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A TECHNIQUE TO SOLVE TRANSSHIPMENT PROBLEM WITH ASYMMETRIC PENTAGONAL FUZZY NUMBERS

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Abstract. In order to draw comparison in largely used fuzzy numbers, ranking of fuzzy numbers becomes important. In this paper, a new method for ranking Asymmetric Pentagonal Fuzzy Numbers (APFN) is proposed. In this method, the pentagonal fuzzy numbers are scored by obtaining their centers of gravity using left and right areas associated with them. Further, we propose a direct technique for solving fuzzy Transshipment Problem (TsP) in which costs are represented as Asymmetric Pentagonal Fuzzy Numbers. The solution so obtained is fuzzy optimal solution. The validity of proposed methodology is illustrated through numerical examples.

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1. Introduction

The cost of moving commodities from origins to destinations is minimized in conventional transportation problems (TP). But occasionally, the products may reach a particular location either directly or *via* intermediary nodes. These nodes can each act as a supply or demand node. Such problems are referred to as Transshipment Problems (TsP). TsP's are a unique class of problems in which, in order to minimize total cost, commodities are moved from one source to another through a number of intermediate nodes (sources/destinations). Finding the least transportation cost that will satisfy supply and demand at every node is the basic goal of a transshipment problem. Such problems have found several applications in the e-commerce era, when the online retail sector is flourishing by making purchase easy for consumers.

TsP are a form of TPs. In order to address the transshipment problems, we must first transform it into an equivalent TP. From there, we may use existing methodologies to identify a fundamental, workable solution to the problem. Furthermore, the ideal solution of equivalent TP can be derived by applying the modified distribution (MODI) approach. The concept of the transshipment problem was given by Orden [18]. He developed the classical TP theory to define the transshipment problem.

The solution methods continued to advance thanks to numerous researchers. Rhody [20] gave a technique by taking a reduced matrix form of TsP's. King and Logan [14] gave a technique for TsP by using an iterative

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TABLE 1. Literature review of pentagonal fuzzy numbers.

Authors	Membership function	Contribution	Application
Srinivasan *et al.* [22]	Linear	Developed direct algorithm to solve transportation problem	Application in transportation problem
Karthikeyan Natarajan [23]	Linear	Defined Centroid based ranking function	Application in transportation problem
Maheswari and Ganesan [15]	Linear	Algorithm to solve fully fuzzy transportation problem	Application in transportation problem
Chakraborty *et al.* [3]	Linear and Non-Linear	Multiple Ranking Approaches	Mathematical modelling, Operational research problems
Mondal *et al.* [16]	Linear	Properties of Pentagonal fuzzy numbers	Application in Fuzzy equation

procedure for obtaining the minimum cost solution. Hurt and Tramel [7] gave a technique without subtracting an artificial variable. Judge *et al.* [8] converted TsP into classical linear programming problem (LPP). They put forth a method in order to solve transshipment problems involving multiple products, plants, and regions. Time minimising TsP was discussed by Garg and Prakash [5]. Their suggested algorithm reduced the amount of time required to transship goods from different sources to different destinations. Here and Tzur [6] studied dynamic TsP. Khurana and Arora [11–13] gave different methodologies to solve TsP.

Knowledge of parameters like demand, supply, associated costs, time, location etc. is necessary when formulating transshipment problems. Incase these parameters are precisely known, the transshipment problem can be solved using standard mathematical methods. However, in real-world scenarios, there are variety of uncertainties that arise when formulating a transshipment problem mathematically because of factors like incomplete or unavailable information, traffic congestion, or weather conditions. The best way to deal with these uncertainties is to denote these parameters in fuzzy form. Zadeh [24] described concept of fuzzy set theory, which is applied when problem parameters have ambiguous data. Fuzzy transshipment problem refers to TsP discussed in fuzzy environment. According to Zimmerman [25], a problem with fuzzy parameters has an efficient solution. Mondal [1] dealt with the uncertainty in optimal harvesting modelling by choosing different possible combinations of fuzzy initial conditions and coefficients to form the fuzzy equation.

In classical fuzzy theory, different types of polygon fuzzy numbers exist namely triangular fuzzy numbers, trapezoidal fuzzy numbers, pentagonal fuzzy numbers and so on. Throughout the literature, Srinivasan [22] gave a technique to solve TP by generalised pentagonal and hexagonal fuzzy numbers. Further, Karthikeyan [23] described a procedure to solve TP by pentagonal fuzzy number. A method to solve fully fuzzy TP using pentagonal fuzzy numbers is given by Maheswari and Ganesan [15]. Pentagonal fuzzy numbers and its representations have been discussed by Chakraborty [3]. The ranking method for generalised fuzzy numbers has been discussed by Barazandeh and Ghazanfari [2]. Khan and Mondal [9] generalised the idea of membership functions for both symmetric and asymmetric hexagonal fuzzy numbers to non-linear membership functions. Mondal [17] gave representations to various non-linear membership functions and found arithmetic operations using the average method. Some of the major contributions solving transportation problems pentagonal fuzzy numbers have been given in Table 1.

The articles in the literature transform the Fuzzy TsP into an equivalent fuzzy TP, find a basic feasible solution using conventional techniques, and then use the MODI (modified distribution) method to find an optimal solution. To the best of our knowledge, no one has created a technique that provides a fuzzy optimal solution to the transshipment problem directly. The underlying motivation behind this article is the ability to obtain a fuzzy optimal solution without first finding a basic feasible solution. Thus, a fuzzy cost transshipment problem is examined in this paper with costs represented as Asymmetric Pentagonal Fuzzy Numbers (APFN).

