RAIRO-Oper. Res. 58 (2024) 2817–2844
https://doi.org/10.1051/ro/2024096

RAIRO Operations Research
www.rairo-ro.org

CHARACTERIZATIONS OF THE SOLUTION SET OF NONSMOOTH SEMI-INFINITE PROGRAMMING PROBLEMS ON HADAMARD MANIFOLDS

Balendu Bhooshan Upadhyay* and Shivani Sain

Abstract. This article is concerned with a class of nonsmooth semi-infinite programming problems on Hadamard manifolds (abbreviated as, (NSIP)). We introduce the Guignard constraint qualification (abbreviated as, (GCQ)) for (NSIP). Subsequently, by employing (GCQ), we establish the Karush-Kuhn-Tucker (abbreviated as, KKT) type necessary optimality conditions for (NSIP). Further, we derive that the Lagrangian function associated with a fixed Lagrange multiplier, corresponding to a known solution, remains constant on the solution set of (NSIP) under geodesic pseudoconvexity assumptions. Moreover, we derive certain characterizations of the solution set of the considered problem (NSIP) in the framework of Hadamard manifolds. We provide illustrative examples that highlight the importance of our established results. To the best of our knowledge, characterizations of the solution set of (NSIP) using Clarke subdifferentials on Hadamard manifolds have not been investigated before.

Mathematics Subject Classification. 90C34, 90C46, 90C48.

Received June 5, 2023. Accepted April 26, 2024.

1. Introduction

Semi-infinite programming problems (abbreviated as, (SIP)) refer to a class of mathematical programming problems that involve a finite number of decision variables but an infinite number of constraints. Haar [16] established the mathematical foundations of (SIP). Due to its diverse applications in several real-world problems, including mathematical physics, engineering, and economics (see, for instance, [11, 25]), (SIP) has been the subject of extensive research in recent years. For further details and an updated survey of (SIP), we refer the readers to [38, 39, 41] and the references cited therein.

It has been observed that a lot of programming problems arising in various fields of engineering, technology, and science can be better formulated in a manifold setting rather than Euclidean space (see, for instance, [1, 2] and the references cited therein). Extending optimization methods from Euclidean spaces to the framework of manifolds has several benefits from both theoretical and practical standpoints. For instance, constrained mathematical optimization problems can be conveniently transformed into unconstrained optimization problems. Further, the framework of Riemannian geometry can be suitably utilized to transform many nonconvex optimization problems into convex optimization problems (see, for instance, [33, 34]). Apart from this, several important constraints that naturally arise in some real-world optimization problems have a relative interior

Keywords. Semi-infinite programming, geodesic pseudoconvexity, guignard constraint qualification, clarke subdifferential, hadamard manifolds.

Department of Mathematics, Indian Institute of Technology Patna, Patna, India.
*Corresponding author: bhooshan@iitp.ac.in

that can be viewed as Hadamard manifolds (see, for instance, [32,33]). Due to the aforementioned advantages, several researchers have explored optimization methods in the framework of manifolds in recent times (see, for instance, [1,4,46,49] and the references cited therein).

The well-known notions of convex sets and convex functions, which are essential tools in optimization, have been extended to the corresponding notions of geodesic convex sets and geodesic convex functions in the setting of manifolds by Rapcsák [35], respectively. Further, Udrişte [40] introduced the concepts of geodesic pseudoconvex and quasiconvex functions in the context of Riemannian manifolds. Several iterative optimization algorithms, namely, gradient-descent line-search algorithms, Newton-based methods, trust-region methods, as well as conjugate gradient and quasi-Newton methods have been developed by Absil *et al.* [1] in the setting of matrix manifolds. Notably, their approach has provided a framework to optimize the efficiency of the numerical algorithms while retaining the convergence properties of their abstract geometric counterparts. Barani [5] has extended the concepts of monotonicity and convexity for locally Lipschitz functions on Hadamard manifolds. Upadhyay *et al.* [41] have investigated optimality criteria, as well as Mond-weir type and Wolfe-type dual models for multiobjective (SIP) on Hadamard manifolds, by using the notions of geodesic pseudoconvexity and geodesic quasiconvexity. Furthermore, Upadhyay *et al.* [42,43] have established the notions of vector variational inequalities and Minty variational principle within the setting of Hadamard manifolds. They have utilized these concepts to establish the relationship among nonsmooth vector optimization problems and vector variational inequalities, utilizing the geodesic approximate convexity. Karush-Kuhn-Tucker type optimality conditions, as well as several duality results for nonsmooth multiobjective (SIP) involving geodesic convex functions, have been explored by Upadhyay *et al.* [44] in the framework of Hadamard manifolds. Various constraint qualifications (which are well-known to play a significant role in establishing KKT-type optimality conditions) for mathematical programming problems with vanishing constraints have been investigated in [45]. Recently, Upadhyay *et al.* [46] have derived second-order KKT-type necessary optimality conditions as well as duality results for multiobjective (SIP) in the setting of Hadamard manifolds using the notions of generalized geodesic convex functions.

The phenomenon of non-smoothness often appears in various real-world problems in the field of science and engineering (see, for instance, [5, 9, 50]). As a result, several scholars have explored nonsmooth optimization problems in the framework of finite as well as infinite-dimensional Banach spaces (see, for instance, [21, 30] and the references cited therein). However, in various real-life optimization problems in different fields, such as control theory [10] and matrix analysis [24], nonsmooth functions are observed to appear on smooth manifolds. In view of this, many researchers have recently studied nonsmooth analysis on manifolds (see [2, 3]) and the references cited therein.

In the field of optimization, we often encountered various problems with multiple optimal solutions (see, for instance, [27, 53]). Characterizations and properties of the solution set for such optimization problems play a very significant role in properly understanding and analyzing the behavior of solution methods. For instance, characterizations of the solution set provide a way to determine the complete solution set of an optimization problem, in case only one solution is known (see, for instance, [29]). Moreover, investigation of the characterizations of the solution set is often seen to provide a very helpful tool in the analysis of various optimization problems arising in the fields of variational inequalities and game theory (see, for instance, [36,52] and the references cited therein).

Mangasarian [27] has presented a simple characterization for the solution set of convex extremum problems with a single known solution in the Euclidean space setting. Further, Burke and Ferris [7] extended such characterizations for nonsmooth optimization problems involving extended real-valued functions. The work of Mangasarian [27] has been extended for a class of generalized convex functions, namely, pseudolinear functions by Jeyakumar and Yang [19]. Zhao and Yang [53] have established the characterizations of the solution set of nonsmooth nonlinear programming problems involving pseudoconvex functions. Moreover, Mishra and Upadhyay [29] have derived the characterizations of the solution set for constrained nonsmooth optimization problems involving pseudolinear functions in the framework of Euclidean space. In the setting of Riemannian manifolds, Barani and Hosseini [5] have derived the characterizations of the solution set of nonlinear programming problems for smooth geodesic convex functions. Further, Tung *et al.* [38] have considered a smooth semi-infinite programming problem involving geodesic convex functions in the setting of Riemannian manifolds. They have established the KKT-type optimality conditions by employing the (GCQ) followed by the characterizations of the solution set of considered problem.

It is worthwhile to note that in sharp contrast to Euclidean spaces, manifolds, in general, are not equipped with a linear structure, though globally diffeomorphic. Therefore, despite being globally diffeomorphic to Euclidean spaces, the development of optimization techniques in the framework of Hadamard manifolds is accompanied by several difficulties. In contrast to the setting of Euclidean space, any two points of a Hadamard manifold can be joined by a unique minimal geodesic, which is not necessarily a line segment. For instance, the Poincaré half-plane $H := \{(p, q) \in \mathbb{R}^2 |\ q > 0\})$ is a well-known Hadamard manifold with a non-positive sectional curvature -1. Evidently, the unique minimal geodesic connecting any two points (p_1, q_1) and (p_2, q_2) in H, when $p_1 \neq p_2$, is a circular arc with center at the p-axis (see, for instance, [35,40]). Further, the exponential map and inverse of the exponential map on Hadamard manifolds are nonlinear (see, for instance, [23]). As a result, new techniques have been developed by researchers in the last few decades to investigate optimization problems on manifolds. For instance, the concept of geodesic convexity is introduced in the manifold setting. Moreover, the concepts of parallel transport and exponential maps on the tangent space of a Hadamard manifold (which has a vector space structure) are employed in order to deal with the nonlinearity of manifolds.

The primary motivation and objective to characterize the solution set of (NSIP) on Hadamard manifolds, rather than Riemannian manifolds, is as follows: Firstly, the infinite injectivity radius (see, for instance, [35,51]) ensures that the exponential map is globally diffeomorphic in the case of Hadamard manifolds. On the other hand, the exponential map is locally diffeomorphic in the setting of Riemannian manifolds. Therefore, the established results in the framework of Hadamard manifolds hold within the totally normal neighborhood of each point in the Riemannian manifolds. Secondly, it is worth noting that KKT-type optimality conditions and characterizations of the solution set for (SIP) on complete Riemannian manifolds using smooth geodesic convex functions have been explored by Tung *et al.* [38]. However, KKT-type optimality conditions and characterizations of the solution set for a general class of problems, namely, (NSIP) have not yet been studied in the setting of Hadamard manifolds using generalized geodesic convex functions.

Motivated by the works of [29,38,53], in this paper, we consider a class of (NSIP) on Hadamard manifolds. We introduce the (GCQ) for (NSIP) and employ it to establish the KKT-type necessary optimality conditions for (NSIP). Moreover, by utilizing the geodesic pseudoconvexity assumptions, we demonstrate that the Lagrangian function associated with a fixed Lagrange multiplier, corresponding to a known solution, remains constant on the solution set. Furthermore, we employ the properties of locally Lipschitz geodesic pseudoconvex functions on Hadamard manifolds to derive certain characterizations of the solution set of (NSIP). We provide some illustrative examples to demonstrate the significance of our established results.

The novelty and contributions of this article are threefold: Firstly, the results presented in this paper extend the corresponding results derived by Tung *et al.* [38] from smooth (SIP) to (NSIP), and further generalize them from geodesic convex functions to geodesic pseudoconvex and geodesic quasiconvex functions in the framework of Hadamard manifolds. Secondly, the results derived in this paper generalize the corresponding results presented by Zhao and Yang [53] in the framework of an even more general space, namely, Hadamard manifolds. Moreover, in view of the fact that the (NSIP) are generalizations of nonlinear programming problems, we generalize the corresponding results derived by Zhao and Yang [53] from Euclidean space to the setting of Hadamard manifolds. Notably, this is the first time that characterizations of the solution set of (NSIP) have been investigated in the setting of Hadamard manifolds.

The paper is organized as follows: In Section 2, we recall the fundamental concepts of Riemannian and Hadamard manifolds. In Section 3, we consider a (NSIP) and establish the KKT-type necessary optimality conditions for (NSIP) by employing (GCQ). Furthermore, we show that the Lagrangian function associated with a fixed Lagrange multiplier corresponding to a known solution remains constant on the solution set of (NSIP). Section 4 explores the characterizations of the solution set for (NSIP) by utilizing the properties of locally Lipschitz geodesic pseudoconvex functions. In Section 5, we conclude our results and discuss directions for future research endeavors.

2. Notations and mathematical preliminaries

In this paper, the standard notation $\mathbb{R}^n$ is used to symbolize the n-dimensional Euclidean space, while $\mathbb{R}^n_+$ represents the non-negative orthant of $\mathbb{R}^n$. Let A be an infinite subset of $\mathbb{R}$, then $\mathbb{R}^{|A|}$ denotes the linear space and is defined as follows:

$$\mathbb{R}^{|A|} := \{s = (s_j)_{j \in A} \mid s_j = 0, \ \forall j \in A, \text{ except } s_j \neq 0 \text{ for finitely many } j \in A\}.$$

The notation $\mathbb{R}^{|A|}_+$, denotes the positive cone of $\mathbb{R}^{|A|}$ and is defined as follows:

$$\mathbb{R}^{|A|}_+ := \{s = (s_j)_{j \in A} \in \mathbb{R}^{|A|} \mid s_j \geq 0, \ \forall j \in A\}.$$

The symbol $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{R}^n$. Let $\mathbb{M}$ be a n_1-dimensional linear subspace of $\mathbb{R}^n$. Then $\mathbb{M}$ inherits the inner product from $\mathbb{R}^n$, denoted by $\langle \cdot, \cdot \rangle_{\mathbb{M}} = \langle \cdot, \cdot \rangle$. Moreover, the topology in $\mathbb{R}^n$ is induced to $\mathbb{M}$. For $\emptyset \neq S \subseteq \mathbb{M}$, the symbols cl (S) and co (S) denote the closure and convex hull of S in $\mathbb{M}$, respectively. The convex cone containing the origin generated by S is called the positive conic hull of S and is defined as:

$$\text{pos } (S) := \left\{ \sum_{j=1}^{m} \alpha_j p_j \mid \alpha_j \geq 0, \quad p_j \in S, \quad m \in \mathbb{N} \right\},$$

where $\mathbb{N}$ is the set of natural numbers. The following notations for inequalities will be used in the subsequent part of this article:

$$p \prec q \Leftrightarrow p_j < q_j, \forall j = 1, 2, \ldots, m.$$

$$p \preceq q \Leftrightarrow \begin{cases} p_j \leq q_j, & \forall j = 1, 2, \ldots, m \\ p_k < q_k, & \text{for at least one } k \in \{1, 2, \ldots, m\}. \end{cases}$$

A topological space H is termed as a topological manifold of dimension n ($n \in \mathbb{N}$), provided that H is Hausdorff, second-countable, and every element of H belongs to some neighborhood, which is homeomorphic to an open subset of $\mathbb{R}^n$. Let U signify an open set in H and ψ be a homeomorphism from U to some open set in $\mathbb{R}^n$. Then, the pair (ψ, U) is referred to as a chart on the manifold H.

