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ON TWO VARIANTS OF SPLIT GRAPHS: 2-UNIPOLAR GRAPH AND k-PROBE-SPLIT GRAPH

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Abstract. A graph is called split if its vertex set can be partitioned into a stable set and a clique. In this article, we studied two variants of split graphs. A graph G is polar if its vertex set can be partitioned into two sets A and B such that $G[A]$ is a complete multipartite graph and $G[B]$ is a disjoint union of complete graphs. A 2-unipolar graph is a polar graph G such that $G[A]$ is a clique and $G[B]$ is the disjoint union of complete graphs with at most two vertices. We present a minimal forbidden induced subgraph characterization for 2-unipolar graphs. In addition, we show that they can be represented as an intersection of substars of special cacti. Let $\mathcal{G}$ be a graph class, the *$\mathcal{G}$-width* of a graph G is the minimum positive integer k such that there exist k independent sets $\mathcal{N}_1, \ldots, \mathcal{N}_k$ such that a set F of nonedges of G, whose endpoints belong to some $\mathcal{N}_i$ with $i = 1, \ldots, k$, can be added so that the resulting graph G' belongs to $\mathcal{G}$. We say that a graph G is k-probe-$\mathcal{G}$ if it has $\mathcal{G}$-width at most k and when $\mathcal{G}$ is the class of split graphs it is denominated k-probe-split. We prove that deciding, given a graph G and a positive integer k, whether G is a h-probe-split graph for some $h \leq k$ is NP-complete. Besides, a characterization by minimal forbidden induced subgraphs for 2-probe-split cographs is presented.

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1. Introduction

We warn the reader that definitions and concepts mentioned here can be found in [12].

A graph is called *split* if its vertex set can be partitioned into a stable set and a clique. Split graphs were introduced by Földes and Hammer in [10], who proved that a graph is split if and only if contains no $2K_2$, C_4 and C_5 as an induced subgraph. In 1981, they showed a nice characterization of split graphs in terms of their degree sequences [13]. A summary for this graph class can be found in [12]. Since split graphs appeared in the specialized literature, many variants in connection with them were studied. In 1994, Maffray and Preissmann considered graphs with no $2K_2$ and C_4 as an induced subgraph, called pseudo-split graphs. It is well-known that split graphs can be represented as the intersection graph of substars of a star. Nevertheless, this intersection graph class contains strictly the class of split graphs. It was not until 2006 when Cerioli and Szwarcfiter characterized those graphs that can be represented as the intersection graph of substars of a star in terms of minimal forbidden induced subgraphs [1]. Clearly, split graphs can be equivalently defined as those graphs G whose vertex set can be partitioned into two sets A and B such that the subgraph $G[A]$, induced by A, contains no $\overline{P_2}$ as an induced

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subgraph, and the subgraph $G[B]$, induced by B, contains no P_2 as an induced subgraph. The class of polar graphs, introduced by Tyshkevich and Cherny [21], generalized split graphs. Polar graphs are those graphs G whose vertex set can be partitioned into two sets A and B, such that $G[A]$ contains no $\overline{P_3}$ as induced subgraphs and $G[B]$ contains no P_3 as an induced subgraph; *i.e.*, $G[A]$ is a complete multipartite graph and $G[B]$ is a disjoint union of complete graphs. They also proved that deciding whether a graph is polar is an NP-complete problem. When $G[A]$ is restricted to be a graph with no $\overline{2K_1}$ as induced subgraph; *i.e.*, an edgeless graph, G is said to be *monopolar* (see [7,8,14,16] and reference therein). When $G[A]$ is restricted to be a graph with no $\overline{P_2}$ as induced subgraph; *i.e.*, a complete graph, G is said to be *unipolar*. Unipolar graphs were considered, for the first time in the literature, under the name of clique split graphs, by Maffray and Szwarcfiter [20]. Late, in 2014, Eschen and Wang found a polynomial-time recognition algorithm for this class [9].

