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AN EFFICIENT ALGORITHM FOR TWO-STAGE CAPACITATED TIME MINIMIZATION TRANSPORTATION PROBLEM WITH RESTRICTED FLOW

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Abstract. This paper discusses a two-stage capacitated time minimization transportation problem with the restricted flow (TSCTMTP-F) in which the transportation takes place in two stages and only a specified amount of commodity is transported in both stages. The total amount F_1 is transported during Stage-I and F_2 during Stage-II, and the objective is to minimize the sum of the transportation times for Stage-I and Stage-II. In 2017, Kaur *et al.* [*RAIRO-Oper. Res.* **51** (2017) 1169–1184] studied this problem and developed a polynomially bounded iterative algorithm (Algorithm-A) to solve TSCTMTP-F. However, their proposed algorithm has some flaws and may not always yield an optimal solution to the problem TSCTMTP-F. An improved iterative algorithm (Algorithm-C) is proposed in this paper that guarantees an optimal solution to the problem. Various theoretical results prove the convergence and efficiency of Algorithm-C to obtain an optimal solution to the problem TSCTMTP-F. Numerical problems are included in the support of theory along with a counter-example for which Algorithm-A fails to obtain its optimal solution. Computational experiments on a variety of test problems have been carried out to validate the convergence and efficiency of Algorithm-C.

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1. Introduction

The transportation problem is an important class of optimization problem that plays a crucial role in supply chain management. In a supply chain, transportation refers to the movement of commodities from various warehouses (sources) to the end users (destinations). In today's competitive world, transportation managers need to keep the supply chain moving with minimal cost and minimal delay. Therefore, the strategic use of appropriate transportation is an important part of an industrial project.

Transportation problem deals with the shipment or transportation of a homogeneous commodity from various sources to various destinations under availability and requirement constraints in order to minimize the cost of transportation or time of transportation or both. If the objective is to find a transportation schedule that

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minimizes the total transportation cost and satisfies all the transportation constraints, the problem is called the cost minimization transportation problem (CMTP).

Sometimes, in emergencies like war, natural disaster, medical assistance, transportation of perishable goods, etc., minimizing the time of transport is more crucial than minimizing the cost of transportation problems. This gives rise to a very significant variant of transportation problems called the time minimization transportation problem (TMTP).

TMTP was initially studied by Grabowski [17], Hammer [21], Ahuja [1]. In 1980, Bansal and Puri [3] proved that TMTP lies in the class of concave minimization problems, thereby restricting the hunt for its optimal solution to the set of basic feasible solutions only. Other variants of TMTP like TMTP with impurities [16, 44], capacitated TMTP with rim conditions [19], fractional bottleneck objective [23] and constrained bottleneck transportation problem [10] have been studied by researchers due to their wide applications in industry.

Both the aforementioned problems, *i.e.*, CMTP and TMTP, have only one objective function which is either minimization of total transportation cost or minimization of transportation time. However, in many real-world situations, it becomes essential to optimize two or more objectives simultaneously. For example, in supply chain management, the objective is to minimize transportation costs and delays in transportation. In the manufacturing industry, the manager would like to minimize the production cost of the vehicle but maximize its efficiency. Therefore, the objectives of a problem may be accordant or conflicting. Various authors have contributed in this direction with a wide variety of objective functions in a crisp and fuzzy environment. Gupta and Arora [18], Sarma [34], Singh and Saxena [40], Singh [39] investigated time-cost trade-off analysis in transportation problems. Ammar and Youness [2], Bula *et al.* [7], Chakraborty *et al.* [9], Gupta *et al.* [20], Sungeeta *et al.* [41], Biswas *et al.* [6] carried out important studies for multi-objective transportation problems.

Another significant variant of CMTP and TMTP arises due to flow restrictions which are attributed to various reasons like budgetary constraints, limited storage capacity at sources or destinations, perishable commodities being shipped, limited vehicle availability, and limited production capacity of sources due to insufficient manpower or limited availability of raw materials, etc. In such a problem, there is a restriction on the total amount of the commodity to be transported from various sources to destinations and consequently, the problem is termed a transportation problem with restricted flow. Khurana *et al.* [29, 30], Thirwani *et al.* [43], Khanna *et al.* [28] developed efficient solution techniques for transportation problems with restricted flow. Here, only a specified amount of the commodity can be transported from sources to destinations in one go. But situations may arise where demand at the destinations has to be satisfied exactly along with a restriction on the total flow in each go. Therefore, to satisfy the total demand transportation has to be done more than once. This leads to a new variant of transportation problems called multi-stage transportation problems. Two-stage time minimization transportation problem (TSTMTP) has been studied as a variant of standard TMTP by various researchers. Sharma *et al.* [37] developed an iterative algorithm to solve a two-stage capacitated time minimization transportation problem where capacity constraints were imposed on each source-destination link. Pandian and Natarajan [32] developed an algorithm to solve a two-stage cost minimization transportation problem that was based on the zero-point method. In 2008, Sharma *et al.* [36] discussed another variant of the two-stage transportation problem, *viz.*, the two-stage interval time minimization transportation problem. Later in 2017, Kaur *et al.* [27] studied a capacitated two-stage time minimization transportation problem with a specified flow in each stage. Recently, Xie *et al.* [45] presented two iterative algorithms for capacitated two-stage time minimization transportation problems. Extensive studies have been carried out by researchers for other variants of two-stage transportation problems, like, the two-stage job-scheduling problem [5, 11, 42, 47], two-stage transportation problem with uncertainty [4, 22, 33, 35], two-stage fixed charge transportation problem [8, 12, 13, 25].

An important variant of the two-stage transportation problem is the two-level hierarchical time minimization transportation problem where the whole set of source-destination links is divided into two disjoint sets termed as Level-1 and Level-2. During Stage-I of transportation, only Level-1 links are used and Level-2 links are used during Stage-II. Sharma *et al.* [38], Kaur *et al.* [26], Xie *et al.* [46] developed efficient iterative algorithms for this problem. In 2021, Jain *et al.* [24] developed a polynomial time algorithm for a three-level time minimization transportation problem.

Motivated by applications of transportation problems with the restricted flow in the iron and steel industry where production capacity is limited due to limited availability of infrastructure/manpower/raw materials, this paper discusses a capacitated two-stage time minimization transportation problem with the restricted flow in each stage. In Stage-I, a specified flow F_1 is transported from various sources to destinations depending upon the minimum requirements at the destinations. Then in Stage-II a flow F_2 is transported depending upon the total requirement of the destinations (refer to [27] for more details on the applications). This problem was initially studied by Kaur *et al.* [27] and a polynomial time iterative algorithm was proposed to find its optimal solution. However, the algorithm proposed by Kaur *et al.* [27] does not guarantee an optimal solution in all scenarios. This motivated us to develop an efficient solution technique that can generate a solution for any given capacitated two-stage time minimization transportation problem with restricted flow.

This paper is organized as follows. In the next section mathematical formulation of two-stage capacitated time minimization transportation problem with the restricted flow (TSCTMTP-F) is given along with an outline of the existing solution technique proposed by Kaur *et al.* [27]. In Section 3, the drawbacks of the existing technique are explained and a contradictory example is given to prove that the existing technique does not always yield an optimal solution of TSCTMTP-F. In Section 4, solution techniques to find a feasible solution for Stage-I and a corresponding feasible solution for Stage-II problem along with various theoretical results justifying the convergence of the proposed algorithm are given. In Section 5, numerical problems of various sizes are solved to explain the working of the proposed algorithm. In Section 6, computational experiments are provided followed by concluding remarks in Section 7.

2. Mathematical formulation of TSCTMTP-F

The following notations have been used throughout the manuscript.

$I = \{1, 2, \ldots, m\}$: Index set of m sources.
$J = \{1, 2, \ldots, n\}$: Index set of n destinations.
$a_i \in \mathbb{Z},\ i \in I$: Maximum availability at ith source.
$b_j \in \mathbb{Z},\ j \in J$: Minimum requirement at jth destination.
F_1: The flow of commodity during Stage-I transportation.
F_2: The flow of commodity during Stage-II transportation.

Here we assume that

(i) Maximum total availability at various sources is greater than or equal to the total minimum requirement at various destinations, *i.e.*, $\sum_{i=1}^{m} a_i \geq \sum_{j=1}^{n} b_j$.
(ii) The specified flow F_1 in Stage-I transportation is greater than or equal to the total maximum requirement of the destinations, *i.e.*, $F_1 \geq \sum_{j=1}^{n} b_j$.
(iii) Total flow in Stage-I and Stage-II transportation is less than or equal to maximum total availability at sources, *i.e.*, $0 < F_1 + F_2 \leq \sum_{i=1}^{m} a_i$.

