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SUPPLY CHAIN MODEL HAVING STOCHASTIC LEAD TIME DEMAND WITH VARIABLE PRODUCTION RATE AND DEMAND DEPENDENT ON PRICE AND ADVERTISEMENT

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Abstract. In this present study, a single-manufacturer and single-retailer supply chain management model are formulated for a single product. This study specifically looks at a supply chain with variable production rate, stochastic lead time demand, and price- and advertisement-dependent demand. By incorporating these complex aspects into a model, which enables to examine their combined effects on supply chain performance, this study adds to the body of knowledge. The study reveals unique insights into the complex interplay between pricing tactics, advertising efforts, production dynamics, and the variability brought on by stochastic lead times through meticulous study and modelling. Finally, the total system profit is calculated and optimized with all the decision variables. A classical approach is performed to obtain the optimized solution of the joint profit function along with the decision variables. Two models are discussed in the study: (1) the model with normally distributed lead time demand and (2) the model with distribution-free lead time demand. The joint profit of the supply chain is found to be lesser by 1% for the normally distributed lead time demand than the distribution free pattern. The comparison of the shipment policies and the safety factors for the different distribution patterns of the lead time demand are shown. Though the huge increment in the safety factor for unknown leadtime demand distribution may help in reducing the uncertainity factor and disruptions in the supply chain, but also it may unnecessarily tie up more capital which can be invested in other sectors of the supply chain.

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1. Introduction

This section consists of the motivation of the study, research questions that arose during the study, contribution of the study, and organization of the manuscript.

1.1. Motivation

Supply chain management (SCM) is the backbone of the society starting from the ecological or natural resources and ending up as finished products to be consumed by the households. The supplier, manufacturer,

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and retailer are the players of SCM. The cost minimization or profit maximization with the required demands in any supply chain (SC) is the key objective of the research. Looking from an organization's point of view, the job of advertising in supply chains has been on the ascent. Putting resources into advertisement can be considered as an empowering agent for working on the picture of and the interest in items. Along these lines, supply chains utilize different strategies to make cooperative energy by thinking about various devices, like publicizing and greening [19]. Hybrid modeling of green and sustainable supply chain with pricing and advertisement factors has been strategized with uncertain demand by Khorshidvand *et al.* [22].

Supply chains should consolidate factors that decidedly sway the attracted demand, like smart items, pricing, and advertisement. Sarkar *et al.* [18] proposed one study on decentralized SCM to manage two types of inventory. Clients focus closer on smart items and are impacted by new advertising modes, *e.g.*, social media and print media. Thus, the simultaneous use of these components, *i.e.*, assessing and publicizing, seems, by all accounts, to be essential to make the conditions applied in insightful assessment closer to genuine cases. Sometimes the longer lead time of any smart product may divert customer attraction to another product of interest. Lead time is the duration between the order placing and receiving. In SCM, lead time can be shortened by incurring some additional expenses. Sometimes the administrative cost can be added by including part-time workers or by the timing of regular employees to the reduction of lead time. Transportation is another component of lead time. For crashing the lead time duration, faster transportation is required, resulting in some additional costs. For crashing the lead time, the cost incurred is called lead time crashing cost (LTCC) [27]. In the SC of smart products, the pricing of a product, advertisement of a product, and investment in waiting time reduction play a key role. The model's necessity and practicality lie in its ability to provide a more realistic representation of SC dynamics, which can lead to better decision-making and improved overall SC performance.

- Necessity of the present work
  - Realistic demand dynamics: In many industries, retailer demand is influenced not only by the product's price but also by the effectiveness of advertising and promotional efforts. By considering these factors, the model captures a more accurate representation of how the retailer demand evolves in response to changes in marketing strategies and pricing.
  - Supply chain coordination: The model's consideration of both manufacturer's production rate and retailer's demand response to pricing and advertising efforts enables better supply chain coordination. This coordination can help prevent imbalances between supply and demand, reduce stockouts, and optimize inventory levels.
  - Pricing and marketing strategy optimization: The model aids in optimizing pricing and advertising strategies to maximize profitability and market share. Understanding how price and advertising levels impact retailer demand allows the manufacturer to find the optimal balance between revenue and costs.
- Practicality of the model
  - Adaptability and flexibility: The model can be adapted to specific industry contexts and SC structures, making it practical for a wide range of businesses.
  - Strategic insights: The insights generated from the model provide strategic direction to improve customer service, revenue generation, and overall SC efficiency.

1.2. Research questions

From a realistic viewpoint, consumers' behavior depends on the pricing of the products. Demand dependent linearly on selling price is often used in different research proposals [37]. The stock of products should be exhausted in time such that the quality and goodwill of the product should remain intact. Thus, different policies and mathematical models are generated on the demand pattern dependent on the time and pricing of the products. Some research questions come up in this view.

Q1. **Optimal pricing and advertising strategies:** What pricing and marketing tactics will optimize the manufacturer's profit while satisfying demands and taking production variability into account?

Q2. **Impact of advertisement effectiveness on demand:** How does the effectiveness of advertising efforts influence retailer demand for the product, and how can manufacturer optimize their advertising budget to achieve the best possible results?

Q3. **Risk analysis of parameters:** What are the supply chain's possible risks and weaknesses under conditions of price- and advertisement-dependent demand and unpredictable output, and how can the SC be strengthened?

In view of such research questions, the following contributions are incorporated into the present study.

1.3. Contribution and organization

Research is done with controllable lead time, price-dependent demand, stock-dependent demand, advertisement-dependent SC, and variable production rate. The major contribution of this study is the modeling of a complex SC system where the lead time demand (LTD) is stochastic, production is variable, and simultaneously the demand is dependent on price and advertisement. Previous works were done either for economic order quantity (EOQ) models or supply chains without the presence of stochastic LTD. In the previous research, the factors of price, advertisement, stochastic LTD, and variable production were not taken in a single model. The production-consumption SC of the smart product justifies this complex model. The novelty of this study is the aggregation of all these factors in a single model.

The association of the rest of this study is as per the following: Section 2 is constructed with past research related to the present study. Section 3 is related to the assumptions made, the notation used, and the problem formulation of the present study. Section 4 shows the methodology used in solving the models. Section 5 gives two numerical examples for justification of the model mathematically. Section 6 gives the expected value of additional information and sensitivity analysis of the parameters, while Section 7 is elaborated with managerial insights related to the research. Section 8 consists of the conclusions and future extension of the research work.

2. Literature review

In this section, the literature on different fields is discussed with the following literature gaps.

2.1. Lead time demand (LTD)

The LTD takes place during the time interval of ordering and receiving the number of products. The integrated vendor–buyer (IVB) model was proposed by Goyal [17] initially. Savvy item acquisition of a solitary client (customer) from a solitary provider (supplier) was examined in that paper. Sarkar *et al.* [47] discussed an advertisement-dependent SCM model without consideration of any lead time and lead time reduction. Mashud *et al.* [30] studied the production-inventory model of newsboy where the product deterioration was incorporated. In an extension of the IVB model proposed by Viswanathan [53], two policies of inventory management were discussed. In one model, the replenishment amount was considered identical and in the other policy, the vendor supplied to the buyer an amount that it contained. Guchhait and Sarkar [42] proposed a model where the LTD was stochastic. In that model, costs were optimized when the LTD followed a distribution free (DF) type. In addition to that study, Wangsa and Wee [8] proposed an integrated vendor–buyer problem where the demand was stochastic and transportation costs were included. In that study, the transportation cost was taken as a function of shipping weights. Habib *et al.* [11], minimized the cost of the SCM considering different environmental issues such as uncertainty but without considering lead time. Dey *et al.* [52] proposed a controllable lead time problem for an SCM where the production rate was variable. In a recent study by Sarkar and Chung [43], LTD was assumed to follow a normal distribution (ND), and a flexible rate of production was considered. Moreover, LTCC was incorporated for the reduction of lead time in supply chain (SC). Roy and Sana [38] proposed a reduction strategy of lead time and setup cost in two-echelon SCM where LTD was stochastic. The trade-credit financing was introduced for seetling down the account of the buyer. In all such studies of the IVB model and lead time demand, no research is done by taking simultaneous price, advertisement-dependent demand, and variable production rate. The present research fulfills such a gap in the research.