Let H denote a n-dimensional connected manifold endowed with the Riemannian metric. The tangent space at $p \in H$ is a real n-dimensional vector space, denoted by T_pH and its dual is denoted T_pH^*. We denote the inner product on T_pH by $\langle \cdot, \cdot \rangle_{T_p}$ with the associated norm $||.||_{T_p}$. The collection of all the tangent spaces at the points of manifold H is called the tangent bundle and is denoted by $TH = \cup_{p \in H} T_pH$. The symbols $\mathbb{B}^*_p(v, \eta)$, $\mathbb{B}_\eta(p)$ represent the ball centered at $v \in T_pH$ of radius $\eta > 0$ and the ball centered at $p \in H$ with radius $\eta > 0$, respectively. A piecewise smooth curve joining p to q in the manifold H is given by $\theta_{p,q} : [a, b] \to H$, such that $\theta_{p,q}(a) = p, \theta_{p,q}(b) = q$. We define the length of curve $\theta_{p,q}$ by:

$$L(\theta_{p,q}) = \int_a^b ||\theta'_{p,q}(t)|| dt.$$

The Riemannian distance between p and q is given by:

$$d(p, q) = \inf\{L(\theta_{p,q}) : \theta_{p,q} \text{ is a smooth curve joining } p \text{ to } q\}, \quad \forall \ p, q \in H.$$

Let $\triangledown$ denote the Levi-Civita connection associated to H. A smooth curve $\theta_{p,q}$ joining p to q is said to be a geodesic if its vector field $(\theta'_{p,q})$ is parallel along $\theta_{p,q}$, that is,

$$\triangledown_{\theta'_{p,q}} \theta'_{p,q} = 0.$$

A geodesic joining any two points p and q in H is said to be a minimal geodesic if its length equal to the Riemannian distance d. For any element $p \in H$, the exponential map $\exp_p : T_pH \to H$ is defined as follows:

$$\exp_p(v) = \theta_{p,\cdot}(1), \quad \forall v \in T_pH,$$

where $\theta_{p,\cdot} : [0,1] \to H$ is a geodesic emanating from p with velocity v, that is, $\theta_{p,\cdot}(0) = p, \theta'_{p,\cdot}(0) = v$. The parallel transport from p to q in H along a geodesic $\theta_{p,q}$ is a map denoted by $P^{p,q}_{\theta p,q} : T_pH \to T_qH$, such that

$$P^{p,q}_{\theta p,q}(v) = V(q), \quad \forall p,q \in H, \quad \forall v \in T_pH,$$

where V is the unique vector field parallel along $\theta_{p,q}$. The parallel transport $P^{p,q}_{\theta_{p,q}}$ satisfies the following properties:

(a) $P^{p,q}_{\theta p,q}$ is an invertible linear map.
(b) $P^{p,q}_{\theta p,q}$ is an isometry from T_pH to T_qH.

Notably, $P^{p,q}$ denotes the parallel transport from p to q for a minimal geodesic $\theta_{p,q}$.

A complete, simply connected Riemannian manifold with non-positive sectional curvature is called a Hadamard manifold. Let $p \in H$. The exponential map given by $\exp_p : T_pH \to H$ is a diffeomorphism and its inverse, $\exp_p^{-1} : H \to T_pH$ satisfies $\exp_p^{-1}(p) = 0_p$, where $0_p \in T_pH$ is the zero vector of the vector space T_pH. Furthermore, for any $p, q \in H$, there exists a unique minimal geodesic $\theta_{p,q} : [0,1] \to H$, joining p and q satisfying

$$\theta_{p,q}(\tau) = \exp_p(\tau \exp_p^{-1}(q)), \quad \forall \tau \in [0,1].$$

Remark 2.1. In view of Theorem 2.1, in Kristaly *et al.* [23], exponential map $\exp : T_pH \to H$ on Hadamard manifolds with zero sectional curvature is a global isometry. Despite this well-known fact that Hadamard manifolds with zero sectional curvature are isometric to Euclidean spaces, there are a lot of problems that one encounters while investigating optimization problems in the framework of Hadamard manifolds with negative sectional curvature. For instance, due to the linear structure of Euclidean space, it is apparent that $p - q = -(q-p)$, for any $p, q \in \mathbb{R}^n$. However, in the framework of Hadamard manifold H, $\exp_q^{-1} p \neq -\exp_p^{-1} q$, $p, q \in H$, due to its nonlinear structure. As a result, the development of optimization techniques on Hadamard manifolds with non-zero sectional curvatures is significantly difficult, as compared to Euclidean spaces.

In the following definition, we recall the notion of a strongly convex set in the framework of the Riemannian manifold from [35, 51].

Definition 2.2. A subset $F \subseteq H$ is said to be strongly convex if for any $p, q \in F$, there exists a unique minimal geodesic in H joining p and q.

Remark 2.3. If H is a Hadamard manifold, then from Hopf-Rinow theorem (see, for instance, [40]), there exists a unique minimal geodesic between any two points in H. Therefore, every subset of a Hadamard manifold is a strongly convex set.

In the following definitions, we recall the notions of injectivity radius and convexity radius in a complete and connected $n-$dimensional Riemannian manifold (see, [12, 35, 51]).

Definition 2.4. The injectivity and convexity radius related to a point $p \in H$ are denoted by η_p and $r_{\mathrm{cvx}}(p)$, respectively and defined as follows:

$$\eta_p := \sup\{\eta > 0 | \ \exp_p : \mathbb{B}^*_p(0_p, \eta) \subset T_pH \to \exp_p(\mathbb{B}^*_p(0_p, \eta)) \text{ is a diffeomorphism}\},$$

$$r_{\mathrm{cvx}}(p) := \sup\{\eta > 0 | \ \text{every ball in } \mathbb{B}_\eta(p) \text{ is strongly convex and every geodesic in } \mathbb{B}_\eta(p) \text{ is a minimal geodesic}\}.$$

Remark 2.5. (i) For any complete and connected Riemannian manifold,

$$\eta_p \geq r_{\mathrm{cvx}}(p) > 0, \ \forall p \in H.$$

(ii) The set $U_p = \exp_p(\mathbb{B}_p^*(0_p, \eta_p))$, a neighborhood of p, is known as totally normal neighborhood of p. In the particular case of Hadamard manifold, $U_p = H, \ \forall p \in H$.

(iii) If H is a Hadamard manifold, then $r_{\mathrm{cvx}}(p) = \eta_p = +\infty, \forall p \in H$. Therefore, H is globally diffeomorphic to Euclidean space, while, a Riemannian manifold is locally diffeomorphic to Euclidean space. Thus, the results derived in this paper in the setting of Hadamard manifolds need not be true globally for a general Riemannian manifold. In particular, the established results within the framework of Hadamard manifolds hold within the totally normal neighborhood of each point in a Riemannian manifold.

Throughout the paper, we consider H to be a n-dimensional Hadamard manifold unless otherwise specified.

Definition 2.6. A function $\phi : H \to \mathbb{R}$ is said to be locally Lipschitz on a given subset F of H, if for every $p \in F$ there exists a constant $K > 0$ and a neighborhood U of p, such that:

$$|\phi(p_1) - \phi(p_2)| \leq K \ d(p, q), \quad \forall p_1, p_2 \in U,$$

where constant K is called the Lipschitz rank of ϕ.

The following definitions are from Barani [5] and Hosseini and Pouryayeveli [17].

Definition 2.7. Let $\phi : H \to \mathbb{R}$ be a locally Lipschitz function on a geodesic convex set $F \subseteq H$. For any $p, q \in H$ the generalized directional derivative of ϕ at p in the direction $v \in T_pH$ is $\phi^o(p; v)$ and is defined as follows :

$$\phi^o(p; v) := \lim_{q \to p} \sup_{t \downarrow 0} \frac{\phi \circ \psi^{-1}(\psi(q) + td\psi(p)v) - \phi \circ \psi^{-1}(\psi(q))}{t},$$

where (ψ, U) is the chart at p. Indeed,

$$\phi^o(p; v) = (\phi \circ \psi^{-1})^o(\psi(p); d\psi(p)v).$$

Considering $0_p \in T_pH$, we have

$$\phi^o(p; v) = (\phi \circ \exp_p)^o(0_p; v).$$

Definition 2.8. Let $\phi : H \to \mathbb{R}$ be a locally Lipschitz function on a geodesic convex set $F \subseteq H$. The Clarke subdifferential of ϕ at $p \in H$, denoted by $\partial_c\phi(p)$, and is defined as follows:

$$\partial_c\phi(p) = \{\xi \in T_pH | \ \phi^o(p; v) \geq \langle \xi, v \rangle, \forall v \in T_pH\}.$$

Remark 2.9. From the diffeomorphism property of exponential map $\exp_p$, it follows that there exist $q \in H$ for every $v \in T_pH$ such that $v = \exp_p^{-1}(q)$. Therefore,

$$\partial_c\phi(p) := \{\xi \in T_pH | \ \phi^\circ(p; \exp_p^{-1}(q)) \geq \langle \xi, \exp_p^{-1}(q) \rangle, \forall q \in H\}.$$

Meanwhile, in the setting of a general Riemannian manifold, the above expression holds for $q \in U_p = \exp_p(\mathbb{B}^*(0_p, \eta_p))$.

Lemma 2.10. *Let $p \in H$ and $\phi : H \to \mathbb{R}$ be a Lipschitz function of rank K near p. Then*

I. $\partial_c\phi(p)$is a non-empty, geodesic convex, weak*-compact subset of T_pH^*, and $||\xi||_{T_pH^*} \leq k$ for every $\xi \in \partial_c\phi(p)$.

II. *For every $v \in T_pH$, the generalized directional derivative of ϕ satisfy the following:*

$$\phi^\circ(p; v) = max\{\langle \xi, v\rangle |\ \xi \in \partial_c\phi(p)\}.$$

III. *If $\{p_n\}_{n=1}^\infty$ and $\{\xi_n\}_{n=1}^\infty$ are sequences in H and tangent bundle TH^* respectively, such that $\xi_n \in \partial_c\phi(p_n)$ for each $n(n \in \mathbb{N})$ and if $\{p_n\}_{n=1}^\infty$ converges to p, and ξ is a weak*-cluster point of the sequence $\{P^{p_n,p}\xi_n\}_{n=1}^\infty$, then we have $\xi \in \partial_c\phi(p)$.*

The definitions of geodesic convex sets and functions on a Riemannian manifold are from Udriste [40] and Rapcsák [35], respectively.

Definition 2.11. A subset $F \subseteq H$ is called a geodesic convex set if for any two distinct points $p, q \in F$, the geodesic $\theta_{p,q}$ joining p to q is contained in F *i.e.*, if $\theta_{p,q} : [0,1] \to H$ satisfies $\theta_{p,q}(0) = p$ and $\theta_{p,q}(1) = q$, then we have

$$\theta_{p,q}(\tau) = \exp_q(\tau \exp_q^{-1}(p)) \in F, \quad \forall \tau \in [0,1].$$

Definition 2.12. Let $F \subseteq H$ be a geodesic convex set. A function $\phi : F \to \mathbb{R}$ is called a geodesic convex function on F if for every elements $p, q \in F$ and for any geodesic $\theta_{p,q} : [0,1] \to H$ the following inequality holds:

$$\phi(\theta_{p,q}(\tau)) \leq (1-\tau)\phi(p) + \tau\phi(q), \quad \forall \tau \in [0,1].$$

The necessary and sufficient condition for a function to be geodesic convex on a Hadamard manifold is given by the following lemma from Barani [5].

Lemma 2.13. *Let $F \subseteq H$ be a geodesic convex subset of H then ϕ is geodesic convex on F if and only if for every pair of points $p, q \in F$ and for every $\xi \in \partial_c\phi(p)$ the following inequality holds:*

$$\phi(q) - \phi(p) \geq \langle \xi, \exp_p^{-1}(q)\rangle.$$

The following definitions taken from Chen and Fang [8], recall the notions of geodesic quasiconvex and pseudoconvex functions .

Definition 2.14. Let F be a geodesic convex subset of H and $\phi : F \to \mathbb{R}$ be a locally Lipschitz function on F. Then, ϕ is called

(i) geodesic quasiconvex on F if for any $p, q \in F$ and a geodesic $\theta_{p,q} : [0,1] \to H$ the following inequality holds:

$$\phi(\theta_{p,q}(\tau)) \leq \max\{\phi(p), \phi(q)\}, \tau \in [0,1].$$

geodesic semistrictly quasiconvex on F, if for any $p, q \in F$ and a geodesic $\theta_{p,q} : [0,1] \to H$ with $\phi(p) \neq \phi(q)$, the following inequality holds:

$$\phi(\theta_{p,q}(\tau)) < \max\{\phi(p), \phi(q)\}, \tau \in [0,1].$$

(ii) geodesic pseudoconvex on F, if for any $p, q \in F$ and for any $\xi \in \partial_c\phi(p)$ the following inequality holds:

$$\phi(q) - \phi(p) < 0 \Rightarrow \langle \xi, \exp_p^{-1}(q)\rangle_{T_p} < 0.$$

In other words, ϕ is geodesic pseudoconvex on F, if for any $p, q \in F$ there exists $\xi \in \partial_c\phi(p)$, such that the following inequality holds:

$$\langle \xi, \exp_p^{-1}(q)\rangle_{T_p} \geq 0 \Rightarrow \phi(q) - \phi(p) \geq 0.$$

Remark 2.15. (i) Let $H = \mathbb{R}^n$, $F \subseteq \mathbb{R}^n$ be a convex set, then the notion of geodesic pseudoconvexity and geodesic quasiconvexity of function $\phi : F \to \mathbb{R}$ reduces to the standard notion of pseudoconvexity and quasiconvexity of ϕ, respectively (see [28]).

(ii) If $H = \mathbb{R}^n$, $F \subseteq \mathbb{R}^n$ is a convex set and $\phi : F \to \mathbb{R}$ is a smooth function, then $\partial_c \phi(p) = \{\operatorname{grad} \phi(p)\} = \{\nabla \phi(p)\}$ and $\exp_p^{-1}(q) = q - p,\ p, q \in \mathbb{R}^n$ then, the above definition of geodesic quasiconvex and pseudoconvex reduce to the corresponding standard notion of differentiable quasiconvex and pseudoconvex from Mangasarian [28].

(iii) Considering the above definitions and Lemma 2.13, it becomes evident that the geodesic convex functions are geodesic quasiconvex and pseudoconvex functions as well.