Let x_{ij} and y_{ij} denote the number of units to be transported on the (i, j)th source-destination link during Stage-I and Stage-II transportation respectively.

Mathematically, the Stage-I problem (P_1) is stated as follows.

$$P_1 \ \min_{X \in S} \left[\max_{I \times J} \{t_{ij}(x_{ij})\} \right] \tag{1}$$

$$S = \left\{ X = \{x_{ij}\} \in \mathbb{Z}^{m \times n} \;\middle|\; \begin{array}{ll} \sum_{j \in J} x_{ij} \leq a_i, & \forall i \in I \\ \sum_{i \in I} x_{ij} \geq b_j, & \forall j \in J \\ \sum_{i \in I} \sum_{j \in J} x_{ij} = F_1 & \\ 0 \leq x_{ij} \leq u_{ij}, & \forall (i, j) \in I \times J \end{array} \right\} \tag{2}$$

where u_{ij} is the capacity of the (i, j)th source-destination link and $t_{ij}(x_{ij})$ is defined as

$$t_{ij}(x_{ij}) = \begin{cases} t_{ij} & x_{ij} > 0 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

where t_{ij} is the time of transportation from source i to destination j.

Here, the objective is to minimize the total transportation time on various source-destination links of $I \times J$ subject to the constraints (2) that denote the supply and demand restrictions. Since time is not additive in nature, so the total transportation time on all source-destination links is equal to the maximum of all the transportation times on all those links where there is a positive flow of the commodity. The first set of inequality constraints implies that the total amount to be transported to various destinations from a particular source is not more than a_i, $\forall i \in I$. Next inequality constraints imply that during Stage-I a minimum demand of b_j is to be fulfilled at the jth destination. The equality constraint denotes that the total amount of the commodity to be transported from various sources to various destinations during Stage-I is exactly F_1.

Corresponding to a feasible solution X of the Stage-I problem, the Stage-II problem (P_2) can be formulated as

$$P_2 \quad \min_{Y \in S(X)} \left[\max_{I \times J} \{t_{ij}(y_{ij})\} \right] \tag{4}$$

$$S(X) = \left\{ Y = \{y_{ij}\} \in \mathbb{Z}^{m \times n} \;\middle|\; \begin{array}{ll} \sum_{j \in J} y_{ij} \leq \bar{a}_i = a_i - \sum_{j \in J} x_{ij}, & \forall i \in I \\ \sum_{i \in I} \sum_{j \in J} y_{ij} = F_2 & \\ 0 \leq y_{ij} \leq \bar{u}_{ij} = u_{ij} - x_{ij}, & \forall (i, j) \in I \times J \end{array} \right\} \tag{5}$$

where $S(X)$ denotes the set of feasible solutions of the Stage-II problem corresponding to a feasible solution X of the Stage-I problem and the objective function $t_{ij}(y_{ij})$ is defined in a similar way as $t_{ij}(x_{ij})$ in (3).

In problem, P_2, first inequality constraints denote the supply constraints that the maximum available amount at a particular source during Stage-II transportation is the left-over amount at that particular source after the completion of Stage-I transportation. The equality constraint denotes that the total amount to be transported to various destinations during Stage-II is exactly F_2. Therefore, a two-stage capacitated time minimization transportation problem with restricted flow can be stated as

$$P_3 \quad \min_{X \in S} \left[\max_{I \times J} \{t_{ij}(x_{ij})\} + \min_{Y \in S(X)} \left[\max_{I \times J} \{t_{ij}(y_{ij})\} \right] \right].$$

Remark 2.1. Problem P_3 is solved by solving an equivalent transportation problem in which the transportation matrix is unimodular. Using the unimodularity property, all the basic feasible solutions take integral values if $a_i, \forall i \in I$ and $b_j, \forall j \in J$ are integers. Therefore, we can relax the condition $x_{ij}, y_{ij} \in \mathbb{Z}\ \forall i \in I, j \in J$ in problem P_3 (refer to [15]).

Remark 2.2. Note that if bounds on some variables in problem P_3 are not given, then using source constraints, for $i \in I$, $x_{ij} \leq a_i$, $\forall j \in J$. Therefore, the problem can be rewritten as a capacitated transportation problem.

Outline of the algorithm proposed by Kaur *et al.* [27]. Kaur *et al.* [27] developed a polynomial time iterative algorithm (call it Algorithm-A) to find an optimal solution of TSCTMTP-F that generates a sequence of pairs of transportation times of Stage-I and Stage-II starting with the minimum time of Stage-I. At each iteration, a balanced capacitated time minimization transportation problem is solved that yields a feasible solution to the Stage-I problem, and a corresponding optimal solution of Stage-II is obtained from it without solving any additional transportation problem.

An optimal solution of Stage-I problem (P_1) is obtained by rewriting it as a capacitated TMTP with bounds on rim conditions defined below (refer to [14])

$$P_1 \quad \min_{X \in S} \left[\max_{I \times J} \{t_{ij}(x_{ij})\} \right]$$

$$S = \left\{ X = \{x_{ij}\} \in \mathbb{R}^{m\times n} \;\middle|\; \begin{array}{ll} 0 \le \sum_{j\in J} x_{ij} \le a_i, & \forall i \in I \\ b_j \le \sum_{i\in I} x_{ij} \le \sum_{i\in I} u_{ij}, & \forall j \in J \\ 0 \le x_{ij} \le u_{ij}, & \forall (i,j) \in I \times J \\ \sum_{i\in I}\sum_{j\in J} x_{ij} = F_1 & \end{array} \right\}.$$

Algorithm A starts with an optimal solution of Stage-I problem (P_1) that is obtained by solving an equivalent balanced capacitated TMTP $(\bar{P}_1)$ defined as

$$\bar{P}_1 \;\min_{Z\in\bar{S}} \left[\max_{\bar{I}\times\bar{J}} (\bar{t}_{ij}(z_{ij}))\right]$$

$$\bar{S} = \left\{ Z = \{z_{ij}\} \in \mathbb{R}^{(m+1)\times(n+1)} \;\middle|\; \begin{array}{ll} \sum_{j\in\bar{J}} z_{ij} = \bar{a}_i & \forall i \in \bar{I} \\ \sum_{i\in\bar{I}} z_{ij} = \bar{b}_j & \forall j \in \bar{J} \\ 0 \le z_{ij} \le \bar{u}_{ij} & \forall (i,j) \in \bar{I}\times\bar{J} \end{array} \right\}$$

where

$$\begin{array}{llll}
\bar{I} = I \bigcup \{m+1\} & & \bar{J} = J \bigcup \{n+1\} & \\
\bar{t}_{ij} = t_{ij} & \forall (i,j) \in I\times J & \bar{t}_{i,n+1} = 0 & \forall i \in I \\
\bar{t}_{m+1,j} = 0 & \forall j \in J & \bar{t}_{m+1,n+1} = M & \\
\bar{a}_i = a_i & \forall i \in I & \bar{a}_{m+1} = \sum_{i\in I}\sum_{j\in J} u_{ij} - F_1 & \\
\bar{b}_j = \sum_{i\in I} u_{ij} & \forall j \in J & \bar{b}_{n+1} = \sum_{i\in I} a_i - F_1 & \\
\bar{u}_{ij} = u_{ij} & \forall (i,j) \in I\times J & \bar{u}_{i,n+1} = a_i & \forall i \in I \\
\bar{u}_{m+1,j} = \sum_{i\in I} u_{ij} - b_j & \forall j \in J & \bar{u}_{m+1,n+1} = \infty. &
\end{array}$$

Algorithm-A scans special types of feasible solutions (namely, corner feasible solution, M-feasible solution, M-feasible solution with respect to P_1, special M-feasible solution [27]) of problem $\bar{P}_1$ and its restricted version $\bar{P}_1(T_2^k)$ at kth iteration corresponding to Stage-II time T_2^k and obtains a feasible solution of Stage-I and Stage-II from these special solutions. Restricted versions $\bar{P}_1(T_2^k)$ of problem $\bar{P}_1$ are obtained by blocking various source-destination links and imposing partial sum constraints so that transportation time of Stage-II strictly decreases at various iterations (for more details refer to Sect. 4.3 in [27] and [15]). For the clarity of the readers, various types of special solutions defined by Kaur *et al.* [27] are reproduced below.

Definition 2.3 (Corner feasible solution [27])**.** A feasible solution $\{z_{ij}\}_{\bar{I}\times\bar{J}}$ of the problem $\bar{P}_1$ is called a corner feasible solution (CFS) if $z_{m+1,n+1} = 0$.

A feasible solution of the problem $\bar{P}_1$ which is not a CFS is called a non-corner feasible solution.