In order to defuzzify the problem, a ranking method based on centroid ranking technique is proposed. These indices are then used to score fuzzy numbers by using the area parameters to derive a fuzzy optimal solution. The technique to obtain fuzzy optimal solution to fuzzy TsP described in this paper has the advantage that without first locating a basic feasible solution, it directly generates a fuzzy optimal solution, making it computationally more efficient. The technique is easily applied to problems involving trapezoidal and triangular fuzzy numbers in fuzzy transshipment problems.

The rest of the paper has been organized as follows: Section 2 gives basic mathematical formulation of fuzzy transshipment problems. Section 3 gives some basic preliminaries of Fuzzy Numbers and its various forms. The proposed method for ranking Asymmetric Pentagonal Fuzzy Numbers based on score function has been discussed in Section 4. In Section 5, a technique for solving fuzzy TsP using proposed ranking method is presented. In Section 6, two numerical examples are provided to illustrate the proposed approach. The conclusion of the work is drawn towards the end.

2. Transshipment problem

A classical Transportation Problem uses the direct shipments, *i.e.*, ships a commodity from sources to the destinations; however, the commodity can reach the desired destination through one or more intermediate nodes. Sequentially, these nodes are further furnished by more intermediate nodes. Thus, the above transshipment of commodity which minimizes the overall transshipment cost and meets the demand and supply limits of the sources and destinations, is known as a TsP. In a classical TsP, back and forth transshipment, including source to source and destination to destination, occurs between the sources and destinations. In most of the cases, transportation cost gets higher in the absence of transshipment. In many applications, allowing transshipments to exchange the data from the supply nodes or demand nodes reduces the transshipment cost. Therefore, transshipment plays an essential role in reducing the overall transportation cost. The mathematical model of TsP used by Khurana [12] is:

$$\text{Min } z = \sum_{i=1}^{m+n} \sum_{j=1}^{m+n} c_{ij} x_{ij} \tag{TsP}$$

subject to

$$\sum_{\substack{j=1\\ j\neq i}}^{m+n} x_{ij} - \sum_{\substack{j=1\\ j\neq i}}^{m+n} x_{ji} = a_i \qquad \forall i = 1, 2, \ldots, m$$

$$\sum_{\substack{i=1\\ i\neq j}}^{m+n} x_{ij} - \sum_{\substack{i=1\\ i\neq j}}^{m+n} x_{ji} = b_j \qquad \forall j = m+1, m+2, \ldots, m+n$$

$$x_{ij} \geq 0 \qquad \forall i, j = 1, 2, \ldots, m+n$$

where a_i denotes the amount shipped by each node, b_j denotes the amount received at each node, c_{ij} be the unit cost of shipment from node i to node j and x_{ij} be the shipments of the commodity from node i to j.

If the cost, supply and demand parameters in the above problem become fuzzy in nature, then problem (TsP) translates to Fuzzy TsP as depicted below:

$$\text{Min } \tilde{z} = \sum_{i=1}^{m+n} \sum_{j=1}^{m+n} \tilde{c_{ij}} \tilde{x_{ij}} \tag{FTsP}$$

subject to

$$\sum_{\substack{j=1\\ j\neq i}}^{m+n} \tilde{x_{ij}} - \sum_{\substack{j=1\\ j\neq i}}^{m+n} \tilde{x_{ji}} = \tilde{a}_i \qquad \forall i = 1, 2, \ldots, m$$

$$\sum_{\substack{i=1\\i\neq j}}^{m+n} \tilde{x_{ij}} - \sum_{\substack{i=1\\i\neq j}}^{m+n} \tilde{x_{ji}} = \tilde{b_j} \qquad \forall j = m+1, m+2, \ldots, m+n$$

$$\tilde{x_{ij}} \geq 0 \qquad \forall i,j = 1,2,\ldots,m+n$$

where $\tilde{a_i}$ denotes the fuzzy amount shipped by each node, $\tilde{b_j}$ denotes the fuzzy amount received at each node, $\tilde{c_{ij}}$ be the fuzzy unit cost of shipment from node i to node j and $\tilde{x_{ij}}$ be the fuzzy shipments of the commodity from node i to j.

3. Preliminaries

This section provides some basic definitions of fuzzy numbers and their various forms, which are used throughout the article.

Definition 3.1 ([21])**.** A *Fuzzy Set* $\tilde{A}$ in the universal set X is defined by a set of ordered pairs $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \in X\}$ where $\mu_{\tilde{A}}(x) \in [0,1]$ is the membership degree of x in $\tilde{A}$ and is called the *membership function* of $\tilde{A}$.