The following lemmas presented by Chen and Fang [8] establish the relationship between geodesic pseudoconvex and quasiconvex functions. These lemmas also provide necessary and sufficient condition for a function to be geodesic quasiconvex on a Riemannian manifold.

Lemma 2.16. *Let $\phi : F \subseteq H \to \mathbb{R}$ be a locally Lipschitz pseudoconvex function on a geodesic convex set F, then ϕ is both quasiconvex and semistrictly quasiconvex on F.*

Lemma 2.17. *A locally Lipschitz function $\phi : F \subseteq H \to \mathbb{R}$ is geodesic quasiconvex on a geodesic convex set F if and only if the following inequality holds:*

$$\phi(q) - \phi(p) \leq 0 \Rightarrow \langle \xi, \exp_p^{-1}(q) \rangle_{T_p} \leq 0, \quad \forall p, q \in F, \xi \in \partial_c \phi(p).$$

The following lemmas from Tung et al. [38] will be utilized to prove the necessary optimality condition. Furthermore, we assume $\mathbb{M}$ is a $n-$dimensional subspace in $\mathbb{R}^n$.

Lemma 2.18. *Suppose $\{C_i \mid i \in I\}$ be an arbitrary collection of nonempty convex sets in $\mathbb{M}$ and let $C = \operatorname{pos}(\cup_{i \in I} C_i)$. Then, every non-zero vector of C can be written as a non-negative linear combination of n elements or some linear independent vectors, each belonging to a different set C_i.*

Lemma 2.19. *Let $B(\neq \emptyset)$ and D be arbitrary index sets not necessarily finite, $b_i = b(i) = (b_1(i),\ b_2(i), \ldots, b_n(i))$ maps B onto $\mathbb{M}$, and b_j maps D onto $\mathbb{M}$. Further, the set*

$$\operatorname{co}\{b_i \mid i \in B\} + \operatorname{pos}\{b_j \mid j \in D\},$$

is closed. Then the following statements are equivalent:

(i) The system of inequalities:

$$\langle b_i, p \rangle < 0, \quad \forall i \in B,$$
$$\langle b_j, p \rangle \leq 0, \quad \forall j \in D,$$

have no solution $p \in \mathbb{M}$.

(ii) The following inclusion holds:

$$0 \in \operatorname{co}\{b_i \mid i \in B\} + \operatorname{pos}\{b_j \mid j \in D\}.$$

For further details and exposition on Hadamard manifolds, we refer the readers to [14, 15, 37, 47, 48] and the references cited therein.

3. Necessary optimality conditions for (NSIP)

In this section, we consider a nonsmooth semi-infinite programming problem (NSIP). We establish the KKT-type necessary optimality conditions using (GCQ). Further, we establish that the Lagrangian function corresponds to a known solution and associated with a fixed Lagrange multiplier remains constant on the solution set

Consider the following nonsmooth semi-infinite programming problem (NSIP):

$$\begin{aligned} (NSIP) \quad & \min \quad \phi(p) \\ & \text{subject to } \psi_l(p) \leq 0, \quad \forall l \in A, \end{aligned}$$

where $\phi,\ \psi_l : H \to \mathbb{R}$ are nonsmooth locally Lipschitz, geodesic pseudoconvex functions on H. The index set A is an arbitrary nonempty set, but not necessarily finite. Let $\Omega \subset H$ be the feasible set of (NSIP), defined as:

$$\Omega := \{p \in H \mid \psi_l(p) \leq 0, \forall l \in A\}.$$

For a given $\bar{p} \in \Omega$, the index set of active inequality constraints is denoted by $A(\bar{p})$ and is defined as follows:

$$A(\bar{p}) := \{l \in A \mid \psi_l(\bar{p}) = 0\}.$$

A point $\bar{p} \in \Omega$ is a locally optimal solution of (NSIP), if there exists a neighborhood U of $\bar{p}$, such that

$$\phi(p) \geq \phi(\bar{p}),\ \forall p \in \Omega \cap U.$$

If $U = H$, then $\bar{p}$ is the globally optimal or optimal solution of (NSIP).
Let S denote the solution set of (NSIP), as follows:

$$S := \{p \in \Omega |\ \phi(p) \leq \phi(q), \forall q \in \Omega\}.$$

Now, we recall the notion of Bouligand tangent cone for a subset of a Hadamard manifold (see, for instance, [41]).

Definition 3.1. Let $\Omega \subseteq H$ and $\bar{p} \in cl(\Omega)$. Then the Bouligand tangent cone (contingent cone) of Ω at $\bar{p}$ is denoted by $\mathscr{T}(\Omega, \bar{p})$, and is given by:

$$\mathscr{T}(\Omega, \bar{p}) := \{p \in T_{\bar{p}}H |\ \exists t_k \downarrow 0, \exists p^k \in T_{\bar{p}}H, p^k \to p, \exp_{\bar{p}}(t_k p^k) \in \Omega, \forall k \in \mathbb{N}\}.$$

In the following definition, we extend the notion of linearizing cone given by Maeda [26] from Euclidean space $\mathbb{R}^n$ to Hadamard manifolds for (NSIP).

Definition 3.2. Let $\bar{p} \in \Omega$ be an arbitrary feasible element of (NSIP). Then, the linearizing cone to the set Ω at $\bar{p}$ is the set defined as follows:

$$\mathscr{L}(\Omega, \bar{p}) := \{p \in T_{\bar{p}}H |\ \langle \eta_l, p\rangle_{T_{\bar{p}}} \leq 0, \forall \eta_l \in \partial_c \psi_l(\bar{p}), \forall l \in A(\bar{p})\}.$$

Now, we extend the Abadie constraint qualification (ACQ) and Guignard constraint qualification (GCQ) presented by Tung and Tam [39] for our considered (NSIP).

Definition 3.3. I. The Abadie constraint qualification (ACQ) holds at $\bar{p} \in \Omega$ if $\mathscr{L}(\Omega, \bar{p}) \subseteq \mathscr{T}(\Omega, \bar{p})$ and the set pos $\cup_{l \in A(\bar{p})} \partial_c \psi_l(\bar{p})$ is a closed set.
II. The Guignard constraint qualification (GCQ) holds at $\bar{p} \in \Omega$ if $\mathscr{L}(\Omega, \bar{p}) \subseteq \mathrm{cl}(\mathrm{co}(\mathscr{T}(\Omega, \bar{p})))$ and the set pos $\cup_{l \in A(\bar{p})} \partial_c \psi_l(\bar{p})$ is a closed set.

Remark 3.4. (ACQ) implies (GCQ) but the converse may not hold true.

The above remark is illustrated by the following example.

Example 3.5. Consider the following Riemannian manifold H (see, [35, 39]) which is defined as follows:

$$H = \{(p_1, p_2) \in \mathbb{R}^2 \mid p_1, p_2 > 0\}.$$

The tangent space at any point $p \in H$ is given by $T_pH = \mathbb{R}^2$. The Riemannian metric on the set H is given by

$$\langle r, s\rangle_{T_q} = \langle \mathscr{G}(q) r, s\rangle, \quad \forall r, s \in T_qH = \mathbb{R}^2,$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^2$ and

$$\mathscr{G}(q) = \begin{bmatrix} \frac{1}{q_1^2} & 0 \\ 0 & \frac{1}{q_2^2} \end{bmatrix}.$$

Moreover, H is a Hadamard manifold because of its zero sectional curvature.
The Riemannian distance between any $p = (p_1, p_2), q = (q_1, q_2) \in H$ is given by:

$$d(p, q) = \left\| \left(\ln \frac{p_1}{q_1}, \ln \frac{p_2}{q_2} \right) \right\|.$$

The exponential map $\exp_p : T_pH \to H$ for any $q \in T_pH$ is defined by:

$$\exp_p(q) = (p_1 e^{\frac{q_1}{p_1}}, p_2 e^{\frac{q_2}{p_2}}).$$

The inverse of exponential map $\exp_p^{-1} : H \to T_pH$ for any $p \in H$ is given by:

$$\exp_p^{-1}(q) = \left(p_1 \ln \frac{q_1}{p_1}, p_2 \ln \frac{q_2}{p_2} \right).$$

Consider the following nonsmooth semi-infinite programming problem:

$$\begin{aligned} (P) \qquad & \min \phi(p) = 2|p_1 - e| + \ln p_2, \\ \text{subject to } & \psi_l(p) = \frac{1}{2} - \frac{(1-l)}{2} \ln p_1 - \frac{l}{2} \ln p_2 \leq 0, \quad l \in A = [0, 1], \\ & \psi_2(p) = \ln p_1 . \ln p_2 - \ln p_1 - \ln p_2 + 1 \leq 0, \end{aligned}$$

where, $\phi, \psi_l, l \in A \cup \{2\}$ are the real valued functions defined on H. The feasible set for the considered problem (P) is:

$$\Omega := \{p \in H \mid p_1 = e, p_2 \geq e\} \cup \{p \in H \mid p_1 \geq e, p_2 = e\}.$$

Notably, the objective function is non-negative on Ω, and it attains its minimum value zero only at $(e, e) \in \Omega$. Thus, (e, e) is an optimal solution and $S = \{(e, e)\}$. Further, it can be shown that

$$\partial_c \phi(p) = \operatorname{co}\{(-2p_1^2, p_2)^T, (2p_1^2, p_2)^T\}$$

$$\partial_c \psi_l(p) = \mathscr{G}(p)^{-1}\{\operatorname{grad} \psi_l(p)\} = \left\{ \left(-p_1 \frac{(1-l)}{2}, -p_2 \frac{l}{2} \right)^T \right\}$$

$$\partial_c \psi_2(p) = \mathscr{G}(p)^{-1}\{\operatorname{grad} \psi_2(p)\} = \left\{ (p_1(\ln p_2 - 1), p_2(\ln p_1 - 1))^T \right\}$$

Also, $A(\bar{p}) = A \cup \{2\}$, where $\bar{p} = (e, e)$. Let p be an arbitrary element of $\mathscr{T}(\Omega, \bar{p})$ then $\exists t_k \downarrow 0$ and $p^k = (p_1^k, p_2^k) \in T_{\bar{p}}H = \mathbb{R}^2$ with $p^k \to p = (p_1, p_2)$, such that

$$\exp_{\bar{p}}(t_k p^k) = (e.e^{\frac{t_k p_1^k}{e}}, e.e^{\frac{t_k p_2^k}{e}}) \in \Omega, \ \forall k.$$

This gives us

$$\begin{cases} e.e^{\frac{t_k p_1^k}{e}} & = e \\ e.e^{\frac{t_k p_2^k}{e}} & \geq e \end{cases} \forall k \text{ or } \begin{cases} e.e^{\frac{t_k p_1^k}{e}} & \geq e \\ e.e^{\frac{t_k p_2^k}{e}} & = e \end{cases} \forall k,$$

which implies

$$\begin{cases} \frac{t_k p_1^k}{e} & = 0 \\ \frac{t_k p_2^k}{e} & \geq 0 \end{cases} \forall k \text{ or } \begin{cases} \frac{t_k p_1^k}{e} & \geq 0 \\ \frac{t_k p_2^k}{e} & = 0 \end{cases} \forall k.$$

Equivalently,

$$\begin{cases} p_1^k & = 0 \\ p_2^k & \geq 0 \end{cases} \forall k \text{ or } \begin{cases} p_1^k & \geq 0 \\ p_2^k & = 0 \end{cases} \forall k.$$

Let $k \to \infty$, we get

$$\begin{cases} p_1 & = 0 \\ p_2 & \geq 0 \end{cases} \text{ or } \begin{cases} p_1 & \geq 0 \\ p_2 & = 0 \end{cases}.$$

Hence, it follows that $\mathscr{T}(\Omega, \bar{p}) \subseteq L = \{p \in \mathbb{R}^2 \mid p_1 = 0, p_2 \geq 0\} \cup \{p \in \mathbb{R}^2 \mid p_1 \geq 0, p_2 = 0\}$. Similarly, it can be proved that $L \subseteq \mathscr{T}(\Omega, \bar{p})$. Thus, we have

$$\mathscr{T}(\Omega, \bar{p}) = L.$$

Further,

$$\partial_c \phi(\bar{p}) = \text{co}\{(-2e^2, e)^T, (2e^2, e)^T\},$$

$$\partial_c \psi_l(\bar{p}) = \left\{ \left(-e\frac{(1-l)}{2}, -e\ \frac{l}{2} \right)^T \right\},$$

$$\partial_c \psi_2(\bar{p}) = \left\{ (0,0)^T \right\}.$$

Let $p \in \mathscr{L}(\Omega, \bar{p})$ which implies $p \in T_{\bar{p}}H = \mathbb{R}^2$ and

$$\langle \eta_l, p \rangle_{T_{\bar{p}}} \leq 0 \Rightarrow -(1-l)ep_1 - lep_2 \leq 0, \forall l \in A \cup \{2\} \Rightarrow p_1, p_2 \geq 0.$$

Thus, we have $\mathscr{L}(\Omega, \bar{p}) \subseteq \mathbb{R}^2_+$ and it is easy to see that $\mathbb{R}^2_+ \subseteq \mathscr{L}(\Omega, \bar{p})$. Thus,

$$\mathscr{L}(\Omega, \bar{p}) = \mathbb{R}^2_+.$$

Further, it follows that

$$\text{pos} \cup_{l \in A(\bar{p})} \partial_c \psi_l(\bar{p}) := \{p \in T_{\bar{p}}H \mid p_1 \leq 0, p_2 \leq 0\},$$

is a closed set.
Thus, (ACQ) does not hold at $\bar{p}$ because

$$\mathscr{L}(\Omega, \bar{p}) = \mathbb{R}^2_+ \not\subseteq \mathscr{T}(\Omega, \bar{p}),$$

whereas (GCQ) holds at $\bar{p}$ because

$$\mathscr{L}(\Omega, \bar{p}) = \mathbb{R}^2_+ \subseteq cl\ co\mathscr{T}(\Omega, \bar{p}).$$

In the following lemma, we prove the geodesic convexity of the feasible set and the solution set of (NSIP) which will be used in the subsequent part of the article.

Lemma 3.6. *Let $\bar{p}$ be an arbitrary element of the solution set S. Then, the following assertions hold:*

(1) Ω is a geodesic convex subset of H.
(2) $S := \{p \in \Omega \mid \phi(p) = \phi(\bar{p})\}$.
(3) S is a geodesic convex subset of Ω.