Definition 2.4 (M-feasible solution [27])**.** A feasible solution $Z = \{z_{ij}\}_{\bar{I}\times\bar{J}}$ of the problem $\bar{P}_1(T_2^k)$ is called an M-feasible solution (MFS) if $z_{ij} = 0 \;\forall (i,j) \in \bar{I}\times\bar{J}$ with $\bar{t}_{ij} = M$.

A feasible solution of the problem $\bar{P}_1$ which is not M-feasible is called a non-M-feasible solution.

Definition 2.5 (M-feasible solution w.r.t P_1 [27])**.** A feasible solution $Z = \{z_{ij}\}_{\bar{I}\times\bar{J}}$ of the problem $\bar{P}_1(T_2^k)$ is called an M-feasible solution with respect to P_1 if

(i) It is CFS of the problem $\bar{P}_1(T_2^k)$.

TABLE 1. Time of transportation along various links.

	J_1	J_2	a_i $\downarrow$
I_1	10	11	300
I_2	9	12	300
b_j $\rightarrow$	100	100	

(ii) $z_{ij} = 0\ \forall (i,j) \in I \times J$ with $\bar{t}_{ij} = M$.

Definition 2.6 (Special M-feasible solution [27]). A feasible solution $Z = \{z_{ij}\}_{\bar{I}\times\bar{J}}$ of the problem $\bar{P}_1(T_2^k)$ is called a Special M-feasible solution (SMFS) if

(i) It is MFS w.r.t. P_1.
(ii) $\sum_{i\in I^k} z_{i,n+1} \geq F_2$ where $I^k = \{i \in I|\min_j\{t_{ij}\} < T_2^k\}$.

At each iteration, Algorithm-A solves a restricted version of problem $\bar{P}_1$ till a stage is reached when no special M-feasible solution of the problem $\bar{P}_1(T_2^k)$ exists for some k.

3. Drawbacks of Algorithm-A and motivation

Algorithm-A terminates when the restricted problem $\bar{P}_1(T_2^k)$ does not possess any special M-feasible solution for some k. The scanning of special M-feasible solutions is not an easy task due to the following reasons.

(1) While searching for special M-feasible solutions of the problem $\bar{P}_1(T_2^k)$ the first condition is that the solution must be MFS with respect to P_1. It means that we have to search for only those solutions $\{\bar{z}_{ij}\}_{\bar{I}\times\bar{J}}$ for which

$$\bar{z}_{ij} = 0\ \forall (i,j) \in I \times J \text{ with } \bar{t}_{ij} = M \text{ and } \bar{z}_{m+1,n+1} = 0.$$

It implies that $\bar{z}_{i,n+1}$ may be positive for some $i = 1, 2, \cdots, m$ with $\bar{t}_{i,n+1} = M$. Therefore, Algorithm-A scans non-M-feasible solutions of the problem $\bar{P}_1(T_2^k)$ for which $\sum_{i\in I^k} \bar{z}_{i,n+1} \geq F_2$. There is no clarity for how long one has to search these non-M-feasible solutions of the problem $\bar{P}_1(T_2^k)$ to get an optimal solution of the problem TSCTMTP-F. Generally, an algorithm terminates in case of non-availability of an M-feasible solution. Further, one has to additionally check the feasibility of the solution for the constraint $\sum_{i\in I^k} \bar{z}_{i,n+1} \geq F_2$. This makes the scanning even more complex.

(2) The strategy to obtain an optimal solution of Stage-II corresponding to a special M-feasible solution of the problem $\bar{P}_1(T_2^k)$ is quite complex. It requires arranging various cells of the transportation matrix in non-decreasing order and then amounts allocated in the $(n+1)$th column is transported to these above-mentioned links starting with the minimum time of transportation till F_2 amount is transported (For more details refer to Sect. 4.2 in [27]).

(3) Due to uncertainty about the termination of Algorithm-A as discussed above, one may not get an optimal solution to the problem TSCTMTP-F. A counter-example is produced below where the solution obtained by Algorithm-A at the terminal step is not optimal.
Consider the following 2×2 transportation problem given in Table 1. Here, $u_{ij} = 200\ \forall i \in I, j \in J$, $F_1 = 300,\ F_2 = 100$.
To find an optimal solution to this problem using Algorithm-A, the corresponding problem $\bar{P}_1$ is formulated and its optimal solution is obtained as given in Table 2. Various boxed entries in Table 2 represent the amount transported along these routes.
From Table 2, we get $T_1^0 = 11$ corresponding to Stage-I solution $x_{12} = 100, x_{21} = 200, x_{ij} = 0$, otherwise, and $T_2^0 = 10$ corresponding to Stage-II solution $y_{11} = 100$, $y_{ij} = 0$, otherwise. Here $T_1^0 + T_2^0 = 21$. Corresponding

TABLE 2. An optimal solution of problem $\bar{P}_1$.

	J_1	J_2	J_3	$\bar{a}_i \downarrow$
I_1	10	11 [100]	0 [200]	300
I_2	9 [200]	12	0 [100]	300
I_3	0 [200]	0 [300]	M	500
$\bar{b}_j \rightarrow$	400	400	300	

TABLE 3. Problem $\bar{P}_1(10)$.

	J_1	J_2	J_3	$\bar{a}_i \downarrow$
I_1	10	11	M	300
I_2	9	M	0	300
I_3	0	0	M	500
$\bar{b}_j \rightarrow$	400	400	300	

to T_2^0, problem $\bar{P}_1(10)$ is formulated and solved to find a special M-feasible solution that must satisfy $z_{21} + z_{23} \leq 200$.
From Table 3, it is clear that there does not exist any special M-feasible solution of the $\bar{P}_1(10)$ because for the existence of special M-feasible solution $z_{22} = 0$. This implies that $z_{21} + z_{23} = 300$ should be satisfied for the existence of a special M-feasible solution which clearly does not satisfy the partial sum constraint $z_{21} + z_{23} \leq 200$. Therefore Algorithm-A terminates yielding 21 as the optimal time of transportation for TSCTMTP-F. But there exists a feasible solution of this TSCTMTP-F given below that yields time as 20.
Stage-I solution: $x_{12} = 200, x_{21} = 100, x_{ij} = 0$, otherwise yielding $T_1 = 11$.
Stage-II solution: $y_{21} = 100$, $y_{ij} = 0$, otherwise yielding $T_2 = 9$.
This justifies that Algorithm-A may not always yield an optimal solution of the problem TSCTMTP-F.

This has motivated us to develop an efficient technique to solve the problem TSCTMTP-F that

- Generates a global optimal solution to the problem.
- Simple strategy to obtain a feasible solution for Stage-II corresponding to a given solution of Stage-I.
- Avoids scanning of special M-feasible solutions and terminates as soon as non-availability of M-feasible solution is attained.

4. Theoretical results and proposed algorithms

In this section, various theoretical results and an algorithm to find an optimal solution to problem TSCMTP-F are discussed. The proposed technique starts with an optimal solution to the Stage-I problem and a corresponding solution to Stage-II. It systematically decreases Stage-II time and generates pairs of Stage-I and Stage-II times that finally yield an optimal solution of the problem TSCTMTP-F.

To find an optimal solution to Stage-I problem (P_1), we solve the following balanced capacitated transportation problem (P_1^*)

$$P_1^* \quad \min_{Z\in S^*}\left[\max_{I^*\times J^*}\{t_{ij}^*(z_{ij})\}\right]$$

$$S^* = \left\{ Z=\{z_{ij}\}\in\mathbb{R}^{(m+1)\times(n+2)} \;\middle|\; \begin{array}{ll} \sum_{j\in J^*} z_{ij} = a_i^*, & \forall i\in I^* \\ \sum_{i\in I} z_{ij} = b_j^*, & \forall j\in J^* \\ 0\le z_{ij}\le u_{ij}^*, & \forall (i,j)\in I^*\times J^* \end{array} \right\}$$

where $I^* = \{1,2,\dots,m,m+1\}$, $J^* = \{1,2,\dots,n,n+1,n+2\}$

$$t_{ij}^* = \left\{ \begin{array}{ll} t_{ij}, & \forall (i,j)\in I\times J \\ 0, & j=n+1,\ j=n+2,\ \forall i\in I \\ 0, & i=m+1,\ \forall j\in J \\ \text{M}, & i=m+1 \text{ and } j=n+1,\ n+2 \end{array} \right\}$$

$$\begin{array}{llll} a_i^* = a_i, & \forall i\in I, & u_{ij}^* = u_{ij}, & \forall (i,j)\in I\times J \\ b_j^* = \sum_{i\in I} u_{ij}, & \forall j\in J & u_{m+1,j}^* = \sum_{i\in I} u_{ij} - b_j, & \forall j\in J \\ a_{m+1}^* = \sum_{i\in I}\sum_{j\in J} u_{ij} - F_1, & & u_{ij}^* = \infty, & \forall i\in I, j=n+1,n+2 \\ b_{n+1}^* = F_2,\ b_{n+2}^* = \sum_{i\in I} a_i - F_1 - F_2 & & u_{m+1,n+1}^* = u_{m+1,n+2}^* = \infty. & \end{array}$$

Remark 4.1. In the problem P_1^*, there is no need to assign any upper bound to the variables in $(n+1)$th and $(n+2)$th columns, for all $i = 1,2,\dots,m$. We can define $u_{i,n+1}^* = u_{i,n+2}^* = \infty$, for all $i = 1,2,\dots,m$ or $u_{i,n+1}^* = u_{i,n+2} = a_i^* = a_i$, for all $i = 1,2,\dots,m$.