Let $\mathcal{G}$ be a family of graphs. A graph G is a *probe-*$\mathcal{G}$ graph if its vertex set can be partitioned into two sets: a set $\mathcal{P}$ of probe vertices and a stable set $\mathcal{N}$ of nonprobe vertices so that a possibly empty set F can be added to obtain a graph G' belonging to $\mathcal{G}$ such that $V(G) = V(G')$ and $E(G') = E(G) \cup F$, where the edges in F have their endpoints in $\mathcal{N}$. Probe graph classes have been considered for many classes $\mathcal{G}$, in particular for split graphs [15]. This concept was generalized in [2] in the following way. The $\mathcal{G}$*-width* of a graph G is the minimum positive integer k such that there exist k independent sets $\mathcal{N}_1, \ldots, \mathcal{N}_k$ and a set F such that the graph G' with $V(G) = V(G')$ and $E(G') = E(G) \cup F$ belongs to $\mathcal{G}$, where the edges in F have their endpoints in some $\mathcal{N}_i$ with $i = 1, \ldots, k$. Notice that when $k = 1$ then G is probe-$\mathcal{G}$. In [2], Chang *et al.* proved that the decision problem related to the $\mathcal{G}$-width is NP-complete when $\mathcal{G}$ either is the class of block graph or is the class of complete graphs. Le and Peng [17] presented some new classes where the decision problem related to $\mathcal{G}$-width when $\mathcal{G}$ is the class of complete graphs remains NP-complete and other ones where it becomes polynomial-time solvable and they also characterized those graphs of small complete width, k at most three. We say that a graph G is k-probe-$\mathcal{G}$ if it has $\mathcal{G}$-width at most k.

In this article, we deal with two superclasses of split graphs. The first one is the subclass of those unipolar graphs whose vertex set can be partitioned into two sets A and B such that $G[A]$ contains no $\overline{P_2}$ as an induced subgraph, and $G[B]$ contains no P_3 and K_3 as an induced subgraph, meaning $G[B]$ is a disjoint union of complete graphs with at most two vertices. We call these graphs 2*-unipolar*. We obtain a characterization by forbidden induced subgraphs. This kind of characterizations related to polar graphs classes can be found for instance in [4]. We also present a representation of these graphs as the intersection graphs of substars of a starlike cactus. The second one is the class of k-probe-split graphs. We prove that the decision problem of $\mathcal{G}$-width is NP-complete when $\mathcal{G}$ is the class of split graphs and characterize by a finite number of forbidden induced subgraphs the 2-probe-split graphs within the class of cographs. In this way, we contribute to studying two variants of split graphs in connection with polar graphs and k-probe classes.

1.1. Definitions and preliminaries

Basic concepts

All graphs considered in this article are simple and finite. Let G be a graph. We use $V(G)$ and $E(G)$ to denote the set of vertices of G and the set of edges of G, respectively. We use $N_G(v)$ to denote the set of neighbors of a vertex $v \in V(G)$ and $N[v] = N_G(v) \cup \{v\}$, we omit the subscript in case the context is clear enough. Let $S \subseteq V(G)$. We use $N_G(S)$ to denote the set of those vertices with at least one neighbor in S and $N[S] = N(S) \cup S$. Two vertices u and v are *true twins* if $N[u] = N[v]$. Let S be a subset of vertices of G, we use $G - S$ to denote the graph obtained from G by deleting each vertex in S and its respective incident edges. We use $G - v$ to denote $G - \{v\}$. Let $X \subseteq V(G)$, we use $G[X]$ to denote the subgraph induced by X in G. A *stable set* (or *independent set*) of a graph is a set of pairwise nonadjacent vertices. By $\overline{G}$ we denoted the *complement graph* of G. The *maximum independent number*, denoted $\alpha(G)$, is the cardinality of an independent set with the maximum number of vertices. A *clique* is a set of pairwise adjacent vertices. A *complete graph* is a graph such that all its vertices are pairwise adjacent. We use C_n, K_n, $K_{1,n-1}$ and P_n to denote the isomorphism classes of cycles, complete graphs, stars, and paths, all of them on n vertices, respectively. Let $\mathcal{H}$ be a set of

FIGURE 1. Thickened edges are the simplicial edges in each graph.

graphs. A graph is said to be $\mathcal{H}$*-free* if it contains none of the graphs in $\mathcal{H}$ as an induced subgraph. If $\mathcal{H}$ has only one element H, we use H*-free* for short. A *simplicial* vertex is a vertex whose neighborhood is a clique. Notice that if u and v are adjacent simplicial vertices then u and v are true twins. The set of pairwise adjacent simplicial vertices is called a *simplicial clique*. Let G and H be two graphs. We use $G + H$ (resp. $G \vee H$) to denote the disjoint union of G and H (resp. the joint between G and H; *i.e.*, $G + H$ plus all edges having an endpoint in $V(G)$ and the other one in $V(H)$). A *cut-vertex* is a vertex whose removal increases the number of connected components. A *block* of a graph is a maximal connected subgraph of it having no cut-vertex. A *cactus* is a connected graph whose blocks are cycles. By nG we denote the graph obtained by the disjoint union of n copies of G. A *matching* in a graph G is a set of edges M without common endpoints. A set of vertices X is said to be *saturated by* M or, equivalently, M saturates X if all vertices in X are endpoints of some edge in M. If $G[V(M)] = tK_2$ for some positive integer t, where $V(M)$ stands for the set of endpoints of M we say that M is an *induced matching*.

