

DIRECTED GRAPHS WITH LOWER ORIENTATION RAMSEY THRESHOLDS

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Abstract. We investigate the threshold $p_{\vec{H}} = p_{\vec{H}}(n)$ for the Ramsey-type property $G(n, p) \rightarrow \vec{H}$, where $G(n, p)$ is the binomial random graph and $G \rightarrow \vec{H}$ indicates that every orientation of the graph G contains the oriented graph \vec{H} as a subdigraph. Similarly to the classical Ramsey setting, the upper bound $p_{\vec{H}} \leq Cn^{-1/m_2(\vec{H})}$ is known to hold for some constant $C = C(\vec{H})$, where $m_2(\vec{H})$ denotes the maximum 2-density of the underlying graph H of \vec{H} . While this upper bound is indeed the threshold for some \vec{H} , this is not always the case. We obtain examples arising from rooted products of orientations of sparse graphs (such as forests, cycles and, more generally, subcubic $\{K_3, K_{3,3}\}$ -free graphs) and arbitrarily rooted transitive triangles.

Dedicated to the memory of Gabriel Ferreira Barros and to Professor Jayme Luiz Szwarcfiter on the occasion of his 80th birthday.

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1. INTRODUCTION

Given a graph G and an oriented graph \vec{H} , we write $G \rightarrow \vec{H}$ to mean that every orientation of G contains a copy of \vec{H} . The *orientation Ramsey number* $\vec{R}(\vec{H}) := \inf\{n : K_n \rightarrow \vec{H}\}$ has been investigated by many authors (see, e.g., [8, 11, 15–17, 23]). It was noted by Stearns (see, e.g., [9]) that a tournament with 2^t vertices must contain a transitive tournament of order t . Therefore, $\vec{R}(\vec{H}) < \infty$ if and only if \vec{H} contains no directed cycle (the “only if” part of the statement follows by considering transitive tournaments).

We study the property $G \rightarrow \vec{H}$ in the context of the *binomial random graph* $G(n, p)$, which is formed from the complete graph K_n by deleting each edge independently of all others with probability $1 - p$. Our goal is to find, given an acyclically oriented graph \vec{H} , the *threshold function* $p_{\vec{H}} = p_{\vec{H}}(n)$ for the property $G(n, p) \rightarrow \vec{H}$;

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that is, a function satisfying

$$\lim_{n \rightarrow \infty} \mathbb{P}(G(n, p) \rightarrow \vec{H}) = \begin{cases} 0 & \text{if } p \ll p_{\vec{H}}, \\ 1 & \text{if } p \gg p_{\vec{H}}, \end{cases}$$

where $a \ll b$ (or, equivalently, $b \gg a$) means $\lim_{n \rightarrow \infty} a(n)/b(n) = 0$ (we speak of “the threshold $p_{\vec{H}}$ ”, since $p_{\vec{H}}$ is unique up to constant factors). We shall say that an event \mathcal{E} holds asymptotically almost surely (a.a.s.) if \mathcal{E} occurs with probability tending to 1 as $n \rightarrow \infty$.

Thresholds for Ramsey-type properties have been widely studied (see, e.g., [13, 19] and the many references therein). If \vec{H} is acyclic, then the property $G(n, p) \rightarrow \vec{H}$ is non-trivial and monotone, and thus has a threshold [5].

Let H be a graph. As usual, let $v(H)$ and $e(H)$ denote the number of vertices and edges in H , respectively. For a graph H with $v(H) \geq 3$, let $\rho_2(H) = (e(H) - 1)/(v(H) - 2)$. Also, let $\rho_2(K_1) = \rho_2(2K_1) = 0$ and $\rho_2(K_2) = 1/2$. An important parameter for estimating $p_{\vec{H}}$ is the *maximum 2-density* $m_2(H)$ of H , which is given by $m_2(H) = \max\{\rho_2(J) : J \subset H\}$. We also consider the *maximum density* $m(H)$ of H , given by $m(H) = \max\{e(J)/v(J) : J \subset H\}$. Analogous definitions are used if \vec{H} is an oriented graph: we denote by H the (undirected) graph we obtain from \vec{H} by ignoring the orientation of its arcs, and let $m_2(\vec{H}) = m_2(H)$ and $m(\vec{H}) = m(H)$.

The following result gives an upper bound for $p_{\vec{H}}$ for any acyclic \vec{H} .

Theorem 1. *For every acyclically oriented graph \vec{H} with $m_2(\vec{H}) > 1/2$, there exists a constant $C = C(\vec{H})$ such that if $p \geq Cn^{-1/m_2(\vec{H})}$, then a.a.s. $G(n, p) \rightarrow \vec{H}$.*

Theorem 1 can be proved using the regularity method (it suffices to combine ideas from Sect. 8.5 of [13] and, say, [7]). We note that, when \vec{H} is a transitive tournament of order at least four, Theorem 1 is implied by a theorem of Rödl and Ruciński [21], but for other oriented graphs this implication is no longer clear. It will be convenient to have a variant of Theorem 1, namely, Lemma 12 (which is a version of Thm. 1 stating exponentially small bounds on the failure probability). Let us also remark that a version of Theorem 1 also appeared in a preprint of the second author, with an almost identical proof as the one presented here (the theorem has not been published elsewhere). For completeness, we note that the condition $m_2(H) > 1/2$ is equivalent to $\Delta(H) > 1$.

Readers familiar with the so called “random Ramsey theorem” of Rödl and Ruciński [20, 21] will find it natural that the edge probability $p = p(n) = n^{-1/m_2(\vec{H})}$ appears in Theorem 1, and may even guess that the threshold $p_{\vec{H}}$ is $n^{-1/m_2(\vec{H})}$ as long as H is not a forest of stars, as this is the case in the classical Ramsey set-up involving colourings. This was confirmed in [4] for cycles and complete graphs, with a single exception.