To find a feasible solution to the Stage-I problem, we have to scan some special types of feasible solutions of the problem P_1^* termed as feasible flow solutions.

Definition 4.2 (Feasible flow solution (FFS))**.** A feasible solution $\{z_{ij}\}_{I^*\times J^*}$ of the problem P_1^* is called a feasible flow solution if $z_{m+1,n+1} = 0$ and $z_{m+1,n+2} = 0$. Otherwise, it is called a non-feasible flow solution of the problem P_1^*.

Let $\{z_{ij}\}_{I^*\times J^*}$ be a FFS of P_1^*. Then a corresponding feasible solution of Stage-I problem (P_1) is given by

$$x_{ij} = z_{ij},\ \forall (i,j)\in I\times J. \tag{6}$$

Equivalence between P_1 and P_1^*. Now we prove that optimizing the problem P_1 is equivalent to optimizing the problem P_1^*. It is proved that optimal values of P_1 and P_1^* are equal.

Theorem 4.3. *A non-feasible flow solution to P_1^* cannot provide a feasible solution to P_1.*

Proof. Let $\{z_{ij}\}_{I^*\times J^*}$ be a non-feasible flow solution of P_1^*. This implies that it satisfies all the following constraints

$$\sum_{j=1}^{n+2} z_{ij} = a_i, \qquad \forall i = 1,2,\dots,m \tag{7}$$

$$\sum_{j=1}^{n+2} z_{m+1,j} = \sum_{i=1}^{m}\sum_{j=1}^{n} u_{ij} - F_1 \tag{8}$$

$$\sum_{i=1}^{m+1} z_{ij} = \sum_{i=1}^{m} u_{ij}, \qquad \forall j = 1,2,\ldots,n \tag{9}$$

$$\sum_{i=1}^{m+1} z_{i,n+1} = F_2 \tag{10}$$

$$\sum_{i=1}^{m+1} z_{i,n+2} = \sum_{i=1}^{m} a_i - F_1 - F_2 \tag{11}$$

$$0 \le z_{ij} \le u_{ij}, \qquad \forall i = 1,2,\ldots,m;\ j = 1,2,\ldots,n \tag{12}$$

$$z_{i,n+1} \ge 0,\ z_{i,n+2} \ge 0, \qquad \forall i = 1,2,\ldots,m \tag{13}$$

$$0 \le z_{m+1,j} \le \sum_{i=1}^{m} u_{ij} - b_j, \qquad \forall j = 1,2,\ldots,n \tag{14}$$

along with the following condition due to non-M-feasibility condition

$$z_{m+1,n+1} + z_{m+1,n+2} = \lambda > 0. \tag{15}$$

Now, define $\{x_{ij}\}_{I\times J}$ using relation (6).

Addition of equations (10) and (11) implies that

$$\sum_{i=1}^{m+1} z_{i,n+1} + \sum_{i=1}^{m+1} z_{i,n+2} = \sum_{i=1}^{m} a_i - F_1$$

$$\Rightarrow \sum_{i=1}^{m} z_{i,n+1} + \sum_{i=1}^{m} z_{i,n+2} + z_{m+1,n+1} + z_{m+1,n+2} = \sum_{i=1}^{m} a_i - F_1$$

$$\text{Using (15), we get } \sum_{i=1}^{m} z_{i,n+1} + \sum_{i=1}^{m} z_{i,n+2} = \sum_{i=1}^{m} a_i - F_1 - \lambda. \tag{16}$$

From equation (7), we have $\sum_{i=1}^{m}\sum_{j=1}^{n+2} z_{ij} = \sum_{i=1}^{m} a_i$.

$$\Rightarrow \sum_{i=1}^{m}\sum_{j=1}^{n} z_{ij} + \left(\sum_{i=1}^{m} z_{i,n+1} + \sum_{i=1}^{m} z_{i,n+2}\right) = \sum_{i=1}^{m} a_i$$

$$\Rightarrow \sum_{i=1}^{m}\sum_{j=1}^{n} z_{ij} + \left(\sum_{i=1}^{m} a_i - F_1 - \lambda\right) = \sum_{i=1}^{m} a_i \text{ (using (16))}$$

$$\Rightarrow \sum_{i=1}^{m}\sum_{j=1}^{n} z_{ij} = F_1 + \lambda.$$

As $x_{ij} = z_{ij},\ \forall i \in I, j \in J$, this implies that

$$\Rightarrow \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} = F_1 + \lambda.$$

It shows that the total transportation flow in Stage-I corresponding to $\{x_{ij}\}_{I\times J}$ is greater than F_1.

Thus, a non-feasible flow solution of P_1^* cannot provide a feasible solution to problem P_1, *i.e.*, Stage-I problem. □

Theorem 4.4. *Corresponding to a feasible flow solution of the problem P_1^*, there exists a feasible solution of problem P_1.*

Proof. Let $\{z_{ij}\}_{I^*\times J^*}$ be a feasible flow solution of problem P_1^*, *i.e.*, $\{z_{ij}\}_{I^*\times J^*}$ satisfies equations (7) to (14). Let $\{x_{ij}\}_{I\times J}$ be defined by (6) as

$$x_{ij} = z_{ij}, \qquad \forall i = 1, 2, \ldots, m \text{ and } j = 1, 2, \ldots, n.$$

As, $0 \le z_{ij} \le u_{ij},\ \forall i \in I,\ j \in J \Rightarrow 0 \le x_{ij} \le u_{ij},\ \forall i \in I,\ j \in J$.

Equation (7) implies that $\sum_{j=1}^{n+2} z_{ij} = a_i,\ \forall i = 1, 2, \ldots, m$ and from equation (13), $z_{i,n+1},\ z_{i,n+2} \ge 0$.

$$\Rightarrow \sum_{j=1}^{n} z_{ij} \le a_i, \qquad \forall i = 1, 2, \ldots, m$$

$$\Rightarrow \sum_{j=1}^{n} x_{ij} \le a_i, \qquad \forall i = 1, 2, \ldots, m.$$

Equation (9) implies that $\sum_{i=1}^{m+1} z_{ij} = \sum_{i=1}^{m} u_{ij},\ \forall j = 1, 2, \ldots, n$

$$\Rightarrow \sum_{i=1}^{m} z_{ij} + z_{m+1,j} = \sum_{i=1}^{m} u_{ij}, \qquad \forall j = 1, 2, \ldots, n$$

$$\Rightarrow \sum_{i=1}^{m} x_{ij} = \sum_{i=1}^{m} u_{ij} - z_{m+1,j}, \qquad \forall j = 1, 2, \ldots, n \tag{17}$$

$$\text{Also, from equation (14)}, \ 0 \le z_{m+1,j} \le \sum_{i=1}^{m} u_{ij} - b_j, \qquad \forall j = 1, 2, \ldots, n \tag{18}$$

$$\text{Therefore, equations (17) and (18)} \ \Rightarrow \sum_{i=1}^{m} x_{ij} \ge b_j, \qquad \forall j = 1, 2, \ldots, n.$$

Further, as $\{z_{ij}\}_{I^*\times J^*}$ is a feasible flow solution, therefore

$$z_{m+1,n+1} = z_{m+1,n+2} = 0. \tag{19}$$

As discussed in Theorem 4.3, from equations (10) and (11), we have

$$\Rightarrow \sum_{i=1}^{m} z_{i,n+1} + \sum_{i=1}^{m} z_{i,n+2} + z_{m+1,n+1} + z_{m+1,n+2} = \sum_{i=1}^{m} a_i - F_1$$

$$\text{Using equation (19), we have} \ \sum_{i=1}^{m} z_{i,n+1} + \sum_{i=1}^{m} z_{i,n+2} = \sum_{i=1}^{m} a_i - F_1. \tag{20}$$

From equation (7), we have $\sum_{i=1}^{m}\sum_{j=1}^{n+2} z_{ij} = \sum_{i=1}^{m} a_i$.

$$\Rightarrow \sum_{i=1}^{m}\sum_{j=1}^{n} z_{ij} + \sum_{i=1}^{m} z_{i\ n+1} + \sum_{i=1}^{m} z_{i,n+2} = \sum_{i=1}^{m} a_i$$

$$\Rightarrow \sum_{i=1}^{m}\sum_{j=1}^{n} z_{ij} + \sum_{i=1}^{m} a_i - F_1 = \sum_{i=1}^{m} a_i \text{ (using (20))}$$

$$\Rightarrow \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} = F_1 \text{ (using (6))}.$$

Thus, $x_{ij} = z_{ij},\ \forall i \in I,\ j \in J$ is a feasible solution of problem P_1. □

TABLE 4. Solution Z^1 of problem (P_1^*).