Graph classes

A graph is *chordal* if it admits no induced cycle with at least four edges. Notice that split graphs are a proper subclass of chordal graphs. A *cograph* is a P_4-free graph. If G is a cograph, then G or $\overline{G}$ is connected [5]. Thus if G is connected cograph then $G = H \vee J$, for two cographs H and J. A graph is *trivially perfect* if for each induced subgraph the maximum cardinality of an independent set agrees with the number of maximal cliques. Indeed, trivially perfect graphs are precisely the $\{P_4, C_4\}$-free graphs [11]. In addition, a graph is trivially perfect if and only if every connected induced subgraph has a universal vertex (see [22]). A graph is *threshold* if it is $\{2K_2, P_4, C_4\}$-free. Observe that threshold graphs are precisely the split cographs. For more details about those graph classes described above, we refer the reader to [12].

2. 2-UNIPOLAR GRAPHS

In order to characterize 2-unipolar graphs we need to introduce the following concept. A *simplicial edge* uv of a graph G is an edge $uv \in E(G)$ such that $(N(u) \cup N(v)) \setminus \{u, v\}$ is a clique (see Fig. 1).

Lemma 1. *If a graph G contains no graph from (iv) to (xvi) of Figure 2 as an induced sugbraph, then every induced C_4 in G has at least one simplicial edge of G.*

Proof. Assume that $C = \{a, b, c, d\}$ induces C_4 in G and $G[C]$ has edges ab, bc, cd and da.

If C is a connected component of G, then all edges in C are simplicial. Let $B_i \subseteq N(C)$ be the set formed by those vertices x of $G - C$ such that $|N(x) \cap C| = i$ for each $i \in \{1, 2, 3, 4\}$. At least one B_i is not empty.

Since G has no $K_{2,3}$ as an induced subgraph, for each $x \in B_2$, the vertices in $N(x) \cap C$ are consecutive in C. Moreover, B_4 is a clique of G otherwise, there exists $x, y \in B_4$ non adjacent and $C \cup \{x, y\}$ induces $\overline{3K_2}$, contradicting that $\overline{3K_2}$ is a forbidden induced subgraph for G.

We claim that B_3 is a clique. Let be $x, y \in B_3$ such that $N(x) \cap C \neq N(y) \cap C$, we claim that $N(x) \cap N(y) \cap C$ are two consecutive vertices in C. Otherwise, assume without losing generality, that $N(x) \cap C = \{a, b, d\}$ and $N(y) \cap C = \{b, c, d\}$. Hence $\{a, b, c, d, x, y\}$ induces $\overline{P_4 + K_2}$ when x is adjacent to y and it induces $\overline{C_4 + K_2}$ otherwise, in both cases we reach a contradiction that comes from supposing that $N(x) \cap N(y) \cap C$ are two nonconsecutive vertices in C. If $N(x) \cap N(y) \cap C$ are two consecutive vertices in C, then x is adjacent to y,