Definition 3.2 ([21])**.** A *Triangular Fuzzy Number* on the real line $\mathbb{R}$ is a fuzzy subset $\tilde{A} = (a_1, a_2, a_3)$ with parameters $a_1 \leq a_2 \leq a_3$ is a fuzzy set with the membership function:

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & x \leq a_1, \\ \frac{x-a_1}{a_2-a_1}, & a_1 \leq x \leq a_2, \\ \frac{a_3-x}{a_3-a_2}, & a_2 \leq x \leq a_3, \\ 0, & x \geq a_3. \end{cases}$$

Definition 3.3 ([21] A Trapezoidal Fuzzy Number (TrFN))**.** $\tilde{A} = (a_1, a_2, a_3, a_4)$ with parameters $a_1 \leq a_2 \leq a_3 \leq a_4$ is a fuzzy set with the membership function:

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{x-a_1}{a_2-a_1}, & a_1 < x \leq a_2, \\ 1, & a_2 \leq x \leq a_3, \\ \frac{a_4-x}{a_4-a_3}, & a_3 \leq x < a_4, \\ 0, & \text{otherwise.} \end{cases}$$

Definition 3.4 ([4] α-cut of a Fuzzy Number)**.** Let $\tilde{A}$ be a fuzzy number. The set $\tilde{A}_\alpha = \{x \in \mathbb{R} : \mu_{\tilde{A}}(x) \geq \alpha\}$ is called the *α-cut* of fuzzy number $\tilde{A}$ for any real number α in the interval $[0,1]$.

Definition 3.5 ([16,19] *Pentagonal Fuzzy Number with k level α-cut*)**.** A fuzzy number $A = (a_1, a_2, a_3, a_4, a_5)$ where $a_1 \leq a_2 \leq a_3 \leq a_4 \leq a_5$ are real numbers is called a pentagonal fuzzy number with k level α-cut when the membership function has the form:

$$\mu_A(x) = \begin{cases} 0 & x \leq a_1 \\ k\left(\frac{x-a_1}{a_2-a_1}\right) & \text{if} \quad a_1 \leq x \leq a_2, \\ k + (1-k)\left(\frac{x-a_2}{a_3-a_2}\right) & \text{if} \quad a_2 \leq x \leq a_3 \\ 1 & \text{if} \quad x = a_3 \\ k + (1-k)\left(\frac{a_4-x}{a_4-a_3}\right) & \text{if} \quad a_3 \leq x \leq a_4 \\ k\left(\frac{a_5-x}{a_5-a_4}\right) & \text{if} \quad a_4 \leq x \leq a_5, \\ 0 & \text{if} \quad x \geq a_5 \end{cases}$$

where k is a real number in interval $[0,1]$.

The membership function of PFN mentioned above satisfies the following conditions:

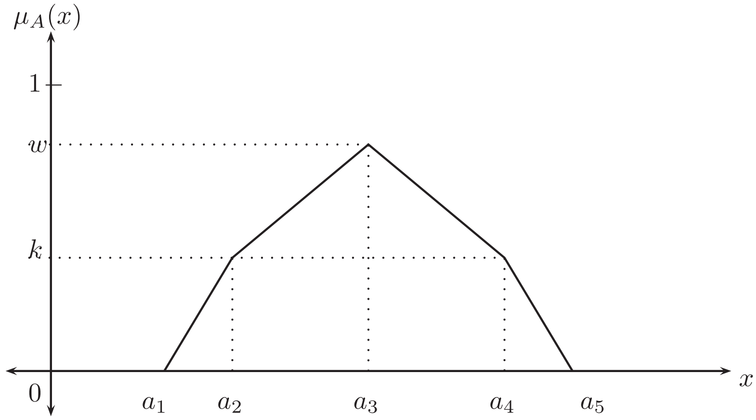

FIGURE 1. Generalised PFN with k-level α-cut.

(1) $\mu_{\tilde{A}}(x)$ is continuous in $[0,1]$.
(2) $\mu_{\tilde{A}}(x)$ is strictly non-decreasing continuous function on the intervals $[a_1,a_2]$ and $[a_2,a_3]$.
(3) $\mu_{\tilde{A}}(x)$ is strictly non-increasing continuous function on the intervals $[a_3,a_4]$ and $[a_4,a_5]$.

The concept of Generalised Fuzzy numbers with different left and right heights was introduced by Garg and Prakash [5]. Generalised Pentagonal Fuzzy Number with k level α-cut defined by Pathinathan [19] is as follows:

Definition 3.6 ([19] Generalised Pentagonal Fuzzy Number with k level α-cut)**.** A fuzzy number $A = (a_1,a_2,a_3,a_4,a_5;k,w)$ is called a generalised pentagonal fuzzy number when the membership function has the form:

$$\mu_A(x) = \begin{cases} 0 & x \le a_1 \\ k\left(\frac{x-a_1}{a_2-a_1}\right) & \text{if} \quad a_1 \le x \le a_2, \\ k+(w-k)\left(\frac{x-a_2}{a_3-a_2}\right) & \text{if} \quad a_2 \le x \le a_3 \\ w & \text{if} \quad x = a_3 \\ k+(w-k)\left(\frac{a_4-x}{a_4-a_3}\right) & \text{if} \quad a_3 \le x \le a_4 \\ k\left(\frac{a_5-x}{a_5-a_4}\right) & \text{if} \quad a_4 \le x \le a_5, \\ 0 & \text{if} \quad x \ge a_5 \end{cases}$$

where a_1,a_2,a_3,a_4,a_5 are real numbers and $0 < w \le 1$ is the new core (not necessarily 1). Figure 1 depicts Generalised PFN with k-level α-cut.

Remark 3.1. Pentagonal fuzzy numbers can be reduced to trapezoidal and triangular as follows:

– If $k = 0$ then the pentagonal fuzzy number reduces to triangular fuzzy number.
– If $k = w = 1$ then the pentagonal fuzzy number reduces to trapezoidal fuzzy number.
– If $k = w = 0$ then pentagonal fuzzy number becomes a constant on the real line.