	J_1	J_2	$J_3 = J_{n+1}$	$J_4 = J_{n+2}$	$a_i^* \downarrow$
I_1		100		400	500
I_2		400	300		700
$I_3 = I_{m+1}$	600	300			900
$b_j^* \rightarrow$	600	800	300	400	

TABLE 5. Solution Z^2 of problem (P_1^*).

	J_1	J_2	$J_3 = J_{n+1}$	$J_4 = J_{n+2}$	$a_i^* \downarrow$
I_1		100	300	100	500
I_2		400		300	700
$I_3 = I_{m+1}$	600	300			900
$b_j^* \rightarrow$	600	800	300	400	

Remark 4.5. The correspondence between feasible solutions of P_1^* and P_1 is not one-to-one. Corresponding to a feasible flow solution of P_1^*, we can uniquely define a feasible solution of problem P_1 using $x_{ij} = z_{ij},\ \forall (i,j) \in I \times J$. But given a feasible solution to problem P_1, we can have more than one solution to the problem P_1^*.

For example, consider two different solutions Z^1 and Z^2 of the problem P_1^* given in Tables 4 and 5 corresponding to a given problem P_1 with $I = J = \{1, 2\}$.

Both the solutions Z^1 and Z^2 provide the same solution of the problem P_1, *i.e.*, $x_{11} = 0, x_{12} = 100, x_{12} = 0, x_{22} = 400$. Therefore, the correspondence between a feasible flow solution of P_1^* and a feasible solution to the Stage-I problem is not one-to-one.

Remark 4.6 (Feasible flow solution of Problem P_1^* corresponding to a feasible solution of problem P_1)**.** Given a feasible solution $\{x_{ij}\}_{I\times J}$ of problem P_1, we can define FFS $\{z_{ij}\}_{I^*\times J^*}$ of the problem P_1^* as

$$z_{ij} = x_{ij},\ \forall (i,j) \in I \times J,\ z_{m+1,j} = \sum_{i=1}^{m} u_{ij} - \sum_{i=1}^{m} x_{ij},$$
$$z_{m+1,n+1} = z_{m+1,n+2} = 0,$$

and $z_{i,n+1}$ and $z_{i,n+2}$ $\forall i = 1, 2, \ldots, m$ can be obtained by solving the following transportation problem given in Table 6.

This problem can be solved by any solution technique available for transportation problems.

Remark 4.7. Corresponding to a feasible solution (X, Y) of problem P_3, we can uniquely define a feasible flow solution of problem P_1^* (refer to Thm. 4.18).

Theorem 4.8. *The value of the objective function of P_1^* at a feasible flow solution $\{z_{ij}\}_{I^*\times J^*}$ is equal to the value of the objective function of problem P_1 at its corresponding feasible solution $\{x_{ij}\}_{I\times J}$ obtained by the transformation $x_{ij} = z_{ij}$ $\forall i \in I, j \in J$.*

TABLE 6. Evalution of $z_{i,n+1}$ and $z_{i,n+2}$ $\forall i = 1, 2, \ldots, m$.

	J_{n+1}	J_{n+2}	a_i^* $\downarrow$
I_1	0	0	$a_1 - \sum_{j=1}^{n} x_{1j}$
		$\vdots$	
I_i	0	0	$a_i - \sum_{j=1}^{n} x_{ij}$
		$\vdots$	
I_{m+1}	M	M	0
b_j^* $\rightarrow$	F_2	$\sum_{i=1}^{m} a_i - F_1 - F_2$	

Proof. From the mathematical formulation of problem P_1^* it is clear that

$$t_{ij}^* = 0, \quad \forall i = 1, 2, \ldots, m, \; j = n+1, n+2; \qquad t_{m+1,j}^* = 0 \quad \forall j \in J \tag{21}$$

and solution $\{z_{ij}\}_{I^* \times J^*}$ is feasible flow solution, *i.e.*,

$$z_{m+1,n+1} = z_{m+1,n+2} = 0. \tag{22}$$

The value of the objective function of P_1^* at $\{z_{ij}\}_{I^* \times J^*}$

$$\begin{aligned}
&= \max_{I^* \times J^*} \left\{ t_{ij}^*(z_{ij}) \right\} \\
&= \left\{ \max_{I \times J} \left\{ t_{ij}^*(z_{ij}) \right\}, \max_{I \times \{n+1, n+2\}} \left\{ t_{ij}^*(z_{ij}) \right\}, \max_{j=1,2,\ldots,n} \left\{ t_{m+1\ j}^*(z_{ij}) \right\} \right\} \\
&\quad (\text{as } z_{m+1,n+1} = z_{m+1,n+2} = 0 \text{ and } t_{i,n+1} = t_{i,n+2} = 0, \; \forall i = 1, 2, \ldots, m) \\
&= \max_{I \times J} \left\{ t_{ij}^*(z_{ij}) \right\} \qquad (\text{using (21) and (22)}) \\
&= \max_{I \times J} \left\{ t_{ij}^*(x_{ij}) \right\} \quad (\text{using the transformation } x_{ij} = z_{ij} \; \forall i \in I, j \in J) \\
&= \text{Objective function value of } P_1 \text{ at the corresponding feasible solution.}
\end{aligned}$$

□

Theorem 4.9. *Optimizing problem P_1 is equivalent to optimizing problem P_1^*, provided P_1 has a feasible solution.*

Proof. Let us assume that the result is not true. Then, there exists a feasible solution $\{x_{ij}^*\}_{I \times J}$ of the problem P_1 yielding time lesser than the optimal value of P_1^* obtained corresponding to an optimal feasible flow solution $\{\hat{z}_{ij}\}_{I^* \times J^*}$.

As discussed in Remark 4.6, we can construct a feasible flow solution $\{z_{ij}^*\}_{I^* \times J^*}$ of problem P_1^* corresponding to this solution $\{x_{ij}^*\}_{I \times J}$. Then by Theorem 4.8, we have

$$\max_{I^* \times J^*} \left\{ t_{ij}^* \left(z_{ij}^* \right) \right\} = \max_{I \times J} \left\{ t_{ij}^*(x_{ij}) \right\} < \max_{I^* \times J^*} \left\{ t_{ij}^*(\hat{z}_{ij}) \right\}$$

which contradicts the optimality of the solution $\{\hat{z}_{ij}\}_{I^* \times J^*}$. Hence, our assumption is wrong. Therefore, optimizing problem P_1 is equivalent to optimizing problem P_1^*, provided P_1 has a feasible solution. □

From the above discussion, it is clear that corresponding to an FFS, Z of problem P_1^*, a feasible solution X of Stage-I problem can be obtained by using the transformation (6). To solve problem P_3, now we need to find a corresponding feasible solution to Stage-II problem. In the next subsection, we have discussed a strategy to find a feasible solution for Stage-II for a given FFS Z of the problem P_1^* and the corresponding feasible solution X of Stage-I.

4.1. Strategy to find a feasible solution of Stage-II corresponding to a feasible solution X of Stage-I

Let $Z = \{z_{ij}\}_{I^* \times J^*}$ be a feasible flow solution of P_1^* and $X = \{x_{ij}\}_{I \times J}$ be the corresponding feasible solution of Stage-I problem P_1. Then a feasible solution $Y = \{y_{ij}\}_{I \times J}$ of Stage-II problem is obtained by simply reallocating the amount $z_{i,n+1} > 0$ corresponding to some specific sources that are currently allocated in $(n+1)$th column of the transportation matrix of problem P_1^*. It is done by allocating the maximum possible amount to various links in the ith row corresponding to various destinations starting with a destination having the minimum time of transportation, then the next higher time, and so on. This strategy can be presented mathematically as given below as Algorithm-B.

Algorithm 1. Algorithm-B.

