alpha(G) = 2$.

Now, suppose $r \geq 4$, and there are two vertices in G, say u_2 and u_3, which are of type $r-1$. By Lemma 4.8, each one of these vertices must be adjacent to u_1. By the girth restriction, vertices u_2 and u_3 are independent. Let F_i be the set of leaves of u_i, $i = 1, 2, 3$, and let I be an MIS of G that contains $\{u_2, u_3\}$. Let $I' = (I - \{u_2, u_3\}) \cup (F_2 \cup F_3)$. Extend I' to a maximal independent set of G. It is clear that $|I'| - |I| \geq |F_2| + |F_3| - 2 = 2r - 4 \geq r$, since $r \geq 4$. This contradicts $\mu_\alpha(G) = r - 1$. Therefore, G has at most one vertex of type $r - 1$. □

5. $\mathcal{M}_r$ AND $\mathcal{I}_r$ CLASSES ON TREES

Staples [20] and Ravindra [18] independently characterized the well-covered bipartite graphs, which include all trees. A tree is in $\mathcal{M}_1$ if and only if all its pendant edges form a perfect matching. Therefore, in a well-covered tree, each vertex is a support vertex with exactly one leaf or a leaf. The trees in $\mathcal{M}_2$ were characterized by Finbow *et al.* [12]. The existing characterizations lead to polynomial time algorithms to decide whether a tree is in $\mathcal{M}_r$, for $r = 1, 2$. Expanding the solvability of the $\mathcal{M}_r$-Membership problem to any positive integer r, we present a recursive algorithm able to compute in polynomial time the set miss(T), given a rooted tree T.

Let us present additional definitions. Let T be a tree rooted at $r \in V(T)$ and $v \in V(T)$. We denote by T_v the subtree of T rooted at v, and by $C_T(v)$ the set of children of v in T. For $v \neq r$, denote by $p_T(v)$ the parent of v in T, and by T_v^+ the subgraph of T induced by $V(T_v) \cup \{p_T(v)\}$. Let $T_r^+ = T$ and $p_T(r) = \emptyset$. When T is clear from the context, we simply write $p(v)$ and $C(v)$.

The algorithm proceeds by keeping track of MIS sizes of T_v or T_v^+, for every $v \in V(T)$, by using the functions:

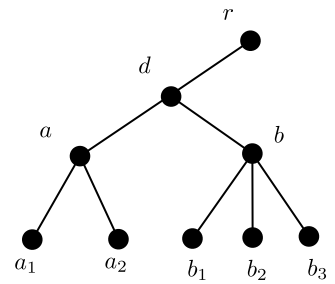

FIGURE 3. Tree T rooted at r with $\text{miss}(T_a) = \{1, 2\}$, $\text{miss}(T_b) = \{1, 3\}$, and $\text{miss}(T_d) = \{2, 3, 4, 6\}$.

– $A(v) = \{|S| : S$ is an MIS of T_v, with $v \in S\}$.
– $B_0(v) = \{|S| : S$ is an IS of T_v^+ where S is also an MIS of T_v, with $v, p(v) \notin S\}$.
– $B_1(v) = \{|S| : S$ is an IS of T_v where $S \cup \{p(v)\}$ is an MIS of T_v^+, with $v \notin S\}$.

Notice that B_1 is not defined for the root r. The function that solves the whole problem is $H(r) = A(r) \cup B_0(r)$.

As an example, let T_d be the subtree of T in Figure 3. Consider $S_1 = \{d, a_1, a_2, b_1, b_2, b_3\}$, $S_2 = \{a_1, a_2, b\}$, $S_3 = \{a, b_1, b_2, b_3\}$, and $S_4 = \{a_1, a_2, b_1, b_2, b_3\}$. We have that $A(d) = \{|S_1|\} = \{6\}$, $B_0(d) = \{|S_2|, |S_3|\} = \{3, 4\}$, and $B_1(d) = \{|S_2|, |S_3|, |S_4|\} = \{3, 4, 5\}$.

We compute the functions in a bottom-up order as follows.

Let v be a leaf in T_r. The unique MIS of T_v containing v is $\{v\}$, thus

– $A(v) = \{1\}$.

Notice that there is no IS S of T_v^+ where S is also an MIS of T_v, with $v, p(v) \notin S$. Then, we define

– $B_0(v) = \emptyset$.