Proof. (1) Let $p,\ q \in \Omega$ be arbitrary elements. Then

$$\psi_l(p) \leq 0, \quad \psi_l(q) \leq 0, \quad \forall l \ \in A.$$

Using geodesic pseudoconvexity of $\psi_l(.)$, $l \in A$ and for any geodesic $\theta_{p,q} : [0,1] \to H$, it follows from Lemma 2.16, that,

$$\psi_l(\theta_{p,q}(t)) \leq \max\{\psi_l(p), \psi_l(q)\} \leq 0, \quad \forall l \in A, t \in [0,1].$$

Thus, Ω is a geodesic convex subset of H.

(2) Let $p,\ \bar{p} \in S$ then,

$$\phi(\bar{p}) \leq \phi(p) \text{ and } \phi(p) \leq \phi(\bar{p}).$$

Thus, $\phi(\bar{p}) = \phi(p)$.

(3) Evidently, $S \subseteq \Omega$. For any $p, q \in S$, we have $\phi(p) = \phi(q)$, by using the proof of above part. For any geodesic $\theta_{p,q} : [0,1] \to H$ and using geodesic pseudoconvexity of ϕ, it follows from Lemma 2.16, that,

$$\phi(\theta_{p,q}(\tau)) \leq \max\{\phi(p), \phi(q)\} = \phi(p), \quad \forall \tau \in [0,1],\ p, q \ \in S.$$

Thus, $\theta_{p,q}(\tau) \in S, \forall \tau \in [0,1]$ implies S is a geodesic convex set.

□

Remark 3.7. (i) In view of geodesic pseudoconvexity assumption on ϕ, Remark 2.9 and Lemma 2.10, one can easily verify that

$$\phi^\circ(p; \exp_p^{-1}(\bar{p})) \leq 0 \text{ and } \phi^\circ(\bar{p}; \exp_{\bar{p}}^{-1}(p)) \leq 0,\ \forall p, \bar{p} \in S.$$

(ii) If $\phi, \psi_l : H \to \mathbb{R}$ are smooth geodesic convex functions, then $\partial_c\phi(\bar{p}) = \{\operatorname{grad}\phi(\bar{p})\}, \partial_c\psi_l(\bar{p}) = \{\operatorname{grad}\psi_l(\bar{p})\}, \forall l \in A$. In view of Remark 2.15, the above Lemma 3.6 reduces to Lemma 3.1 deduced by Tung *et al.* [38].

(iii) If $H = \mathbb{R}^n$, then $\exp_p^{-1}(q) = q - p$. Moreover, if A is a finite set, then in view of Remark 2.15, Lemma 3.6 reduces to Lemma 2.3 discussed by Zhao and Yang [53].

(iv) It is worthwhile to note that in the framework of Euclidean spaces, we have $p - q = -(q - p)$, for any $p, q \in \mathbb{R}^n$. However, such properties cannot be extended in the framework of Hadamard manifolds, which makes the task of optimization on manifolds significantly difficult.

In the following proposition, we deduce necessary conditions for efficiency of a feasible element of (NSIP).

Proposition 3.8. *Let $\overline{p} \in \Omega$ be an optimal solution of (NSIP), such that (GCQ) holds at $\overline{p}$. Suppose that the objective and constraint functions ϕ, ψ_l, $l \in A$ are locally Lipschitz geodesic pseudoconvex functions on Ω. Then, the system of inequalities given below:*

$$\begin{aligned} \langle \xi, q \rangle_{T_{\overline{p}}} &< 0, \quad \forall \xi \in \partial_c\phi(\overline{p}), \\ \langle \eta_l, q \rangle_{T_{\overline{p}}} &\leq 0, \quad \forall \eta_l \in \partial_c\psi_l(\overline{p}),\ \forall l \in A(\overline{p}), \end{aligned}$$

have no solution in $T_{\overline{p}}H$.

Proof. On the contrary, we assume that there exists some vector $q^* \in T_{\overline{p}}H$ satisfying:

$$\begin{aligned} \langle \xi, q^* \rangle_{T_{\overline{p}}} &< 0, \quad \forall \xi \in \partial_c\phi(\overline{p}), \\ \langle \eta_l, q^* \rangle_{T_{\overline{p}}} &\leq 0, \quad \forall \eta_l \in \partial_c\psi_l(\overline{p}),\ \forall l \in A(\overline{p}), \end{aligned}$$

From the definition of the linearized cone, it follows from the above inequalities that $q^* \in \mathscr{L}(\Omega, \overline{p})$.
From the given hypothesis, (GCQ) holds at $\overline{p} \in \Omega$. As a result, we have

$$q^* \in \operatorname{cl}(\operatorname{co}(\mathscr{T}(\Omega, \overline{p}))).$$

Consequently, we have some sequence $\{q_m\}_{m=1}^{\infty} \in \text{co}\,\mathscr{T}(\Omega, \overline{p})$ satisfying the following:

$$\lim_{m\to\infty} q_m = q^*.$$

Then, for any element $q_m (m \in \mathbb{N})$ of the sequence, $\exists N_m \in \mathbb{N}, \lambda_{mk} \geq 0$ and $q_{mk} \in \mathscr{T}(\Omega, \overline{p})$, for each $k = 1, 2, \dots, N_m$, which satisfies the following :

$$\sum_{k=1}^{N_m} \lambda_{mk} = 1, \sum_{k=1}^{N_m} \lambda_{mk} q_{mk} = q_m.$$

For every $m \in \mathbb{N}$ and $k = 1, 2, \dots, N_m$. As we know $q_{mk} \in \mathscr{T}(\Omega, \overline{p})$, by using the definition of contingent cone, there exists a sequence $\{q_{mk}^n\}_{n=1}^{\infty}, q_{mk}^n \in T_{\overline{p}}H,\ \forall n \in \mathbb{N}$ and $\{t_{mk}^n\}_{n=1}^{\infty}, t_{mk}^n \in \mathbb{R},\ \ \forall n \in \mathbb{N}$, with $t_{mk}^n \downarrow 0$, such that

$$\lim_{n\to\infty} q_{mk}^n = q_{mk}, \exp_{\overline{p}}(t_{mk}^n q_{mk}^n) \in \Omega.$$

For the sake of simplicity, we write

$$\exp_{\overline{p}}(t_{mk}^n q_{mk}^n) = p_{mk}^n,\ \forall n \in \mathbb{N}.$$

Using optimality of $\overline{p}$ and the feasibility of p_{mk}^n the following inequalities can be obtained for every $n \in \mathbb{N}$:

$$\begin{aligned} \phi(p_{mk}^n) &= \phi(\exp_{\overline{p}}(t_{mk}^n q_{mk}^n)) \geq \phi(\overline{p}), \\ \psi_l(p_{mk}^n) &= \psi_l(\exp_{\overline{p}}(t_{mk}^n q_{mk}^n)) \leq 0 = \psi_l(\overline{p}), \qquad \forall l \in A(\overline{p}). \end{aligned}$$

By using geodesic pseudoconvexity of $\phi, \psi_l, l \in A(\overline{p})$, it follows from Lemmas 2.16 and 2.17, that,

$$\begin{aligned} \langle \zeta_{mk}^n, \exp_{p_{mk}^n}^{-1}(\overline{p}) \rangle_{T_{p_{mk}^n}} &\leq 0, \qquad \forall \zeta_{mk}^n \in \partial_c \phi(p_{mk}^n), \\ \langle \eta_l, q_{mk}^n \rangle_{T_{\overline{p}}} &\leq 0, \qquad \forall \eta_l \in \partial_c \psi_l(\overline{p}), \quad l \in A(\overline{p}). \end{aligned} \tag{1}$$

Using parallel transport from p_{mk}^n to $\overline{p}$, we have

$$\langle P^{p_{mk}^n, \overline{p}} \zeta_{mk}^n, q_{mk}^n \rangle_{T_{\overline{p}}} \geq 0.$$

Since ϕ is locally Lipschitz, it follows from Lemma 2.10(i) there exists $k > 0$, such that for sufficiently large n, $||\zeta_{mk}^n||_{T_{p_{mk}^n}} \leq k$. It follows that

$$||P^{p_{mk}^n, \overline{p}} \zeta_{mk}^n||_{T_{\overline{p}}} \leq k,$$

and so there is a subsequence $P^{p_{mk}^{n_j}, \overline{p}} \zeta_{mk}^{n_j} \to \hat{\xi}$ in *weak** topology. It follows from Lemma 2.10(ii) that $\hat{\xi} \in \partial_c \phi(\overline{p})$. Therefore, there exists $\hat{\xi} \in \partial_c \phi(\overline{p})$, such that

$$\langle \hat{\xi}, q_{mk}^n \rangle_{T_{\overline{p}}} \geq 0.$$

By invoking the continuity of inner product, one can have

$$\begin{aligned} &\langle \hat{\xi}, q^* \rangle_{T_{\overline{p}}} \geq 0, \\ &\langle \eta_l, q^* \rangle_{T_{\overline{p}}} \leq 0,\ \ \forall \eta_l \in \partial_c \psi_l(\overline{p}), l \in A(\overline{p}), \end{aligned}$$

which is a contradiction, and thus, we claim our proof. □

In the following theorem, we deduce necessary optimality conditions for (NSIP).

Theorem 3.9. *Suppose that $\bar{p}$ is an optimal solution of (NSIP), such that (GCQ) holds at $\bar{p}$. Assume that the objective function ϕ and constraint functions $\psi_l,\ l \in A$ are locally Lipschitz geodesic pseudoconvex on Ω. Then, there exists some real numbers $\bar{\lambda}_l,\ l \in A$, such that the following conditions hold:*

$$0 \in \partial_c \phi(\bar{p}) + \sum_{l \in A} \bar{\lambda}_l \partial_c \psi_l(\bar{p}),$$
$$\psi_l(\bar{p}) \leq 0, \quad \forall l \in A,$$
$$\bar{\lambda}_l \geq 0, \quad \bar{\lambda}_l \psi_l(\bar{p}) = 0,\ \forall l \in A.$$

Proof. Given that $\bar{p} \in \Omega$ is an optimal solution of (NSIP). Since (GCQ) holds at $\bar{p}$ and the functions $\phi, \psi_l,\ l \in A$ are geodesic pseudoconvex on Ω, it follows from Proposition 3.8, that the following system of inequalities:

$$\langle \xi, q^* \rangle_{T_{\bar{p}}} < 0, \qquad \forall \xi \in \partial_c \phi(\bar{p}),$$
$$\langle \eta_l, q^* \rangle_{T_{\bar{p}}} \leq 0, \forall \eta_l \in \partial_c \psi_l(\bar{p}), l \in A(\bar{p}),$$

have no solution in $T_{\bar{p}}H$. By using Lemma 2.19, it follows that there exists $\lambda > 0,\ \sigma_l \geq 0, l \in A(\bar{p})$, such that

$$0 \in \lambda \partial_c \phi(\bar{p}) + \sum_{l \in A(\bar{p})} \sigma_l \partial_c \psi_l(\bar{p}).$$

Setting $\sigma_l = 0, l \in A \setminus A(\bar{p})$, we have

$$0 \in \lambda \partial_c \phi(\bar{p}) + \sum_{l \in A} \sigma_l \partial_c \psi_l(\bar{p}). \tag{2}$$

Using positivity of λ, one can rewrite the above equation

$$0 \in \partial_c \phi(\bar{p}) + \sum_{l \in A} \frac{1}{\lambda} \sigma_l \partial_c \psi_l(\bar{p}). \tag{3}$$

Setting $\frac{\sigma_l}{\lambda} = \bar{\lambda}_l$, we have

$$0 \in \partial_c \phi(\bar{p}) + \sum_{l \in A} \bar{\lambda}_l \partial_c \psi_l(\bar{p}). \tag{4}$$

Moreover,

$$\psi_l(\bar{p}) = 0, \forall l \in A(\bar{p}), \bar{\lambda}_l = 0, \forall l \in A \setminus A(\bar{p}).$$

Hence, we have

$$\bar{\lambda}_l \psi_l(\bar{p}) = 0, \forall l \in A.$$

This completes the proof. □

Remark 3.10. (i) Theorem 3.10 and Proposition 3.8 generalize the corresponding Proposition 2.6 in [38] from smooth (SIP) to (NSIP), as well as generalizes it for more general class of nonconvex functions, namely, geodesic pseudoconvex and quasiconvex functions.

Let us define the set of non-zero Lagrange multipliers at $\bar{p} \in S$ as:

$$\widetilde{A}(\bar{p}) := \{l \in A(\bar{p}) \mid \bar{\lambda}_l > 0\}.$$

Now, we establish that the Lagrangian function associated with a Lagrange multiplier corresponding to a solution is a constant function on the solution set S.

Theorem 3.11. *Let $\bar{p} \in S$ be a known solution of the problem (NSIP). Assume that (GCQ) holds at $\bar{p}$, and there exists $\bar{\lambda} \in \mathbb{R}_+^{|A|}$, such that*

$$0 \in \partial_c \phi(\bar{p}) + \sum_{l \in A} \bar{\lambda}_l \partial_c \psi_l(\bar{p}),$$

$$\bar{\lambda}_l \psi_l(\bar{p}) = 0.$$

If ϕ, ψ_l, $l \in A$ are geodesic pseudoconvex on Ω. Then for every $p \in S$,

$$\sum_{l \in A(\bar{p})} \bar{\lambda}_l \psi_l(p) = 0,$$

$$L(., \bar{\lambda}) = \phi(.) + \sum_{l \in A(\bar{p})} \bar{\lambda}_l \psi_l(.),$$

is constant on S.