Step 1. Construct $\bar{I} = \{i \in I | z_{i,n+1} > 0\} = \{i_1, i_2, \ldots, i_r\}$ and go to step 2.
Step 2. For each $i \in \bar{I}$, find $\bar{J}_i = \{j \in J | \bar{u}_{ij} > 0\}$.
Let $\bar{J}_i = \{j_1^i, j_2^i, \ldots, j_{r_i}^i\}$ such that $t_{ij_1^i} \le t_{ij_2^i} \le \cdots \le t_{ij_{r_i}^i}$.
If $t_{ij_\alpha^i} = t_{ij_\beta^i}$ for some $\alpha,\ \beta \in \{1, 2, \ldots, r_i\}$, $\alpha \neq \beta$, then arrange with respect to js increasing value.
i.e., If $t_{ij_\alpha^i} = t_{ij_\beta^i}$ with $j_\alpha^i < j_\beta^i$ (otherwise we can arrange randomly also for equal time entries). Go to step 3.
Step 3. Set $\beta = 1$ and proceed.
(a) Set $\alpha = 1$, $i = i_\beta$, $\hat{z}_{i,n+1} = z_{i,n+1}$ and go to step 3(b).
(b) Set $y_{ij_\alpha^i} = \min\{\bar{u}_{ij_\alpha^i}, \hat{z}_{i,n+1}\}$.
(b.i) If $y_{ij_\alpha^i} = \bar{u}_{ij_\alpha^i}$, then update $\hat{z}_{i,n+1} = \hat{z}_{i,n+1} - \bar{u}_{ij_\alpha^i}$ and $\alpha := \alpha + 1$ and go to Step 3(b).
(b.ii) If $y_{ij_\alpha^i} = \hat{z}_{i,n+1}$ and $\beta = r$, then go to step 4 otherwise, set $\beta := \beta + 1$ and go to step 3(a).
Step 4. For $i \notin \bar{I}, j \in J$, set $y_{ij} = 0$ and for $i \in \bar{I},\ j \notin \bar{J}_i,\ y_{ij} = 0$ and proceed.
Step 5. After implementing Step 3 for each $i \in \bar{I}$ and Step 4 for each $(i,j) \in (I \times J) \setminus (\bar{I} \times \bar{J}_i)$ we have a feasible solution $Y = \{y_{ij}\}_{I \times J}$ of Stage-II corresponding to the feasible solution $X = \{x_{ij}\}_{I \times J}$ of Stage-I.

Remark 4.10. Note that in Step 2 for each $i \in \bar{I}$, the set $\bar{J}_i$ is always non-empty. This is because of the fact that the partial sum constraint is being satisfied.

Theorem 4.11. *Let $Y = \{y_{ij}\}_{I \times J}$ be obtained by using Algorithm-B. Then it is a feasible solution to Stage-II problem corresponding to Stage-I solution $X = \{x_{ij}\}_{I \times J}$.*

Proof. $Y = \{y_{ij}\}_{I \times J}$ will be a feasible solution of Stage-II if it lies in $S(X)$.

$$S(X) = \left\{ Y = \{y_{ij}\} \in \mathbb{Z}^{m \times n} \middle| \begin{array}{ll} \sum_{j \in J} y_{ij} \le \bar{a}_i = a_i - \sum_{j \in J} x_{ij}, & \forall i \in I \\ \sum_{i \in I} y_{ij} \ge 0, & \forall j \in J \\ 0 \le y_{ij} \le \bar{u}_{ij} = u_{ij} - x_{ij}, & \forall (i,j) \in I \times J \\ \sum_{i \in I} \sum_{j \in J} y_{ij} = F_2 & \end{array} \right\}.$$

Clearly, from the construction of Y, $0 \le y_{ij} \le \bar{u}_{ij}$ as

$$y_{ij} = \begin{cases} \min\{\bar{u}_{ij}, \hat{z}_{i\ n+1}\} & \forall (i,j) \in \bar{I} \times \bar{J}_i \\ 0 & \text{otherwise.} \end{cases}$$

From Algorithm-B, Step 3(b), we have

$$\sum_{j \in \bar{J}_i} y_{ij} = z_{i,n+1}, \qquad \forall i \in \bar{I}$$

$$\Rightarrow \sum_{j\in \bar{J}_i} y_{ij} + \sum_{j\in J\setminus \bar{J}_i} y_{ij} = z_{i,n+1}, \qquad \forall i \in \bar{I} \ \left(\because y_{ij} = 0\ \forall i \in \bar{I}, j \in J \setminus \bar{J}_i\right)$$

$$\Rightarrow \sum_{j\in J} y_{ij} = z_{i,n+1}, \qquad \forall i \in \bar{I} \tag{23}$$

$$\Rightarrow \sum_{j\in J} y_{ij} = z_{i,n+1} \le z_{i,n+1} + z_{i,n+2} \ (\because z_{i,n+2} \ge 0)$$

$$= a_i - \sum_{j=1}^{n} z_{ij} = a_i - \sum_{j=1}^{n} x_{ij}$$

$$\left(\text{using } \sum_{j=1}^{n} z_{ij} + z_{i,n+1} + z_{i,n+2} = a_i\right)$$

$$\Rightarrow \sum_{j\in J} y_{ij} \le a_i - \sum_{j=1}^{n} x_{ij}, \qquad \forall i \in \bar{I} \text{ (using (6))}$$

$$\text{Also } \sum_{j\in J} y_{ij} = 0, \qquad \forall i \in I \setminus \bar{I} \tag{24}$$

$$\text{which implies that } \sum_{j\in J} y_{ij} \le a_i - \sum_{j=1}^{n} x_{ij}, \qquad \forall i \in I \setminus \bar{I}$$

$$\text{As } y_{ij} \ge 0,\ \forall (i,j) \in I \times J \Rightarrow \sum_{i=1}^{m} y_{ij} \ge 0, \qquad \forall j \in J$$

$$\text{Equation (24) implies that } \sum_{j\in J} y_{ij} = 0 = z_{i,n+1}, \qquad \forall i \in I \setminus \bar{I}. \tag{25}$$

Therefore, equations (23) and (25) imply that $\sum_{j\in J} y_{ij} = z_{i,n+1},\ \forall i \in I$.

$$\Rightarrow \sum_{i\in I}\sum_{j\in J} y_{ij} = \sum_{i\in I} z_{i,n+1} = F_2 \text{ (using } b^*_{n+1} = F_2). \tag{26}$$

This proves that $Y = \{y_{ij}\}_{I\times J}$ obtained by using Algorithm-B is a feasible solution to Stage-II transportation problem. □

4.2. Strategy to find an optimal feasible solution of problem P_3

To find a feasible solution of Problem P_3 we solve problem P_1^*. From a feasible flow solution $\{z_{ij}\}_{I^*\times J^*}$ of problem P_1^*, we obtain a feasible solution of Stage-I as

$$x_{ij} = z_{ij},\ \forall (i,j) \in I \times J$$

and Stage-II solution is obtained by using Algorithm-B discussed in Section 4.1, which yields a feasible solution of P_3.

To obtain an optimal feasible solution to problem P_3, we solve various restricted versions of the problem P_1^*. Optimal solutions of these restricted versions will generate pairs such that T_1, Stage-I time increases, and T_2, Stage-II time decreases strictly.

Construction of the restricted version of problem P_1^*. Following restrictions are imposed along various source-destination links in the problem P_1^* to generate its restricted version $P_1^*(T_2^k)$ corresponding to the current in-hand pair (T_1^k, T_2^k) of Stage-I and Stage-II transportation times.

Restriction (i). Set $t^*_{i,n+1} = M$, $\forall i \in I$ such that $\min_{j\in J}\{t^*_{ij}\} \geq T_2^k$ where M is a large positive number.
Restriction (ii). Set $t_{ij} = M$, $\forall (i,j) \in I \times J$ for which

$$t_{ij} + T_2^L \geq \min_{h=0,1,2,\dots,k}\{T_1^h + T_2^h\}$$

where T_2^L is the lower bound on Stage-II time (refer to Rem. 4.12).
Restriction (iii). Define $I^k = \{i \in I | \min_{j\in J} t_{ij} < T_2^k\}$.
For each $i \in I^k$, define $J_i^k = \{j \in J | t_{ij} < T_2^k\}$ and construct the set

$$S^k = \cup_{i\in I^k}\big(\{i\} \times J_i^k\big).$$

Define $\bar{I}^k = \{i \in I^k | \sum_{j\in J_i^k} u_{ij} < a_i\}$. For $i \in \bar{I}^k$, apply the following partial sum constraint

$$\sum_{j\in J_i^k} z_{ij} + z_{i,n+1} \leq \sum_{j\in J_i^k} u_{ij}. \tag{27}$$

This partial sum constraint ensures that the Stage-II time will be strictly less than T_2^k, the current in-hand time of Stage-II. The partial sum constraints can be handled by the technique discussed by Dantzig [15].