Theorem 2 ([4]). *If \vec{H}_t is an acyclic orientation of a clique or of a cycle with t vertices, then*

$$p_{\vec{H}_t} = \begin{cases} n^{-1/m(K_4)} & \text{if } t = 3, \\ n^{-1/m_2(H_t)} & \text{if } t \geq 4. \end{cases}$$

Let $\overrightarrow{\text{TT}}_3$ be the transitive tournament on 3 vertices. Theorem 2 with $t = 3$ tells us that $p_{\overrightarrow{\text{TT}}_3} = n^{-1/m(K_4)} = n^{-2/3} \ll n^{-1/2} = n^{-1/m_2(\overrightarrow{\text{TT}}_3)}$. Therefore, the heuristic mentioned above based on the classical colouring case fails for $\overrightarrow{\text{TT}}_3$. This phenomenon allows us to give a family of oriented graphs \vec{H} (which are not forests of stars) for which $p_{\vec{H}} \ll n^{-1/m_2(\vec{H})}$. In order to define this family, we consider *rooted oriented graphs* \vec{H} , that is, oriented graphs \vec{H} with a distinguished vertex r , called the *root*. Given an oriented graph \vec{F} , we denote by $\vec{F} \circ \vec{H}$ the *rooted product* of \vec{F} and \vec{H} , defined as $\vec{F} \circ \vec{H} = (V, E)$ where

$$\begin{aligned} V &= V(\vec{F}) \times V(\vec{H}), \text{ and} \\ E &= \left\{ ((f, r), (f', r)) : (f, f') \in E(\vec{F}) \right\} \cup \bigcup_{x \in V(\vec{F})} \left\{ ((x, h), (x, h')) : (h, h') \in E(\vec{H}) \right\}. \end{aligned}$$

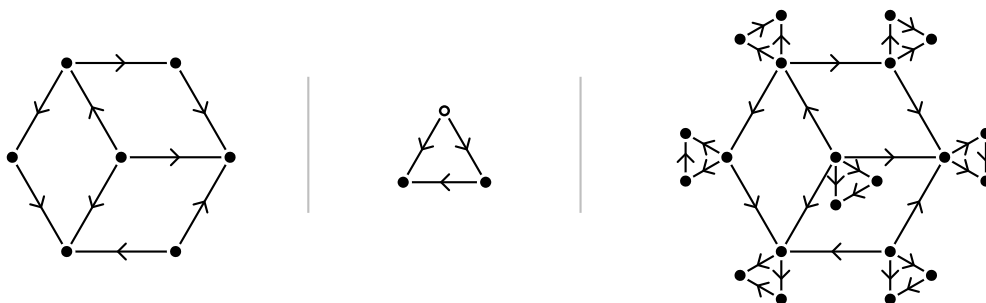


FIGURE 1. A triangle-free subcubic graph \vec{F} , a rooted $\vec{\text{TT}}_3$ and their rooted product $\vec{F} \circ \vec{\text{TT}}_3$.

See Figure 1 for an example of a rooted product. Rooted products have been considered before for undirected graphs [2].

In what follows, we shall consider oriented graphs of the form $F \circ \vec{H}$ with \vec{H} a rooted $\vec{\text{TT}}_3$. For brevity, $\vec{\text{TT}}'_3$ below stands for any rooted $\vec{\text{TT}}_3$. We prove that the rooted product $\vec{F} \circ \vec{\text{TT}}'_3$ for some oriented graphs \vec{F} is such that $p_{\vec{F} \circ \vec{\text{TT}}'_3} \ll n^{-1/m_2(\vec{F} \circ \vec{\text{TT}}'_3)}$.

Theorem 3. *Let \vec{F} be an acyclically oriented graph with $1 < m_2(\vec{F}) < 2$. Then*

$$p_{\vec{F} \circ \vec{\text{TT}}'_3} \ll n^{-1/m_2(\vec{F} \circ \vec{\text{TT}}'_3)}. \tag{1}$$

One can check that if F is a $\{K_3, K_{3,3}\}$ -free graph with maximum degree at most three that is not a forest, then any acyclic orientation \vec{F} of F satisfies the hypothesis of Theorem 3 (see Prop. 11 (b)), and hence (1) holds. While the hypothesis of Theorem 3 requires that F should contain a cycle ($m_2(\vec{F}) > 1$ is required), the conclusion of Theorem 3 also holds when F is a forest: to see this, it suffices to take $\delta < 1/6$ in the result below. In fact, the result below is more general in the sense that it allows us to consider forests F with growing order.

Theorem 4. *Let $p = n^{\delta-2/3}$ where $2/21 < \delta \leq 1/6$ is a constant. Then a.a.s. $G(n, p) \rightarrow \vec{F} \circ \vec{\text{TT}}'_3$ for every oriented forest \vec{F} with $v(\vec{F}) \leq bn^{7\delta-2/3}/(\log n)^2$, where b is some positive absolute constant.*

Theorem 4 guarantees the presence of somewhat large oriented trees in every orientation of $G(n, p)$. If $\delta < 1/6$, then for any oriented forest \vec{F} we have $n^{\delta-2/3} \ll n^{-1/2} = n^{-1/m_2(\vec{F} \circ \vec{\text{TT}}'_3)}$, and indeed Theorem 4 extends Theorem 3. Also, if $\delta > 2/21$, then $7\delta - 2/3 > 0$ and the forest \vec{F} of Theorem 4 may be chosen to have more than n^c vertices for some constant $c > 0$. And if $\delta = 1/6$, then $v(\vec{F} \circ \vec{\text{TT}}'_3) = \Omega(n^{1/2}/(\log n)^2)$.

Lemma 12 (a variant of Thm. 1) and Theorem 3 are proved in Section 3 using lemmas from Section 2. Theorem 4 is proved in Section 4.

Given a set V and a positive integer ℓ , we denote by $\binom{V}{\ell}$ the collection of ℓ -element subsets of V , by 2^V the collection of subsets of V , and by $[\ell]$ the set $\{1, 2, \dots, \ell\}$. To avoid uninteresting technicalities, we omit floor and ceiling signs whenever they are not important. All unqualified logarithms are base e (where e denotes Euler’s constant).

The main results of this work were announced in the extended abstract [3].

2. A CONTAINER LEMMA

Let \vec{H} be an acyclically oriented graph. In this section, we derive a container lemma for graphs G that admit \vec{H} -free orientations from the celebrated container method of Balogh *et al.* [1] and Saxton and Thomason [22].

2.1. Preliminaries

We begin with a simple saturation result, somewhat in the spirit of [10].