Asymmetric pentagonal fuzzy numbers have been discussed in detail in literature by Chakraborty [3]. A modified version of the definition is as follows:

Definition 3.7 (Asymmetric Pentagonal Fuzzy Number (APFN))**.** A fuzzy number $A = (a_1,a_2,a_3,a_4,a_5;w_L,w_R)$ where a_1,a_2,a_3,a_4,a_5 are real numbers is called asymmetric pentagonal fuzzy

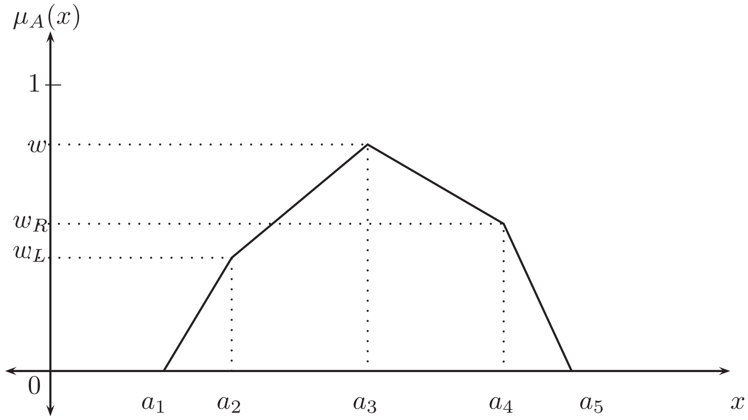

FIGURE 2. Generalised APFN.

number when the membership function has the form:

$$\mu_A(x) = \begin{cases} 0 & x \leq a_1 \\ w_L\left(\frac{x-a_1}{a_2-a_1}\right) & \text{if} \quad a_1 \leq x \leq a_2, \\ w_L + (1-w_L)\left(\frac{x-a_2}{a_3-a_2}\right) & \text{if} \quad a_2 \leq x \leq a_3 \\ 1 & \text{if} \quad x = a_3 \\ w_R + (1-w_R)\left(\frac{a_4-x}{a_4-a_3}\right) & \text{if} \quad a_3 \leq x \leq a_4 \\ w_R\left(\frac{a_5-x}{a_5-a_4}\right) & \text{if} \quad a_4 \leq x \leq a_5, \\ 0 & \text{if} \quad x \geq a_5 \end{cases}$$

where w_L and w_R denotes left and right heights such that $w_L \neq w_R$.

Definition 3.8 (Generalised Asymmetric Pentagonal Fuzzy Number (APFN))**.** A fuzzy number $A = (a_1, a_2, a_3, a_4, a_5; w_L, w_R, w)$ where a_1, a_2, a_3, a_4, a_5 are real numbers is called asymmetric pentagonal fuzzy number when the membership function has the form:

$$\mu_A(x) = \begin{cases} 0 & x \leq a_1 \\ w_L\left(\frac{x-a_1}{a_2-a_1}\right) & \text{if} \quad a_1 \leq x \leq a_2, \\ w_L + (w-w_L)\left(\frac{x-a_2}{a_3-a_2}\right) & \text{if} \quad a_2 \leq x \leq a_3 \\ w & \text{if} \quad x = a_3 \\ w_R + (w-w_R)\left(\frac{a_4-x}{a_4-a_3}\right) & \text{if} \quad a_3 \leq x \leq a_4 \\ w_R\left(\frac{a_5-x}{a_5-a_4}\right) & \text{if} \quad a_4 \leq x \leq a_5, \\ 0 & \text{if} \quad x \geq a_5 \end{cases}$$

where w_L and w_R denotes left and right heights such that $w_L \neq w_R$ and $0 < w \leq 1$ is the new core. The generalised APFN is depicted in Figure 2.

The arithmetic operations on PFNs introduced by Pathinathan [19] are as follows.

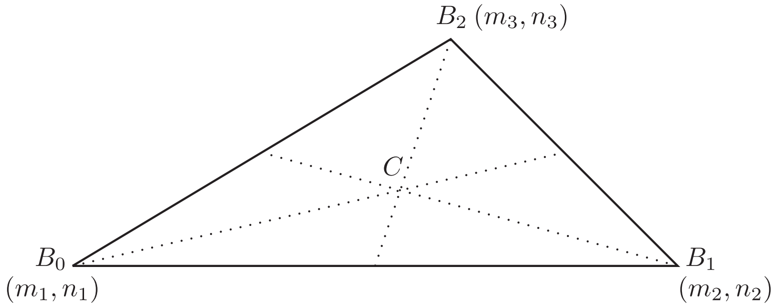

FIGURE 3. Centre of gravity of triangle $B_0B_1B_2$.

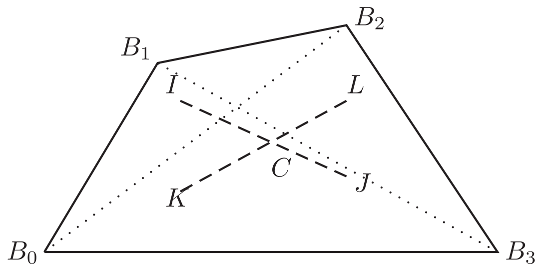

FIGURE 4. Center of gravity of quadrilateral $B_0B_1B_2B_3$.