2.2. Variable production rate

The production rate of any product affects the supply of products and sometimes the rate of the product. In an earlier study by Taleizadeh *et al.* [45], multi-product from a single-machine production was considered with constant production rate. With the presumption that every product is produced on a single machine with a constrained capacity, a unique cycle length was taken into account for every generated item. In another study, Sarkar and Guchhait [46] proposed multi-retailer, single-manufacturer, and single-product problem with linearly varied production rate. In an advance study by sarkar *et al.* [44], an SCM model with a constant production and carbon emissions were discussed. A sustainable model for production was proposed by Mishra *et al.* [34] where the production model was run with and without shortages. Production rate is sometimes related to the environmental effect. Production rate may be constant, or variable depending on the item and its demand in the market. The smart production system is variable in nature. The production rate sometimes controls the cost of production which was first proposed by Khouja and Mehrez [23]. In an extended study by Aldurgam *et al.* [4], demand was taken as uncertain, and the production cost was taken as dependent on the production rate. The model's goal was to coordinate the product's manufacturing and distribution in order to minimize the supply chain's overall expenses. Aldurgam [3] proposed a model with a variable production rate with integrated stochastic dynamic lot-sizing. Another remarkable study on stochastic modeling of remanufacturing control was done by Li *et al.* [26]. Ullah and Sarkar [51] proposed a hybrid manufacturing and remanufacturing product model with product quality. An imperfect production inventory was studied by Giri and Dash [16] where the demand function was dependent on the greening level of the product and selling price. In order to decide on the best pricing, advertising, and inventory options, two models – one centralized and one decentralized – based on a Stackelberg game approach were constructed. A very recent study by Saranya and Chandrasekhar [41] proposed the study of the initial nonzero demand due to the advertisement in a depleted demand inventory model. The product of those models varied with the demand-production rate and corresponding costs. With variable production rate and optimal replenishment cycle of vendor-managed three layers SC, Salas-Navarro *et al.* [39] incorporated the study of probabilistic demand rate of the retailers.

2.3. Demand pattern and different models

SC models or EOQ models are mostly illustrated with different demand patterns and different cost parameters. Taleizadeh *et al.* [48] proposed a stochastic demand led model where multi-chance constraints were introduced. The objective was to establish the reorder point and order amount of each product for each customer in order to reduce the SC overall cost. The price discount policy has an impact on the SC model. Khan *et al.* [21] discussed a model where the demand was price and stock dependent and the discount policies on full or partial advance payments. Another study on lot-sizing was developed by Leuveano *et al.* [25] where the total cost of vendor–buyer was optimized with delivery quantity, process quality, and several shipments. Another study by Kim and Van Oyen [24] was done on dynamic production and corresponding financial benefits. In another study by Darom *et al.* [9], a branch and bound heuristic method was applied to solve the SC disruption where the transportation cost was included. Moreover, the model was constructed with one vendor and one buyer. It was underlined that an increase in setup and ordering costs will result in an increase in the ideal safety stock quantity within the same number of recovery cycles. Another hierarchical heuristic approach was taken by Aazami and Saidi-Mehrabad [1] to solve a problem with competitive factors upon perishable goods. Due to the perishability of the product, freshness is seen as an essential competitive criterion in addition to price. An efficient algorithm was developed by San-José *et al.* [40] to optimize the profit function of an EOQ model under the advertisement-dependent and selling price-dependent demand. An auto-regressive demand was taken in a study by Michna *et al.* [32] where the bullwhip effect was quantified for revealing the stochastic lead time forecasting and impact of random demand. Several inferences about the bullwhip behavior were made about the demand auto-correlation, the quantity of historical lead times, and the demands employed in the forecasts. Ma *et al.* [28] proposed a model where two heuristics were applied to optimize the cost of the production process. Another study was developed by Noori-Daryan *et al.* [36] where the demand was dependent on price and delivery LTD.

Table 1. Table of comparison with previous researches.

Authors	Model	Demand rate		Production rate	Lead time demand	Safety factor	Transport policy
		Price	Advertisement				
Viswanathan [53]	IVB	○	○	◇	○	○	○
Taleizadeh *et al.* [48]	MBSV	○	○	◇	Variable	○	SSMD
Sarkar and Majumder [42]	IVB	○	○	◇	ND and DF	□	SSMD
Wangsa and Wee [8]	IVB	○	○	◇	ND	□	SSMD
Darom *et al.* [9]	IVB	○	○	◇	○	□	○
San-José *et al.* [40]	EOQ	□	□	○	ND	○	○
Khan *et al.* [21]	EOQ	□	○	○	○	○	○
Aldurgam *et al.* [3]	MDP	○	○	⊞	○	○	○
Dey *et al.* [13]	EMQ	○	○	⊞	○	○	○
Gautam *et al.* [15]	EOQ	□	□	○	○	○	○
Saha *et al.* [29]	EOQ	□	□	○	○	○	○
Giri *et al.* [16]	SC	□	□	Uniform	○	○	○
Chaudhari *et al.* [5]	SC	□	○	⊞	○	○	○
Saranya and Chandrasekaran [41]	EOQ	□	□	○	○	○	○
Mirzaei *et al.* [33]	O2O CLSC	□	□	◇	○	○	○
Salas-Navarro *et al.* [39]	Three-echelon SC	○	○	⊞	○	○	○
This paper	IVB	□	□	⊞	ND and DF	□	SSMD

Notes. ○ – Not applicable, MBSV – Multiple-buyer single-vendor, MDP – Markov decision process, EOQ – Economic order quantity, □ – Yes, ◇ – Constant, PD – Poisson distribution, ⊞ – variable, NV – Newsvendor.

In that study, with unknown demand response times, the best options made by a global chain of pharmaceutical producers with capacity constraints and one retailer were examined. An algorithm was developed by Cui *et al.* [7] with stochastic LTD where ordering cost, holding cost, lost sale cost, and transportation costs were optimized. That study extended the multi-item joint replenishment-distribution problem (JRD) with stochastic lead time and demands to examine their mutual impacts on the JRD system, assuming a Business-to-consumer (B2C) e-business organization with many regional distribution centers. A linear demand function, dependent on time over an infinite planning horizon, was discussed by Udayakumar [50] to obtain a new replenishment policy. By identifying the ideal interval and the ideal order quantity, the article helped the shop reduce the overall inventory cost. In a recent study by Gautam *et al.* [15] developed a model where demand was dependent on the price and advertisement but only optimization of selling price and batch size were considered. A model with stochastic market demand and used product's return rate was discussed by Sarkar *et al.* [6]. In all research works, the demand pattern is taken as either price-dependent or advertisement-dependent. Sometimes LTD is taken into consideration, sometimes not. In the view of the literature discussed, the gap is found in the integrated study of price-advertisement-dependent demand with variable production rate, and stochastic LTD. The proposed study deals with the fulfillment of such literature gaps. Some comparison with previous research is given in Table 1.

In most of the literature discussed in Section 2, the works are done regarding either selling price, advertisement, or both. The model's sophistication lies in its realistic representation of real-world SC complexities. By considering price and advertisement-dependent demand, variable production, and a single setup multiple single-setup-multi-delivery (SSMD) system, the research acknowledges the intricacies that manufacturers and retailers face in today's dynamic markets. Another contribution of the study is to analyze the safety factor of the inventory in different distribution cases of LTD. Investigating inventory management with variable production rates enhances the supply chain's adaptability to changing market conditions and potential disruptions is another contribution to this work. Overall, the novelty of this research lies in its comprehensive and integrated approach to addressing multifaceted challenges in SCM.

3. Problem description

This section incorporates the study of assumptions to be made for building up the model, the notation to be used in the mathematical expressions, and the mathematical model. The components of the section are following.

3.1. Assumptions

The accompanying presumptions are taken for the numerical model.

A1. An integrated SSMD model of single-manufacturer and single-retailer is considered.
A2. Demand is selling price and advertisement-dependent.
A3. After getting an order M from the retailer, the manufacturer produces νM amount with a finite but variable rate of production $P(P > d)$ but the delivery of M quantity occurs over ν times.
A4. The reorder point is $R_1 = dl + k\sigma\sqrt{l}$ where $dl =$ the average (expected) LTD, $k\sigma\sqrt{l} =$ safety stock, $k =$ safety factor. LTD is a random variable (rv).
A5. Shortages are allowed.
A6. The lead time l has s mutually independent parts. Place the assumption A8 here. $e_i =$ least time duration, $f_i =$ standard duration and $g_i =$ crashing cost per unit time.
A7. Total transportation cost is negligible as the cost per unit time is taken constant from manufacturer to retailer.

3.2. Notation

The accompanying documentation is utilized to detail the numerical model (Tab. 2).