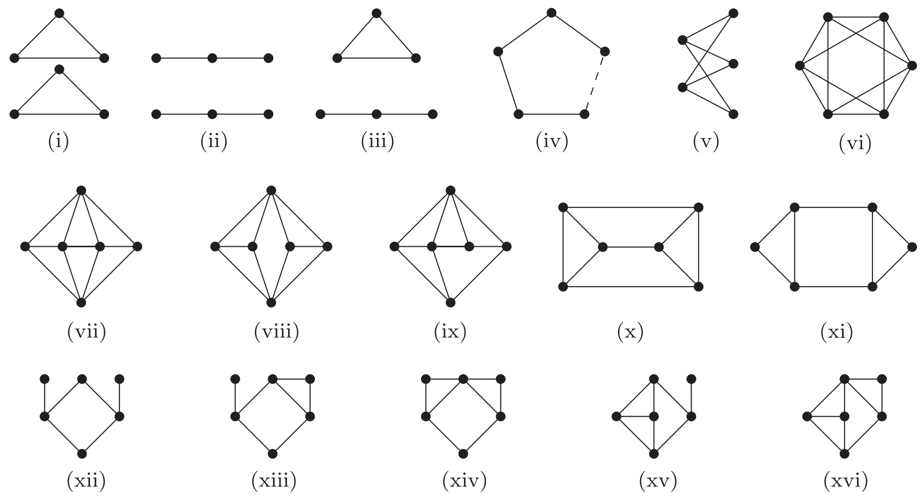

FIGURE 2. Forbidden subgraphs for 2-unipolar graphs. (i) $2K_3$. (ii) $2P_3$. (iii) $K_3 + P_3$. (iv) $C_k, k \geq 5$. (v) $K_{2,3}$. (vi) $\overline{3K_2}$. (vii) $\overline{P4 + K2}$. (viii) $\overline{C4 + K2}$. (ix) $\overline{P6}$. (x) $\overline{C6}$. (xi) $\overline{domino}$. (xii) F_1. (xiii) F_2. (xiv) F_3. (xv) F_4. (xvi) F_5.

otherwise $\{a, b, c, d, x, y\}$ induces $\overline{P_6}$. Finally if $N(x) \cap C = N(y) \cap C$, say for example $\{a, b, c\}$, then x and y are neighbors, otherwise $\{a, c, d, x, y\}$ induces $K_{2,3}$. In consequence, B_3 is a clique.

Moreover, $B_3 \cup B_4$ is a clique in G. It remains to observe that if $x \in B_3$ and $y \in B_4$, then x is adjacent to y, otherwise $C \cup \{x, y\}$ induces $\overline{P_4 + K_2}$.

If $x, y \in B_2$ then $N(x) \cap N(y) \cap C \neq \emptyset$. Otherwise $C \cup \{x, y\}$ would induce the complement of the domino (when x is nonadjacent to y) or $\overline{C_6}$ (when x is adjacent to y). See domino in Figure 1. Besides, it must be $N(x) \cap C = N(y) \cap C$. Otherwise, assume without lose of generality, that $N(x) \cap C = \{a, b\}$ and $N(y) \cap C = \{b, c\}$. If x is adjacent to y, then $\{a, c, d, x, y\}$ would induce C_5. And if x is nonadjacent to y, then $C \cup \{x, y\}$ induces F_3 (see Fig. 2). We conclude that $N(x) \cap C = N(y) \cap C$ for all $x, y \in B_2$.

If $x, y \in B_1$, then either $N(x) \cap C = N(y) \cap C$ or their neighbors are adjacent in C, because otherwise F_1 (see Fig. 2) would be an induced subgraph of G (when x is nonadjacent to y) or C_5 would be an induced subgraph of G (when x is adjacent to y).

Recall that $B_3 \cup B_4$ is a clique. Since G is $\{K_{2,3}, \overline{P_6}, F_2, F_4, F_5\}$-free, $N(B_1) \cap C \subseteq N(B_2) \cap C \subseteq N(x) \cap C$ for all $x \in B_3$. Thus there exists an edge in C having no endpoint in $(B_1 \cup B_2) \cap C$ and at most one extreme in $N(x) \cap C$ for all $x \in B_3$. This edge is a simplicial edge of G. □

Proposition 1. *Let G be a graph containing none of the graphs from (iv) to (xvi) of Figure 2 as an induced subgraph. If G has at least one induced C_4, then there exists an induced matching M of simplicial edges of G such that each induced C_4 has at least one edge in M.*

Proof. We are going to construct M. By Lemma 1, every induced C_4 in G has at least one simplicial edge. Let $e_0 = a_0b_0$ be a simplicial edge of G in an induced C_4 of G. Recursively, let $e_k = a_kb_k$ be a simplicial edge of G in an induced C_4 of $H_k = G - \bigcup_{i=0}^{k-1}\{a_ib_i\}$ for every $k \geq 1$ such that H_k contains C_4 as induced subgraph. Let $M = \bigcup_{i=0}^{m}\{a_ib_i\}$, where m is the maximum positive integer such that H_{m-1} contains C_4 as induced subgraph and H_m is C_4-free, in case H_1 is chordal we set $M = \{e_0\}$. Notice that M is a matching. Observe that the endpoints of two edges in M do not induce C_4 in G, because of the definition of simplicial edge; *i.e.*, each induced C_4 in G has at most one of them as an edge. Notice also that if C is a C_4 in G, there exists a nonnegative integer $n \leq m$ such that C belong to H_{n-1} and not to H_n. Since H_n arises from H_{n-1} by deleting a simplicial edge ab, then ab should be an edge of C. Consequently, each induced C_4 of G has one edge in the matching M. Moreover, since every edge in M is simplicial in G, it can be proved that C has exactly one edge in M.