Arithmetic operations on APFNs

Consider two APFN's $\tilde{A} = (a_1, a_2, a_3, a_4, a_5; w_{L1}, w_{R1})$ and $\tilde{B} = (b_1, b_2, b_3, b_4, b_5; w_{L2}, w_{R2})$, then

(i) $\tilde{A} \oplus \tilde{B} = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4, a_5 + b_5; \min(w_{L_1}, w_{L_2}), \min(w_{R_1}, w_{R_2}))$.
(ii) $\tilde{A} \ominus \tilde{B} = (a_1 - b_5, a_2 - b_4, a_3 - b_3, a_4 - b_2, a_5 - b_1; \min(w_{L_1}, w_{L_2}), \min(w_{R_1}, w_{R_2}))$.
(iii) $c\tilde{A} = (ca_1, ca_2, ca_3, ca_4, ca_5)$; where $c > 0$ is a real constant.
(iv) $c\tilde{A} = (ca_5, ca_4, ca_3, ca_2, ca_1)$; where $c < 0$ is a real constant.

4. A proposed method for ranking Asymmetric Pentagonal Fuzzy Numbers

In this section, a score function to rank generalised APFN is proposed. This score function is based on centre of gravity of APFN and areas associated with them.

Definition 4.1 ([10] Center of Gravity of m-sided polygon)**.** A point C is called centre of gravity of m-sided polygon $B_0B_1B_2 \dots B_{m-1}$ if the following conditions hold true:

(i) For $m = 3$ (triangle $B_0B_1B_2$), C is the point of intersection of all medians. The cartesian coordinates of C in Figure 3 is $\left(\frac{m_1+m_2+m_3}{3}, \frac{n_1+n_2+n_3}{3}\right)$.
(ii) For $m = 4$ (quadrilateral $B_0B_1B_2B_3$), C is the intersection of centroid of the triangles $B_0B_1B_2, B_0B_2B_3$, $B_0B_1B_3, B_1B_2B_3$ respectively. Let I, J, K and L be the centers of triangles $B_0B_1B_2$, $B_0B_2B_3$, $B_0B_1B_3$, and $B_1B_2B_3$ respectively. The cartesian coordinates of C for quadrilateral $B_0B_1B_2B_3$ is the point of intersection of the lines IJ and KL as depicted in Figure 4.
(iii) For $m = 5$ (pentagon $B_0B_1B_2B_3B_4$), we set C $= IJ \cap KL$ where I is the centroid of the 3-gon $B_0B_1B_2$; J is the centroid of the 4-gon $B_0B_2B_3B_4$; L is the centroid of the 4-gon $B_0B_1B_2B_3$; K is the centroid of the 3-gon $B_0B_3B_4$ as depicted in Figure 5.

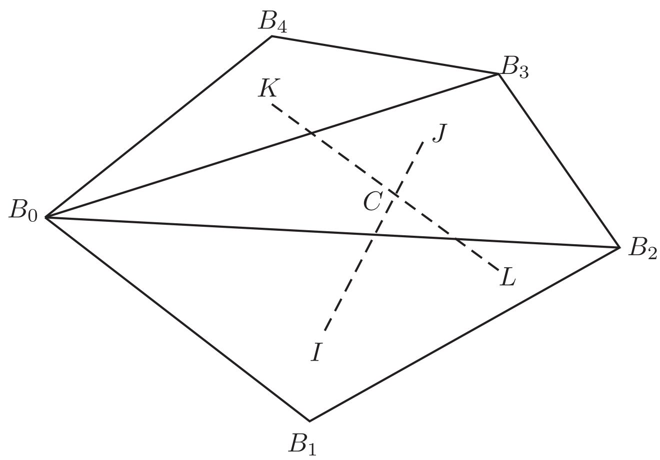

FIGURE 5. The centre of gravity of pentagon $B_0B_1B_2B_3B_4$.

Proposed method

Consider n Generalised Pentagonal Fuzzy Numbers $\tilde{A}_i = (a_{1i}, a_{2i}, a_{3i}, a_{4i}, a_{5i}; w_{Li}, w_{Ri}, w_i)$, $i = 1, 2, \ldots, n$ in the inerval $[0, w_i]$, $0 \leq w_i \leq 1$ where w_{Li} represents left height, w_{Ri} represents right height. The following steps are followed for ranking n Generalised Pentagonal Fuzzy Numbers with different left and right heights of APFN:

Step 1: convert generalised APFN to Standard Generalised APFN
Standard generalised APFN is determined as follows:

$$A_i = \left(\frac{a_{1i}}{M}, \frac{a_{2i}}{M}, \frac{a_{3i}}{M}, \frac{a_{4i}}{M}, \frac{a_{5i}}{M}, w_{Li}, w_{Ri}, w_i\right) = (b_{1i}, b_{2i}, b_{3i}, b_{4i}, b_{5i}; w_{Li}, w_{Ri}, w_i)$$

where $M = \max\{\lceil|a_{1i}|\rceil, \lceil|a_{2i}|\rceil, \lceil|a_{3i}|\rceil, \lceil|a_{4i}|\rceil, \lceil|a_{5i}|\rceil, 1\}$.