3.3. Problem formulation

The current model is a SC model with a single-manufacturer, single-retailer, and single-product in flow. The retailer's advertising budget and selling price have an impact on demand of the SC model. The manufacturer's manufacturing pace is regarded as changeable. Following SSMD policy is how the finished products are shipped. In two circumstances, LTD is taken into account because ordering and delivery take time. The first situation is taken into consideration as the LTD is normally distributed, but the second case is taken into consideration as the LTD is DF. When reducing setup costs during the SC flow, the initial setup cost expenditure is taken into account. The dependent demand function and variable production rate integratedly affect the overall profit of the SC.

In Figure 1, single-retailer has demand which is dependent on the price of the product and the advertisement factor. The LTD is stochastic and follows either normal distribution or no distribution. The manufacturer has a variable production and sends the finished product to the retailer by SSMD policy. Manufacturer demand is the same as the retailer's demand. In an SC model with advertisement and price-dependent demand of the retailer, several factors can influence the manufacturer's demand. The manufacturer's demand refers to the quantity of products the retailer orders from the manufacturer. Understanding these factors is crucial for the manufacturer to effectively manage its inventory, production, and overall SC operations. The retailer's price sensitivity is a critical factor in the demand function. If the retailer is highly sensitive to price changes, small variations in the product's price can lead to significant changes in the quantity demanded by the retailer. The effectiveness of the retailer's advertising efforts can impact demand. Effective advertising can lead to increased product awareness and desire, resulting in higher demand for the manufacturer's products. In Figure 1, the retailer's demand depends on price and advertisement which consequently affects the manufacturer demand as explained.

TABLE 2. Notation for all decision variables and parameters for the mathematical model.

Decision	Variables
Se	Manufacturer setup cost ($/setup)
M	Retailers order quantity (units/cycle)
l	Retailer's lead time (time unit)
ν	Shipment number (integer)
p_r	Selling price of unit product by retailer ($/unit)
P	Rate of production of the manufacturer (units/year)
ξ	Advertisement variable
k	Safety factor
Parameters	
d	Demand of the retailer (units/year)
O	Ordering cost of the retailer ($/order)
p_m	Selling price of unit product by manufacturer ($/unit)
Se_0	Setup cost of the manufacturer initially ($/setup)
C_p	Cost of production of unit item for manufacturer ($/unit)
h_m	Manufacturer holding cost ($/unit time/unit)
h_b	Retailers holding cost ($/unit time/unit)
τ	Unit backlogging cost of retailer ($/unit)
$C(l)$	LTCC ($/unit)
σ	Standard deviation of LTD
$x+$	max value of 0 and x
R_1	Reordering point of retailer (units)
θ	Portion of annual cost of capital investment (%)
p_{maxr}	Maximum selling price of the item by retailer (retailer) ($/unit)
p_{minr}	Minimum selling price of the item by retailer (retailer) ($/unit)
θ_2	Shape parameter for advertisement variable
α	Scaling parameter of one time advertisement cost function
a_1, a_2	Scaling parameters of the demand
f_i	Standard duration of lead time
e_i	Minimum duration of lead time
g_i	Unit crashing cost of lead time for each component i

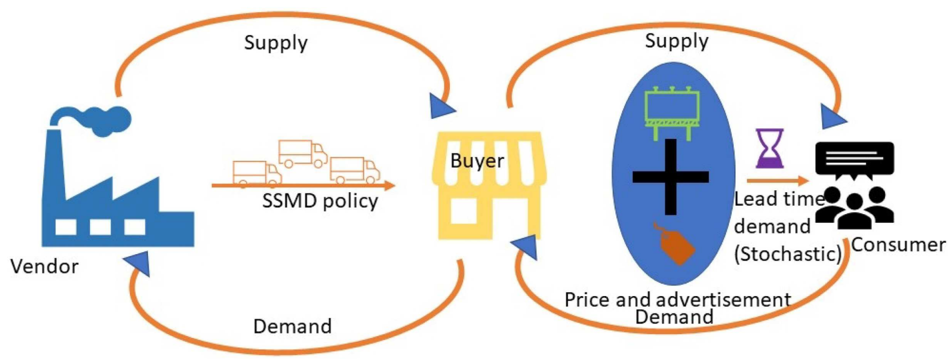

FIGURE 1. Diagram of the model with price and advertisement-dependent demand and variable production rate.

3.4. Mathematical illustration of the model

This section expounds on the mathematical model of both manufacturer and retailer inventory and respective costs. The section is divided into two parts. The first part has the LTD following ND and the second part contains the DF approach. In each part of the solution, the joint profit function is obtained and finally optimized. This section is categorized into retailer's profit component and manufacturer's profit component.

3.4.1. Buyers profit component

In the present study, only one retailer considered. The retailer's demand is price and advertisement-dependent and is given by

$$d = a_1 \frac{p_{maxr} - p_r}{p_r - p_{minr}} + a_2 \xi^{\theta_2}. \tag{1}$$

The demand of the retailer is given as where $a_1 =$ scaling parameter for selling price, a_2 is the scaling parameter of the advertisement variable, and θ_2 is the shape parameter. The following costs are generated.

3.4.1.1. Ordering and purchasing cost. The ordering and purchasing cost of a retailer in a SC is the cost associated with placing and processing orders to replenish inventory. This cost is incurred each time the retailer initiates a purchase order to the manufacturer to restock goods. The ordering cost is an essential consideration in inventory management as it influences the retailer's overall SC costs and impacts the ordering and replenishment policies. The cycle length for the retailer is $\frac{M}{d}$. The retailer orders the finished goods in each time cycle from the manufacturer and incurs ordering costs. Then, the retailer buys those ordered goods. Hence, the ordering and purchasing cost per cycle cost is given by [49]

$$\mathrm{OC}_b = \frac{Od}{M} + p_m d\cdot \tag{2}$$

3.4.1.2. Holding cost. The inventory holding cost for the retailer in a SC is the cost associated with storing and managing inventory over a specific period. This cost arises from various factors related to inventory management and includes both direct and indirect expenses. The inventory holding cost is an important consideration for the retailer as it impacts their overall SC costs and profitability. The reordering point of the retailer is fixed at R_1. The inventory of the retailer before replenishment is $R_1 - dl$ and after replenishment is $M + (R_1 - dl)$. Hence, the expected holding cost is

$$\mathrm{EHC}_b = h_b \left[\frac{M}{2} + (R_1 - dl) \right]. \tag{3}$$

3.4.1.3. Shortage cost. Shortage cost, known as stockout cost, is the cost incurred by the retailer in a SC when there is an insufficient inventory level to meet customer demand. Stockouts occur when the retailer runs out of stock and cannot fulfill customer orders in a timely manner. The shortage cost is an essential consideration in inventory management, as it can have significant financial and non-financial consequences for the retailer. The LTD X is a RV. If $X > R$, then a shortage is taken into consideration. The expected shortage cost of the retailer per unit of time is

$$\mathrm{ESC}_b = \frac{\tau d E(X - R_1)^+}{M}\cdot \tag{4}$$

3.4.1.4. Lead time crashing cost (LTCC). LTCC refers to the additional expenses incurred when the retailer takes measures to reduce the lead time of a particular process or project. Crashing, in project management or SC context, refers to the act of accelerating the schedule or reducing the time required to complete a task or deliver a product. LTCC per unit time is given as

$$\mathrm{LTCC}_b = \frac{dC(l)}{M}\cdot \tag{5}$$

3.4.1.5. Advertisement cost. The one-time advertisement for any SC depends on the specific advertising strategy and objectives set by the company, as well as the market conditions and available resources. One time advertisement cost is given by [20]

$$\mathrm{AC}_b = \alpha \frac{\xi^2}{2}\cdot \tag{6}$$

Therefore the total expected cost of the retailer is

$$\begin{aligned}\mathrm{TCB}\,(M,k,l,p_r,\xi) &= \text{holding cost} + \text{ordering cost} + \text{purchasing cost} + \text{shortage cost} + \text{LTCC} + \text{advertisement cost}\\ &= \frac{d}{M}\Big(O + C(l) + \tau E(X-R_1)^+\Big) + h_b\left[\frac{M}{2} + (R_1 - dl)\right] + p_m d + \alpha\frac{\xi^2}{2}\\ &= \frac{\left(a_1\frac{p_{maxr}-p_r}{p_r-p_{minr}} + a_2\xi^{\theta_2}\right)}{M}(O + C(l) + \tau E(X-R_1)^+) + h_b\left[\frac{M}{2} + (R_1 - dl)\right] + p_m d + \alpha\frac{\xi^2}{2}\cdot \end{aligned}\tag{7}$$

The profit of the retailer is

$$\begin{aligned}&\mathrm{TPB}\,(M,l,k,p_r,\xi)\\ &= p_r d - \left[\frac{\left(a_1\frac{p_{maxr}-p_r}{p_r-p_{minr}} + a_2\xi^{\theta_2}\right)}{M}\Big(O + C(l) + \tau E(X-R_1)^+\Big) + h_b\left[\frac{M}{2} + (R_1 - dl)\right] + p_m d + \alpha\frac{\xi^2}{2}\right]\cdot \end{aligned}\tag{8}$$

3.4.2. Manufacturer's profit component

In this study, an SSMD model is considered with a single-manufacturer and a single-retailer with a single type of product. When the retailer orders M quantity of product, the manufacturer produces νM amount of product in one production cycle. Therefore the cycle lengths for the retailer and manufacturer are $\frac{M}{d}$ and $\frac{\nu M}{d}$, respectively.