Let us prove now that M is an induced matching. Let $ab, cd \in M$. Suppose, towards a contradiction, that $G[\{a, b, c, d\}]$ has at least three edges, $e_i = ab$ and $e_j = cd$ with $i < j$. Assume, without losing generality, that $bc \in E(G)$. By construction of M, there exists a C_4 containing a and b and not c and d. Let a' be adjacent to a and nonadjacent to b. Since ab is a simplicial edge, a' is adjacent to c. Since $bc \in E(G)$ and cd is a simplicial edge, a' is adjacent to b, contradicting that a' is nonadjacent to b. The contradiction arose from supposing that $G[\{a, b, c, d\}]$ has at least three edges. Therefore, M is an induced matching and, by construction, also satisfies the conditions' statement. □

The following lemma presents an important property of simplicial vertices in connection with chordal graphs. A proof of this fact can be found in [12].

Lemma 2. *Every chordal graph which is not a complete graph has at least two nonadjacent simplicial vertices.*

The following proposition characterizes, by a finite list of minimal forbidden induced subgraphs, those chordal 2-polar graphs. It is worth pointing out that Ekim *et al.* [6] observed the existence of an infinite list of minimal forbidden induced subgraphs for polar graphs within the class of chordal graphs.

Proposition 2. *Let G be a chordal graph, then G is 2-unipolar if and only if G contains none of the graphs in* $\{2P_3,\ 2K_3,\ P_3 + K_3\}$ *as an induced subgraph.*

Proof. Clearly G does not contain any of $2P_3$, $2K_3$, and $P_3 + K_3$ as an induced subgraph.

Conversely, assume that G is a chordal graph having no $2P_3$, $2K_3$, and $P_3 + K_3$ as an induced subgraph. Let us proceed by induction on $|V(G)|$.

First assume that G is a disconnected graph, whose connected components are $G_1, \ldots, G_k$. Since G contains no $2P_3$ as an induced subgraph, at most one of its connected components is not a complete graph. If all of them are complete graphs the result holds, because G is $2K_3$-free. Assume that one of them is not a complete graph, say G_1. Since G_1 contains P_3 as induced subgraph and G is $P_3 + K_3$-free, G_i is a complete graph with at most two vertices, for every $2 \leq i \leq k$. By the inductive hypothesis, there exists a clique C in G_1, such that $G_1 - C$ is a disjoint union of complete graphs with at most two vertices. Therefore, $G - C$ is also the disjoint union of complete graphs with at most two vertices.

Finally, assume that G is connected and it is not a complete graph in which case the result holds. Let S be the set of simplicial vertices of G. Set $R = V(G) \setminus S$.

First, assume that $G[R]$ is a complete graph. Notice that $G[S]$ is a disjoint union of complete graphs (see "Graph classes" in Sect. 1.1). Since G is $2K_3$-free, at most one of them has more than two vertices. If all of them are complete graphs with at most two vertices the result holds because $G - R$ is the disjoint union of cliques with at most two vertices. Otherwise, exactly one of them would be a complete graph with at least three vertices, say G_1. Since $C = N[V(G_1)]$ is a clique, $|V(G_1)| \geq 3$ and G is $\{2K_3, K_3 + P_3\}$-free, $G - C$ is the disjoint union of complete graphs with at most two vertices and the result holds.

Assume now that $G[R]$ is not a complete graph. Let S' be the set of simplicial vertices of $G[R]$. Set $R' = R \setminus S'$. Let us denote by $S'_1, S'_2, \ldots, S'_l$ the simplicial cliques of $G[S']$. We assert that R' is a clique. Suppose, towards a contradiction, that R' is not a clique. Hence, by Lemma 2, there exist two nonadjacent simplicial vertices a and b in $G[R']$. Since a and b are not simplicial vertices of $G[R]$, each of them has at least a neighbor in exactly one simplicial clique S'_i. Notice also that if a has a neighbor in S'_i and b has a neighbor in S'_j, then $i \neq j$. Assume that a' and b' are neighbors of a and b in S_i and S'_j respectively, with $i \neq j$. Let $S_1, S_2, \ldots, S_k$ be the simplicial cliques of G. Analogously, since a' and b' are nonsimplicial vertices of G, there exist two distinct indices r and s such that a' has a neighbor a'' in S_r and b' has a neighbor b'' in S_s. Hence $\{a, a', a'', b, b', b''\}$ induces one of the graphs $2P_3$, $P_3 + K_3$ and $2K_3$, a contradiction. Therefore, R' is a clique of G.