Finally, $S = \emptyset$ is an IS of T_v where $S \cup \{p(v)\}$ is an MIS of T_v^+, with $v \notin S$, then

– $B_1(v) = \{0\}$.

For instance, let T be the tree of Figure 3 again. With the base cases stated above, for every $u \in S_4 = \{a_1, a_2, b_1, b_2, b_3\}$, we have that $A(u) = \{1\}$, $B_0(u) = \emptyset$, and $B_1(u) = \{0\}$.

Now, let v be a non-leaf in T_r. We exploit the approach of "combining" through union and Minkowski sums, previously computed functions for $C(v)$, for the recursive steps.

To compute $A(v)$, *i.e.*, all sizes $|S|$, such that v belongs to an MIS S of T_v, $|\{v\}|$ contributes with one size Minkowski-summed up with all previously computed sizes of MIS's S' of T_c^+, for every child c of v. Since $p(c) = v \in S'$, we make use of function $B_1(c)$, obtaining

– $A(v) = \{1\} \oplus \Big(\bigoplus_{c \in C(v)} B_1(c) \Big)$.

For an example, recall T from Figure 3. The set $A(a)$ can be computed by $\{|\{a\}|\} \oplus B_1(a_1) \oplus B_1(a_2) = \{1\} \oplus \{0\} \oplus \{0\} = \{1\}$ and, similarly, $A(b) = \{1\}$. Further, having computed $B_0(a) = B_1(a) = \{2\}$, and $B_0(b) = B_1(b) = \{3\}$, $A(d)$ can be obtained by $\{1\} \oplus B_1(a) \oplus B_1(b) = \{1\} \oplus \{2\} \oplus \{3\} = \{6\}$.

For $B_0(v)$, we compute the sizes of IS's S of T_v^+ not containing v nor $p(v)$, that are also MIS's of T_v. In this case, since S must be an MIS of T_v, we have that $v, p(v) \notin S$ implies to consider only the sets S such that $S \cap C(v) \neq \emptyset$. To ensure this, we mark the sizes of IS's S in T_v^+ that do not correspond to MIS's of T_v (*i.e.*, the sets S that $S \cap C(v) = \emptyset$), for later removal. That marking assures to identify specific sizes that might not occur (notice if there are a size $\dot{k}$ and k, we should remove only $\dot{k}$, since k is a valid size for T_v). We then compute, for every child c of v, the MIS's sizes containing c, returned by $A(c)$, union with the IS's sizes not containing c, provided by $\dot{B}_0(c)$. Recall that the operation $\dot{A}$ on a set A returns a copy of A by marking every unmarked element of A. Then,

– $B_0(v) = \Big(\bigoplus_{c \in C(v)} \big(A(c) \cup \dot{B}_0(c)\big)\Big) - \Big(\bigoplus_{c \in C(v)} \dot{B}_0(c)\Big).$

For instance, consider T as depicted in Figure 3 once more. We have that $B_0(a) = ((A(a_1) \cup \dot{B}_0(a_1)) \oplus (A(a_2) \cup \dot{B}_0(a_2))) - (\dot{B}_0(a_1) \oplus \dot{B}_0(a_2)) = ((\{1\} \cup \dot{\emptyset}) \oplus (\{1\} \cup \dot{\emptyset})) - (\dot{\emptyset} \oplus \dot{\emptyset}) = \{2\}$. Similarly, we obtain $B_0(b) = \{3\}$. Then, $B_0(d)$ can be computed by $(A(a) \cup \dot{B}_0(a)) \oplus (A(b) \cup \dot{B}_0(b)) - (\dot{B}_0(a) \oplus \dot{B}_0(b)) = (\{1, \dot{2}\} \oplus \{1, \dot{3}\}) - (\{\dot{2}\} \oplus \{\dot{3}\}) = \{2, 3, 4, \dot{5}\} - \{\dot{5}\} = \{2, 3, 4\}$.