3.4.2.1. Setup cost. The setup cost for the manufacturer in the supply chain SC is the cost incurred each time the manufacturing process is set up to produce a specific product or a batch of products. It is known as the production setup cost or changeover cost. The setup cost per unit of time is given as

$$\mathrm{SC}_m = \frac{Sed}{\nu M}\cdot \tag{9}$$

Setup cost is taken fixed in many cases [12], but an investment is incorporated for the setup cost reduction. The investment is given as

$$\mathrm{SCI}_m = \beta_v \ln\left(\frac{Se_0}{Se}\right)\cdot \tag{10}$$

$\beta = \frac{1}{\Lambda}$ and Λ is the portion of reduction of setup cost upon per unit increase in investment.

3.4.2.2. Holding cost. The manufacturer produces finished goods and sends them to the retailer with SSMD policy and stores those products for further delivery which incurs the holding cost. The average inventory per cycle for the manufacturer is

$$\begin{aligned}\mathrm{Inv}_v &= \left[\left\{\nu M\left(\frac{M}{P} + (\nu-1)\frac{M}{d}\right) - \frac{\nu^2 M^2}{2P}\right\} - \left\{\frac{M^2}{2}(1+2+3+\ldots+(\nu-1))\right\}\right]\frac{d}{\nu M}\\ &= \frac{M}{2}\left[\nu\left(1-\frac{d}{P}\right) - 1 + \frac{2d}{P}\right]\cdot \end{aligned}\tag{11}$$

Hence, the holding cost of the manufacturer is

$$\mathrm{HC}_m = h_m \frac{M}{2}\left[\nu\left(1-\frac{d}{P}\right)-1+\frac{2d}{P}\right]. \tag{12}$$

3.4.2.3. Production cost. Unit production cost of the manufacturer is $(a_3P+\frac{a_4}{P}+a_5)$, where a_3 is the tool or die cost, a_4 is the development cost, and a_5 is the material cost. Therefore, the total production cost per unit time [10] is

$$\mathrm{PC}_m = \left(a_3P+\frac{a_4}{P}+a_5\right)d. \tag{13}$$

Therefore, the expected total cost of the manufacturer is

$$\begin{aligned}\mathrm{TCV} &= \text{setup cost} + \text{holding cost} + \text{production cost}\\ &= \frac{Sed}{\nu M}+\beta_v \ln\left(\frac{Se_0}{Se}\right)+h_m\frac{M}{2}\left[n\left(1-\frac{d}{P}\right)-1+\frac{2d}{P}\right]+\left(a_3P+\frac{a_4}{P}+a_5\right)d. \end{aligned}\tag{14}$$

The selling price of the item per unit by the manufacturer is p_m and the revenue generated is $p_m d$. Hence, the profit generated by manufacturer is

$$\begin{aligned}\mathrm{TPV}\,(M,P,Se,p_r,\xi,n) &= \text{revenue} - (\text{setup cost} + \text{holding cost} + \text{production cost})\\ &= p_m d-\left[\frac{Sed}{\nu M}+\beta_v\ln\left(\frac{Se_0}{Se}\right)+\left(a_3P+\frac{a_4}{P}+a_5\right)d+h_m\frac{M}{2}\left[\nu\left(1-\frac{d}{P}\right)-1+\frac{2d}{P}\right]\right]. \end{aligned}\tag{15}$$

Therefore, the joint profit of the SC is

$$\begin{aligned}\mathrm{JTP}\,(M,P,Se,p_r,\xi,v,k,l) &= (p_m+p_r)\left(a_1\frac{p_{maxr}-p_r}{p_r-p_{minr}}+a_2\xi^{\theta_2}\right)\\ &-\left[\begin{array}{l}\frac{d}{M}\left(O+C(l)+\tau E(X-R)^+\right)+h_b\left[\frac{M}{2}+\left(k\sigma\sqrt{l}\right)\right]+\alpha\frac{\xi^2}{2}+\frac{Sed}{\nu M}+\beta_v\ln\left(\frac{Se_0}{Se}\right)\\ +\left(a_3P+\frac{a_4}{P}+a_5\right)d+p_m d+h_m\frac{M}{2}\left[\nu\left(1-\frac{d}{P}\right)-1+\frac{2d}{P}\right]\end{array}\right]. \end{aligned}\tag{16}$$

In this study, for the LTD once the RV follows a normal distribution, and in another case, it follows the DF approach. Two different approaches are elaborated on in the next section.

3.5. LTD follows normal distribution (ND)

When the LTD X for the retailer follows the ND, the mean is taken as $dC(l)$ and the standard deviation is $\sigma\sqrt{l}$. The cumulative distribution function (CDF) is G and the reorder point $R_1 = dl + k\sigma\sqrt{l}$. For the shortage, the expectation of the RV is given as $E(X-R_1)^+ = \int_{R_1}^{\infty}(x-R_1)\,\mathrm{d}G(x) = \sigma\sqrt{l}\psi(k)$, where, $\psi(k)=\phi(k)-k[1-\Phi(k)]$. The standard normal probability density function is ϕ and the cumulative distribution function of the standard normal variable is Φ. k is the safety factor. The joint expected profit of the SC problem becomes

$$\begin{aligned}\mathrm{JTP}\,(M,P,Se,p_r,\xi,\nu,k,l) &= (p_m+p_r)\left(a_1\frac{p_{maxr}-p_r}{p_r-p_{minr}}+a_2\xi^{\theta_2}\right)\\ &-\left[\begin{array}{l}\frac{d}{M}\left(O+C(l)+\tau k\sigma\sqrt{l}\psi(k)\right)+h_b\left[\frac{M}{2}+\left(k\sigma\sqrt{l}\right)\right]+\alpha\frac{\xi^2}{2}+\frac{Sed}{\nu M}+\beta_v\ln\left(\frac{Se_0}{Se}\right)\\ +\left(a_3P+\frac{a_4}{P}+a_5\right)d+p_m d+h_m\frac{M}{2}\left[\nu\left(1-\frac{d}{P}\right)-1+\frac{2d}{P}\right]\end{array}\right]. \end{aligned}\tag{17}$$

The problem is further written as

$$\begin{aligned}\text{Max JTP}\,(M,P,Se,p_r,\xi,\nu,k,l) &= (p_m+p_r)\left(a_1\frac{p_{maxr}-p_r}{p_r-p_{minr}}+a_2\xi^{\theta_2}\right)\\ &-\left[\begin{array}{l}\dfrac{d}{M}\Big(O+C(l)+\tau k\sigma\sqrt{l}\psi(k)\Big)+h_b\left[\dfrac{M}{2}+\Big(k\sigma\sqrt{l}\Big)\right]+\alpha\dfrac{\xi^2}{2}+\dfrac{Sed}{\nu M}+\beta_v\ln\left(\dfrac{Se_0}{Se}\right)\\ +\Big(a_3P+\dfrac{a_4}{P}+a_5\Big)d+p_md+h_m\dfrac{M}{2}\left[\nu\left(1-\dfrac{d}{P}\right)-1+\dfrac{2d}{P}\right]\end{array}\right].\end{aligned} \tag{18}$$

Equation (18) is a non-linear programming problem and the problem can be settled by relaxing the constraint $0 \le Se \le Se_0$.

3.6. LTD is distribution free

As this approach does not include any conventional distribution, no assumptions are made for the distribution of LTD X. The CDF G belongs to a class ς with mean dl and variance $\sigma^2 l$.As the value of $E(X-R_1)^+$ is not known, one min–max DF approach is applied to it. The accompanying recommendation is utilized to inexact the worth of $E(X-R_1)^+$.