Step 2: calculate the geometric centre of gravity of standard APFN
Consider the standard generalised asymmetric pentagonal fuzzy number $A_i = (b_{1i}, b_{2i}, b_{3i}, b_{4i}, b_{5i}; w_{Li}, w_{Ri}, w_i)$ where $b_{1i} \leq b_{2i} \leq b_{3i} \leq b_{4i} \leq b_{5i}$. Using Definition 4.1, the centroid I and K of the 3-gon's $B_OB_1B_2$ and $B_0B_3B_4$ are as follows:

$$I = (I_{x_i}, I_{y_i}) = \left(\frac{b_{1i} + b_{2i} + b_{3i}}{3}, \frac{w_{Li} + w_i}{3}\right)$$
$$K = (K_{x_i}, K_{y_i}) = \left(\frac{b_{1i} + b_{4i} + b_{5i}}{3}, \frac{w_{Ri}}{3}\right).$$

The centroids J and L of the 4-gon's $B_0B_2B_3B_4$ and $B_0B_1B_2B_3$ are as follows:
$J = (J_{x_i}, J_{y_i})$ can be obtained using the equations:

$$J_{y_i} - \frac{w_i}{b_{1i} - b_{5i}}\left[J_{x_i} - \frac{b_{1i} + b_{4i} + b_{5i}}{3}\right] = \frac{w_{Ri}}{3}$$
$$J_{y_i} - \frac{w_{Ri}}{b_{4i} - b_{1i}}\left[J_{x_i} - \frac{b_{1i} + b_{3i} + b_{5i}}{3}\right] = \frac{w_i}{3}$$

and $L = (L_{x_i}, L_{y_i})$ can be obtained as follows:

$$L_{y_i} - \frac{w_{Ri} - w_{Li}}{b_{4i} - b_{2i}}\left[L_{x_i} - \frac{b_{1i} + b_{3i} + b_{4i}}{3}\right] = \frac{w_i + w_{Ri}}{3}$$

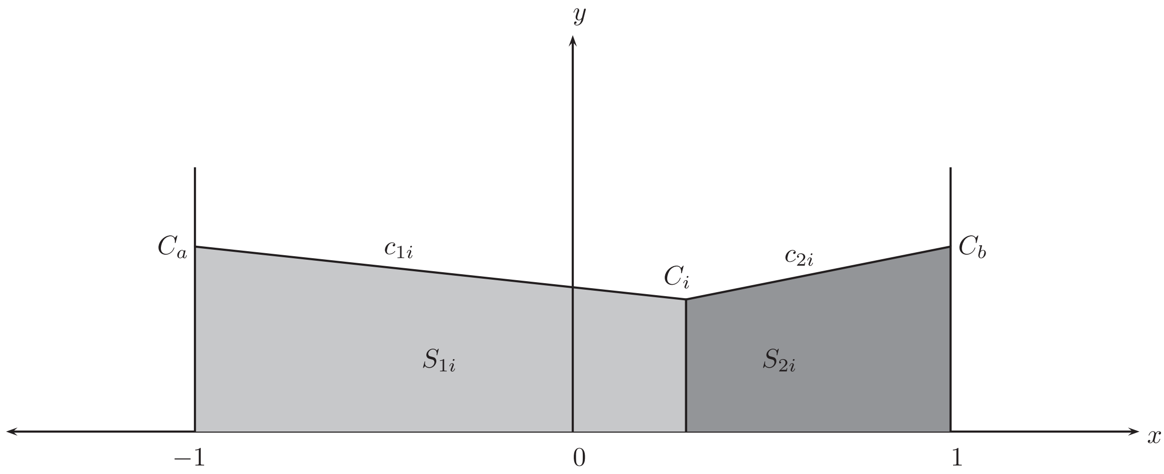

FIGURE 6. The areas C_{1i} and C_{2i} for APFN A_i.

$$L_{y_i} - \frac{w_i}{b_{3i} - b_{1i}}\left[L_{x_i} - \frac{b_{1i} + b_{2i} + b_{4i}}{3}\right] = \frac{w_{Li} + w_{Ri}}{3}.$$

If $A_i = (b_{1i}, b_{2i}, b_{3i}, b_{4i}, b_{5i}; w_{Li}, w_{Ri}, w_i)$ be a standard generalised pentagonal fuzzy number in which $b_{1i} \leq b_{2i} \leq b_{3i} \leq b_{4i} \leq b_{5i}$, then using Definition 4.1, the centre of gravity $C = (C_{x_i}, C_{y_i})$ of A_i is obtained by solving the following system of equations:

$$C_{y_i} - J_{y_i} = \frac{J_{y_i} - \frac{w_{Li}+w_i}{3}}{J_{x_i} - \frac{b_{1i}+b_{2i}+b_{3i}}{3}}[C_{x_i} - J_{x_i}]$$
$$C_{y_i} - L_{y_i} = \frac{L_{y_i} - \frac{w_{Ri}}{3}}{L_{x_i} - \frac{b_{1i}+b_{4i}+b_{5i}}{3}}[C_{x_i} - L_{x_i}].$$

If $A_i = (b_{1i}, b_{2i}, b_{3i}, b_{4i}, b_{5i}; w_{Li}, w_{Ri}, w_i)$ be a APFN with $b_{1i} = b_{2i} = b_{3i} = b_{4i} = b_{5i} = b_i$ and $w_{Li} = w_{Ri} = w_i$ then the centre of gravity of A_i is $C_i = (C_{xi}, C_{yi}) = (b_i, \frac{w_i}{2})$.