Remark 4.12 (Computing T_2^L)**.** The technique used by Kaur *et al.* [27] is used to find a lower bound for Stage-II transportation time. It is reproduced below.
Find $\min_{I\times J} t_{ij} = t_{r_1 s_1}$.
If $F_2 \leq u_{r_1 s_1}$, then $T_2^L = t_{r_1 s_1}$ else, find $\min_{I\times J\setminus\{(r_1,s_1)\}} t_{ij} = t_{r_2 s_2}$.
If $F_2 \leq u_{r_1 s_1} + u_{r_2 s_2}$, then $T_2^L = t_{r_2 s_2}$ else, find $\min_{I\times J\setminus\{(r_1,s_1),(r_2,s_2)\}} t_{ij} = t_{r_3 s_3}$.
Continuing this way, we get $T_2^L = t_{r_{k+1} s_{k+1}}$ where $\min_{I\times J\setminus\{(r_1,s_1),(r_2,s_2)\dots.(r_k,s_k)\}} t_{ij} = t_{r_{k+1} s_{k+1}}$ if $\sum_{l=1}^{l=k} u_{r_l s_l} < F_2$ and $\sum_{l=1}^{l=k+1} u_{r_l s_l} \geq F_2$.

Mathematically, this restricted version $P_1^*(T_2^k)$ can be stated as

$$P_1^*(T_2^k) \quad \min_{Z\in S^{**}} \left[\max_{I^*\times J^*}\left\{t^{**}_{ij}(z_{ij})\right\}\right] \tag{28}$$

$$S^{**} = S^* \cup \left\{\sum_{j\in J_i^k} z_{ij} + z_{i,n+1} \leq \sum_{j\in J_i^k} u_{ij},\ \forall i \in \bar{I}^k\right\}.$$

Let $L = \big\{(i,j) \in I \times J | : t_{ij} + T_2^L < \min_{h=0,1,2,\dots,k}\{T_1^h + T_2^h\}\big\}$

$$t^{**}_{ij} = \left\{\begin{array}{ll} t^*_{ij} = t_{ij}, & \forall (i,j) \in L \\ t^*_{ij} = M, & \forall (i,j) \in (I\times J)\setminus L \\ 0, & j = n+1,\ \forall i \in I^k \\ M, & j = n+1,\ \forall i \in I\setminus I^k \\ 0, & j = n+2,\ \forall i \in I \\ 0, & i = m+1,\ \forall j \in J \\ M, & i = m+1 \text{ and } j = n+1, n+2 \end{array}\right\}.$$

Definition 4.13 (M-feasible solution)**.** A feasible solution of the problem $P_1^*(T_2^k)$ is called an M-feasible solution (MFS) if $z_{ij} = 0,\ \forall (i,j) \in I^* \times J^*$ with $t^{**}_{ij} = M$.

Algorithm 2. Algorithm-C.

Step 1. Given problem P_3 formulate the problem P_1^* and find its optimal feasible solution $Z^0 = \{z_{ij}\}_{I^* \times J^*}$. If this solution is an MFS, then set $k = 0$ and go to Step 2. Otherwise, go to Step 4(a).

Step 2. (a) Corresponding to an MFS, $Z^k = \{z_{ij}\}_{I^* \times J^*}$ evaluate Stage-I solution $X^k = \{x_{ij}\}_{I \times J}$ using (6) and the corresponding feasible solution of Stage-II $Y^k = \{y_{ij}\}_{I \times J}$ using Algorithm-B. Go to Step 2(b).

(b) Note $T_1^k = \max_{I \times J}\{t_{ij}(x_{ij})|x_{ij} > 0\}$ and $T_2^k = \max_{I \times J}\{t_{ij}(y_{ij})|y_{ij} > 0\}$. Record the pair (T_1^k, T_2^k) and Go to Step 3.

Step 3. (a) If $T_2^k = T_2^L$, then go to Step 4(b). Otherwise, go to next Step 3(b).

(b) Corresponding to the the pair (T_1^k, T_2^k), construct the problem $P_1^*(T_2^k)$ as defined by (28) and find its optimal feasible solution $Z^{k+1} = \{z_{ij}\}_{I^* \times J^*}$. If this solution is MFS, then set $k := k + 1$ and go to Step 2. Otherwise, go to Step 4(b).

Step 4. (a) Problem P_3 has no feasible solution. Algorithm terminates.

(b) The optimal objective function value of problem P_3 is $T_1^\alpha + T_2^\alpha = \min_{h=0,1,\ldots,k}\{T_1^h + T_2^h\}$ with X^α and Y^α as the corresponding solutions of Stage-I and Stage-II respectively. Algorithm terminates.

Remark 4.14. From Definitions 4.2 and 4.13, we can simply say that an M-feasible solution is always a feasible flow solution but not conversely.

Note. If an optimal feasible solution of $P_1^*(T_2^k)$ is not M-feasible, then the proposed algorithm terminates.

Remark 4.15. Algorithm-B, Algorithm-C, and Restriction (iii) ensure that solution $Y = \{y_{ij}\}_{I \times J}$ obtained at Step 2(a) of Algorithm-C always yields a time of Stage-II strictly less than T_2^k.

Remark 4.16. Algorithm-C is a polynomially bound algorithm as it solves a balanced capacitated time minimization transportation at each iteration. Any polynomially bound solution technique available in literature can be applied to solve this balanced transportation problem [14,15,31]. Algorithm-C terminates in a finite number of steps and solves at the most N number of such transportation problems where $N = \min\{s - r + 1, p - q + 1\}$. Here, s represents the total number of distinct time entries in the transportation matrix $(t^1 < t^2 < \cdots < t^s)$ and $p, q, r \in \{1, 2, \ldots, s\}$ such that $T_1^0 = t^r$, $T_2^0 = t^p$, $T_2^L = t^q$.

4.3. Theoretical justification of Algorithm-C

In this subsection, various theoretical results establish the convergence and efficiency of Algorithm-C for solving problem P_3.

Theorem 4.17. *An optimal M-feasible solution of problem $P_1^*(T_2^k)$ yields Stage-I and Stage-II times as T_1^{k+1} and T_2^{k+1} respectively such that*

$$T_1^k \leq T_1^{k+1} \text{ and } T_2^{k+1} < T_2^k.$$

Proof. Let $Z^{k+1} = \{z_{ij}\}_{I^* \times J^*}$ be an optimal M-feasible solution of $P_1^*(T_2^k)$ yielding optimal time T_1^{k+1}. As $S^* \subset S^{**}$, this implies that $Z^{k+1} = \{z_{ij}\}_{I^* \times J^*}$ is a feasible solution of problem P_1^*. Further, $Z^{k+1} = \{z_{ij}\}_{I^* \times J^*}$ is an M-Feasible solution, so $z_{m+1,n+1} = z_{m+1,n+2} = 0$. This implies that $Z^{k+1} = \{z_{ij}\}_{I^* \times J^*}$ is a FFS of problem of the problem P_1^*.

Let $X = \{x_{ij}\}_{I \times J}$ be obtained by using the transformation $x_{ij} = z_{ij}$, $\forall (i, j) \in I \times J$ and $Y = \{y_{ij}\}_{I \times J}$ be obtained by using Algorithm-B.

Then by Theorems 4.4 and 4.8, we can say that $X = \{x_{ij}\}_{I \times J}$ is a feasible solution of Stage-I problem yielding time T_1^{k+1}.

By Theorem 4.11 and Remark 4.15, $Y = \{y_{ij}\}_{I \times J}$ is a feasible solution of Stage-I problem yielding time $T_2^{k+1} < T_2^k$.

Now, we prove that $T_1^{k+1} \geq T_1^k$.

If possible let $T_1^{k+1} < T_1^k$. From the construction of restricted versions of problem P_1^*, it is clear that $P_1^*(T_2^k)$ is obtained by putting more restrictions along various source-destination links as compared to $P_1^*(T_2^{k-1})$ and hence, all the links which are blocked in $P_1^*(T_2^{k-1})$ remain blocked in $P_1^*(T_2^k)$ as well. Then $Z^{k+1} = \{z_{ij}\}_{I^* \times J^*}$ would be an M-feasible solution of the problem $P_1^*(T_2^{k-1})$ provided it satisfies partial sum constraints of the problem $P_1^*(T_2^{k-1})$.

Now, we check feasibility of Z^{k+1} for the partial sum constraints of the problem $P_1^*(T_2^{k-1})$.

Pair (T_1^k, T_2^k) is obtained from an M-feasible solution of problem $P_1^*(T_2^{k-1})$ yielding Stage-I time as T_1^k and T_2^k as the corresponding Stage-II time.

If $T_1^{k+1} < T_1^k$, this implies that links $(i,j) \in I \times J$ with $t_{ij} = T_1^{k+1}$ are not blocked in the problem $P_1^*(T_2^{k-1})$ and $t_{i,n+1} = 0$, $\forall i \in I$, with $\min_{j \in J} t_{ij} < T_2^{k-1}$.