Lemma 5. *Let \vec{H} be an acyclically oriented graph on h vertices, and let $R := \vec{R}(\vec{H})$. The following holds for every $n \geq R$. For every $F \subseteq E(K_n)$, if there exists an orientation \vec{F} of F such that \vec{F} has at most $(2\binom{R}{h})^{-1} \binom{n}{h}$ copies of \vec{H} , then*

$$|E(K_n) \setminus F| \geq (2R^2)^{-1} n^2.$$

Proof. For convenience, let $\varepsilon := (2\binom{R}{h})^{-1}$. Let $F \subseteq E(K_n)$ be such that there exists an orientation \vec{F} of F with at most $\varepsilon \binom{n}{h}$ copies of \vec{H} . Let \vec{K} be an orientation of K_n which agrees with the orientation \vec{F} of F . Let

$$\mathcal{S} := \left\{ S \in \binom{V(\vec{K})}{R} : E(\vec{K}[S]) \subseteq \vec{F} \right\}.$$

That is, the family \mathcal{S} is the collection of all R -element subsets S of $V(\vec{K})$ such that every arc of $\vec{K}[S]$ is contained in \vec{F} . By definition of R , every R -element subset of the vertices of \vec{K} contains at least one copy of \vec{H} . This means that, for every $S \in \mathcal{S}$, there exists one copy of \vec{H} in $E(\vec{K}[S])$. Moreover, every copy of \vec{H} in \vec{K} is contained in at most $\binom{n-h}{R-h}$ R -element subsets. Double-counting the pairs (S, \vec{H}') where $S \in \mathcal{S}$ and \vec{H}' is a copy of \vec{H} contained in S yields

$$|\mathcal{S}| \leq \varepsilon \binom{n-h}{R-h} \binom{n}{h} = \frac{1}{2} \frac{\binom{n-h}{R-h}}{\binom{R}{h}} \binom{n}{h} = \frac{1}{2} \binom{n}{R}.$$

This implies that the set $\bar{\mathcal{S}}$ defined as $\bar{\mathcal{S}} := \binom{V(\vec{K})}{R} \setminus \mathcal{S}$ satisfies $|\bar{\mathcal{S}}| \geq (1/2) \binom{n}{R}$. Every set $S \in \bar{\mathcal{S}}$ induces at least one arc $e \in E(\vec{K}) \setminus \vec{F}$. Moreover, every arc $e \in E(\vec{K}) \setminus \vec{F}$ is contained in at most $\binom{n-2}{R-2}$ R -element subsets. Double-counting the pairs (S, e) where $S \in \bar{\mathcal{S}}$ and $e \in E(\vec{K}[S]) \setminus \vec{F}$ we obtain

$$|E(\vec{K}) \setminus \vec{F}| \geq \frac{|\bar{\mathcal{S}}|}{\binom{n-2}{R-2}} \geq \frac{1}{2} \frac{\binom{n}{R}}{\binom{n-2}{R-2}} \geq \frac{1}{2R^2} n^2.$$

The desired result now follows by observing that $|E(K_n) \setminus F| = |E(\vec{K}) \setminus \vec{F}|$. □

We now recall the hypergraph container lemma. We need to introduce some notation first. Let \mathcal{H} be an l -uniform hypergraph, and let $v \in V(\mathcal{H})$. For each $J \subseteq V(\mathcal{H})$, we call $d(J) := |\{e \in E(\mathcal{H}) : J \subseteq e\}|$ the *degree of J* , and write $d(v)$ for $d(\{v\})$. For each $j \in [l]$, the *maximum j -degree* of v is $d^{(j)}(v) := \max\{d(J) : v \in J \in \binom{V(\mathcal{H})}{j}\}$. We also let

$$d_j := \frac{1}{v(\mathcal{H})} \sum_{v \in V(\mathcal{H})} d^{(j)}(v),$$

and note that d_1 is the average degree of \mathcal{H} . Finally, for $\tau > 0$, let

$$\delta_j := \frac{d_j}{d_1 \tau^{j-1}}$$

and define the *co-degree function* $\delta(\mathcal{H}, \tau)$ by

$$\delta(\mathcal{H}, \tau) := 2^{\binom{l}{2}-1} \sum_{j=2}^l 2^{-\binom{j-1}{2}} \delta_j, \tag{2}$$

where $\binom{1}{2} = 0$. We shall apply the following hypergraph container lemma, given in [12].

Theorem 6 ([12], Thm. 2.1). *Let $0 < \varepsilon, \tau < 1/2$ and $l \geq 2$ be given. There exist integers $K = K(l)$ and $s = s(l, \varepsilon)$ such that the following holds. Let $\mathcal{H} = (V, E)$ be an l -uniform hypergraph and suppose τ is such that $\delta(\mathcal{H}, \tau) \leq \varepsilon/(12l!)$. Then, for every independent set $I \subseteq V$ in \mathcal{H} , there exist an s -tuple $T = (T_1, \dots, T_s)$ of subsets of V and a subset $C = C(T) \subseteq V$ depending only on T such that*

- (a) $\bigcup_{i \in [s]} T_i \subseteq I \subseteq C$,
- (b) $e(\mathcal{H}[C]) \leq \varepsilon e(\mathcal{H})$, and
- (c) $|T_i| \leq K\tau|V|$ for every $i \in [s]$.

Theorem 6 above is a version of Corollary 3.6 from [22]. For completeness, we mention that, in [12], explicit values are given for the constants $K = K(l)$ and $s = s(l, \varepsilon)$: $K = 800l(l!)^3$ and $s = \lfloor K \log(1/\varepsilon) \rfloor$. We now define the hypergraph on which we shall apply Theorem 6.

Definition 7. Let $n \in \mathbb{N}$ be given. The complete digraph \vec{D}_n is the digraph with vertex set $[n]$ and arc set $E(\vec{D}_n) := ([n] \times [n]) \setminus \{(v, v) : v \in [n]\}$.

Definition 8. Let \vec{H} be an oriented graph with l arcs and let $n \in \mathbb{N}$ be given. The hypergraph $\mathcal{D}(n, \vec{H}) = (\mathcal{V}, \mathcal{E})$ is the l -uniform hypergraph with vertex set $\mathcal{V} := E(\vec{D}_n)$ and edge set

$$\mathcal{E} := \left\{ B \in \binom{\mathcal{V}}{l} : \text{the arcs of } B \text{ form a digraph isomorphic to } \vec{H} \right\}.$$

We close this section estimating the quantity $\delta(\mathcal{D}(n, \vec{H}), \tau)$ defined in (2) for a certain relevant value of τ .