Proof. We observe that $\bar{\lambda}_l = 0,\ \forall l \in A \setminus \widetilde{A}(\bar{p})$, therefore for any $p \in S$, we have

$$\sum_{l \in A(\bar{p})} \bar{\lambda}_l \psi_l(p) = 0 \Leftrightarrow \sum_{l \in \widetilde{A}(\bar{p})} \bar{\lambda}_l \psi_l(p) = 0 \Leftrightarrow \psi_l(p) = 0, \forall l \in \widetilde{A}(\bar{p}). \tag{5}$$

If $\widetilde{A}(\bar{p}) = \emptyset$, the proof is trivial. If $\widetilde{A}(\bar{p})$ is non-empty, *i.e.* there exists $i \in \widetilde{A}(\bar{p})$ and $q^* \in S$, such that $\psi_i(q^*) < 0 = \psi_i(\bar{p})$. Using the geodesic pseudoconvexity of function ψ_i, we have

$$\langle \eta_i, \exp_{\bar{p}}^{-1}(q^*) \rangle_{T_{\bar{p}}} < 0, \forall \eta_i \in \partial_c \psi_i(\bar{p}). \tag{6}$$

Moreover,

$$\psi_l(q^*) \leq 0 = \psi_l(\bar{p}), \forall l \in \widetilde{A}(\bar{p}) \setminus \{i\}.$$

Using the geodesic pseudoconvexity of $\psi_l, \forall l \in \widetilde{A}(\bar{p}) \setminus \{i\}$, along with Lemmas 2.16 and 2.17, we have

$$\langle \eta_l, \exp_{\bar{p}}^{-1}(q^*) \rangle_{T_{\bar{p}}} \leq 0, \forall \eta_l \in \partial_c \psi_l(\bar{p}). \tag{7}$$

From (6) and (7), we deduce

$$\sum_{l \in A(\bar{p})} \bar{\lambda}_l \langle \eta_l, \exp_{\bar{p}}^{-1}(q^*) \rangle_{T_{\bar{p}}} < 0, \forall \eta_l \in \partial_c \psi_l(\bar{p}). \tag{8}$$

Using necessary optimality condition at $\bar{p}$, there exist $\hat{\xi} \in \partial_c \phi(\bar{p})$ and $\hat{\eta}_l \in \partial_c \psi_l(\bar{p})$, $l \in A(\bar{p})$ satisfies the following

$$\langle \hat{\xi}, \exp_{\bar{p}}^{-1}(q^*) \rangle_{T_{\bar{p}}} + \sum_{l \in A(\bar{p})} \bar{\lambda}_l \langle \hat{\eta}_l, \exp_{\bar{p}}^{-1}(q^*) \rangle_{T_{\bar{p}}} = 0. \tag{9}$$

From (8) and (9), it follows that

$$\langle \hat{\xi}, \exp_{\bar{p}}^{-1}(q^*) \rangle_{T_{\bar{p}}} > 0. \tag{10}$$

Since $q^*, p \in S$, it follows from Lemma 3.6, that,

$$\phi(q^*) = \phi(\bar{p}).$$

In view of the geodesic pseudoconvexity of ϕ, it follows from Lemmas 2.16 and 2.17, that,

$$\langle \xi, \exp_{\bar{p}}^{-1}(q^*) \rangle_{T_{\bar{p}}} \leq 0, \forall \xi \in \partial_c \phi(\bar{p}),$$

contradiction to (10).
It follows from (5),

$$L(p,\bar{\lambda}) = \phi(p) + \sum_{l\in A(\bar{p})} \bar{\lambda}_l \psi_l(p) = \phi(\bar{p}), \quad \forall p \in S.$$

Thus, the Lagrangian function $L(p,\bar{\lambda}) = \phi(p) + \sum_{l\in A(\bar{p})} \bar{\lambda}_l \psi_l(p)$ is constant on the solution set S.

□

Remark 3.12. (i) In view of Remark 2.15 (ii) and (iii), Theorem 3.11 generalizes Proposition 3.1 established by Tung *et al.* [38] for a more general class of nondifferentiable nonconvex functions, namely, nondifferentiable geodesic pseudoconvex and geodesic quasiconvex functions.

(ii) In view of Remark 2.15 (i), Theorem 3.12 generalizes Theorem 3.1 derived by Zhao and Yang [53] from the setting of Euclidean space to a more general space, namely, Hadamard manifolds, provided A is a finite set.

4. Characterization of the solution set of (NSIP)

This section deals with the characterization of solution set of (NSIP) on Hadamard manifolds in terms of Clarke's subdifferentials and Lagrange multipliers. Further, we consider A is an arbitrary non-empty set but not necessarily finite and the objective, constraint functions are locally Lipschitz nonsmooth geodesic pseudoconvex. We always assume that the solution set is non-empty *i.e.*, there exists some $\bar{p} \in S$ and Theorem 3.9 holds with a Lagrange multiplier $\bar{\lambda} = (\bar{\lambda}_1, \bar{\lambda}_2, \dots, \bar{\lambda}_{|A|}) \in R_+^{|A|}$. Define the following sets which will be used to characterize the solution set of (NSIP):
$\Omega_1 := \{p \in H |\ \psi_l(p) = 0, \forall l \in \widetilde{A}(\bar{p}), \psi_l(p) \leq 0, \forall l \in A \setminus \widetilde{A}(\bar{p})\}$,
$\Omega_2 := \{p \in H |\ \psi_l(p) \leq 0, \forall l \in A \setminus \widetilde{A}(\bar{p})\}$,
$C(\bar{p}) := \{\xi \in \partial_c \phi(\bar{p}) |\ \langle \xi, \exp_{\bar{p}}^{-1}(p)\rangle_{T_{\bar{p}}} \geq 0, \forall p \in \Omega\}$.

Remark 4.1. (i) It is worth noting that Ω_1 is composed of points within H for which the active constraints corresponding to positive Lagrange multipliers at solution $\bar{p}$ remain active and for remaining indices in set A, these points satisfy the feasibility condition. Further, $\Omega_2 \subseteq H$ consists of those points that satisfy the inequality constraints corresponding to indices in A that are not in $\tilde{A}(\bar{p})$. Moreover,

$$\Omega_1 \subseteq \Omega \subseteq \Omega_2.$$

(ii) In view of Remark 2.9 and Lemma 2.10, $C(\bar{p})$ can be considered as a collection of all elements in $\partial_c \phi(\bar{p})$ for which the generalized directional derivative of ϕ is non-negative in every direction at $\bar{p}$.
(iii) In view of the geodesic pseudoconvexity of ϕ, one can claim that $\phi(\bar{p}) \leq \phi(p)$ for some $\xi \in C(\bar{p})$, and for every $p \in \Omega$. Therefore, $C(\bar{p})$ is defined to ensure the optimality of $\bar{p}$.
(iv) If $\phi : H \to \mathbb{R}$ is a differentiable function and $\bar{p} \in S$. Then for any given $p \in \Omega$, $\langle \operatorname{grad} \phi(\bar{p}), \exp_{\bar{p}}^{-1}(p)\rangle \geq 0$ holds and $C(\bar{p}) = \{\operatorname{grad} \phi(\bar{p})\}$.
(v) The aforementioned sets Ω_1, Ω_2 and $C(\bar{p})$ play a very significant role in deriving the main results of the paper.
(vi) In view of Remarks, Remarks 2.5 and 2.9, we observe that $\exp_{\bar{p}}^{-1}(p)$ may not be well-defined in the entire feasible set Ω, unless $p \in U_{\bar{p}}$ and $U_{\bar{p}} = \Omega$. Therefore, $C(\bar{p})$ may not be well-defined in the setting of a general Riemannian manifold. However, if H is a complete Riemannian manifold such that its exponential map is a diffeomorphism map onto H, which is true in the case of Hadamard manifolds, then $C(\bar{p})$ can be well-defined.

The following theorem characterize the solution set for (NSIP) by utilizing the defined set Ω_1, a subset of Ω.

Theorem 4.2. *Suppose that $\bar{p}$ be an arbitrary element of S, such that (GCQ) holds at $\bar{p}$ and there exists Lagrange multiplier $\bar{\lambda} = (\bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_{|A|}) \in \mathbb{R}_+^{|A|}$, such that*

$$0 \in \partial_c \phi(\bar{p}) + \sum_{l \in A} \bar{\lambda}_l \partial_c \psi_l(\bar{p}).$$

Then the following condition holds:

$$S = S_1 = S_2 = S_3 = S_4 = S_5,$$

where

(i) $S_1 := \{p \in \Omega_1 \mid \langle \zeta, \exp_p^{-1}(\bar{p}) \rangle_{T_p} = 0 \text{ for some } \zeta \in \partial_c \phi(p)\}$,
(ii) $S_2 := \{p \in \Omega_1 \mid \langle \zeta, \exp_p^{-1}(\bar{p}) \rangle_{T_p} \geq 0 \text{ for some } \zeta \in \partial_c \phi(p)\}$,
(iii) $S_3 := \{p \in \Omega_1 \mid \langle \zeta, \exp_p^{-1}(\bar{p}) \rangle_{T_p} = \langle \xi, \exp_{\bar{p}}^{-1}(p) \rangle_{T_{\bar{p}}} \text{ for some } \zeta \in \partial_c \phi(p) \text{ and } \xi \in C(\bar{p})\}$,
(iv) $S_4 := \{p \in \Omega_1 \mid \langle \xi, \exp_{\bar{p}}^{-1}(p) \rangle_{T_{\bar{p}}} \leq \langle \zeta, \exp_p^{-1}(\bar{p}) \rangle_{T_p} \text{ for some } \zeta \in \partial_c \phi(p) \text{ and } \xi \in C(\bar{p})\}$,
(v) $S_5 := \{p \in \Omega_1 \mid \langle \xi, \exp_{\bar{p}}^{-1}(p) \rangle_{T_{\bar{p}}} = \langle \zeta, \exp_p^{-1}(\bar{p}) \rangle_{T_p} = 0 \text{ for some } \zeta \in \partial_c \phi(p) \text{ and } \xi \in C(\bar{p})\}$.

Proof. Evidently, $S_5 \subseteq S_1 \subseteq S_2$ and $S_5 \subseteq S_3 \subseteq S_4$. Hence, we only need to prove that $S \subseteq S_5, S_2 \subseteq S$ and $S_4 \subseteq S$.
We begin by proving $S \subseteq S_5$. Let p be an arbitrary element of S and by Theorem 3.11, we have

$$\sum_{l \in A(\bar{p})} \bar{\lambda}_l \psi_l(p) = 0. \tag{11}$$

From (11), it follows that $\psi_l(p) = 0$ for any $l \in \widetilde{A}(\bar{p})$ and $\psi_l(p) \leq 0$ for any $l \in A \setminus \widetilde{A}(\bar{p})$. Using geodesic convexity of the solution set S, we have $\exp_p(\tau \exp_p^{-1}(\bar{p})) \in S$, $\forall \tau \in [0,1]$. Thus, for any $\tau \in [0,1]$

$$\phi(\exp_p(\tau \exp_p^{-1}(\bar{p})) = \phi(p). \tag{12}$$

Choosing $\delta > 0$ small enough, since $\phi \circ \exp_p$ is locally Lipschitz in a neighborhood of 0_p in T_pH, then,

$$\begin{aligned}
\sup_{||w||<\delta, 0<t<\delta} & \frac{\phi \circ \exp_p(w + t \exp_p^{-1}(\bar{p})) - \phi \circ \exp_p(w)}{t} \\
& \geq \frac{\phi \circ \exp_p(0 + \delta \exp_p^{-1}(\bar{p})) - \phi \circ \exp_p(0)}{\delta} \\
& = \frac{\phi(\exp_p(\delta \exp_p^{-1}(\bar{p}))) - \phi(p)}{\delta} = 0.
\end{aligned}$$

Thus, the directional derivative of ϕ at p is given by:

$$\begin{aligned}
\phi^o(p; \exp_p^{-1}(\bar{p})) &= (\phi \circ \exp_p)^o(0_p; \exp_p^{-1}(\bar{p})) \\
&= \lim_{w \to 0} \sup_{t \downarrow 0} \frac{\phi \circ \exp_p(w + t \exp_p^{-1}(\bar{p})) - \phi \circ \exp_p(w)}{t} \geq 0.
\end{aligned}$$

Therefore, there exists $\zeta' \in \partial_c \phi(p)$, such that

$$\langle \zeta', \exp_p^{-1}(\bar{p}) \rangle_{T_p} \geq 0. \tag{13}$$

By using geodesic pseudoconvexity of ϕ and $\phi(p) = \phi(\bar{p})$, it follows from Lemmas 2.16 and 2.17 that for any $\zeta \in \partial_c \phi(p)$,

$$\langle \zeta, \exp_p^{-1}(\bar{p}) \rangle_{T_p} \leq 0. \tag{14}$$

Using (13) and (14),

$$\langle \zeta^{'}, \exp_p^{-1}(\bar{p}) \rangle_{T_p} = 0.$$

Since $\phi(p) = \phi(\bar{p})$ and ϕ is geodesic pseudoconvex, it follows from Lemmas 2.16 and 2.17,

$$\langle \xi, \exp_{\bar{p}}^{-1}(p) \rangle_{T_{\bar{p}}} \leq 0, \forall \xi \in \partial_c \phi(\bar{p}).$$

From the necessary optimality condition at $\bar{p}$, there exists $\hat{\xi} \in \partial_c \phi(\bar{p})$ and $\hat{\eta}_l \in \partial_c \psi_l(\bar{p}), l \in A$, such that

$$\langle \hat{\xi}, \exp_{\bar{p}}^{-1}(p) \rangle_{T_{\bar{p}}} + \sum_{l \in A} \bar{\lambda}_l \langle \hat{\eta}_l, \exp_{\bar{p}}^{-1}(p) \rangle_{T_{\bar{p}}} = 0. \tag{15}$$

From (15) and geodesic pseudoconvexity of $\psi_l, l \in A$, we have

$$\langle \hat{\xi}, \exp_{\bar{p}}^{-1}(p) \rangle_{T_{\bar{p}}} \geq 0.$$

Thus, there exist $\hat{\xi} \in C(\bar{p})$, such that

$$\langle \hat{\xi}, \exp_{\bar{p}}^{-1}(p) \rangle_{T_{\bar{p}}} = 0.$$

Hence, $p \in S_5$ and $S \subseteq S_5$.

Let us prove $S_2 \subseteq S$. Let p be an arbitrary element of S_2 which implies $p \in \Omega_1$ and there exists $\zeta \in \partial_c \phi(p)$, such that

$$\langle \zeta, \exp_p^{-1}(\bar{p}) \rangle_{T_p} \geq 0.$$

Using geodesic pseudoconvexity of ϕ, we have

$$\phi(\bar{p}) \geq \phi(p).$$

As a result $p \in S$. Hence, $S_2 \subseteq S$.