Finally, for $B_1(v)$, we compute the sizes MIS's S of T_v, with the property that $S^+ = S \cup \{p(v)\}$ is an MIS of T_v^+. Notice that $p(v) \in S^+$ allows $S \cap C(v)$ to be empty. Then, we compute

– $B_1(v) = \bigoplus_{c \in C(v)} \big(A(c) \cup B_0(c)\big).$

For the tree T from Figure 3, we have $B_1(a) = (A(a_1) \cup B_0(a_1)) \oplus (A(a_2) \cup B_0(a_2)) = (\{1\} \cup \emptyset) \oplus (\{1\} \cup \emptyset) = \{2\}$. Similarly, $B_1(b) = \{3\}$. Consequently, $B_1(d)$ is equal to $(A(a) \cup B_0(a)) \oplus (A(b) \cup B_0(b)) = (\{1, 2\}) \oplus (\{1, 3\}) = \{2, 3, 4, 5\}$. Finally, given $A(d) = \{6\}$ and $B_0(d) = \{2, 3, 4\}$, we obtain $\text{miss}(T_d) = H(d) = A(d) \cup B_0(d) = \{2, 3, 4, 6\}$.

We prove in Theorem 5.1 that the above functions hold.

Theorem 5.1. *Let T be a tree rooted at $r \in V(T)$. The function $H(r)$ computes correctly* $\text{miss}(T)$*.*

Proof. We will prove by induction that $A(v)$, $B_0(v)$, and $B_1(v)$ correctly compute the sizes stated in their definition, for every $v \in V(T)$.

The base is the case when v is a leaf in T. Since $T_v \cong K_1$ which is well-covered, the unique MIS of T_v is $S = \{v\}$, then $A(v) = \{1\}$ is correct. Function $B_0(v)$ should compute an IS S of T_v^+ which is also an MIS of T_v, with $v, p(v) \notin S$, then S does not exist. Hence there are no sizes of MIS's not containing v and $B_0(v) = \emptyset$ is correct. For $B_1(v)$ we should compute an IS S of T_v, not containing v, such that $S \cup \{p(v)\}$ is an MIS of T_v^+. Since $S = \emptyset$ holds, $|S| = 0$. Then $B_1(v) = \{0\}$ is correct.

Now, by induction hypothesis, assume that for every child c of v, functions $A(c)$, $B_0(c)$, and $B_1(c)$ are correct according to their definitions. From now on, we fix $\ell = d_T(v) - 1$ and $C(v) = \{c_1, \dots, c_\ell\}$. When $B_1(c_i) \neq \emptyset$, for every $i \in [\ell]$, we assign a positive integer t_i to index the sets in $B_1(c_i)$. Then, we write $B_1(c_i) = \{|S_{i,j_i}| : 1 \leq j_i \leq t_i\}$.

First, consider function $A(v)$, *i.e.*, the sizes of MIS's S of T_v such that $v \in S$. We show that $A(v)$ is correct in Claim 5.2.

Claim 5.2. *S is an* MIS *of T_v containing v if and only if $|S| \in A(v)$.*

Proof (of Claim 5.2). Consider S an MIS as described. Since $v \in S$, it is clear that $S \cap C(v) = \emptyset$. Then we make use of $B_1(c_i)$, for every $i \in [\ell]$, to compute $|S|$. Recall that S is an MIS of T_v and, by induction hypothesis, $S_{i,j_i} \cup \{v\}$ is an MIS of $T_{c_i}^+$, for every $i \in [\ell]$, and for every $j_i \in [t_i]$. This implies that there exist $j_1 \in [t_1], j_2 \in [t_2], \dots, j_\ell \in [t_\ell]$, such that $S = \{v\} \cup S_{1,j_1} \cup S_{2,j_2} \cup \cdots \cup S_{\ell,j_\ell}$. Then, $|S| = 1 + |S_{1,j_1}| + |S_{2,j_2}| + \cdots + |S_{\ell,j_\ell}| \in \{1\} \oplus B_1(c_1) \oplus B_1(c_2) \oplus \cdots \oplus B_1(c_\ell) = A(v)$.

For the converse, let $|S| \in A(v) = \{1\} \oplus B_1(c_1) \oplus B_1(c_2) \oplus \cdots \oplus B_1(c_\ell)$. By definition of $B_1(v)$, there exist $j_1 \in [t_1], j_2 \in [t_2], \dots, j_\ell \in [t_\ell]$, such that $|S|$ can be written as $1 + |S_{1,j_1}| + |S_{2,j_2}| + \cdots + |S_{\ell,j_\ell}|$. By induction hypothesis, S_{i,j_i} is an IS of T_{c_i} where $S_{i,j_i} \cup \{v\}$ is an MIS of $T^+_{c_i}$, for every $i \in [\ell]$. Then it is clear that $S = \{v\} \cup S_{1,j_1} \cup S_{2,j_2} \cup \cdots \cup S_{\ell,j_\ell}$ is an MIS of T_v.