Lemma 9. *Let \vec{H} be an oriented graph with order h and $l \geq 2$ arcs. Let $\tau := Dn^{-1/m_2(\vec{H})}$, where $D \geq 1$ is a constant. We have*

$$\delta(\mathcal{D}(n, \vec{H}), \tau) \leq 2^{\binom{l}{2}} h^{h-2} D^{-1}. \tag{3}$$

Proof. For convenience, set $\mathcal{H} := \mathcal{D}(n, \vec{H})$. Note that \mathcal{H} is regular, i.e., $d(e) = d_1$ for all $e \in E(\vec{H})$. Given $J \subseteq V(\mathcal{H})$ with $J \neq \emptyset$ and $2 \leq |J| \leq l$, let

$$V_J := \bigcup_{(a,b) \in J} \{a, b\} \subseteq [n].$$

Note that (V_J, J) is the subdigraph of \vec{D}_n induced by the set of arcs J . Given $J \subseteq V(\mathcal{H})$ and an edge $e_0 \in J$, choose a hyperedge $F \in E(\mathcal{H})$ containing e_0 uniformly at random. By symmetry, $V_F \setminus e_0$ is a uniformly-chosen $(h - 2)$ -subset of $[n] \setminus e_0$. Since $J \subseteq F$ implies $V_J \subseteq V_F$, we have

$$\frac{d(J)}{d(e_0)} = \mathbb{P}(J \subseteq F) \leq \mathbb{P}(V_J \subseteq V_F) = \frac{\binom{n - |V_J|}{h - |V_J|} \binom{n - 2}{h - 2}^{-1}}{\binom{h}{h - 2}} \leq \left(\frac{h}{n}\right)^{|V_J| - 2}. \tag{4}$$

For all $\ell \in [l]$, let

$$f(\ell) := \min \left\{ v(\vec{H}') : \vec{H}' \subseteq \vec{H} \text{ and } e(\vec{H}') = \ell \right\}.$$

It follows from (4) that, for every $2 \leq j \leq l$, we have $d^{(j)}(v)/d_1 \leq (h/n)^{f(j) - 2}$ for any $v \in V(\mathcal{H})$. Since $f(j) \leq h$, this gives us that

$$\frac{d_j}{d_1} = \frac{1}{v(\mathcal{H})} \sum_{v \in V(\mathcal{H})} \frac{d^{(j)}(v)}{d_1} \leq \frac{1}{v(\mathcal{H})} \sum_{v \in V(\mathcal{H})} h^{f(j) - 2} n^{2 - f(j)} = h^{f(j) - 2} n^{2 - f(j)} \leq h^{h - 2} n^{2 - f(j)}.$$

We furthermore have that

$$\delta_j = \frac{d_j}{d_1 \tau^{j-1}} \leq h^{h-2} n^{2-f(j)} \tau^{1-j} \leq h^{h-2} n^{2-f(j)+(j-1)/m_2(\vec{H})} D^{1-j}. \tag{5}$$

Observe now that, by definition, we have $m_2(\vec{H}) \geq (j-1)/(f(j)-2)$ for all $j \geq 2$. From this we may derive $2-f(j)+(j-1)/m_2(\vec{H}) \leq 0$. Therefore, we can conclude from (5) that

$$\delta_j \leq h^{h-2} D^{1-j} \leq h^{h-2} D^{-1}.$$

We can finally bound the co-degree function $\delta(\mathcal{H}, \tau)$ by observing that

$$\delta(\mathcal{H}, \tau) = 2^{\binom{l}{2}-1} \sum_{j=2}^l 2^{-\binom{j-1}{2}} \delta_j \leq 2^{\binom{l}{2}-1} h^{h-2} D^{-1} \sum_{j=2}^l 2^{-\binom{j-1}{2}} \leq 2^{\binom{l}{2}} h^{h-2} D^{-1},$$

which establishes (3). □

2.2. A container lemma for graphs with \vec{H} -free orientations

Let \vec{H} be an acyclically oriented graph. Applying Theorem 6 to the hypergraph $\mathcal{D}(n, \vec{H})$ from Definition 8 gives us a container lemma for \vec{H} -free digraphs. We need something a little different: we need a container lemma for graphs G that admit \vec{H} -free orientations. This is given in the lemma below.

Lemma 10. *Let \vec{H} be an acyclically oriented graph with at least two arcs. There exist positive real numbers $\alpha < 1$ and c and a positive integer s such that the following holds for every large enough n . Let $r = \lfloor cn^{2-1/m_2(\vec{H})} \rfloor$. If G is a graph of order n and $G \not\rightarrow \vec{H}$, then there exist an s -tuple $T = (T_1, \dots, T_s) \in (2^{E(G)})^s$ and a set $\mathcal{C} = \mathcal{C}(T) \subseteq 2^{E(K_n)}$ of size at most 2^r , depending only on T , such that*

- (a) $\bigcup_{i \in [s]} T_i \subseteq E(G) \subseteq C$ for some $C \in \mathcal{C}$,
- (b) $\bigcup_{i \in [s]} T_i \subseteq C$ for every $C \in \mathcal{C}$.
- (c) $|E(K_n) \setminus C| \geq \alpha n^2$ for every $C \in \mathcal{C}$, and
- (d) $|\bigcup_{i \in [s]} T_i| \leq r$.

Proof. We apply Theorem 6. Suppose \vec{H} has h vertices and $l \geq 2$ arcs and let $\alpha = (2R^2)^{-1}$, where $R := \vec{R}(\vec{H})$. In what follows, we assume that n is large enough for our inequalities to hold. Set $\mathcal{H} := \mathcal{D}(n, \vec{H})$ and $\varepsilon := (2^{\binom{R}{h}} e(\mathcal{D}(h, \vec{H})))^{-1}$ and let $\tau := Dn^{-1/m_2(\vec{H})}$, where

$$D := \frac{12l! 2^{\binom{l}{2}} h^{h-2}}{\varepsilon}.$$

By Lemma 9, we have $\delta(\mathcal{H}, \tau) \leq \varepsilon/(12l!)$. Theorem 6 now gives us numbers $K = K(l)$ and $s = s(l, \varepsilon)$ satisfying the conclusion of that theorem.