Proceeding further, we prove $S_4 \subseteq S$. Let p be an arbitrary element of S_4, this implies that $p \in \Omega_1$ and there exist $\zeta \in \partial_c \phi(p)$ and $\xi \in C(\bar{p})$, such that

$$\langle \xi, \exp_{\bar{p}}^{-1}(p) \rangle_{T_{\bar{p}}} \leq \langle \zeta, \exp_p^{-1}(\bar{p}) \rangle_{T_p}. \tag{16}$$

From (16) along with the definition of $C(\bar{p})$ and $\Omega_1 \subseteq \Omega$,

$$\langle \zeta, \exp_p^{-1}(\bar{p}) \rangle_{T_p} \geq 0.$$

Using geodesic pseudoconvexity of ϕ, we have

$$\phi(\bar{p}) \geq \phi(p),$$

therefore, $p \in S$ and we conclude that $S_4 \subseteq S$. □

Remark 4.3. (i) From the above theorem, the generalized directional derivative of ϕ at $p \in S$ is non-negative, that is

$$\phi^{\circ}(p; \exp_p^{-1}(q)) \geq 0, \forall q \in \Omega.$$

Therefore, in view of Lemma 2.10, $C(\bar{p})$ is non-empty for any $\bar{p} \in S$.

(ii) The aforementioned theorem shows that the active constraints with positive Lagrange multipliers at a solution remain active at every solution of the considered problem (NSIP).

(iii) In view of the geodesic pseudoconvexity assumption on ϕ, the condition in the aforementioned set S_1, that is, $\langle \zeta, \exp_p^{-1}(\bar{p})\rangle_{T_p} = 0$ for some $\zeta \in \partial_c \phi(p)$ ensure that:

$$\phi(p) \leq \phi(\bar{p}).$$

The above inequality has played a significant role in characterizing the solution set S in Theorem 4.2.

(iv) Notably, the defined set $C(\bar{p})$ ensure that $\langle \zeta, \exp_p^{-1}(\bar{p}) \geq 0$, for some $\zeta \in \partial_c \phi(p)$ in the aforementioned sets S_3, S_4, S_5.

(v) In view of Remark 4.1, it is significant to note that Theorem 4.2 may not necessarily hold in the framework of a general Riemannian manifold. Moreover, from the Remark 2.5, we observe that $\exp_p^{-1}(\bar{p})$ may not be well-defined in the entire Riemannian manifold unless $\bar{p} \in U_p$.

(vi) If $\partial_c \phi(\bar{p}) = \{\operatorname{grad} \phi(\bar{p})\}, \partial_c \psi_l(\bar{p}) = \{\operatorname{grad} \psi_l(\bar{p})\}, l \in A$. Moreover, if functions $\phi, \psi_l, l \in A$ are geodesic convex functions, then in view of Remark 2.15, Theorem 4.2 reduces to the Proposition 3.2 derived by Tung *et al.* [38].

(vii) If A is a finite set and $H = \mathbb{R}^n$, then in view of Remark 2.15, the above Theorem 4.2 reduces to Theorem 4.1 derived by Zhao and Yang [53].

(viii) If $H = \mathbb{R}^n$ and A is a finite set. Moreover, if $\partial_c \phi(\bar{p}) = \{\operatorname{grad} \phi(\bar{p})\} = \{\nabla \phi(\bar{p})\}, \partial_c \psi_l(\bar{p}) = \{\operatorname{grad} \psi_l(\bar{p})\} = \{\nabla \psi_l(\bar{p})\}, l \in A$ and each function becomes convex. In this particular case, the aforementioned theorem reduces to Theorem 1a in [27].

Example 4.4. Let us consider the manifold H as defined in Example 3.5.

Consider the following nonsmooth semi-infinite programming problem:

$$\begin{aligned} (P) \quad & \min \ \phi(p) = 2|p_1 - e|, \\ \text{subject to} \ \ & \psi_l(p) = \frac{1}{2} - \frac{(1-l)}{2} \ln p_1 - \frac{l}{2} \ln p_2 \leq 0, \quad l \in A = [0,1]. \end{aligned}$$

where, $\phi : H \to \mathbb{R}$, $\psi_l : H \to \mathbb{R}$. The feasible set is defined as:

$$\Omega := \{p \in H \mid p_1 \geq e, p_2 \geq e\}.$$

Clearly, $(e,e) \in S$ and we denote it by $\bar{p} = (e,e)$.

Now, we obtain the subdifferential of $\phi, \psi_l, l \in A$

$$\partial_c \phi(p) = \operatorname{co}\{(-2p_1^2, 0)^T, (2p_1^2, 0)^T\},$$

$$\partial_c \psi_l(p) = \mathscr{G}(p)^{-1}\{\operatorname{grad} \psi_l(p)\} = \left\{ \left(-p_1 \frac{(1-l)}{2}, -p_2 \frac{l}{2} \right)^T \right\}.$$

From the Example 3.5, (GCQ) is satisfied at (e,e).

Let $\bar{\lambda} : A \to \mathbb{R}$ be defined as follows:

$$\bar{\lambda}(l) = \begin{cases} 4e, & l = 0 \\ 0, & \text{otherwise} \end{cases}.$$

Thus, there exist $\xi = (2e^2, 0) \in \partial_c \phi(\bar{p})$ and $\eta_l = \left(-e\frac{(1-l)}{2}, -e\frac{l}{2}\right)^T \in \partial_c \psi_l(\bar{p})$, such that

$$\xi + \sum_{l \in A} \bar{\lambda}_l \eta_l = \left(2e^2, 0\right)^T + 4e\left(-e\frac{1}{2}, 0\right)^T = (0,0)^T.$$

Moreover, $\widetilde{A}(\bar{p}) = \{0\}$ and

$$\Omega_1 = \{p \in \Omega \mid \psi_0(p) = 0, \psi_l(p) \leq 0, \quad l \in A \setminus \widetilde{A}(\bar{p})\} = \Omega,$$

$$C(\bar{p}) = \left\{((2t-1)2e^2, 0) \mid \frac{1}{2} \leq t \leq 1\right\}.$$

By Theorem 4.2, the solution set is characterized by

$$S = S_1 = S_2 = S_3 = S_4 = S_5$$

We determine the set S_5 which is defined as

$$S_5 = \{p \in \Omega_1 \mid \langle \zeta, \exp_p^{-1}(\bar{p})\rangle_{T_p} = \langle \xi, \exp_{\bar{p}}^{-1}(p)\rangle_{T_{\bar{p}}} = 0 \text{ for some } \zeta \in \partial_c\phi(p) \text{ and } \xi \in C(\bar{p})\}.$$

Evidently, $(e, e) \in S_5$. Now,

$$\begin{aligned}\langle \zeta, \exp_p^{-1}(\bar{p})\rangle_{T_p} &= \zeta_1 p_1(1 - \ln p_1) + \zeta_2 p_2(1 - \ln p_2)\\ &= \zeta_1 p_1(1 - \ln p_1).\end{aligned}$$

Thus,

$$\langle \zeta, \exp_p^{-1}(\bar{p})\rangle_{T_p} = 0 \Leftrightarrow \zeta_1 p_1(1 - \ln p_1) = 0 \Leftrightarrow p_1 = e.$$

Moreover,

$$\langle \xi, \exp_{\bar{p}}^{-1}(p)\rangle_{T_{\bar{p}}} = (2t-1)2e^3(\ln p_1 - 1).$$

Then,

$$\langle \xi, \exp_{\bar{p}}^{-1}(p)\rangle_{T_{\bar{p}}} = 0 \Leftrightarrow t = \frac{1}{2} \text{ or } p_1 = e.$$

Therefore, the solution set is

$$S = S_5 = \{(e, p_2) \mid p_2 \geq e\} = S_2 = S_3 = S_4.$$

In the following theorem, we characterize the solution set of (NSIP) with the use of set Ω_2, a superset of Ω.

Theorem 4.5. *Suppose that $\bar{p}$ be an arbitrary element of S, such that (GCQ) holds at $\bar{p}$ and there exists Lagrange multiplier $\bar{\lambda} = (\bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_{|A|}) \in \mathbb{R}_+^{|A|}$, such that*

$$0 \in \partial_c\phi(\bar{p}) + \sum_{l \in A} \bar{\lambda}_l \psi_l(\bar{p}).$$

Then, the solution set is characterized by

$$S = S_6 = S_7 = S_8 = S_9 = S_{10} = S_{11} = S_{12} = S_{13} = S_{14} = S_{15},$$

where

(i) $S_6 := \{p \in \Omega_2 \mid \langle \eta_l, \exp_p^{-1}(\bar{p})\rangle_{T_p} = 0 \text{ for some } \eta_l \in \partial_c\psi_l(p),\ \forall l \in \widetilde{A}(\bar{p});$ $\langle \zeta, \exp_p^{-1}(\bar{p})\rangle_{T_p} = 0 \text{ for some } \zeta \in \partial_c\phi(p)\},$

(ii) $S_7 := \{p \in \Omega_2 \mid \langle \eta_l, \exp_p^{-1}(\bar{p})\rangle_{T_p} = 0 \text{ for some } \eta_l \in \partial_c\psi_l(p),\ \forall l \in \widetilde{A}(\bar{p});$ $\langle \zeta, \exp_p^{-1}(\bar{p})\rangle_{T_p} \geq 0 \text{ for some } \zeta \in \partial_c\phi(p)\},$

(iii) $S_8 := \{p \in \Omega_2 \mid \langle \eta_l, \exp_p^{-1}(\bar{p})\rangle_{T_p} \geq 0 \text{ for some } \eta_l \in \partial_c\psi_l(p),\ \forall l \in \widetilde{A}(\bar{p});$ $\langle \zeta, \exp_p^{-1}(\bar{p})\rangle_{T_p} = 0 \text{ for some } \zeta \in \partial_c\phi(p)\},$

(iv) $S_9 := \{p \in \Omega_2 \mid \langle \eta_l, \exp_p^{-1}(\bar{p})\rangle_{T_p} \geq 0$ for some $\eta_l \in \partial_c \psi_l(p),\ \forall l \in \widetilde{A}(\bar{p});$
$\langle \zeta, \exp_p^{-1}(\bar{p})\rangle_{T_p} \geq 0$ for some $\zeta \in \partial_c \phi(p)\}$,

(v) $S_{10} := \{p \in \Omega_2 \mid \langle \eta_l, \exp_p^{-1}(\bar{p})\rangle_{T_p} = 0$ for some $\eta_l \in \partial_c \psi_l(p),\ \forall l \in \widetilde{A}(\bar{p});$
$\langle \xi, \exp_{\bar{p}}^{-1}(p)\rangle_{T_{\bar{p}}} = \langle \zeta, \exp_p^{-1}(\bar{p})\rangle_{T_p}$ for some $\zeta \in \partial_c \phi(p)$ and $\xi \in C(\bar{p})\}$,

(vi) $S_{11} := \{p \in \Omega_2 \mid \langle \eta_l, \exp_p^{-1}(\bar{p})\rangle_{T_p} = 0$ for some $\eta_l \in \partial_c \psi_l(p),\ \forall l \in \widetilde{A}(\bar{p});$
$\langle \xi, \exp_{\bar{p}}^{-1}(p)\rangle_{T_{\bar{p}}} \leq \langle \zeta, \exp_p^{-1}(\bar{p})\rangle_{T_p}$ for some $\zeta \in \partial_c \phi(p)$ and $\xi \in C(\bar{p})\}$,

(vii) $S_{12} := \{p \in \Omega_2 \mid \langle \eta_l, \exp_p^{-1}(\bar{p})\rangle_{T_p} \geq 0$ for some $\eta_l \in \partial_c \psi_l(p),\ \forall l \in \widetilde{A}(\bar{p});$
$\langle \xi, \exp_{\bar{p}}^{-1}(p)\rangle_{T_{\bar{p}}} = \langle \zeta, \exp_p^{-1}(\bar{p})\rangle_{T_p}$ for some $\zeta \in \partial_c \phi(p)$ and $\xi \in C(\bar{p})\}$,

(viii) $S_{13} := \{p \in \Omega_2 \mid \langle \eta_l, \exp_p^{-1}(\bar{p})\rangle_{T_p} \geq 0$ for some $\eta_l \in \partial_c \psi_l(p),\ \forall l \in \widetilde{A}(\bar{p});$
$\langle \xi, \exp_{\bar{p}}^{-1}(p)\rangle_{T_{\bar{p}}} \leq \langle \zeta, \exp_p^{-1}(\bar{p})\rangle_{T_p}$ for some $\zeta \in \partial_c \phi(p)$ *and* $\xi \in C(\bar{p})\}$,

(ix) $S_{14} := \{p \in \Omega_2 \mid \langle \eta_l, \exp_p^{-1}(\bar{p})\rangle_{T_p} \geq 0$ for some $\eta_l \in \partial_c \psi_l(p),\ \forall l \in \widetilde{A}(\bar{p});$
$\langle \xi, \exp_{\bar{p}}^{-1}(p)\rangle_{T_{\bar{p}}} = \langle \zeta, \exp_p^{-1}(\bar{p})\rangle_{T_p} = 0$ for some $\zeta \in \partial_c \phi(p)$ and $\xi \in C(\bar{p})\}$,

(x) $S_{15} := \{p \in \Omega_2 \mid \langle \eta_l, \exp_p^{-1}(\bar{p})\rangle_{T_p} = 0$ for some $\eta_l \in \partial_c \psi_l(p),\ \forall l \in \widetilde{A}(\bar{p});$
$\langle \xi, \exp_{\bar{p}}^{-1}(p)\rangle_{T_{\bar{p}}} = \langle \zeta, \exp_p^{-1}(\bar{p})\rangle_{T_p} = 0$ for some $\zeta \in \partial_c \phi(p)$ and $\xi \in C(\bar{p})\}$.