Consider next $B_0(v)$, *i.e.*, the set of sizes of IS's of T^+_v not containing v nor $p(v)$, that are also MIS's of T_v. For every $i \in [\ell]$ and $t^a_i, t^b_i \in \mathbb{N}$, define $A(c_i) = \{|S^a_{i,j^a_i}| : 1 \leq j^a_i \leq t^a_i\}$ and $B_0(c_i) = \{|S^b_{i,j^b_i}| : 1 \leq j^b_i \leq t^b_i\}$. Let us proceed with Claim 5.3.

Claim 5.3. *S is an* IS *of T^+_v where S is also an* MIS *of T_v, with $v, p(v) \notin S$, if and only if $|S| \in B_0(v)$.*

Proof (of Claim 5.3). Let S be such a set. Given that $v, p(v) \notin S$ and S is an MIS of T_v, we have that $S \cap C(v) \neq \emptyset$. For every $i \in [\ell]$, $j^a_i \in [t^a_i]$, and $j^b_i \in [t^b_i]$, by induction hypothesis, S^a_{i,j^a_i} is an MIS of T_{c_i} containing c_i and S^b_{i,j^b_i} is an MIS of T_{c_i} not containing c_i. Then, for every $i \in [\ell]$, there exist $x_i \in \{a, b\}$ and $j^{x_i}_i \in [t^{x_i}_i]$ such that $S = S^{x_1}_{1,j^{x_1}_1} \cup S^{x_2}_{2,j^{x_2}_2} \cup \cdots \cup S^{x_\ell}_{\ell,j^{x_\ell}_\ell}$. Recall that, for every $i \in [\ell]$, $|S^{x_i}_{i,j^{x_i}_i}| \in A(c_i) \cup B_0(c_i)$. Then $|S| \in (A(c_1) \cup B_0(c_1)) \oplus (A(c_2) \cup B_0(c_2)) \oplus \cdots \oplus (A(c_\ell) \cup B_0(c_\ell))$. Furthermore, $S \cap C(v) \neq \emptyset$ implies that $S \neq S^b_{1,j^b_1} \cup S^b_{2,j^b_2} \cup \cdots \cup S^b_{\ell,j^b_\ell}$, for every $j^b_i \in [t^b_i]$. Thus, $|S| \notin (B_0(c_1) \oplus B_0(c_2) \oplus \cdots \oplus B_0(c_\ell))$. Therefore, $|S| \in (A(c_1) \cup \dot{B}_0(c_1)) \oplus \cdots \oplus (A(c_\ell) \cup \dot{B}_0(c_\ell)) - (\dot{B}_0(c_1) \oplus \cdots \oplus \dot{B}_0(c_\ell))$.

Now, let $|S| \in B_0(v)$. By definition of $B_0(v)$, we have that $|S| \in (A(c_1) \cup \dot{B}_0(c_1)) \oplus \cdots \oplus (A(c_\ell) \cup \dot{B}_0(c_\ell))$ and $|S| \notin (\dot{B}_0(c_1) \oplus \cdots \oplus \dot{B}_0(c_\ell))$. Then $|S|$ can be obtained by $|S^{x_1}_{1,j^{x_1}_1}| + |S^{x_2}_{2,j^{x_2}_2}| + \cdots + |S^{x_\ell}_{\ell,j^{x_\ell}_\ell}|$, such that $|S^{x_i}_{i,j^{x_i}_i}| \in A(c_i) \cup B_0(c_i)$, $i \in [\ell]$, where there exists $\mathbf{i} \in [\ell]$ such that $c_{\mathbf{i}} \in S^{x_{\mathbf{i}}}_{\mathbf{i},j^{x_{\mathbf{i}}}_{\mathbf{i}}}$. It is clear that $c_{\mathbf{i}}$ dominates v, consequently S is an MIS of T_v.

Finally, consider $B_1(v)$, *i.e.*, the set of sizes of IS's S of T_v where $S \cup \{p(v)\}$ is an MIS of T^+_v, with $v \notin S$. The correctness of $B_1(v)$ is done by Claim 5.4.