Step 3: determine the areas on left and right sides of standard generalized APFN

Let $C_i = (C_{xi}, C_{yi}), C_a = (-1, \frac{1}{2}), C_b = (1, \frac{1}{2})$ be the centres of gravity of $A_i, (-1, -1, -1, -1, -1; 1, 1)$ and $(1, 1, 1, 1, 1; 1, 1)$ respectively. Let c_{1i} and c_{2i} be the lines joining points C_a, C_i and C_b, C_i respectively. The areas associated with left side of A_i is $S_{1i} = \int_{-1}^{C_{xi}} c_{1i}(x)\,\mathrm{d}x$ and the area to the right side of A_i is $S_{2i} = \int_{C_{xi}}^{1} c_{2i}(x)\,\mathrm{d}x$ as shown in Figure 6.

Step 4: determine score($\tilde{A}_i$) to rank standard generalised APFN

Let score$(A_i) = S_{1i} - S_{2i}$. The rank of APFN is given as follows:

(i) score$(A_i) <$ score(A_j) implies $A_i \prec A_j$.
(ii) score$(A_i) >$ score(A_j) implies $A_j \prec A_i$.
(iii) score$(A_i) =$ score$(A_j) = 0$ and $C_{yi} < C_{yj}$ implies $A_i \prec A_j$.
(iv) score$(A_i) =$ score$(A_j) \neq 0$ implies $A_i \approx A_j$.

5. COMPUTATIONAL METHODOLOGY FOR SOLVING BALANCED FUZZY TSP WITH GENERALISED APFN

The optimal solution of Balanced Fuzzy Transshipment Problem based on score function is obtained using the following steps:

Table 2. Transshipment problem.

Destination → Origins ↓	A_1	A_2	Y_1	Y_2	Supply
A_1	$(0,0,0,0,0)$	$(2,3,4,5,6)$	$(2,3,5,6,7)$	$(1,4,5,6,7)$	43
A_2	$(1,2,3,4,6)$	$(0,0,0,0,0)$	$(3,4,5,7,9)$	$(5,6,9,11,13)$	57
Y_1	$(1,4,5,6,7)$	$(3,4,5,7,9)$	$(0,0,0,0,0)$	$(2,3,4,5,6)$	–
Y_2	$(1,5,7,9,10)$	$(6,7,9,12,14)$	$(3,4,5,7,9)$	$(0,0,0,0,0)$	–
Demand	–	–	51	49	

Step 1. Represent the Balanced Fuzzy Transshipment Problem in the form of matrix table.
Step 2. Using the score function described in Section 3, defuzzify the Fuzzy TsP into crisp TsP.
Step 3. Select the minimum cost from each row of the defuzzified matrix and then deduct it from each value in that row.
Step 4. Similarly select minimum cost from each column of the matrix obtained in preceding step and deduct it from each value of that column.
Step 5. Every row and column of the defuzzified matrix include at least one zero value using Steps 3 and 4. Use the following formula to determine the zero-centered value R_{ij} that corresponds to each cell having value zero:
.

$$R_{ij} = \frac{\text{Sum of costs in adjacent cells of the zero} - \text{valued cell } ij}{\text{Number of Non-zero adjacent rank value cells}}.$$

Step 6. Choose the maximum zero centered value R_{ij} and assign the minimum units of demand or supply to the corresponding cell. Incase of same zero centered value, allocate the cell which assigns maximum possible supply/demand.
Step 7. Incase the supply or demand is exhausted, delete that row or column to obtain the reduced table. If both the row and column are exhausted simultaneously, assign a dummy cell with minimum cost in that row/column.
Step 8. Repeat Steps 3–7 until all the allotments are made.
Step 9. Determine the total fuzzy cost of transshipment by adding product of allocations and the respective fuzzy cost in the original table. The fuzzy cost so obtained gives fuzzy optimal solution.

6. Numerical example

In this section, two fuzzy Transshipment problems with APFN are solved to show the applicability of the proposed ranking function and methodology.

6.1. Example 1

Consider a balanced fuzzy TsP with two origins(A_1 and A_2) and two destinations(Y_1 and Y_2) as represented in Table 2. The fuzzy cost is represented as APFN with crisp supply and demand values.

Note that $\sum_{i=1}^{2} a_i = \sum_{j=1}^{2} b_j = 100$. Since the given transshipment problem is balanced $T = \max(\sum a_i, \sum b_j) = 100$. The balanced transshipment problem is obtained by adding Buffer Stock T to each supply and demand node as shown in Table 3.

Note that throughout the example the left and right heights of APFN are considered to be $\frac{1}{3}$ and $\frac{2}{3}$ respectively and $w = 1$.

TABLE 3. Balanced transshipment problem.

Destination → Origins ↓	A_1	A_2	Y_1	Y_2	Supply
A_1	$(0,0,0,0,0)$	$(2,3,4,5,6)$	$(2,3,5,6,7)$	$(1,4,5,6,7)$	143
A_2	$(1,2,3,4,6)$	$(0,0,0,0,0)$	$(3,4,5,7,9)$	$(5,6,9,11,13)$	157
Y_1	$(1,4,5,6,7)$	$(3,4,5,7,9)$	$(0,0,0,0,0)$	$(2,3,4,5,6)$	100
Y_2	$(1,5,7,9,10)$	$(6,7,9,12,14)$	$(3,4,5,7,9)$	$(0,0,0,0,0)$	100
Demand	100	100	151	149	

TABLE 4. Defuzzified transshipment problem.