We know that if $a_i \in S'_i$ then there exists at least an index j such that a_i has a neighbor in S_j. First, suppose that a simplicial clique, say S_1, of G has at least three vertices. Since G is $\{K_3 + P_3, 2K_3\}$-free, $G - (N[S_1])$ is

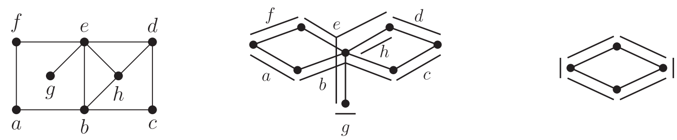

FIGURE 3. From *left* to *right*: a 2-unipolar graph, a representation of it as substars of a cactus, and a representation as substars of a cactus of $\overline{\textit{domino}}$.

$\{P_3, K_3\}$-free, and thus the result holds. Assume now that every simplicial clique of G has at most two vertices. Suppose that some of the simplicial cliques of $G - S$ together with their corresponding simplicial cliques of G, in which it has neighbors, have at least three vertices and say that the simplicial clique is S'_1. Hence, since G is $\{K_3 + P_3, 2K_3\}$-free and every simplicial clique of G has at most two vertices, $G - (N_{G-S}[S'_1])$ is $\{P_3, K_3\}$-free and thus the result holds. In the remaining case, $G - R'$ is a disjoint union of cliques with at most two vertices. □

Now we are ready to prove the main result of this section.

Theorem 1. *A graph G is a 2-unipolar graph if and only if G contains none of the graphs depicted in Figure 2 as an induced subgraph.*

Proof. It is easy to prove that graphs in Figure 2 are minimal forbidden induced subgraphs for the class of 2-unipolar.

Conversely, if G is a chordal graph the result follows from Proposition 2. Otherwise, since G is C_k-free for every $k \geq 5$, G contains at least a C_4 as an induced subgraph. By Proposition 1, G contains an induced matching M of simplicial edges such that $G - V(G[M])$ is chordal. Set $S = V(G[M])$ as the set of those vertices saturated by M. As a consequence of Proposition 2, there exists a clique C such that $H = (G - S) - C$ has no P_3 and K_3 as induced subgraphs. If there are no edges with an endpoint in S and the other one in $V(H)$, then the result holds. Otherwise, it can be proved that there exists exactly one clique Q in H having neighbors in S. In that case, since G is $\{2K_3, 2P_3, K_3 + P_3\}$-free, $G - N_C(Q)$ is $\{P_3, K_3\}$-free. □

Intersection model for 2-polar graphs

It is well known that split graphs are the intersection graphs of substars of a star. Nevertheless, this intersection graph class, known as starlike graphs, properly contains split graphs and they were characterized by forbidden induced subgraphs by Cerioli and Szwarcfiter [1]. We note that 2-unipolar graphs have a similar intersection model. A *starlike cactus* is a cactus having at most one cut-vertex. Given a 2-polar graph, it is easy to find an intersection model as substars of a starlike cactus, indeed this starlike cactus can be chosen in such a way that its blocks are cycles of length at most four (*e.g.*, Fig. 3). Besides, 2-unipolar graphs are properly contained in this intersection graph class. Interesting enough is to study which graphs have a representation as the intersection graph of substars of a cactus. Indeed, this intersection graph class generalizes the widely studied graph class of circular-arc graphs [12].

3. SPLIT-WIDTH

A *k-probe split graph* (resp. k-probe complete graph) is a k-probe $\mathcal{G}$ graph where $\mathcal{G}$ is the class of split graphs (resp. complete graphs). It is easy to show that if G is a k-probe split graph if and only if there exists an independent set S such that $G - S$ is k-probe complete. The *split width* (resp. complete width) of a graph, denoted $spw(G)$ (resp. $cow(G)$), is the $\mathcal{G}$-width where $\mathcal{G}$ is the class of split graphs (resp. complete graphs).

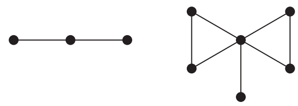

FIGURE 4. $P_3 + \overline{W_4 + K_1}$, a minimal forbidden induced subgraph for 2-probe split cographs.

An edge clique cover of a graph G is a family of complete subgraphs such that each edge of G is also an edge of at least one member of the family. The minimal cardinality of an edge clique cover is the edge clique cover number, denoted by $\theta_e(G)$. Chang *et al.* observed the following fact.

Proposition 3 ([3]). $cow(G) = \theta_e(\overline{G})$.