Proposition 1. *If G is the CDF of any distribution X and G belongs to ς, then the following in equality holds*

$$E(X-R_1)^+ \le \frac{1}{2}\left[\sqrt{\sigma^2 l+(R_1-dl)^2}-(R_1-dl)\right] \tag{19}$$

where the equation (19) has tight upper bound. The joint profit function can be written as

$$\begin{aligned}\text{Max JTP}_{df}(M,P,Se,p_r,\xi,\nu,k,l) &= (p_m+p_r)\left(a_1\frac{p_{maxr}-p_r}{p_r-p_{minr}}+a_2\xi^{\theta_2}\right)\\ &-\left[\begin{array}{l}\dfrac{d}{M}\left(O+C(l)+\dfrac{\tau\sigma\sqrt{l}\big(\sqrt{k^2+1}-k\big)}{2}\right)+h_b\left[\dfrac{M}{2}+\Big(k\sigma\sqrt{l}\Big)\right]+\alpha\dfrac{\xi^2}{2}+\dfrac{Sed}{\nu M}+\beta_v\ln\left(\dfrac{Se_0}{Se}\right)\\ +\Big(a_3P+\dfrac{a_4}{P}+a_5\Big)d+p_md+h_m\dfrac{M}{2}\left[\nu\left(1-\dfrac{d}{P}\right)-1+\dfrac{2d}{P}\right]\end{array}\right].\end{aligned} \tag{20}$$

JTP_{df} *is the joint profit in DF case.*

4. Solution method

The solution method is described in this section for two types of LTD. One follows ND and the other is the DF approach. Methodologies are as follows:

4.1. Case I: LTD follows ND

The equation (18) is partially differentiated with respect to M,k,Se,l,P,p_r, and ξ and equated to zero. For a fixed positive integer ν, the differentiation are given by

$$\frac{\partial \text{JTP}(M,P,Se,k,p_r,\xi,\nu,l)}{\partial M} = \frac{\chi_1}{M^2}-\chi_2 \tag{21}$$

$$\frac{\partial \text{JTP}(M,P,Se,p_r,\xi,\nu,k,l)}{\partial P} = \left[a_4-\frac{2h_m(\nu-2)}{M}\right]\frac{d}{P^2}-a_3 \tag{22}$$

$$\frac{\partial \text{JTP}(M,P,Se,p_r,\xi,\nu,k,l)}{\partial Se} = \frac{\beta}{Se}-\frac{d}{\nu M} \tag{23}$$

$$\frac{\partial \text{JTP}(M,P,Se,p_r,\xi,\nu,k,l)}{\partial \kappa} = \sigma\sqrt{l}\left[h_b - \frac{\tau d(1-\Phi(k))}{M}\right] \tag{24}$$

$$\frac{\partial \text{JTP}(M,P,Se,p_r,\xi,\nu,k,l)}{\partial p_r} = d - \frac{a_1(p_{\max r} - p_{\min r})}{(p_r - p_{\min r})^2} \times \left[p_r + \frac{dh_m M(\nu-1)}{2d^2} - \frac{\left(O + C(l) + \tau\sigma\sqrt{l}\psi(k) + \frac{Se}{\nu}\right)}{M}\right] \tag{25}$$

$$\frac{\partial \text{JTP}(M,P,Se,p_r,\xi,\nu,k,l)}{\partial \xi} = a_2\theta_2\xi^{\theta_2-1}\left[p_r - \frac{O + C(l) + \tau\sigma\sqrt{l\psi(k)}}{M} + \frac{dh_m(\nu-1)M}{2d^2}\right] \tag{26}$$

$$\frac{\partial \text{JTP}(M,P,Se,p_r,\xi,\nu,k,l)}{\partial l} = -\frac{\sigma}{2\sqrt{l}}[kh_b + \tau d\psi(k)] - \frac{d}{M}\frac{dC(l)}{dl} = -\frac{d}{M}\left(\frac{\tau\sigma\psi(k)}{2\sqrt{l}} - g_i\right) - \frac{h_b k\sigma}{2\sqrt{l}}. \tag{27}$$

Differentiating twice the profit function with respect to l, one has

$$\frac{\partial^2 \text{JTP}(M,P,Se,p_r,\xi,\nu,k,l)}{\partial l^2} = \frac{d}{M}\left(\tau\sigma\psi(k)l^{-\frac{3}{2}}\right) + \frac{h_b k\sigma l^{-\frac{3}{2}}}{4} > 0. \tag{28}$$

Therefore, the maximum value of the profit is obtained with fixed values of M, P, Se, p_r, k, ν, and ξ in the boundary points of the interval $[l_j, l_{j-1}]$. Keeping the value of v fixed, the values of the decision variables are obtained as follows:

$$M = \sqrt{\frac{\chi_1}{\chi_2}} \tag{29}$$

$$P = \frac{\beta\nu M}{d} \tag{30}$$

$$p_r = p_{\min r} + \sqrt{\frac{(p_{\max r} - p_{\min r})\left(p_r + \frac{h_m d^2 M(\nu-1)}{2d^2} + \frac{1}{M}\left(O + C(l) + \tau\sigma\sqrt{l}\psi(k)\right)\right)}{d}} \tag{31}$$

$$Se = \frac{\beta\nu M}{d} \tag{32}$$

$$\xi = \left(\frac{a_2\theta_2\left(p_r + \frac{h_m d^2 M(\nu-1)}{2d^2} + \frac{1}{M}\left(O + C(l) + \tau\sigma\sqrt{l}\psi(k)\right)\right)}{\alpha}\right)^{\left(\frac{1}{2-\theta_2}\right)}. \tag{33}$$

The proposition 2 is used to find the maximum profit function.

Proposition 2. *When the constraint* $0 \le Se \le Se_0$ *is relaxed for the joint profit function* $\text{JTP}(M,P,Se,p_r,\xi,\nu,k,l)$*, and* $M^*, P^*, p_r^*, Se^*, \xi^*$ *acts as the optimal values for the decision variables* M, P, p_r, Se*, and* ξ *respectively obtained from equations* (29) *to* (33) *then, for fixed* ν *and* l *belongs to* $[l_j, l_{j-1}]$ *the joint expected profit become maximum.*

The integer variable ν will attain its optimal value when the following inequality is satisfied.

$$\text{JTP}(\nu^* - 1) <= \text{JTP}(\nu^*) >= \text{JTP}(\nu^* + 1).$$

If the optimal value of the setup cost is greater than the initial value of the setup cost then, investment is no more required. The reorder point optimality is obtained by getting the optimal value of κ. An algorithm is generated for the attainment of the optimal values of the decision variables.

Algorithm 1.

Step 1. Initialize the value of the integer variable. Value of ν is set 1.
Step 2. Initiate a loop with the counter as the lead time duration. Steps 3–8 are to be repeated for every l_j, $j = 1, 2, \ldots, n$.
Step 3. Initiate the values of the decision variables setup cost reduction investment and safety factor. Set $Se_{j1} = 0$, $\kappa_{j1} = 0$.
Step 4. Evaluating the safety factor value in first step after finding the order quantity at initial step. Find out M_{i1} from equation (29) and evaluate k_{i2}.
Step 5. Evaluate the setup cost investment, production rate, advertisement variable value using the order quantity value at first step. Using the values of M_{i1} evaluate Se_{i1}, P_{i1}, ξ_{i1}.
Step 6. Condition to be applied for stop iteration and finding optimal values of the decision variables and optimal joint cost. Iterate the Steps 4 and 5 until the values of M^*, p_r^*, P^*, Se^*, and k^* are unchanged.
Step 7. After obtaining optimal values, the setup cost investment should be compared with initial value. If it is less then go to step 8, else the optimal setup cost investment will be the initial investment. If $se^* < se_0$ then goto Step 8. Else set $Se_i^* = Se_0$ and find out optimal values of M^*, P^*, ξ^*.
Step 8. For the Step 1, maximum profit value is obtained. If $\text{JTP}(M^*, k^*, Se^*, l^*, \nu, p_r^*, P^*, \xi^*) = \text{Max}_{i=1,2,\ldots,n}\ \text{JTP}(M^*, k^*, Se^*, l^*, \nu, p_r^*, P^*, \xi^*)$, then optimal solution is reached.
Step 9. Increasing the number of shipment ν, profit value is measured with previous value. If profit comes greater, Steps 2–8 will be iterated for getting maximum profit. Set $\nu = \nu + 1$. If $\text{JTP}(M^*, k^*, Ke^*, l^*, \nu, p_r^*, P^*, \xi^*) \leq \text{JTP}(M^{**}, k^*, Se^*, l^*, \nu + 1, p_r^*, P^*, \xi^*)\nu + 1$, then reiterate Steps 2–8.
Step 10. The step counter will stop if the optimality is reached. Set $m + 1$ as m^*. Obtain the maximum profit.