Claim 5.4. *S is an* IS *of T_v where $S \cup \{p(v)\}$ is an* MIS *of T^+_v, with $v \notin S$, if and only if $|S| \in B_1(v)$.*

Proof (of Claim 5.4). Let S be a set satisfying the hypothesis. Since $S \cup \{p(v)\}$ is an MIS of T^+_v and, by induction hypothesis, $A(c_i) = \{|S^a_{i,j^a_i}| : 1 \leq j^a_i \leq t^a_i\}$ and $B_0(c_i) = \{|S^b_{i,j^b_i}| : 1 \leq j^b_i \leq t^b_i\}$ are correct, we have that $S = S^{x_1}_{1,j^{x_1}_1} \cup S^{x_2}_{2,j^{x_2}_2} \cup \cdots \cup S^{x_\ell}_{\ell,j^{x_\ell}_\ell}$, for every $i \in [\ell]$, for some $x_i \in \{a, b\}$, for some $j^{x_i}_i \in [t^{x_i}_i]$. Consequently, $|S| \in (A(c_1) \cup B_0(c_1)) \oplus (A(c_2) \cup B_0(c_2)) \oplus \cdots \oplus (A(c_\ell) \cup B_0(c_\ell)) = B_1(v)$.

For the other direction, let $|S| \in B_1(v)$. Then, $|S| \in (A(c_1) \cup B_0(c_1)) \oplus (A(c_2) \cup B_0(c_2)) \oplus \cdots \oplus (A(c_\ell) \cup B_0(c_\ell)) = B_1(v)$. We may write $|S|$ as $|S^{x_1}_{1,j^{x_1}_1}| + |S^{x_2}_{2,j^{x_2}_2}| + \cdots + |S^{x_\ell}_{\ell,j^{x_\ell}_\ell}|$, for every $i \in [\ell]$, for some $x_i \in \{a, b\}$, for some $j^{x_i}_i \in [t^{x_i}_i]$. Notice that, by induction hypothesis, if $x_i = a$, $S^{x_i}_{i,j^{x_i}_i}$ is an MIS of T_{c_i} containing c_i, and if $x_i = b$, $S^{x_i}_{i,j^{x_i}_i}$ is an IS of $T^+_{c_i}$ not containing $\{c_i, v\}$ where $S^{x_i}_{i,j^{x_i}_i}$ is also an MIS of T_{c_i}. In both cases, we conclude that $v \notin S$. Then $S^{x_1}_{1,j^{x_1}_1} \cup S^{x_2}_{2,j^{x_2}_2} \cup \cdots \cup S^{x_\ell}_{\ell,j^{x_\ell}_\ell} \cup \{p(v)\}$ is an MIS of $T^+_{c_i}$, as desired. $\square$

The time to compute $H(r)$ is stated in Theorem 5.5.

Theorem 5.5. *Let T be an n-vertex tree rooted at $r \in V(T)$. The function $H(r)$ can be computed in $O(n^4)$ time.*

Proof. Recall that functions $A(v)$ and $B_0(v)$ are computed for every $v \in V(T)$ and $B_1(v)$ is computed for every $v \in V(T) - \{r\}$. Then each of the three functions should be computed $O(n)$ times. So, let us analyze the time of computing each one for a vertex v.

If v is a leaf, then $A(v), B_0(v)$, and $B_1(v)$ are fixed sets. Then each of the three functions can be computed in constant time.

Otherwise, let $A_1, A_2, \ldots, A_p \subseteq \mathbb{N}$, for some $p \geq 2$. We discuss the time required to compute the Minkowski sum $A_1 \oplus A_2 \oplus \cdots \oplus A_p$.

Recall that, for any graph G, the sizes of MIS of G are bounded by 1 and $|V(G)|$, then $|\text{miss}(G)| = O(|V(G)|)$. Given that $A_i \subseteq \text{miss}(T)$, for every $i \in [p]$, as well as $A_1 \oplus A_2 \oplus \cdots \oplus A_j \subseteq \text{miss}(T)$, for every $2 \leq j \leq p$, we have that $|A_1 \oplus \cdots \oplus A_j| = O(n)$. This implies that $A_1 \oplus A_2 \oplus \cdots \oplus A_p$ requires to compute $p-1$ Minkowsky sums, each one running in $O(n^2)$ time. Then $A_1 \oplus A_2 \oplus \cdots \oplus A_p$ can be computed in $O(p \cdot n^2)$ time, for $p \geq 2$.