Let G be a graph on n vertices such that $G \not\rightarrow \vec{H}$. There exists an orientation \vec{G} of G that is \vec{H} -free. Therefore, the set $E(\vec{G})$ is an independent set of \mathcal{H} . Let $\vec{T} = (\vec{T}_1, \dots, \vec{T}_s)$ be an s -tuple of sets of arcs and $\vec{C} = \vec{C}(\vec{T})$ a set of arcs such as given by Theorem 6 applied to $E(\vec{G})$. For $i \in [s]$, let T_i be the set of underlying (undirected) edges of the arcs in \vec{T}_i . Let

$$T = (T_1, \dots, T_s) \in (2^{E(K_n)})^s. \tag{6}$$

Define C analogously from \vec{C} , i.e., let C be the set of edges that we obtain by replacing each arc of \vec{C} by its underlying edge (we remark in passing that it may happen that $|C| < |\vec{C}|$, as \vec{C} may contain digons). By Theorem 6(a), we have

$$\bigcup_{i \in [s]} T_i \subseteq E(G) \subseteq C. \tag{7}$$

Clearly, $e(\mathcal{D}(h, \vec{H}))$ is the number of copies of \vec{H} in any subset of h vertices of \vec{D}_n , whence

$$e(\mathcal{H}) = \binom{n}{h} e(\mathcal{D}(h, \vec{H})).$$

Therefore, by Theorem 6 (b) we conclude that \vec{C} spans at most $\varepsilon e(\mathcal{H}) = (2 \binom{R}{h})^{-1} \binom{n}{h}$ copies of \vec{H} . Let \vec{F} be an orientation of C such that $\vec{F} \subseteq \vec{C}$ (equivalently, let \vec{F} be obtained from \vec{C} by dropping an arbitrary arc from each digon in \vec{C}). Note that the number of copies of \vec{H} in \vec{F} is at most the number of copies of \vec{H} in \vec{C} . Since the underlying graph of \vec{F} is C , Lemma 5 gives, by the choice of α , that

$$|E(K_n) \setminus C| \geq \alpha n^2. \tag{8}$$

Finally, by letting $c := sKD$, we get by Theorem 6 (c) that

$$\sum_{i \in [s]} |T_i| \leq s \max_{i \in [s]} |T_i| \leq \lfloor sK\tau v(\mathcal{H}) \rfloor \leq \lfloor cn^{2-1/m_2(\vec{H})} \rfloor = r. \tag{9}$$

We are supposed to produce a set $\mathcal{C} = \mathcal{C}(T) \subseteq 2^{E(K_n)}$ with $|\mathcal{C}| \leq 2^r$ satisfying (a), (b), (c) and (d) of our lemma. Note that, because of (7), if $C \in \mathcal{C}$, we are in good shape with respect to (a) in our lemma. Owing to (9), we are also fine with respect to (d) in our lemma. We shall now see how to produce \mathcal{C} .

We know that $\vec{C} = \vec{C}(\vec{T})$ depends solely on \vec{T} , and we have produced $C \subseteq E(K_n)$ from \vec{C} . Hence C depends only on \vec{T} . However, in the procedure above, T as defined in (6) may arise from other s -tuples $\vec{T}' = (\vec{T}'_1, \dots, \vec{T}'_s)$ with $\vec{C}(\vec{T}') \neq \vec{C}(\vec{T})$, and hence we cannot say that C is solely determined by T . The solution is simple: we consider all possible $\vec{T}' = (\vec{T}'_1, \dots, \vec{T}'_s)$ that give rise to T and for which $\vec{C}(\vec{T}')$ is defined, to produce the set \mathcal{C} of all the corresponding containers $C' \subseteq E(K_n)$ coming from $\vec{C}(\vec{T}')$. Owing to (9), the number of such \vec{T}' is at most 2^r (each edge $\{x, y\}$ in each T_i in T can be oriented as \vec{xy} or \vec{yx} to produce \vec{T}'_i in \vec{T}' and \vec{T}'_i cannot contain both \vec{xy} and \vec{yx}). Hence the set \mathcal{C} of the corresponding containers C' coming from all the $\vec{C}(\vec{T}')$ is such that $|\mathcal{C}| \leq 2^r$. Because of (8), condition (c) of our lemma is satisfied. Finally, any other s -tuple \vec{T}' that gives rise to T is such that $\bigcup_{i \in [s]} T_i = \bigcup_{i \in [s]} T'_i \subseteq C'$, where $C' \in \mathcal{C}$ is the container produced by \vec{T}' . This implies (b). Therefore, there is an s -tuple T and a set $\mathcal{C} = \mathcal{C}(T)$ that depends only on T with the desired requirements. \square

3. SMALL GRAPHS WITH LOW THRESHOLDS

We prove Theorem 3 in this section. We begin with a simple proposition.

Proposition 11. *The following statements hold.*

(a) *Let \vec{F} be an oriented graph with $m_2(\vec{F}) < 2$ and let $\vec{\text{TT}}'_3$ be a rooted $\vec{\text{TT}}_3$. Then*

$$m_2(\vec{F} \circ \vec{\text{TT}}'_3) = 2.$$

(b) *Let F be a $\{K_3, K_{3,3}\}$ -free graph F with $\Delta(F) \leq 3$ that is not a forest. Then*

$$1 < m_2(F) < 2.$$

Proof. Let J be a graph with $v(J) \geq 3$ and $\rho_2(J) = (e(J) - 1)/(v(J) - 2) < 2$ and let T be a K_3 with $|V(J) \cap V(T)| \leq 1$. Let $J' = J \cup T'$ where T' is a subgraph of T . One can check that $\rho_2(J') < 2$ (roughly speaking, J' can be obtained from J by adding h new vertices and at most $3h/2$ edges for some h , and hence $\rho_2(J') < 2$ follows from a simple calculation). If $v(J) = 2$, it is again true that $\rho_2(J') < 2$. These observations imply that $m_2(\vec{F} \circ \vec{\text{TT}}'_3) \leq 2$ for any \vec{F} as in (a) of our proposition. Since $m_2(\vec{F} \circ \vec{\text{TT}}'_3) \geq m_2(\vec{\text{TT}}'_3) = 2$, assertion (a) follows.

Now let F be as in (b). Since F contains a cycle, we have $m_2(F) > 1$. Using the fact that F is $\{K_3, K_{3,3}\}$ -free, one can check that $m_2(F) < 2$ if $v(F) \leq 6$. For any order t graph J with $\Delta(J) \leq 3$, we have $\rho_2(J) \leq f(t)$, where $f(t) = (3t/2 - 1)/(t - 2)$. It now suffices to observe that $f(6) = 2$ and that $f(t)$ is a strictly decreasing function. \square

We shall need the following variant of Theorem 1.