Proof. One can easily see that the following inclusions hold:

$$S_{15} \subseteq S_{14} \subseteq S_{12} \subseteq S_{13},$$

$$S_{15} \subseteq S_{14} \subseteq S_8 \subseteq S_9,$$

$$S_{15} \subseteq S_{10} \subseteq S_{11} \subseteq S_{13},$$

$$S_{15} \subseteq S_6 \subseteq S_7 \subseteq S_9.$$

Hence, we only need to prove that

$$S \subseteq S_{15},$$

$$S_9 \subseteq S,$$

$$S_{13} \subseteq S.$$

We start by proving $S \subseteq S_{15}$. Let p be an arbitrary element of S and from Theorem 4.2 we have $p \in S_5$ and therefore,

(i) $p \in \Omega_1 \subseteq \Omega_2$,
(ii) there exists $\zeta \in \partial_c \phi(p)$ and $\xi \in C(\bar{p})$, such that $\langle \xi, \exp_{\bar{p}}^{-1}(p)\rangle_{T_{\bar{p}}} =$ $\langle \zeta, \exp_p^{-1}(\bar{p})\rangle_{T_p} = 0$,
(iii) for any $l \in \widetilde{A}(\bar{p}), \psi_l(p) = 0$.

Geodesic convexity of solution set implies $\exp_p(\tau \exp_p^{-1}(\bar{p})) \in S$ for any $\tau \in [0,1]$. From Theorem 3.11, it follows that for any $l \in \widetilde{A}(\bar{p})$,

$$\psi_l(\exp_p(\tau \exp_p^{-1}(\bar{p})) = \psi_l(p) = 0. \tag{17}$$

From (17) and proof of Theorem 4.2, it follows that

$$\psi_l^o(p; \exp_p^{-1}(\bar{p})) \geq 0.$$

For any $l \in \widetilde{A}(\bar{p})$, there exists $\hat{\eta}_l \in \partial_c \psi_l(p)$, such that

$$\langle \hat{\eta}_l, \exp_p^{-1}(\bar{p})\rangle_{T_p} \geq 0. \tag{18}$$

Moreover,

$$\psi_l(\bar{p}) = 0 = \psi_l(p), \quad \forall l \in \widetilde{A}(\bar{p}).$$

From the geodesic pseudoconvexity of $\psi_l, l \in \widetilde{A}(\bar{p})$, along with Lemmas 2.16 and 2.17, we have

$$\langle \eta_l, \exp_p^{-1}(\bar{p})\rangle_{T_p} \leq 0, \quad \forall \eta_l \in \partial_c \psi_l(p), l \in \widetilde{A}(\bar{p}). \tag{19}$$

Combining (18), (19) there exist $\hat{\eta}_l \in \partial_c \psi_l(p)$ for any $l \in \widetilde{A}(\bar{p})$, such that

$$\langle \hat{\eta}_l, \exp_p^{-1}(\bar{p})\rangle_{T_p} = 0.$$

Hence, $p \in S_{15}$ and $S \subseteq S_{15}$.

We prove $S_9 \subseteq S$. Let p be an arbitrary element of S_9 and therefore,

(iv) $p \in \Omega_2 \subseteq H$,
(v) there exists $\zeta \in \partial_c \phi(p)$, such that $\langle \zeta, \exp_p^{-1}(\bar{p})\rangle_{T_p} \geq 0$,
(vi) for any $l \in \widetilde{A}(\bar{p})$, there exist $\eta_l \in \partial_c \psi_l(p)$, such that $\langle \eta_l, \exp_p^{-1}(\bar{p})\rangle_{T_p} \geq 0$.

From (iv), (vi) along with the geodesic pseudoconvexity of ψ_l,

$$\psi_l(p) \leq \psi_l(\bar{p}) = 0, \quad \forall l \in \widetilde{A}(\bar{p}).$$

Thus, $p \in \Omega$. Furthermore, from (v) along with the geodesic pseudoconvexity of ϕ, it follows that $p \in S$ and thus, $S_9 \subseteq S$.

Finally, we prove $S_{13} \subseteq S$. Let p be an arbitrary element of S_{13}. As we have shown in the above part, it can be shown that $p \in \Omega$. Moreover, there exist $\zeta \in \partial_c \phi(p)$ and $\xi \in C(\bar{p})$, such that

$$\langle \xi, \exp_{\bar{p}}^{-1}(p)\rangle_{T_{\bar{p}}} \leq \langle \zeta, \exp_p^{-1}(\bar{p})\rangle_{T_p}. \tag{20}$$

From (20) along with definition of $C(\bar{p})$ and $p \in \Omega$ implies $p \in S_2 \subseteq S$. Thus, $p \in S$, claims the proof. □

Remark 4.6. (i) It is worthwhile to note that, in order to characterize a point in the solution set S, here we have chosen a point from the superset of feasible set Ω and ensured its feasibility by utilizing the geodesic pseudoconvexity of constraints, $\psi_l, l \in \tilde{A}(\bar{p})$. In particular, $\langle \eta_l, \exp_p^{-1}(\bar{p})\rangle_{T_p} \geq 0$ for some $\eta_l \in \partial_c \psi_l(p)$, then geodesic pseudoconvexity of ψ_l implies $\psi_l(p) \leq \psi_l(\bar{p}) = 0$.

(ii) In view of the definition of set $C(\bar{p})$, in the aforementioned set S_{14}, one can replace

$$\langle \xi, \exp_{\bar{p}}^{-1}(p)\rangle_{T_{\bar{p}}} = \langle \zeta, \exp_p^{-1}(\bar{p})\rangle_{T_p} = 0,$$

by the following condition:

$$\langle \xi, \exp_{\bar{p}}^{-1}(p)\rangle_{T_{\bar{p}}} = \langle \zeta, \exp_p^{-1}(\bar{p})\rangle_{T_p} \leq 0,$$

for some $\zeta \in \partial_c \phi(p)$ and $\xi \in C(\bar{p})$.

(iii) In view of Remarks 4.1(vi), 4.3(v), Theorem 4.5 may not necessarily hold for all Riemannian manifolds, unless it is complete Riemannian manifold and the exponential map is a diffeomorphism onto H.

(iv) Theorem 4.5 generalizes Theorem 4.2 derived by Zhao and Yang [53] from the framework of Euclidean spaces to the setting of Hadamard manifolds, provided A is a finite set.

Example 4.7. Consider the following Riemannian manifold H known as Poincaré half-plane (see [40]) and is defined as follows :

$$H := \{q = (q_1, q_2) \in \mathbb{R}^2 \mid q_2 > 0\}.$$

The tangent space at any point $q \in H$ is given by $T_q H = \mathbb{R}^2$. The Riemannian metric on considered manifold H is given by

$$\langle r, s\rangle_{T_q} = \langle \mathscr{G}(q) r, s\rangle, \quad \forall r, s \in T_q H = \mathbb{R}^2,$$

where $\mathscr{G}(q) = \begin{bmatrix} \frac{1}{q_2^2} & 0 \\ 0 & \frac{1}{q_2^2} \end{bmatrix}$. Furthermore, H is a Hadamard manifold as it has sectional curvature value -1.

For $p \in H$ and $s \in T_pH$, the exponent map $\exp_p : T_pH \to H$ is defined by

If $s_1 = 0$,

$$\exp_p(s) = \left(p_1, p_2 e^{\frac{s_2}{p_2}}\right).$$

If $s_1 \neq 0$, $\exp_p(s)$ is given by

$$\left(p_1 + \frac{s_2}{s_1} + \sqrt{1 + \left(\frac{s_2}{s_1}\right)^2} \tanh\left(u_{s_1,s_2}(1)\right), p_2\sqrt{1 + \left(\frac{s_2}{s_1}\right)^2} \frac{1}{\cosh\left(v_{s_1,s_2(1))}\right)}\right),$$

where

$$u_{s_1,s_2}(t) = \begin{cases} t\sqrt{s_1^2 + s_2^2} - \operatorname{arcsinh}\frac{s_2}{s_1}, & \text{if } s_1 > 0, \\ -t\sqrt{s_1^2 + s_2^2} - \operatorname{arcsinh}\frac{s_2}{s_1}, & \text{if } s_1 < 0, \end{cases}$$

$$v_{s_1,s_2}(t) = \begin{cases} t\frac{\sqrt{s_1^2+s_2^2}}{p_2} - \operatorname{arcsinh}\frac{s_2}{s_1}, & \text{if } s_1 > 0, \\ -t\frac{\sqrt{s_1^2+s_2^2}}{p_2} - \operatorname{arcsinh}\frac{s_2}{s_1}, & \text{if } s_1 < 0. \end{cases}$$

Furthermore, the inverse exponential map $\exp_p^{-1} : H \to T_pH$ is given by:

$$\exp_p^{-1}(q) = \begin{cases} \left(0, p_2 \ln\frac{q_2}{p_2}\right), & \text{if } p_1 = q_1, \\ \frac{p_2}{a}\left(\operatorname{arctanh}\frac{b-p_1}{a} - \operatorname{arctanh}\frac{b-q_1}{a}\right)(p_2, b - p_1), & \text{if } p_1 \neq q_1, \end{cases}$$

where

$$b = \frac{p_1^2 + p_2^2 - \left(q_1^2 + q_2^2\right)}{2\left(p_1 - q_1\right)}, \quad a = \sqrt{\left(p_1 - b\right)^2 + p_2^2}.$$

Considering the following open geodesic convex set on H as follows:

$$F := \{(p_1, p_2) \in H \mid p_1 > -\frac{1}{2}\}.$$

Consider the following nonsmooth semi-infinte programming problem:

$$\begin{aligned} (P) \quad \min \phi(p) &= \frac{|p_1|}{2p_2} + \frac{p_2}{2}, \\ \text{subject to } \ \psi_l(p) &= \frac{l}{2p_2} - l \leq 0, \ l \in A = [0,1] \end{aligned}$$

where, $\phi : F \to \mathbb{R}$, $\psi_l : F \to \mathbb{R}$, $l \in [0,1]$ are locally Lipschitz and geodesic pseudoconvex real valued functions defined on F. The feasible set Ω for the problem is:

$$\Omega := \{p \in H \mid p_1 > -\frac{1}{2}, p_2 \geq \frac{1}{2}\}.$$

Clearly, $\left(0, \frac{1}{2}\right)^T \in S$ and we denote it by $\bar{p} = (0, \frac{1}{2})^T$. Then, it can be verified that

$$\mathscr{T}(\Omega, \bar{p}) = \{(p_1, p_2) \in T_{\bar{p}}F \mid p_2 \geq 0\}.$$

Moreover, we obtain the following

$$\partial_c\phi(p) = \text{co}\left\{\left(-\frac{p_2}{2}, \frac{p_1+p_2^2}{2}\right)^T, \left(\frac{p_2}{2}, \frac{-p_1+p_2^2}{2}\right)^T\right\},$$

$$\partial_c\psi_l(p) = \{\text{grad}\,\psi_l(p)\} = \left\{\left(0, -\frac{l}{2}\right)^T\right\}, \quad l \in A = [0,1].$$

Then, the linearizing cone to the feasible set Ω at $\bar{p}$ is given by:

$$\mathscr{L}(\Omega,\bar{p}) := \{p=(p_1,p_2) \in T_{\bar{p}}F \mid p_2 \geq 0\} \subseteq \text{cl}(\text{co}(\mathscr{T}(\Omega,\bar{p}))) = \mathscr{T}(\Omega,\bar{p}).$$

Since, $A(\bar{p}) = \{1\}$, it follows that

$$\text{pos}\cup_{l\in A(\bar{p})}\partial_c\psi_l(\bar{p}) = \{p=(p_1,p_2) \in T_{\bar{p}}H \mid p_1 = 0, p_2 \leq 0\}.$$

which is a closed set. Thus, (GCQ) is satisfied at the feasible point $\bar{p} = \left(0, \frac{1}{2}\right)$. Let $\bar{\lambda} : A \to \mathbb{R}$ be defined as follows:

$$\bar{\lambda}(l) = \begin{cases} \frac{1}{4}, & l = 1 \\ 0, & \text{otherwise} \end{cases},$$

Then, there exist $\xi = \left(0, \frac{1}{8}\right)^T \in \partial_c\phi(\bar{p}),\ \eta_l = \left(0, -\frac{l}{2}\right)^T \in \partial_c\psi_l(\bar{p}),\ l \in A$, such that

$$\xi + \sum_{l\in A}\bar{\lambda}_l\eta_l = \left(0, \frac{1}{8}\right)^T + \left(0, -\frac{1}{8}\right)^T = (0,0)^T.$$

Furthermore, $\widetilde{A}(\bar{p}) = \{1\}$ and

$$\Omega_2 = \{p \in H \mid \psi_l(p) \leq 0, \forall l \in A \setminus \widetilde{A}(\bar{p})\} = \Omega,$$

$$C(\bar{p}) = \left\{\left(\frac{2t-1}{4}, \frac{1}{8}\right) \mid 0 < t < 1\right\}.$$

By Theorem 4.5, the solution set is characterized by

$$S = S_6 = S_7 = S_8 = S_9 = S_{10} = S_{11} = S_{12} = S_{13} = S_{14} = S_{15}.$$

We determine the solution set S_{10} which is defined as
$S_{10} := \{p \in \Omega_2 \mid \langle\eta_l, \exp_p^{-1}(\bar{p})\rangle_{T_p} = 0$ for some $\eta_l, l \in \widetilde{A}(\bar{p}), \langle\zeta, \exp_p^{-1}(\bar{p})\rangle_{T_p} = \langle\zeta, \exp_{\bar{p}}^{-1}(p)\rangle_{T_{\bar{p}}} = 0$ for some $\zeta \in \partial_c\phi(p)$ and $\xi \in C(\bar{p})\}$.
One can immediately observe that $(0, \frac{1}{2}) \in S_{10}$. Suppose, there exist $(p_1,p_2) \in S_{10}$, such that $p_1 \neq 0$ then,

$$\langle\eta_1, \exp_p^{-1}(\bar{p})\rangle_{T_p} = \frac{p_2}{a}\left(\text{arctanh}\,\frac{b-p_1}{a} - \text{arctanh}\,\frac{b}{a}\right)\left(0p_2 - \frac{1}{2}(b-p_1)\right),$$

$$\langle\eta_1, \exp_p^{-1}(\bar{p})\rangle_{T_p} = 0,$$

if and only if $-\frac{1}{2}(b-p_1) = 0 \Rightarrow b = p_1$.
Moreover,

$$\langle\xi, \exp_{\bar{p}}^{-1}(p)\rangle_{T_{\bar{p}}} = \langle\zeta, \exp_p^{-1}(\bar{p})\rangle_{T_p},$$

$$
\begin{aligned}
&\frac{1}{2a}\left(\operatorname{arctanh}\frac{b}{a}-\operatorname{arctanh}\frac{b-p_1}{a}\right)\left(\xi_1\frac{1}{2}+\xi_2(b-0)\right)\\
&=\frac{p_2}{a}\left(\operatorname{arctanh}\frac{b-p_1}{a}-\operatorname{arctanh}\frac{b}{a}\right)\left(\zeta_1p_2+\zeta_2(b-p_1)\right),
\end{aligned}
$$

$$
\frac{1}{2}\left(\xi_1\frac{1}{2}+\xi_2 b\right)=-p_2\left(\zeta_1p_2+\zeta_2(b-p_1)\right).
$$