4.2. Case II: Distribution free approach

The joint profit of the SC model is optimized. This non-linear program is solved by making the constraint relaxation of $0 \leq Se \leq Se_0$. The integer n is taken as a fixed positive integer and the joint profit function $\text{JTP}_{df}(M, k, Se, l, \nu, p_r, P, \xi)$ is differentiated with respect to M, k, Se, l, P, p_r, and ξ and equated to zero for obtaining the optimal solution. Since the LTD X is a RV with unknown distribution, the DF approach is taken for the finding of an optimal solution. Only the mean, the standard deviation, and the CDF of the RV X are known. The mean is dl, the standard deviation is $k\sigma\sqrt{l}$, and the CDF is F. Thus, the min–max DF approach is taken to obtain the maximum profit of the joint profit function $\text{JTP}_{df}(M, k, Se, l, \nu, p_r, P, \xi)$. The min–max problem is given as

$$\text{Min} - \max_{F\epsilon\tau} \text{JTP}_{df}(M, k, Se, l, \nu, p_r, P, \xi), \quad 0 \leq Se \leq Se_0. \tag{34}$$

The expected value of $E(X - R_1)^+$ is obtained from the proposition given by Gallego and Moon [14].

Proposition 3. *For any CDF, G belongs to the class τ, then the following inequality consistently holds*

$$E(X - R_1)^+ \leq \frac{1}{2}\Big[\sqrt{\sigma^2 l + (R_1 - dl)^2} - (R_1 - dl)\Big]. \tag{35}$$

Applying the inequality (35) in the nonlinear profit function (34), the joint profit function becomes

$$\text{Min JTP}_{df}(M,k,Se,l,\nu,p_r,P,\xi) = p_m d + p_r\left(a_1\frac{p_{\max r}-p_r}{p_r-p_{\min r}}+a_2\xi^{\theta_2}\right) \tag{36}$$
$$-\left[\begin{array}{l}\dfrac{O\left(a_1\frac{p_{\max r}-p_r}{p_r-p_{\min r}}+a_2\xi^{\theta_2}\right)}{M}+h_b\left[\dfrac{M}{2}+k\sigma\sqrt{l}\right]+\dfrac{\pi\left(a_1\frac{p_{\max r}-p_r}{p_r-p_{\min r}}+a_2\xi^{\theta_2}\right)\sigma\sqrt{l}\left(\sqrt{k^2+1}-k\right)}{2M}\\ +\dfrac{\left(a_1\frac{p_{\max r}-p_r}{p_r-p_{\min r}}+a_2\xi^{\theta_2}\right)C(l)}{M}+\dfrac{Sed}{\nu M}+\beta\ln\left(\dfrac{Se_0}{Se}\right)+d\left(a_3+\dfrac{a_4}{P}+a_5P\right)+p_m d+\dfrac{h_m M}{2}\left[\nu\left(1-\dfrac{d}{P}\right)-1+\dfrac{2d}{P}\right]\end{array}\right].$$

The partial differentiation of the profit function are given as follows:

$$\frac{\partial \text{JTP}_{df}(\cdot)}{\partial M} = \frac{\zeta_1}{M^2}-\zeta_2 \tag{37}$$

$$\frac{\partial \text{JTP}_{df}(\cdot)}{\partial k} = \sigma\sqrt{l}\left[h_b+\frac{\pi\left(\frac{k}{\sqrt{k^2+1}}-1\right)}{2M}\left(a_1\frac{p_{\max r}-p_r}{p_r-p_{\min r}}+a_2\xi^{\theta_2}\right)\right] \tag{38}$$

$$\frac{\partial \text{JTP}_{df}(\cdot)}{\partial Se} = \frac{\beta}{Se}-\frac{d}{\nu M} \tag{39}$$

$$\frac{\partial \text{JTP}_{df}(\cdot)}{\partial P} = -\frac{(\nu-2)h_m Md-2a_4}{2P^2}+a_3 \tag{40}$$

$$\frac{\partial \text{JTP}_{df}(\cdot)}{\partial p_r} = \left(a_1\frac{p_{\max r}-p_r}{p_r-p_{\min r}}+a_2\xi^{\theta_2}\right)+a_1\frac{p_{\max r}-p_{\min r}}{\left(p_r-p_{\min r}\right)^2}$$
$$\times\left[-p_r+\frac{O}{M}+\frac{\pi\sigma\sqrt{l}\left(\sqrt{k^2+1}-k\right)}{2M}+\frac{C(l)}{M}-\left(a_3P+\frac{a_4}{P}+a_5\right)\right] \tag{41}$$

$$\frac{\partial \text{JTP}_{df}(\cdot)}{\partial \xi} = a_2\theta_2\xi^{\theta_2-1}\left(p_r-\frac{O}{M}-\frac{\pi\sigma\sqrt{l}\left(\sqrt{k^2+1}-k\right)}{2M}-\frac{C(l)}{M}\right)-\alpha\xi. \tag{42}$$

$$\frac{\partial \text{JTP}_{df}(\cdot)}{\partial l} = -\frac{k\sigma h_b}{2\sqrt{l}}+\left(a_1\frac{p_{\max r}-p_r}{p_r-p_{\min r}}+a_2\xi^{\theta_2}\right)\left[-\frac{\pi\sigma\left(\sqrt{k^2+1}-k\right)}{4M\sqrt{l}}+\frac{g_i}{M}\right]. \tag{43}$$

The function $\text{JTP}_{df}(M,k,Se,l,\nu,p_r,P,\xi)$ *is concave in* l *for fixed values of* $M,k,Se,\nu,p_r,$ *and* ξ *as*

$$\frac{\partial^2 \text{JTP}_{df}(\cdot)}{\partial l^2} = \frac{k\sigma h_b}{4l^{-\frac{3}{2}}}+\left(a_1\frac{p_{\max r}-p_r}{p_r-p_{\min r}}+a_2\xi^{\theta_2}\right)\left[\frac{\pi\sigma\left(\sqrt{k^2+1}-k\right)}{8Ml^{-\frac{3}{2}}}\right]>0. \tag{44}$$

Thus, from equations (38) to (42), the optimized decision variables obtained are the following

$$M=\sqrt{\frac{\zeta_1}{\zeta_2}} \tag{45}$$

$$k=\sqrt{k^2+1}\left[1-\frac{2h_bM}{\pi\left(a_1\frac{p_{\max r}-p_r}{p_r-p_{\min r}}+a_2\xi^{\theta_2}\right)}\right] \tag{46}$$

$$Se=\frac{\beta\nu M}{d} \tag{47}$$

$$P=\sqrt{\frac{a_4-\frac{(\nu-2)Mh_md}{2}}{a_3}} \tag{48}$$

$$p_r = p_{\min r} + \sqrt{\frac{A_1}{A_2}} \tag{49}$$

$$\xi = \left[\frac{a_2\theta_2\left(p_r - \frac{O}{M} - \frac{\pi\sigma\sqrt{l}(\sqrt{k^2+1}-k)}{2M}\right) - \frac{C(l)}{M}}{\alpha}\right]^{\frac{1}{2-\theta_2}}. \tag{50}$$

The values of $\zeta_1, \zeta_2, A_1,$ and A_2 are elaborated in Appendix A.

Proposition 4. *If the values $M^*, k^*, Se^*, P^*,$ and ξ^* are taken as the optimal values then, for fixed ν and $l\varepsilon[l_j, l_{j-1}]$ the joint profit function $\text{JTP}_{df}(M, k, Se, l, \nu, p_r, P, \xi)$ has global maximum at $(M^*, k^*, Se^*, P^*, \xi^*)$ which is obtained from equations (45) to (50) when the constraint $0 \le Se \le Se_0$ is relaxed. The optimal value of ν (say ν^*) is obtained from the inequality*

$$\text{JTP}_{df}(\nu^* - 1) \ge <= \text{JTP}_{df}(\nu^*) \le >= \text{JTP}_{df}(\nu^* + 1).$$

In a similar way of Algorithm 1, Algorithm 2 is stated below.

Algorithm 2.