Lemma 12. *For every acyclically oriented graph \vec{H} with $m_2(\vec{H}) > 1/2$, there are constants B and $\beta > 0$ such that if $p = p(n) \geq Bn^{-1/m_2(\vec{H})}$, then*

$$\mathbb{P}(G(n, p) \not\rightarrow \vec{H}) \leq e^{-\beta pn^2}. \tag{10}$$

Proof. We first show that there exist constants A and $\gamma > 0$ such that if $q = q(n) = An^{-1/m_2(\vec{H})}$ (note the equality) then

$$\mathbb{P}(G(n, q) \not\rightarrow \vec{H}) \leq e^{-\gamma qn^2}. \tag{11}$$

Let \vec{H} as in the statement of our lemma be given. Let α, c and s be as given by Lemma 10 for \vec{H} . Examining the statement of Lemma 10, one sees that we may suppose that $c \geq e2^{s+2}$. Let A be such that

$$\frac{\log A}{A} \leq \frac{\alpha}{2c} \quad \text{and} \quad A \geq \frac{c}{2^{s-1}}, \tag{12}$$

and let $q = q(n) = An^{-1/m_2(\vec{H})}$. We show that if $\gamma = \alpha/2$, then (11) holds for every large enough n . Let $r := \lfloor cn^{2-1/m_2(\vec{H})} \rfloor$ and note that $r \rightarrow \infty$ as $n \rightarrow \infty$ because $m_2(\vec{H}) > 1/2$.

Let G be a graph on $[n]$ such that $G \not\rightarrow \vec{H}$. By Lemma 10, there exist an s -tuple $T = (T_1, \dots, T_s) \in (2^{E(K_n)})^s$ and a set $\mathcal{C}(T) \subseteq 2^{E(K_n)}$ with $|\mathcal{C}(T)| \leq 2^r$, depending only on T , such that Lemma 10 (a), (b), (c) and (d) hold. Let $C \in \mathcal{C}(T)$ be as in Lemma 10 (a). Then

$$\bigcup_{i \in [s]} T_i \subseteq E(G) \subseteq C \tag{13}$$

and

$$|E(K_n) \setminus C| \geq \alpha n^2.$$

Let $D_C := E(K_n) \setminus C$. Since $E(G) \subseteq C$, we have

$$E(G) \cap D_C = \emptyset. \tag{14}$$

For convenience, let \mathcal{G} be the family of all graphs G on $[n]$ such that $G \not\rightarrow \vec{H}$. Let us summarise what we did above: given $G \in \mathcal{G}$, we found certain objects $T = (T_1, \dots, T_s) \in (2^{E(K_n)})^s$, $\mathcal{C}(T) \subseteq 2^{E(K_n)}$, and $C \in \mathcal{C}(T)$. Let \mathcal{T}_n be the set of all T that arise in this fashion when we consider all G in \mathcal{G} . By construction, for each $T \in \mathcal{T}_n$, we have a certain associated family of sets $\mathcal{C}(T)$.

We now proceed as follows. For $T = (T_1, \dots, T_s) \in \mathcal{T}_n$, let

$$\mathcal{G}'_T := \{G \in \mathcal{G} : T_i \subseteq E(G) \text{ for all } i \in [s]\}.$$

For any $C \subseteq E(K_n)$, let

$$\mathcal{G}''_C := \{G \in \mathcal{G} : E(G) \cap D_C = \emptyset\}.$$

Our constructions above of $T \in \mathcal{T}_n$ and $C \in \mathcal{C}(T)$ given $G \in \mathcal{G}$ (see, in particular, (13) and (14)) show that

$$\mathcal{G} \subseteq \bigcup_{T \in \mathcal{T}_n} \bigcup_{C \in \mathcal{C}(T)} \mathcal{G}'_T \cap \mathcal{G}''_C.$$

For any $T \in \mathcal{T}_n$, the sets T_i occurring in T and the D_C for $C \in \mathcal{C}(T)$ are disjoint (see Lem. 10 (b)). Therefore, the events $\{G(n, q) \in \mathcal{G}'_T\}$ and $\{G(n, q) \in \mathcal{G}''_C\}$ are independent for any $T \in \mathcal{T}_n$ and any $C \in \mathcal{C}(T)$. We conclude that

$$\mathbb{P}(G(n, q) \in \mathcal{G}) \leq \sum_{T \in \mathcal{T}_n} \sum_{C \in \mathcal{C}(T)} \mathbb{P}(G(n, q) \in \mathcal{G}'_T) \mathbb{P}(G(n, q) \in \mathcal{G}''_C).$$

Since $|D_C| \geq \alpha n^2$ for every $C \in \bigcup_{T \in \mathcal{T}_n} \mathcal{C}(T)$, we have

$$\mathbb{P}(G(n, q) \in \mathcal{G}''_C) \leq (1 - q)^{\alpha n^2} \leq \exp(-\alpha n^2 q).$$

Moreover, we also have

$$\sum_{T \in \mathcal{T}_n} \mathbb{P}(G(n, q) \in \mathcal{G}'_T) \leq \sum_{T \in \mathcal{T}_n} q^{|\bigcup_{i \in [s]} T_i|}.$$

Since $|\mathcal{C}(T)| \leq 2^r$, it follows that

$$\mathbb{P}(G(n, q) \in \mathcal{G}) \leq 2^r \exp(-\alpha n^2 q) \sum_{T \in \mathcal{T}_n} q^{|\bigcup_{i \in [s]} T_i|}. \tag{15}$$

We now proceed to bound the sum in (15). For every integer k with $0 \leq k \leq r$, let

$$\mathcal{T}_n(k) := \left\{ T = (T_1, \dots, T_s) \in \mathcal{T}_n : \left| \bigcup_{i \in [s]} T_i \right| = k \right\}.$$

Note that $\mathcal{T}_n = \bigcup_{0 \leq k \leq r} \mathcal{T}_n(k)$. Observe that $|\mathcal{T}_n(k)| \leq \binom{n}{k} (2^s - 1)^k$. Indeed, there are $\binom{n}{k}$ ways of choosing k edges from $E(K_n)$, and $(2^s - 1)^k$ ways of assigning each of those edges to the sets T_i of the s -tuples $T = (T_1, \dots, T_s)$. Therefore,

$$\sum_{T \in \mathcal{T}_n} q^{|\bigcup_{i \in [s]} T_i|} = \sum_{k=0}^r |\mathcal{T}_n(k)| q^k \leq \sum_{k=0}^r \binom{n}{k} (2^s)^k q^k \leq 1 + \sum_{k=1}^r \left(\frac{e 2^{s-1} n^2 q}{k} \right)^k. \tag{16}$$