The definition of $A(v)$ requires computing $|C(v)|$ Minkowski sums. As discussed above, it gives a complexity of $O(|C(v)| \cdot n^2) = O(n^3)$ time.

The computation of $B_0(v)$ calls $A(c)$ and $\dot{B}_0(c)$ in a total of $|C(v)| = O(n)$ times. Given that $|A(c)| \leq n$ and $|\dot{B}_0(c)| \leq 2n$, we obtain that $|A(c) \cup \dot{B}_0(c)| \leq 2n = O(n)$. Since $A(c) \cup \dot{B}_0(c)$ is Minkowski summed up $|C(v)| - 1 = O(n)$ times, we obtain a complexity of $O(n^3)$ time. If $X = \left(\bigoplus_{c \in C(v)} \left(A(c) \cup \dot{B}_0(c)\right)\right)$ and $Y = \left(\bigoplus_{c \in C(v)} \dot{B}_0(c)\right)$, it is easy to see that $X - Y$ it runs in $O(n^2)$ time.

Finally, for $B_1(v)$ we know that $|A(c) \cup B_0(c)| \leq n = O(n)$. Since $A(c) \cup B_0(c)$ is Minkowski summed up $|C(v)| - 1 = O(n)$ times, we also obtain a complexity of $O(n^3)$.

Since the functions $A(v), B_0(v), B_1(v)$ are computed $O(n)$ times, and each runs in $O(n^3)$ time, the overall complexity for computing $H(r)$ is $O(n^4)$. □

Let T be a tree rooted at r. Theorems 5.1 and 5.5 imply that, by computing $H(r)$, we obtain miss(T) in $O(n^4)$ time. Then, $\mathcal{M}_r$-MEMBERSHIP can be solved in polynomial time for the class of trees. We remark that recognizing whether $T \in \mathcal{I}_r$ is linear time solvable, since $\alpha(T), i(T)$ can be determined in linear time and $T \in \mathcal{I}_r$ if and only if $\alpha(T) - i(T) + 1 = r$. Despite this result, we present further results on the structure of trees in $\mathcal{I}_3$.

If G is a tree and $G \in \mathcal{I}_r$, for $r \geq 3$, then, by Corollary 4.7, G has at most one vertex of type r. Next, we present a result related to the trees in $\mathcal{I}_3$ with exactly one type 3 vertex and exactly two type 2 vertices.

Theorem 5.6. *Let T be a tree with exactly one vertex u_1 of type 3, exactly two vertices u_2 and u_3 of type 2, and with no type k vertices for $k \geq 4$. Then $T \in \mathcal{I}_3$ if and only if $\{u_2, u_3\} \subseteq N_T(u_1)$ and any other vertex of T is of type 1 or a leaf.*

Proof. Let T be a tree as described. First, suppose $T \in \mathcal{I}_3$. Hence $\mu_\alpha(T) = 2$. By Lemma 4.8, $\{u_2, u_3\} \subseteq N_T(u_1)$. By Lemma 4.3, u_1 does not have a neighbor that is an internal vertex with no leaves. Now, we show that the neighbors of u_2 and u_3 other than their leaf vertices, if they exist, are support vertices. Without loss of generality, suppose that v is a vertex with no leaves adjacent to u_2. Let I' be the set of vertices that are simultaneously at distance 3 from u_2 and 2 from v and let I'' be the set of vertices that are simultaneously at distance 3 from u_1 and 2 from u_3 (I'' is possibly empty). Since T is a tree, $I = I' \cup I''$ is independent. Now, consider the component T' of $T - N_T[I]$ containing the vertices u_1, u_2 and u_3. Observe that in T' the vertex v is a leaf of u_2, and then u_2 has 3 leaves. The tree T' is formed by the three support vertices u_1, u_2, u_3 and their respective leaves. It is easy to verify that $\mu_\alpha(T') = 3$ with miss$(T') = \{5, 6, 7, 8\}$, which implies by Lemma 2.1 that $\mu_\alpha(T) \geq 3$, contradicting that $T \in \mathcal{I}_3$. Thus, all vertices adjacent to either u_2 or u_3, excluding their respective leaves and vertex u_1, are vertices of type 1.