Step 1. Initialize the value of the integer variable. Initialize $\nu = 1$.
Step 2. Initiate a loop with the counter as the lead time duration. For each l_j, $j = 1, 2, \ldots, n$; repeat the Steps 3–6.
Step 3. Initiate the values of the decision variables setup cost reduction investment and safety factor. Set $p_{r_{i1}} = 0$ and $k_{i1} = 0$.
Step 4. Evaluating the safety factor value in the first step after finding the order quantity at the initial step. Find out M_{i1} from equation (45) and evaluate k_{i2}.
Step 5. Evaluate the setup cost investment, production rate, and advertisement variable value using the order quantity value at the first step. Using the values of M_{i1} evaluate Se_{i1}, P_{i1}, ξ_{i1}.
Step 6. Condition to be applied for stop iteration and finding optimal values of the decision variables and optimal joint cost. Iterate the Steps 4 and 5 until no changes occur in the values of M^{**}, p_r^{**}, P^{**}, Se^{**}, and k^{**}.
Step 7. After obtaining optimal values, the setup cost investment should be compared with the initial value. If it is less then go to Step 8, else the optimal setup cost investment will be the initial investment. If $Se^{**} < Se_0$ then go to Step 8. Else set $Se_i^{**} = Se_0$ and find out optimal values of M^{**}, P^{**}, ξ^{**}.
Step 8. For the Step 1, maximum profit value is obtained. If $\text{JTP}_{df}(M^{**}, k^{**}, Se^{**}, l^{**}, \nu, p_r^{**}, P^{**}, \xi^{**}) = \text{Min}_{i=1,2,\ldots,n}\ \text{JTP}_{df}(M^{**}, k^{**}, Se^{**}, l^{**}, \nu, p_r^{**}, P^{**}, \xi^{**})$, then optimal solution is reached.
Step 9. Increasing the number of shipments ν, profit value is measured with previous value. If profit comes greater, Steps 2–8 will be iterated for getting maximum profit. Set $\nu = \nu + 1$. If $\text{JTP}_{df}(M^{**}, k^{**}, Se^{**}, l^{**}, \nu, p_r^{**}, P^{**}, \xi^{**}) \le \text{JTP}_{df}(M^{**}, k^{**}, Se^{**}, l^{**}, \nu, p_r^{**}, P^{**}, \xi^{**})_{\nu+1}$, then reiterate Steps 2–8.
Step 10. The step counter will stop if the optimality is reached. Set $m + 1$ as m^{**}. Obtain optimal profit function.

5. Numerical analysis

This section includes two numerical examples of the present study. The numerical is performed on two approaches: (1) LTD follows a ND and (2) LTD does not follow any conventional distribution (DF approach). The values of the parameters are the same for both the numerical. The optimization of the decision variables is obtained by coding in Matlab 2020a. Table 3 represents the parameter values of the numerical examples. The numerical data is collected from Sarkar and Guchhait [47] and Debnath and Sarkar [10]. The LTCC is illustrated

TABLE 3. Parameters values for numerical analysis.

Retailer's parameters			
O (\$/order)	20	τ (\$/unit)	1.1
h_b (\$/unit/unit time)	4.6	$p_{\max r/unit}$ (\$)	195
θ_2	1.5	$p_{\min r/unit}$ (\$)	70
a_1	1	α	10
σ	50		
Manufacturer's parameters			
h_m (\$/unit/unit time)	0.001	a_2	0.01
a_3	0.003	a_4	139
a_5	60	Se_0 (\$/setup)	380
p_m (\$/unit)	75	β	2

TABLE 4. Parameters values of retailer for LTCC (in days).

Components of lead time (i)	f_i	e_i	g_i
1	20	6	0.4
2	20	6	1.2
3	16	9	5

in Table 4. The optimal values of the decision variables and the joint profit function are illustrated in Table 5 when the LTD follows a ND. All data are given for different values of shipments by the manufacturer.

The values of the decision variables and the joint profit values from the Table 5 becomes infeasible when $\nu^* > 2$. The decision variables become optimum at these values of a number of shipments. The profit is slightly higher when the manufacturer adapts the SSMD policy. This holds for both the ND approach and the DF approach.

Table 3 shows all the parameter values of the manufacturer and the retailer. The average price range of the product is from \$70 to \$195 per unit at the retailer's end. With a variable production rate, the manufacturer has a selling price of \$75 for the unit product. The initial setup cost for the production is considered as \$380. The data of LTCC data is taken from Guchhait and Sarkar [42] for Table 4. The standard duration, minimum duration, and unit crashing costs are illustrated in Table 4. Tables 5 and 6 give the optimal values of the decision variables used in the model. The objective is to maximize the profit function. Using normal distribution in LTD, Table 5 shows that whatever the number of shipments, the optimal price is \$76.15, the optimal safety factor k is 1.5, and the optimal advertisement variable value is 272.6. As the number of shipments increases, the rate of production increases, investment in setup cost increases, and joint profit increases. Moreover, there is a little increment in the optimal quantity ordered. The maximum profit becomes \$32 391.5/cycle for 2 shipments and the function reaches its optimality as it becomes infeasible as the number of shipments increases. In a similar way, Table 6 shows that the maximum joint profit becomes \$32 635/cycle at 3 number of shipments (in the case of LTD is DF). The optimal price of the product and the optimal advertisement value are the same as for Table 5. There is a huge jump in the safety factor from 1.5 to 13. The optimal value of the order quantity of DF pattern LTD is way higher than the optimal quantity of normally distributed LTD. Optimal initial setup cost is came high for the DF approach.

The infeasibility condition occurs during the repetitive states of the Algorithm 1. The inverse normal function becomes infeasible when the argument inside becomes either greater than 1 or less than 0. Thus, the

Table 5. Optimal values of decision variables and the joint profit function at different SSMD policies with normally distributed LTD.

ν^{**}	$P^{**}(unit/year)$	$p_r^{**}(\$/unit)$	M^{**} (unit/cycle)	ξ^{**}	k^{**}	Se^{**} ($/setup)	JTP** ($/cycle)
1	182.41	76.15	396.6	273.02	1.9	25.03	32 294.93
2	163.2	76.15	859.1	272.6	1.5	108.6	32 391.5
3	○	○	○	○	○	○	○

Notes. ○ – Infeasible.

Table 6. Optimal values of decision variables and the joint profit function at different SSMD policies with distribution-free LTD.

ν^{**}	P^{**} (unit/year)	p_r^{**} ($/unit)	M^{**} (unit/cycle)	ξ^{**}	k^{**}	Se^{**} ($/setup)	JTP** ($/cycle)
1	187.3	76.2	504.8	271.7	3.62	32.02	32 275
2	163.3	76.2	968	272.6	2.55	122.4	32 379
3	164.4	76.1	651	273.71	13.26	115.7	32 635
4	166.4	76.1	168	273.8	23.26	3.6	32 544

mathematical formulation for the infeasible condition in Table 5 is given as

$$h_b M_i \geq \tau\left(a_1 \frac{p_{\max r} - p_{r_i}}{p_{r_i} - p_{\min r}} + a_2 \xi_i^{\theta_2}\right). \tag{51}$$

6. Expected value of additional information and sensitivity

6.1. Expected value of additional information (EVAI)

In a stochastic SC model, the expected value of additional information (EVAI) refers to the potential benefit from collecting more information or enhancing the quality of the information already available to make better decisions. In other words, it measures the value of having access to more precise, trustworthy, or timely information on ambiguous SC issues. The difference between the anticipated total profit with and without the additional information is used to compute the EVAI. By accounting for the associated expenses of collecting or upgrading that information, it assesses the effect of the new knowledge on predicted profit. The EVAI was first incorporated by Moon and Gallego [35]. The comparison is made between the results of the ND and the DF approaches. From Tables, 5 and 6 the obtained values of optimum decision variables are $(\nu^*, P^*, p_r^*, M^*, \xi^*, k^*, Se^*) = (2, 163.3, 76.15, 859.1, 272.6, 1.5, 108.6)$ and $(\nu^{**}, P^{**}, p_r^{**}, M^{**}, \xi^{**}, k^{**}, Se^{**}) = (3, 164.4, 76.1, 651, 273.71, 13.26, 115.7, 32635)$. The extra profit thus obtained is JTP** − JTP* = \$32 379 − \$32 635 = −\$255.5 which is sufficiently smaller than the 1% of the joint profit of the DF approach. This result shows that this smaller amount the retailer needs to earn profit for gaining the knowledge for the distribution of the LTD.

6.2. Sensitivity of parameters

In this section, the sensitive parameters of the present model is discussed. The sensitivity of the parameters is taken in the range of (−50%, −25%, +25%, +50%). The sensitive parmeters shows the range of sensitivity for both normally distributed LTD and distribution free LTD same, and is illustrated in the Table 7.