Let $b = 2^{s-1} n^2 q$ and $f(x) = (eb/x)^x$ for all $x > 0$. Observe that f is unimodal and achieves its maximum at $x = b$. Moreover, by (12) we obtain

$$r = \left\lfloor cn^{2-1/m_2(\vec{H})} \right\rfloor \leq 2^{s-1} A n^{2-1/m_2(\vec{H})} = 2^{s-1} n^2 q = b.$$

Thus, by (16),

$$\sum_{T \in \mathcal{T}_n} q^{|\bigcup_{i \in [s]} T_i|} \leq 1 + r \left(\frac{eb}{r} \right)^r = 1 + r \left(\frac{e 2^{s-1} n^2 q}{\lfloor cn^{2-1/m_2(\vec{H})} \rfloor} \right)^r \leq 2r \left(\frac{e 2^s A}{c} \right)^r. \tag{17}$$

Recalling (15), we obtain

$$\begin{aligned} \mathbb{P}(G(n, q) \in \mathcal{G}) &\leq 2^{r+1} r e^{-\alpha n^2 q} \left(\frac{e 2^s A}{c} \right)^r \leq e^{-\alpha n^2 q} \left(\frac{e 2^{s+2} A}{c} \right)^r \\ &\leq A^r e^{-\alpha n^2 q} \leq \left(A^{cn^{2-1/m_2(\vec{H})}/n^2 q} e^{-\alpha} \right)^{n^2 q} = \left(A^{c/A} e^{-\alpha} \right)^{n^2 q}. \end{aligned} \tag{18}$$

Owing to (12), we have $A^{c/4}e^{-\alpha} \leq e^{-\alpha/2}$. Thus, equation (18) tells us that

$$\mathbb{P}(G(n, q) \in \mathcal{G}) \leq e^{-(\alpha/2)n^2q} = e^{-\gamma n^2q},$$

as promised in (11).

To conclude the proof, we show, using a standard multiple exposition argument, that Lemma 12 follows from (11) with $B = 2A$ and $\beta = \gamma/2$. Let $p \geq Bn^{-1/m_2(\vec{H})} = 2q$. Note that $G \not\rightarrow \vec{H}$ is a decreasing property in G . Let $t = \lfloor p/q \rfloor \geq \lfloor B/A \rfloor = 2$ and let $G = G_1 \cup \dots \cup G_t$, where each G_i is an independent copy of $G(n, q)$. It is a simple fact that G is a $G(n, p')$ with $p' \leq p$. Also, since $p/q \geq 2$, we have that $t \geq p/q - 1 \geq p/2q$. Thus

$$\begin{aligned} \mathbb{P}(G(n, p) \not\rightarrow \vec{H}) &\stackrel{(\dagger)}{\leq} \mathbb{P}(G \not\rightarrow \vec{H}) \stackrel{(\dagger)}{\leq} \mathbb{P}\left(\bigcap_{i \in [t]} \{G_i \not\rightarrow \vec{H}\}\right) \stackrel{(\sharp)}{=} \prod_{i \in [t]} \mathbb{P}(G_i \not\rightarrow \vec{H}) \stackrel{(11)}{\leq} \exp(-t\alpha q n^2) \\ &\leq \exp(-\alpha p n^2/2) \\ &= \exp(-\beta p n^2), \end{aligned}$$

where (\dagger) follow by monotonicity and (\sharp) follows from the independence of the G_i . This concludes the proof of Lemma 12. \square

The strategy for proving Theorem 3 is simple: we find many vertex-disjoint copies of $\vec{\text{TT}}_3$, and then locate a copy of \vec{F} in the subgraph induced by the roots of those copies. For the first step, since $K_4 \rightarrow \vec{\text{TT}}_3$, it suffices to find many vertex-disjoint copies of K_4 .

Lemma 13. *If $0 < \delta \leq 1/6$ and $p \geq n^{\delta-1/m(K_4)} = n^{\delta-2/3}$, then a.a.s. $G(n, p)$ contains at least $cn^{6\delta}$ vertex-disjoint copies of K_4 , where $c > 0$ is an absolute constant.*

Lemma 13 is a particular case of Theorem 4 in [14] (see also [13], Thm. 3.29). For the second step in the proof of Theorem 3, we apply Lemma 12.

Proof of Theorem 3. Let \vec{F} be as in the statement of the theorem. By Proposition 11 (a), we have $m_2(\vec{F} \circ \vec{\text{TT}}'_3) = 2$. We will show that there exists

$$p \ll n^{-1/2} = n^{-1/m_2(\vec{F} \circ \vec{\text{TT}}'_3)} \tag{19}$$

such that a.a.s. we have $G(n, p) \rightarrow \vec{F} \circ \vec{\text{TT}}'_3$. Since $m_2(\vec{F}) < 2$, we can fix positive constants γ and ε such that

$$1 - \gamma = \frac{1}{2}m_2(\vec{F}) + \varepsilon > \frac{1}{2}m_2(\vec{F}). \tag{20}$$

Let

$$\delta = \frac{1}{6} \left(1 - \frac{6\varepsilon}{m_2(\vec{F})} \right). \tag{21}$$

By choosing ε small enough, we may suppose that

$$6\delta \geq 1 - \gamma. \tag{22}$$

Let B and β be the constants given by Lemma 12 for \vec{F} , and let c be the constant in Lemma 13. Let $B' = c^{-1/m_2(\vec{F})}B$. We may assume that $B' \geq 1$. Let $p = p(n) = B'n^{-(1-\gamma)/m_2(\vec{F})}$ and note that, by (20), relation (19) holds. We show that this choice of p will do.

We first claim that, a.a.s., $G = G(n, p)$ is such that

(A) any $U \subseteq V(G)$ with $|U| \geq cn^{6\delta}$ is such that $G[U] \rightarrow \vec{F}$.