For $i = 1, 2, 3$, let F_i be the set of leaves of u_i, and let $L_i = N_T(u_i) - (\{u_1, u_2, u_3\} \cup F_i)$. Since T is a tree, L_i is independent. Moreover, all vertices in L_i are of type 1. See Figure 4 for an illustration. Let L'_i be the set of all leaves of vertices in L_i, for $i = 1, 2, 3$. Note that $|L_i| = |L'_i|$. Now, let $I = \bigcup_{i \in \{1,2,3\}} (F_i \cup L'_i)$ and let $T'' = T - N_T[I]$. We claim that T'' is well-covered. Observe that $\alpha(T) = \alpha(T'') + |I| = \alpha(T'') + |L'_1| + |L'_2| + |L'_3| + 7$ and $i(T) = i(T'') + |L'_1| + |L'_2| + |L'_3| + 5$. If T'' is not well-covered, $i(T'') < \alpha(T'')$, and then $\alpha(T) - i(T) \geq 3$. By Theorem 2.3, each component of T'' is isomorphic to K_1 or to a graph with a perfect matching formed by its pendant edges. Since T has no cycles, it is easy to see that no component of T'' is isomorphic to K_1. Therefore, T'' has a perfect matching formed by its pendant edges.

Now, we show that T'' does not have a component with only vertices that do not have leaves in T. Suppose the contrary. Then, this component has at least two leaves, say z and z', which are not leaves in T. Without loss of generality, we may assume that there exist vertices $a, b \in (L_1 \cup L_2 \cup L_3)$ such that z is adjacent to a and

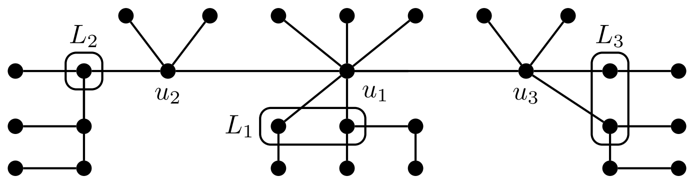

FIGURE 4. Tree T with exactly one vertex of type 3 and exactly two vertices of type 2. $T \in \mathcal{I}_3$ with miss$(T) = [14, 16]$.

z' is adjacent to b. Since in T there exists a path between a and b and another path between z and z', T has a cycle containing these four vertices, a contradiction. Therefore, $V(T)$ contains exactly the vertices u_1, u_2 and u_3, their leaves $(F_1 \cup F_2 \cup F_3)$, and any other vertex of T is of type 1 or a leaf.

Conversely, let T be a tree with exactly one vertex u_1 of type 3 and exactly two vertices, u_2 and u_3, of type 2, such that u_2 and u_3 are adjacent to u_1, and any other vertex of T is of type 1 or a leaf. Let F_i, L_i, L'_i, and T'' be as defined above. Let I be an MIS of T and $s = |L_1| + |L_2| + |L_3| + |V(T'')|/2$. If $I \cap \{u_1, u_2, u_3\} = \emptyset$, I contains the leaves in $F_1 \cup F_2 \cup F_3$ and $|I| = s + 7$. If $I \cap \{u_1, u_2, u_3\} = \{u_1\}$, I contains the leaves in $F_2 \cup F_3$ and $|I| = s + 5$. If $I \cap \{u_1, u_2, u_3\} = \{u_2, u_3\}$, I contains the leaves in F_1 and $|I| = s + 5$. If $I \cap \{u_1, u_2, u_3\} = \{u_2\}$ or $I \cap \{u_1, u_2, u_3\} = \{u_3\}$, I contains the leaves in F_1 and the leaves in F_2 or the ones in F_3. In both cases, $|I| = s + 6$. Therefore, miss$(T) = [s + 5, s + 7]$, which implies that $T \in \mathcal{I}_3$. □

6. Concluding remarks

We have investigated the different sizes of maximal independent sets in some graphs. We have shown that recognizing whether a graph G belongs to $\mathcal{I}_r$ is NP-complete, for $r \geq 3$, expanding the complexity results for $r = 1, 2$ known in the literature. A further work in that direction would be, given a graph $G \in \mathcal{M}_r$ determining when $G \in \mathcal{I}_r$.

On the positive side, we have shown that when G is a tree, all distinct sizes of MIS of G can be determined in polynomial time, for every $r \geq 1$.

In addition, we have shown conditions for graphs of girth at least 6 have $\mu_\alpha(G) = r - 1$, for $r \geq 2$, generalizing a result by Ekim *et al.* [9] for almost well-covered graphs. We have shown some implications of this result for the graphs in $\mathcal{I}_r$. For graphs having girth at least 6, only graphs with girth exactly 6 in $\mathcal{I}_2 \cup \mathcal{I}_3$, can have exactly two vertices with exactly r leaves.