Since $b=p_1$, we get

$$
\xi_1\frac{1}{4}+\frac{\xi_2p_1}{2}=-p_2^2\zeta_1. \tag{21}
$$

Moreover, $\xi\in C(\bar p)$ and $p_1\neq 0$ implies $\zeta_1=-\frac{p_2}{2}$ for $p_1<0$.
Thus,

$$
\frac{2t-1}{16}+\frac{p_1}{16}=\frac{p_2^3}{2},
$$

gives $t>1$ which is not possible.
For $p_1>0$, it follows from equation (21),

$$
\frac{2t-1}{16}+\frac{p_1}{16}=-\frac{p_2^3}{2},
$$

gives $t<0$, not possible. In both cases, we arrive at contradiction, therefore,

$$
S_{10}=\left\{\left(0,\frac{1}{2}\right)\right\}=S=S_6=S_7=S_8=S_9=S_{11}=S_{12}=S_{13}=S_{14}=S_{15}.
$$

5. Conclusion and future directions

In this article, we have investigated a class of (NSIP), involving locally Lipschitz geodesic pseudoconvex functions, in the framework of Hadamard manifolds. The generalized constraint qualification, namely, Guignard constraint qualification (GCQ) has been utilized to establish the KKT-type necessary optimality conditions, instead of Abadie constraint qualification (ACQ) as (GCQ) imposes weaker restrictions on both the constraints and their gradients as compared to (ACQ). It has been established that the Lagrange function, associated with a fixed Lagrange multiplier corresponding to a known solution, remains constant on the solution set. Various characterizations of the solution set for (NSIP) have been established by utilizing the properties of locally Lipschitz geodesic pseudoconvex functions. Moreover, it has been highlighted that despite the globally diffeomorphism of exponential maps in Hadamard manifolds, yet the development of optimization techniques in the framework of Hadamard manifolds is significantly difficult in comparison to Euclidean spaces.

The results established in this paper extend the corresponding results deduced by Tung *et al.* [38] from smooth (SIP) to (NSIP), as well as generalize them for a broader class of nonconvex functions, namely, geodesic pseudoconvex functions on Hadamard manifolds. Furthermore, the results derived in this paper generalize the corresponding results derived by Zhao and Yang [53] to a more general space, namely, Hadamard manifolds.
The results presented in this paper pave various scopes for future research. For instance, we intend to extend the results derived in this paper in the framework of Riemannian manifolds. Moreover, it is well-known that the limiting subdifferential [30,31] offers a better Lagrange multiplier rule as compared to the Clarke subdifferential and is the smallest among all robust subdifferentials. Considering this well-established fact and the research work of Farrokhiniya and Barani [12], the results of this paper could be further sharpened by employing the limiting subdifferential on Hadamard manifolds.

Acknowledgements

The authors would like to thank the anonymous referees for their careful reading of the manuscript and constructive suggestions, which have substantially improved the paper in its present form. The second author would also like to thank the Ministry of Education, Government of India, for their financial support *via* Prime Minister Research Fellowship (PMRF), awarded under PMRF-ID, 2703571.

References

[1] P.A. Absil, R. Mahony and R. Sepulchre, Optimization Algorithms on Matrix Manifolds. Princeton University Press, Princeton, NJ (2009).
[2] D. Azagra, J. Ferrera and F. López-Mesas, Nonsmooth analysis and Hamilton- Jacobi equations on Riemannian manifolds. *J. Funct. Anal.* **220** (2005) 304–361.
[3] D. Azagra and J. Ferrera, Applications of proximal calculus to fixed point theory on Riemannian manifolds. *Nonlinear Anal.* **67** (2007) 154–174.
[4] A. Barani and S. Hosseini, Characterization of solution sets of convex optimization problems in Riemannian manifolds. *Arch. Math.* **114** (2020) 215–225.
[5] A. Barani, Generalized monotonicity and convexity for locally Lipschitz functions on Hadamard manifolds. *Differ. Geom. Dyn. Syst.* **15** (2013) 26–37.
[6] G.C. Bento, J.X. Cruz Neto and L.V. Meirless, Proximal point algorithms for locally Lipschitz functions in multiobjective optimization on Hadamard manifolds. *J. Optim. Theory Appl.* **179** (2018) 37–52.
[7] J. Burke and M.C. Ferris, Characterization of solution set of convex programs. *Oper. Res. Lett.* **10** (1991) 57–60 .
[8] S. Chen and C. Fang, Vector variational inequality with pseudoconvexity on Hadamard manifolds. *Optimization* **65** (2016) 2067–2080.
[9] F.H. Clarke, Optimization and Nonsmooth Analysis. Wiley Interscience, New York (1983).
[10] I. Chryssochoos and R.B. Vinter, Optimal control problems on manifolds: a dynamic programming approach. *J. Math. Anal. Appl.* **287** (2003) 118–140.
[11] T.D. Chuong and D.S. Kim, Nonsmooth semi-infinite multiobjective optimization problems. *J. Optim. Theory Appl.* **160** (2014) 748–762.
[12] M. Farrokhiniya and A. Barani, Limiting subdifferential calculus and perturbed distance function in Riemannian manifolds. *J. Global Optim.* **77** (2020) 661–685.
[13] O.P. Ferreira, Proximal subgradient and a characterization of Lipschitz function on Riemannian manifolds. *J. Math. Anal. Appl.* **313** (2006) 587–597.
[14] A. Ghosh, B.B. Upadhyay and I.M. Stancu-Minasian, Pareto efficiency criteria and duality for multiobjective fractional programming problems with equilibrium constraints on Hadamard Manifolds. *Mathematics* **11** (2023) 1–28.
[15] A. Ghosh, B.B. Upadhyay and I.M. Stancu-Minasian, Constraint qualifications for multiobjective programming problems on Hadamard manifolds. *Aust. J. Math. Anal. Appl.* **20** (2023) 1–17.
[16] A. Haar, Über lineare ungleichungen. *Acta Sci. Math. (Szeged).* **2** (1924) 1–14.
[17] S. Hosseini and M.R. Pouryayevali, Generalized gradients and characterization of epi-Lipschitz sets in Riemannian manifolds. *Nonlinear Anal.* **74** (2011) 3884–3895.
[18] V. Ivanov, Characterizations of the solution sets of differentiable quasiconvex programming problems. *J. Optim. Theory Appl.* **181** (2019) 144–162.
[19] V. Jeyakumar and X.Q. Yang, On characterizing the solution sets of pseudolinear programs. *J. Optim. Theory Appl.* **87** (1995) 747–755.
[20] V. Jeyakumar, G.M. Lee and N. Dinh, Lagrange multiplier conditions characterizing the optimal solution sets of cone-constrained convex programs. *J. Optim. Theory Appl.* **123** (2004) 83–103.
[21] N. Kanzi, G. Caristi and A. Sadeghieh, Optimality conditions for semi-infinite programming problems involving generalized convexity. *Optim. Lett.* **13** (2019) 113–126.
[22] M.M. Karkhaneei and N. Mahdavi-Amiri, Nonconvex weak sharp minima on Riemannian manifolds. *J. Optim. Theory Appl.* **183** (2019) 85–104.
[23] A. Kristály, C. Li, G. López-Acedo and A. Nicolae, What do 'convexities' imply on Hadamard manifolds?. *J. Optim. Theory Appl.* **170** (2016) 1068–1074.

[24] Y.S. Ledyaev and Q.J. Zhu, Nonsmooth analysis on smooth manifolds. *Trans. Amer. Math. Soc.* **359** (2007) 3687–3732.

[25] M.A. López and G. Still, Semi-infinite programming. *Eur. J. Oper. Res.* **180** (2007) 491–518.

[26] T. Maeda, Constraint qualifications in multiobjective optimization problems: differentiable case. *J. Optim. Theory Appl.* **80** (1994) 483–500.

[27] O.L. Mangasarian, A simple characterization of solution sets of convex programs. *Oper. Res. Lett.* **7** (1988) 21–26.

[28] O.L. Mangasarian, Nonlinear Programming. SIAM (1969).

[29] S.K. Mishra, B.B. Upadhyay and L.T.H. An, Lagrange multiplier characterizations of solution sets of constrained nonsmooth pseudolinear optimization problems. *J. Optim. Theory Appl.* **160** (2014) 763–777.

[30] B.S. Mordukhovich, Variational Analysis and Generalized Differentiation. I. Basic Theory. Springer Science & Business Media (2006).

[31] B.S. Mordukhovich, Variational Analysis and Generalized Differentiation. II. Applications. Springer Science & Business Media (2006).

[32] Y.E. Nesterov and M.J. Todd, On the Riemannian geometry defined by self-concordant barrier and interior point methods. *Found. Comput. Math.* **2** (2002) 333–361.

[33] E.A. Papa Quiroz and P.R. Oliveira, Proximal point methods for quasiconvex and convex functions with Bregman distances on Hadamard manifolds. *J. Convex Anal.* **16** (2009) 49–69.

[34] E.A. Papa Quiroz, E.M. Quispe and P.R. Oliveira, Steepest descent method with a generalized Armijo search for quasiconvex functions on Riemannian manifolds. *J. Math. Anal. Appl.* **341** (2008) 467–477.

[35] T. Rapcsák, Smooth Nonlinear Optimization in $\mathbb{R}^n$. Springer Science & Business Media (2013).

[36] U. Ravat and U.V. Shanbhag, On the characterization of solution sets of smooth and nonsmooth convex stochastic Nash games. *SIAM J. Optim.* **21** (2011) 1168–1199.

[37] S. Treanţă, B.B. Upadhyay, A. Ghosh and K. Nanlaopon, Optimality conditions for multiobjective mathematical programming problems with equilibrium constraints on Hadamard manifolds. *Mathematics* **10** (2022) 1–20.

[38] L.T. Tung, D.H. Tam and V. Singh, Characterization of solution sets of geodesic convex semi-infinite programming on Riemannian manifolds. *Appl. Set-Valued Anal. Optim.* **5** (2023) 1–18.

[39] L.T. Tung and D.H. Tam, Optimality conditions and duality for multiobjective semi-infinite programming on Hadamard manifolds. *Bull. Iranian Math. Soc.* **48** (2022) 2191–2219.

[40] C. Udrişte, Convex Functions and Optimization Methods on Riemannian Manifolds. Springer Science & Business Media (2013).

[41] B.B. Upadhyay, A. Ghosh, P. Mishra and S. Treanţă, Optimality conditions and duality for multiobjective semi-infinite programming problems on Hadamard manifolds using generalized geodesic convexity. *RAIRO:RO* **56** (2022) 2037–2065.

[42] B.B. Upadhyay, I.M. Stancu-Minasian, P. Mishra and R.N. Mohapatra, On generalized vector variational inequalities and nonsmooth vector optimization problems on Hadamard manifolds involving geodesic approximate convexity. *Adv. Nonlinear Var. Inequal.* **25** (2022) 1–25.

[43] B.B. Upadhyay, S. Treanţă and P. Mishra, On Minty variational principle for nonsmooth multiobjective optimization problems on Hadamard manifolds. *Optimization* **71** (2022) 1–19.

[44] B.B. Upadhyay, A. Ghosh and S. Treanţă, Optimality conditions and duality for nonsmooth multiobjective semi-infinite programming problems on Hadamard manifolds. *Bull. Iranian Math. Soc.* **49** (2023) 1–36.

[45] B.B. Upadhyay and A. Ghosh, On constraint qualifications for mathematical programming problems with vanishing constraints on Hadamard manifolds. *J. Optim. Theory Appl.* **199** (2023) 1–35.

[46] B.B Upadhyay, A. Ghosh and I.M. Stancu-Minasian, Second-order optimality condition and duality for multiobjective semi-infinite programming problems on Hadamard manifolds. *Asia-Pac. J. Oper. Res.* (2023).

[47] B.B. Upadhyay, A. Ghosh and S. Treanţă, Optimality conditions and duality for nonsmooth multiobjective semi-infinite programming problems with vanishing constraints on Hadamard manifolds. *J. Math. Anal. Appl.* **531** (2023) 127785.

[48] B.B. Upadhyay, A. Ghosh and S. Treanţă, Constraint qualifications and optimality criteria for nonsmooth multiobjective programming problems on Hadamard manifolds. *J. Optim. Theory Appl.* **200** (2024) 794–819.

[49] B.B. Upadhyay, A. Ghosh and S. Treanţă, Efficiency conditions and duality for multiobjective semi-infinite programming problems on Hadamard manifolds. *J. Global Optim.* (2024).

[50] B.B. Upadhyay, I.M. Stancu-Minasian and P. Mishra, On relations between nonsmooth interval-valued multiobjective programming problems and generalized Stampacchia vector variational inequalities. *Optimization* **71** (2022) 2635–2659.

[51] J. Wang, X. Wang, C. Li and J.C. Yao, Convergence analysis of gradient algorithms on Riemannian manifolds without curvature constraints and application to Riemannian mass. *SIAM J. Optim.* **31** (2021) 172–199.

[52] Z.L. Wu and S.Y. Wu, Characterizations of solution sets of convex programs and variational inequality problems. *J. Optim. Theory Appl.* **130** (2006) 341–360.

[53] K.Q. Zhao and X.M. Yang, Characterizations of the solution set for a class of nonsmooth optimization problems. *Optim. Lett.* **7** (2013) 685–694.