In Table 7, the following predictions can be made regarding the sensitivity of parameters used in the present model:

TABLE 7. Sensitive parameters of the model for normally distributed and distribution free LTD.

	% change	% change in JTP*		% change	% change in JTP*
O	−50	+0.025	h_b	−50	+0.044
	−25	+0.011		−25	+0.02
	+25	−0.011		+25	−0.019
	+50	−0.022		+50	−0.037
$p_{\max r}$	−50	−69.84	$p_{\min r}$	−50	−83.36
	−25	−23.99		−25	−35.31
	+25	+15.75		+25	+26.21
	+50	+27.06		+50	+45.52
a_1	−50	−51.21	a_2	−50	−45.39
	−25	−20.32		−25	−29.32
	+25	+14.41		+25	+31.91
	+50	+25.17		+50	+56.65
a_5	−50	+29.96			
	−25	+16.26			
	+25	−19.24			
	+50	−41.9			

- Ordering cost and holding cost of the retailer show very less sensitivity with respect to the change of JTP*. Another observation is that the change is in the reverse direction as the % change of parameter takes place the profit is changing in the reverse direction.
- Both the maximum price and minimum price range of the product which are affecting the demand function of the SC are highly sensitive and show an increasing order. The change in the percentage of the parameters may cause a significant change in the profit function when the parameter cost is cut by 50%.
- On the other side, the parameter for the production cost of the manufacturer has a reverse trend with the joint profit function.
- The scaling and shape parameters for the demand function are moderately sensitive and show a trend with the profit function. Increments in the shape and scale parameters will help increase the profit margin but sometimes may lead to infeasibility.

The other cost parameters like setup cost, backordering cost, and LTCC are all very negligibly sensitive and are not reflected in the table. The following figures are shown for the graphical presentation of the sensitivity analysis of the parameters. Figure 2 depicts the sensitivity of the extremely sensitive parameters of the model.

7. Managerial insight and implications

This section is constructed with some points related to the managerial decisions based on the proposed model.

Insight 1

For any product, it is very much needed to fix the price of the product and the attraction of customers. SCM must decide the price as a key factor to increase the demand. If the price of the product and corresponding investment in the advertisement of the product is integrated to determine the demand pattern, then it is very difficult to adjust the factors to optimize the profit. The proposed study solves this problem. In both stochastic cases of LTD, the optimal price and advertisement variable remain unchanged irrespective of other decision variables and joint profit. For an increasing the range of price $(p_{maxr} - p_{minr})$ (decreasing quantity discount), profit cannot be optimized. Thus, decision-makers should narrow down the price range of products as much as possible for profit optimization. Price-dependent demand is a boost for profit-making [40]. Advertisement elasticity has a direct relationship with optimal quantity and demand [22]. The proposed study integrates

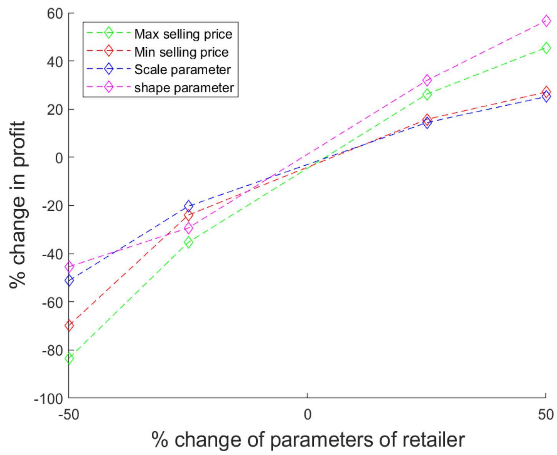

FIGURE 2. Extremely sensitive parameters of the retailer.

advertisements for more profit generation. A huge difference in safety factors is noticed and the manager must keep a good amount of safety stock if the LTD does not follow any conventional statistical distribution. If the statistical distribution of LTD is normal, then the management can decide to keep a less amount of safety stock. From the present study it is observed that irrespective of LTD, the joint profit remains very closer in both cases. But the safety factor optimality changes due to the distribution pattern. Thus managers of retail companies can decide from the study whether to increase or reduce the safety factor so as to maintain the inventory balance. Moreover, for the production house, the logistics manager can take a decision on a number of shipments as the result shows the optimality reaches on multiple shipments.

Insight 2

In a manufacturing unit, sometimes the variable production rate may create complicacy in a supply-demand chain. If the demand is not deterministic and the production rate is variable, then there may be a chance of complexity in setup cost reduction or determining quantity to be ordered in a cycle for profit maximization. As an example of a garments factory, its production is variable depending on the consumer's demand. For home delivery products food and beverages, lead time is a factor. The proposed model considers both the stochastic LTD and variable production and maximizes the profit. Research on taking production variables and controllable lead time profit was optimized Dey *et al.* [52]. But in all such cases, shipment number was not taken into consideration. As the SSMD policy is taken for the shipment of finished items, the management must decide which number of shipments can be done for profit maximization and to keep a rational amount of order quantity as per requirement. The numerical results for the optimal decision variables say that the management can select order shipment numbers according to their required order quantity and setup cost investment with maximum profit. In all cases, the price and advertising always remain optimal, which is a plus point for the management. From the present study one deep insight can be drawn on the profit maximization of a retail company as the price range parameters are very much sensitive to the joint profit. The demand function is dependent on the price range and advertisement variable. Thus, the change in price parameter may affect significantly the change of the total profit in a positive direction. Moreover, the manufacturer's selling price has to be decided accordingly to increase the profit in the whole SC.

8. CONCLUSIONS

The present study was based upon an SC model. The demand was taken as dependent on the selling price of the product and advertisement variable. The production rate of the model was variable. Two cases were considered where the in the first case, LTD followed ND and in the second case, the LTD followed the DF approach. Setup cost investment was incorporated, and the total profit of the supply chain was maximized. The numerical results showed in both cases the profit becomes higher for choosing the SSMD policy rather than the single-setup-single-delivery (SSSD) policy. The results showed that the optimization converged at some integer value of several shipments and after that the solution became infeasible. In both cases, the joint profit had a very negligible difference. The production rate was variable to suit the realistic approach as the demand was dependent on two variables. To be precise, from this study some findings were derived and illustrated as follows:

– The optimal value of the safety factor was deviated about 70% when the LTD had no definite distribution pattern. Although there were no very big differences in the joint profit value in both the cases, but the safety factor change indicated the need of increment of inventory in the distribution free case.
– Another finding was observed from the sensitivity analysis that the parameters which were attached to the demand function were extremely sensitive to the profit function. Moreover, the selling price of the manufacturer was very sensitive. Thus, this observation may lead to the conclusion of keeping a good balance of price range of the product to increase the profit amount overall.

In this study, mostly the impact of the price-dependent, advertisement-dependent demand, and variable manufacturing were discussed as a conclusion. There are several new ideas that can be incorporated into this study for future extension. Partial trade-credit policy is one of the important factors which can be added as a future scope of this study. It may happen that the retailer may receive full trade-credit whereas the customer may receive partial trade-credit from the retailer. The possible extension of this model can be done by taking the approach of partial trade-credit discussed by Mashud *et al.* [31]. In the present study, the production rate is only considered with perfect items. Another future extension can be incorporated as imperfect production which may become more relatable to the real-time study [2].

Appendix A.

$$
\begin{aligned}
\chi_1 &= d\Big(O + C(l) + \tau k\sigma\sqrt{l}\psi(k)\Big)\\
\chi_2 &= \frac{1}{2}\left[h_b + h_m\left\{n\left(1-\frac{d}{P}\right) - 1 + \frac{2d}{P}\right\}\right]\\
\zeta_1 &= d\left(O + C(l) + \pi\sigma\sqrt{l}\left[\sqrt{k^2+1} - k\right] - \frac{Se}{n}\right)\\
\zeta_2 &= \frac{1}{2}\left[h_b + h_m\left\{n\left(1-\frac{d}{P}\right) - 1 + \frac{2d}{P}\right\}\right]\\
A_1 &= a_1(p_{\max r} - p_{\min r})\begin{bmatrix}(P_m + P_r) - \frac{O+C(l)}{M}\\ -\frac{\pi\sigma\sqrt{l}\left(\sqrt{k^2+1}-k\right)}{2M}\\ -\left(a_3 + \frac{a_4}{P} + a_5\right)\end{bmatrix}\\
A_2 &= a_1\frac{p_{\max r} - p_r}{p_r - p_{\min r}} + a_2\xi^{\theta_2}.
\end{aligned}
$$

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