The characterizations, both those in [9] and those presented in this work, contribute to the ongoing effort of fully characterizing graphs in $\mathcal{I}_2 \cup \mathcal{I}_3$ with girth at least 6. The characterizations provided also yield polynomial-time algorithms. This prompts the question of whether graphs in $\mathcal{I}_2 \cup \mathcal{I}_3$ with a girth of at least 6 can be efficiently recognized.

References

[1] R. Barbosa and B. Hartnell, Some problems based on the relative sizes of the maximal independent sets in a graph. *Congr. Numer.* **01** (1998) 115–121.

[2] R. Barbosa, M.R. Cappelle and D. Rautenbach, On graphs with maximal independent sets of few sizes, minimum degree at least 2, and girth at least 7. *Discrete Math.* **313** (2013) 1630–1635.

[3] J.A. Bondy and U.S.R. Murty, Graph Theory with Applications. Elsevier, New York (1976).

[4] M.R. Cappelle and D. Rautenbach, Extending berge's and favaron's results about well-covered graphs. *Discrete Math.* **313** (2013) 2742–2747.

[5] M.R. Cappelle, M.C. Dias and V.G. Santos, Independence gap in graphs of girth at least 6. In: *Anais do Simpósio Brasileiro de Pesquisa Operacional* (2020).

[6] Y. Caro, A. Sebő and M. Tarsi, Recognizing greedy structures. *J. Algorithms* **20** (1996) 137–156.

[7] V. Chvátal and P.J. Slater, A note on well-covered graphs. *Ann. Discrete Math.* **55** (1993) 179–181.

[8] Z. Deniz, A classification of 1-well-covered graphs. *Turk. J. Math.* **45** (2021) 2817–2829.

[9] T. Ekim, D. Gözüpek, A. Hujdurović, and M. Milanič, On Almost Well-Covered Graphs of Girth at Least 6. *Discrete Math. Theor. Comput. Sci.* **20** (2018).

[10] A. Finbow and B. Hartnell, A game related to covering by stars. *Ars Comb.* **16** (1983) 189–198.

[11] A. Finbow, B. Hartnell and R.J. Nowakowski, A characterization of well-covered graphs of girth 5 or greater. *J. Comb. Theory Ser. B* **57** (1993) 44–68.

[12] A. Finbow, B. Hartnell and C. Whitehead, A characterization of graphs of girth eight or more with exactly two sizes of maximal independent sets. *Discrete Math.* **125** (1994) 153–167.

[13] B.L. Hartnell, Well-covered graphs. *J. Comb. Math. Comb. Comput.* **29** (1999) 107–115.

[14] B.L. Hartnell and D. Rall, On graphs having maximal independent sets of exactly t distinct cardinalities. *Graphs and Combin.* **29** (2013) 519–525.

[15] K. Kimura, M. Pournaki, S. Seyed Fakhari, N. Terai and S. Yassemi, A glimpse to most of the old and new results on very well-covered graphs from the viewpoint of commutative algebra. *Res. Math. Sci.* **9** (2022) 29.

[16] M.D. Plummer, Well-covered graphs. *J. Comb. Theory* **8** (1970) 91–98.

[17] M.D. Plummer, Well-covered graphs: a survey. *Quaest. Math.* **16** (1993) 253–287.

[18] G. Ravindra, Well-covered graphs. *J. Comb. Inf. Syst. Sci.* **2** (1977) 20–21.

[19] R.S. Sankaranarayana and L.K. Stewart, Complexity results for well-covered graphs. *J. Comb. Inf. Syst. Sci.* (1997).

[20] J. Staples, *On Some Subclasses of Well-Covered Graphs.* Ph.D. thesis, Vanderbilt University (1975).

Please help to maintain this journal in open access!

This journal is currently published in open access under the Subscribe to Open model (S2O). We are thankful to our subscribers and supporters for making it possible to publish this journal in open access in the current year, free of charge for authors and readers.

Check with your library that it subscribes to the journal, or consider making a personal donation to the S2O programme by contacting subscribers@edpsciences.org.

More information, including a list of supporters and financial transparency reports, is available at https://edpsciences.org/en/subscribe-to-open-s2o.