Destination → Origins ↓	A_1	A_2	Y_1	Y_2	Supply
A_1	0	0.5255	0.4819	0.5719	143
A_2	0.4507	0	0.5351	0.5889	157
Y_1	0.5719	0.5351	0	0.5255	100
Y_2	0.484	0.587	0.5351	0	100
Demand	100	100	151	149	

TABLE 5. Optimal solution.

Destination → Origins ↓	A_1	A_2	Y_1	Y_2	Supply
A_1	$(0,0,0,0,0)$ 100	$(2,3,4,5,6)$	$(2,3,5,6,7)$ 43	$(1,4,5,6,7)$	143
A_2	$(1,2,3,4,6)$	$(0,0,0,0,0)$ 100	$(3,4,5,7,9)$ 8	$(5,6,9,11,13)$ 49	157
Y_1	$(1,4,5,6,7)$	$(3,4,5,7,9)$	$(0,0,0,0,0)$ 100	$(2,3,4,5,6)$	100
Y_2	$(1,5,7,9,10)$	$(6,7,9,12,14)$	$(3,4,5,7,9)$	$(0,0,0,0,0)$ 100	100
Demand	100	100	151	149	

For a real number, $w_L = w_R = w = 1$. Defuzzify the cost of transshipment in Table 3 into crisp form using the score function to rank generalised APFN. The defuzzified problem is shown in Table 4.

Applying Steps 3–8 of the proposed methodology, the final optimal solution so obtained is as shown in Table 5.

The fuzzy optimal solution is as follows:$\tilde{X}_{A_1Y_1} = 43$, $\tilde{X}_{A_2Y_1} = 8$, $\tilde{X}_{A_2Y_2} = 49$. The fuzzy optimal value of the given fuzzy TsP with Asymmetric Pentagonal Fuzzy Cost given in Table 5 is $(2,3,5,6,7)\otimes 43 \oplus (3,4,5,7,9)\otimes 8 \oplus (5,6,7,11,13)\otimes 49 = (355,406,598,853,1010)$. The optimal solution in Table 5 has $m+n-1 = 4+4-1 = 7$ allocations. Using MODI method, it has been verified that the solution so obtained is a fuzzy optimal solution.

6.2. Example 2

Table 6 shows a Balanced transshipment problem with two origins O_1 and O_2 and two destinations D_1, D_2. The fuzzy cost are represented as APFN and demand and supply remain crisp.

TABLE 6. Initial Transshipment Problem.

Destination → Origins ↓	O_1	O_2	D_1	D_2	Supply
O_1	$(0,0,0,0,0)$	$(1,2,3,4,6)$	$(1,2,3,4,5)$	$(2,3,4,5,6)$	41
O_2	$(1,2,3,4,5)$	$(0,0,0,0,0)$	$(1,4,5,6,7)$	$(1,2,5,7,9)$	57
D_1	$(1,4,5,6,7)$	$(1,2,3,4,6)$	$(0,0,0,0,0)$	$(2,3,4,5,6)$	–
D_2	$(1,2,5,7,9)$	$(1,4,5,6,7)$	$(1,2,3,4,6)$	$(0,0,0,0,0)$	–
Demand	–	–	49	49	

TABLE 7. Optimal Transshipment Table.

Destination → Origins ↓	O_1	O_2	D_1	D_2	Supply
O_1	$(0,0,0,0,0)$ 98	$(1,2,3,4,6)$	$(1,2,3,4,5)$	$(2,3,4,5,6)$ 41	139
O_2	$(1,2,3,4,5)$	$(0,0,0,0,0)$ 98	$(1,4,5,6,7)$ 49	$(1,2,5,7,9)$ 8	155
D_1	$(1,4,5,6,7)$	$(1,2,3,4,6)$	$(0,0,0,0,0)$	$(2,3,4,5,6)$	98
D_2	$(1,2,5,7,9)$	$(1,4,5,6,7)$	$(1,2,3,4,6)$	$(0,0,0,0,0)$ 98	98
Demand	98	98	47	49	

Assume left and right heights of APFN to be $\frac{1}{4}$ and $\frac{3}{4}$ respectively. The given transshipment problem is balanced as $T = \max(\sum a_i, \sum b_j) = 98$. Following the steps of the proposed algorithm to obtain an optimal solution as shown in Table 7.

The fuzzy optimal solution to the given transshipment problem is $\tilde{X}_{O_1D_2} = 41$, $\tilde{X}_{O_2D_1} = 49$, $\tilde{X}_{O_2D_2} = 8$. The optimal fuzzy cost of transportation problem with intermediate nodes having Asymmetric Pentagonal Fuzzy Cost as shown in Table 7 is $(2,3,4,5,6) \otimes 41 \oplus (1,4,5,6,7) \otimes 49 \oplus (1,2,5,7,9) \otimes 8 = (139, 335, 449, 555, 661)$.

7. CONCLUSION

In this paper, a solution technique to find fuzzy optimal solution for balanced fuzzy transshipment problem with asymmetric pentagonal fuzzy numbers is given. A new ranking approach is proposed in order to defuzzify the transshipment problem. In order to defuzzify the fuzzy numbers, we use centroid based ranking approach in which scores are obtained using centre of gravity based on left and right areas associated with given APFN. The technique given in this paper is a direct technique that provides the fuzzy solution to fuzzy transshipment problem which is very useful for the decision maker since it provides the option to choose optimal solution to possibilistic situations. Two numerical examples have been given to illustrate the proposed approach. The proposed algorithm works for both asymmetric as well as symmetric pentagonal fuzzy numbers.

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