Indeed, let $t = cn^{1-\gamma}$ and note that $p = B'n^{-(1-\gamma)/m_2(\vec{F})} = B'c^{1/m_2(\vec{F})}t^{-1/m_2(\vec{F})} = Bt^{-1/m_2(\vec{F})}$. By our choice of B and β , for each $W \subset V(G)$ with $|W| = t$, we have

$$\mathbb{P}\left(G[W] \not\rightarrow \vec{H}\right) \leq e^{-\beta pt^2}. \tag{23}$$

Moreover, the number of $W \subset V(G)$ with $|W| = t$ is $\binom{n}{t} \leq n^t$. Since $m_2(\vec{F}) > 1$, we see that

$$\beta pt^2 = \beta t B'n^{-(1-\gamma)/m_2(\vec{F})} cn^{1-\gamma} = \beta B'ctn^{(1-\gamma)(1-1/m_2(\vec{F}))} \gg t \log n. \tag{24}$$

Therefore, inequalities (23) and (24) and the union bound tell us that, a.a.s., for every $W \subset V(G)$ with $|W| = t$, we have $G[W] \rightarrow \vec{F}$. This implies that $G[U] \rightarrow \vec{F}$ for every U with $|U| \geq cn^{6\delta}$, because $t = cn^{1-\gamma}$ and we have (22), concluding the proof of claim (A).

Now note that, by (20), we have $p = B'n^{-(1-\gamma)/m_2(\vec{F})} \geq n^{-(1-\gamma)/m_2(\vec{F})} = n^{-1/2-\varepsilon/m_2(\vec{F})}$. By (21), we have $-1/2 - \varepsilon/m_2(\vec{F}) = \delta - 2/3 = \delta - 1/m(K_4)$. Thus, Lemma 13 tells us that, a.a.s.,

(B) G contains at least $cn^{6\delta}$ vertex-disjoint copies of K_4 .

Suppose G satisfies (A) and (B). We show that $G \rightarrow \vec{F} \circ \overrightarrow{\text{TT}}'_3$. Since (A) and (B) hold a.a.s., this will conclude the proof. Let \vec{G} be an orientation of G . Noticing that $K_4 \rightarrow \overrightarrow{\text{TT}}_3$, property (B) implies that \vec{G} contains $cn^{6\delta}$ vertex-disjoint copies of $\overrightarrow{\text{TT}}'_3$. Let $U \subset V(G)$ be the set of $cn^{6\delta}$ roots of those copies of $\overrightarrow{\text{TT}}'_3$. Since (A) holds, there is a copy of \vec{F} in $\vec{G}[U]$. Therefore, \vec{G} contains a copy of $\vec{F} \circ \overrightarrow{\text{TT}}'_3$. This shows that $G \rightarrow \vec{F} \circ \overrightarrow{\text{TT}}'_3$, as required. \square

4. ROOTED PRODUCTS WITH LARGER TREES

We now prove Theorem 4. We shall use the following corollary of Theorem 3 from [18].

Theorem 14 ([18]). *If G is a graph and \vec{T} is an oriented tree of order $\lfloor \chi(G)/\lceil \log_2 v(G) \rceil \rfloor$, then $G \rightarrow \vec{T}$.*

Proof of Theorem 4. Let $b = c(\log 2)/4$, where c is the constant in Lemma 13. Recall that $p = n^{\delta-2/3}$ and that $2/21 < \delta \leq 1/6$. Let \vec{F} be an oriented tree with $v(\vec{F}) \leq bn^{7\delta-2/3}/(\log n)^2$. As in the proof of Theorem 3, it suffices to prove that, for $p = n^{\delta-2/3}$ as in the statement of Theorem 4, a.a.s., $G = G(n, p)$ has the following two properties:

- (A) $G[U] \rightarrow \vec{F}$ for any $U \subseteq V(G)$ with $|U| \geq cn^{6\delta}$.
- (B) G contains at least $cn^{6\delta}$ vertex-disjoint copies of K_4 .

The fact that (B) holds a.a.s. is Lemma 13. We derive that (A) holds a.a.s. from Theorem 14. An easy application of the first moment method shows that a.a.s. $\alpha(G) < 3p^{-1} \log n$, and hence we suppose that this inequality holds for G . Fix $U \subseteq V(G)$ with $|U| \geq cn^{6\delta}$. Then

$$\chi(G[U]) \geq \frac{|U|}{\alpha(G(n, p))} > \frac{cn^{6\delta}p}{3 \log n} = \frac{cn^{7\delta-2/3}}{3 \log n}. \tag{25}$$

Thus, for any large enough n ,

$$\left\lfloor \frac{\chi(G[U])}{\lceil \log_2 |U| \rceil} \right\rfloor \stackrel{(25)}{>} \frac{cn^{7\delta-2/3}}{3(\log n)(\log_2 n + 1)} - 1 \geq \left(\frac{c \log 2}{4} \right) \frac{n^{7\delta-2/3}}{(\log n)^2} = \frac{bn^{7\delta-2/3}}{(\log n)^2}.$$

Therefore $v(\vec{F}) \leq bn^{7\delta-2/3}/(\log n)^2 \leq \chi(G[U])/\lceil \log_2(G[U]) \rceil$ and Theorem 14 tells us that, indeed, $G[U] \rightarrow \vec{F}$. \square

5. CONCLUDING REMARKS

We believe that finding the threshold $p_{\vec{H}}$ for the property $G(n, p) \rightarrow \vec{H}$ for a given acyclically oriented graph \vec{H} is a natural problem. Methods developed in this general line of research show that $p_{\vec{H}} \leq n^{-1/m_2(\vec{H})}$, but we have shown here that we may have $p_{\vec{H}} \ll n^{-1/m_2(\vec{H})}$ for certain \vec{H} . It would be interesting to characterise those \vec{H} for which $p_{\vec{H}} = n^{-1/m_2(\vec{H})}$.

A related problem is establishing sharp thresholds for the case in which \vec{H} is a fixed-size oriented tree \vec{T} . Indeed, it can be shown that $p_{\vec{T}} = \Theta(1/n)$ (this follows from a theorem of Burr [6]: if \vec{T} is an oriented tree of order t and G is a graph with chromatic number $(t - 1)^2$, then $G \rightarrow \vec{T}$). And yet, determining the precise constant for an arbitrary tree might be challenging, possibly even for paths. We remark that the orientation of \vec{T} plays a role in the threshold, as distinct orientations of a t -vertex path can be shown to have distinct thresholds (this shall be discussed in future work by a subset of the authors).

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