In particular, G is k-probe split if and only if there exists an independent set S of G such that $cow(G-S) = \theta_e(\overline{G} - S) \leq k$. Consequently, since the problem of deciding whether $\theta_e(G) \leq k$ given a graph G and an integer k, is NP-complete [19], the corresponding decision problem in connection with $cow(G)$ is also NP-complete. We would like to mention that the complexity of the decision problem, related to clique cover, when the graph of the input is a cograph, is unknown. The next lemma, whose simple proof is omitted, will allow us to prove the NP-completeness of the decision problem related to the split-width of a graph.

Lemma 3. *Given a graph G and a positive integer k, then $spw(G) \leq k$ if and only if there exists a maximal independent set S of G (resp. a maximal clique S of $\overline{G}$) such that $\theta_e(\overline{G} - S) \leq k$.*

Lemma 3 plays a central role in the following result related to the complexity of the decision problem in connection with the split width.

Theorem 2. *Let G be a graph and k be a positive integer. Then, the problem of deciding whether $spw(G) \leq k$ is NP-complete.*

Proof. We will prove the result by means of a reduction from the edge clique-cover problem. Given a graph G and a positive integer k, set $H = \overline{G} \vee \overline{K_{1,k+1}}$ and thus $\overline{H} = G + K_{1,k+1}$. We will proceed to demonstrate that $\theta_e(G) \leq k$ if and only if $spw(H) \leq k$. First, assume that $\theta_e(G) \leq k$. If S is the maximal clique of $\overline{H}$ formed by the vertex of degree $k+1$ of the copy of K_{k+1} and any other vertex of it, then it is easy to prove that $\theta_e(\overline{H} - S) = \theta_e(G) \leq k$. Therefore, by Lemma 3, $spw(H) \leq k$. Finally, assume now that $spw(H) \leq k$. Thus there exists a maximal clique S of $\overline{H}$ such that $\theta_e(\overline{H} - S) \leq k$. Necessarily, this maximal clique is comprised of the maximum degree vertex of the copy of K_{k+1} plus any other vertex of it, because otherwise $\theta_e(\overline{H} - S) \geq \theta_e(K_{k+1}) = k+1$. Therefore, $\theta_e(G) = \theta_e(\overline{H} - S) \leq k$. □

The following lemma is a consequence of Theorem 12 from [17].

Lemma 4. *Let G be a cograph; then G is a 2-probe complete graph if and only if $\overline{G}$ contains none of the graphs in $\{C_4, K_{1,3}, P_3 + K_2, 3K_2\}$ as an induced subgraph.*

The following lemma is a consequence of Theorem 4 from [15].

Lemma 5. *Let G be a cograph. Then, G is a 1-probe split graph if and only if $\overline{G}$ contains none of the graphs in $\{K_{2,3}, 2P_3\}$ as an induced subgraph.*

In [18], k-probe complete graphs are characterized by forbidden induced subgraphs, for $k = 2, 3$. When $k = 2$, it turns to be a subclass of cographs. In the following result, we deal with those cographs which are 2-probe split.

Theorem 3. *Let G be a cograph. Then, G is 2-probe split if and only if $\overline{G}$ contains none of the graphs in $\{4K_2,\ K_2 + 2P_3,\ K_2 + K_{2,3},\ 2K_{1,3},\ K_{2,4},\ K_{3,3},\ C_4 + P_3,\ C_4 + 2K_2,\ P_3 + \overline{W_4 + K_1}\}$ as an induced subgraph.*

Proof. It is easy to prove that all of the complement of: $4K_2$, K_2+2P_3, $K_2+K_{2,3}$, $2K_{1,3}$, $K_{2,4}$, $K_{3,3}$, C_4+P_3, C_4+2K_2 and $P_3+\overline{W_4+K_1}$ (see Figure 4) are minimal forbidden induced subgraphs for the class of 2-probe split graphs within the class of cographs.

Conversely, assume that G is a cograph such that $\overline{G}$ contains no graph in the list of the statement as an induced subgraph. We will prove that there exists a maximal independent set S of G such that $\theta_e(\overline{G}-S)\leq 2$. We proceed by induction on the number of vertices of G. We will split the proof into two cases.

Case 1. $\overline{G}$ is disconnected with connected components $G_1,\ldots,G_k$ such that at least two of them are nontrivial. Since $\overline{G}$ has no $4K_2$ as an induced subgraph, $\overline{G}$ has at most three nontrivial connected components.

Case 1a. $\overline{G}$ has exactly one nontrivial connected component.

The proof follows from the inductive hypothesis.

Case 1b. $\overline{G}$ has exactly two nontrivial connected components.

Assume, without losing generality, that G_1 and G_2 are nontrivial. Since $\overline{G}$ does not contain $2K_{1,3}$ as an induced subgraph, either G_1 or G_2 does not contain $3K_1$ as an induced subgraph (see "Graph classes" in Sect. 1.1). Hence, by Lemma 5, either $\overline{G_1}$ or $\overline{G_2}$ is a 1-probe split graphs, say $\overline{G_1}$. First, assume that G_1 contains P_3 as an induced subgraph. Hence, since $\overline{G}$ does not contain C_4+P_3 as an induced subgraph, G_2 is a trivially perfect graph. Consequently (see "Graph classes" in Sect. 1.1), either G_2 is a complete graph, in which case the result trivially holds because $\overline{G_1}$ is a 1-probe split graph, or there exists a complete graph Uwhose vertices are universal in G_2, such that $G_2=U+W$, where W is a disconnected graph whose components are $W_1,\ldots,W_k$. Since $\overline{G}$ is $4K_2$-free, at most two of $W_1,\ldots,W_k$ are non trivial graphs. Assume, without losing generality, that the possible nontrivial components are $W_1,\ldots,W_j$ with $1\leq j\leq 2$. Suppose first that $j=2$. On the one hand, since $\overline{G}$ is $4K_2$-free, G_1 is a threshold graph. On the other hand, since $\overline{G}$ is $\{K_2+2P_3\}$-free, W_1 and W_2 are complete graph. Therefore, there exists a clique C in G_1 such that G_1-C is an edgeless graph. In addition, since $\overline{G}$ is $\overline{W_4+K_1}$-free, $k=j=2$. Therefore $\overline{G}-C$ can be covered with two cliques. Suppose now that $j=1$. Since $1<k$, following the same argument as before, since $\overline{G}$ is $\overline{W_4+K_1}$-free, we conclude that W_1 is a threshold graph, implying that W_1 is split. Hence there exists a clique $C=U\cup C_{W_1}$ in G_2 such that G_2-C is an edgeless graph. Besides, since G_2 contains P_3 as induced subgraph, $\overline{G}$ is $\{C_4+P_3, K_2+2P_3, 4K_2\}$-free and $\overline{G_1}$ is $3K_1$-free, it follows from Lemma 4 that G_1 is 2-probe complete, $\theta(G_1)\leq 2$, and the result holds.

Case 1c. $\overline{G}$ has exactly three nontrivial connected components.

On the one hand, since $\overline{G}$ is K_2+2P_3-free, at least two of G_1, G_2 and G_3 are complete graphs, say G_1 and G_2. On the other hand, since $\overline{G}$ is $4K_2$-free, G_3 is a threshold graph, and thus there exists a clique C in $\overline{G}$ such that G_3-C is an edgeless graph. Therefore, $\overline{G}-C$ is the disjoint union of two complete graphs plus some isolated vertices.

Case 2. $\overline{G}$ is connected.

Hence $\overline{G}=U\vee V$.

Case 2a. $\alpha(U)\geq 4$.

Since $\overline{G}$ does not contain $K_{2,4}$ as an induced subgraph, V is a complete graph. By the inductive hypothesis, there exists a clique C in U such that $\theta_e(U-C)\leq 2$. Therefore, $\theta_e(\overline{G}-(C\cup G[V]))\leq 2$.

Case 2b. $\alpha(U)\leq 3$.

Assume first that $\alpha(U)=3$. Since $\overline{G}$ is $K_{3,3}$-free, V is $3K_1$-free. Hence, by Lemma 5, $\overline{V}$ is 1-probe split and since V is $3K_1$-free, there exists a clique C_V such that $V-C_V$ is a complete graph. By the inductive hypothesis, U is a 2-probe split graph and thus there exists a clique C_U such that $U-C_U$ can be covered with at most two cliques. Therefore, $\overline{G}-(C_U\cup C_V)$ can be covered with at most two cliques and $C_U\cup C_V$ is a clique. Assume now that $\alpha(U)\leq 2$, again by Lemma 5, U is a 1-probe split graph and the result follows by combining this fact with the inductive hypothesis.

□

We characterize 2-probe split cographs through a finite list of minimal forbidden induced subgraphs. In fact, we can conclude that deciding whether a cograph is 2-probe split can be solved in $O(n^8)$-time, as the largest of these forbidden induced subgraphs contains 8 vertices. However, we believe that this time complexity can be improved by finding a more efficient algorithm.

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