Using Remark 4.15 we have $T_2^k < T_2^{k-1}$. It implies that $J_i^k \subset J_i^{k-1}$, $\forall i \in I$.

Thus, any feasible solution $Z^{k+1} = \{z_{ij}\}$ of $P_1^*(T_2^k)$ satisfying

$$\sum_{j \in J_i^k} z_{ij} + z_{i,n+1} \le \sum_{j \in J_i^k} u_{ij} \qquad \forall i \in I^k$$

must also satisfy the partial sum constraints of the problem $P_1^*(T_2^{k-1})$

$$\sum_{j \in J_i^{k-1}} z_{ij} + z_{i,n+1} \le \sum_{j \in J_i^{k-1}} u_{ij} \qquad \forall i \in I^{k-1} \quad \left(\text{as } I^k \subset I^{k-1}\right).$$

This implies that Z^{k+1} is a feasible solution of the problem $P_1^*(T_2^{k-1})$ yielding time $T_1^{k+1} < T_1^k$. This contradicts the optimality of T_1^k. Therefore, $T_1^k \le T_1^{k+1}$. $\square$

Theorem 4.18. *If $\tilde{T}_1 + \tilde{T}_2 = \min_{k \ge 0}\{T_1^k + T_2^k\}$, where T_1^k, T_2^k are Stage-I and Stage-II times respectively, corresponding to an optimal M-feasible solution of the problem $P_1^*(T_2^{k-1})$, $k = 1, 2, 3, \ldots$ and (T_1^0, T_2^0) is the pair obtained corresponding to an optimal solution of the problem P_1^*. Then $\tilde{T}_1 + \tilde{T}_2$ is the optimal value of the objective function of Problem P_3.*

Proof. Suppose there exists a feasible solution $(\bar{X}, \bar{Y})$ of problem P_3 such that

$$\bar{T}_1 + \bar{T}_2 = T_1(\bar{X}) + T_2(\bar{Y}) < \tilde{T}_1 + \tilde{T}_2. \tag{29}$$

Here, $\bar{X}$ is a feasible solution of Stage-I and $\bar{Y}$ is the corresponding feasible solution of Stage-II.

Define $\bar{Z} = \{z_{ij}\}_{I^* \times J^*}$ as

$$z_{ij} = \left\{ \begin{array}{ll} x_{ij} & \forall (i,j) \in I \times J \\ \sum_{j=1}^{n} y_{ij} & j = n+1,\ \forall i = 1, 2, \ldots, m \\ a_i - \sum_{j \in J}(x_{ij} + y_{ij}) & j = n+2,\ \forall i = 1, 2, \ldots, m \\ \sum_{i \in I} u_{ij} - \sum_{i \in I} x_{ij} & i = m+1,\ \forall j = 1, 2, \ldots, n \\ 0 & i = m+1,\ j = n+1, n+2 \end{array} \right\}. \tag{30}$$

We will prove that $\bar{Z}$ defined by (30) is a feasible flow solution of problem P_1^*.

$$\begin{aligned} \sum_{i=1}^{m+1} z_{ij} = \sum_{i=1}^{m} z_{ij} + z_{m+1,j} &= \sum_{i=1}^{m} x_{ij} + \left(\sum_{i=1}^{m} u_{ij} - \sum_{i=1}^{m} x_{ij}\right) \\ \Rightarrow \sum_{i=1}^{m+1} z_{ij} &= \sum_{i=1}^{m} u_{ij}, \qquad \forall j = 1, 2, \ldots, n \end{aligned}$$

$$\text{Next, } \sum_{i=1}^{m+1} z_{i,n+1} = \sum_{i=1}^{m} z_{i,n+1} \quad (\text{as } z_{m+1,n+1} = 0)$$
$$= \sum_{i=1}^{m} \left(\sum_{j=1}^{n} y_{ij} \right) = F_2$$
$$\sum_{i=1}^{m+1} z_{i,n+2} = \sum_{i=1}^{m} z_{i,n+2} \quad (\text{as } z_{m+1,n+1} = 0)$$
$$= \sum_{i=1}^{m} \left(a_i - \sum_{j \in J} (x_{ij} + y_{ij}) \right)$$
$$= \sum_{i=1}^{m} a_i - \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} + \sum_{i=1}^{m} \sum_{j=1}^{n} y_{ij}$$
$$\Rightarrow \sum_{i=1}^{m+1} z_{i,n+2} = \sum_{i=1}^{m} a_i - (F_1 + F_2)$$
$$\text{Next } \forall i = 1, 2, \ldots, m \quad \sum_{j=1}^{n+2} z_{ij} = \sum_{j=1}^{n} z_{ij} + z_{i,n+1} + z_{i,n+2}$$
$$= \sum_{j=1}^{n} x_{ij} + \sum_{j=1}^{n} y_{ij} + a_i - \left(\sum_{j=1}^{n} x_{ij} + \sum_{j=1}^{n} y_{ij} \right)$$
$$\sum_{j=1}^{n+2} z_{ij} = a_i, \qquad \forall i = 1, 2, \ldots, m$$
$$\sum_{j=1}^{n+2} z_{m+1,j} = \sum_{j=1}^{n} z_{m+1,j} = \sum_{j=1}^{n} \left(\sum_{i=1}^{m} u_{ij} - \sum_{i=1}^{m} x_{ij} \right)$$
$$= \sum_{j=1}^{n} \sum_{i=1}^{m} u_{ij} - \sum_{j=1}^{n} \sum_{i=1}^{m} x_{ij} = \sum_{j=1}^{n} \sum_{i=1}^{m} u_{ij} - F_1$$
$$\text{Next } \forall i = 1, 2, \ldots, m \quad \sum_{j=1}^{n+2} z_{ij} = \sum_{j=1}^{n} z_{ij} + z_{i,n+1} + z_{i,n+2}$$
$$= \sum_{j=1}^{n} x_{ij} + \sum_{j=1}^{n} y_{ij} + a_i - \left(\sum_{j=1}^{n} x_{ij} + \sum_{j=1}^{n} y_{ij} \right)$$
$$\sum_{j=1}^{n+2} z_{ij} = a_i, \qquad \forall i = 1, 2, \ldots, m$$
$$\sum_{j=1}^{n+2} z_{m+1,j} = \sum_{j=1}^{n} z_{m+1,j} = \sum_{j=1}^{n} \left(\sum_{i=1}^{m} u_{ij} - \sum_{i=1}^{m} x_{ij} \right)$$
$$= \sum_{j=1}^{n} \sum_{i=1}^{m} u_{ij} - \sum_{j=1}^{n} \sum_{i=1}^{m} x_{ij} = \sum_{j=1}^{n} \sum_{i=1}^{m} u_{ij} - F_1$$

and bounds on z_{ij} are also satisfied. This proves that $\bar{Z}$ is a feasible solution to problem P_1^*.

Clearly, $\bar{T}_2 \geq T_2^L$ as T_2^L is the minimum possible time of Stage-II.

Also, T_1^0 is the optimal time of P_1^* this implies that $\bar{T}_1 \geq T_1^0$.

But $\bar{T}_1 + \bar{T}_2 < T_1^0 + T_2^0$ which implies that $\bar{T}_2 < T_2^0$. Therefore $T_2^L \leq \bar{T}_2 < T_2^0$.

Let (T_1^r, T_2^r) be a pair of Stage-I and Stage-II time such that either $T_2^r = T_2^L$ or the problem $P_1^*(T_2^r)$, $r \geq 0$ is not M-feasible. That is (T_1^r, T_2^r) is the last pair recorded by Algorithm-C.

Then, $T_2^r \leq \bar{T}_2 \leq T_2^0$.

Note that $\bar{T}_2 < T_2^r$ is not possible, because if $\bar{T}_2 < T_2^r$, then the relation $\bar{T}_1 + \bar{T}_2 < T_1^r + T_2^r$ implies that $\bar{Z}$ defined by (30) becomes an M-feasible solution of the problem $P_1^*(T_2^r)$ as explained below.

By Restriction (i), $t_{i,n+1} = 0$, $\forall i \in I$ for which $\bar{z}_{i,n+1} = \sum_{j=1}^n y_{ij}$ as $\bar{T}_2 < T_2^r$.

Further, $t_{i,n+1} = M$, $\forall i$ for which $\min_{j=1,2,\dots,n}\{t_{ij}\} \geq T_2^r$.

Also, for $(i,j) \in I \times J$, $t_{ij}^{**} = M$ only if

$$\begin{aligned} & t_{ij} + T_2^L \geq \min_{h=0,1,2,\dots,r}\{T_1^h + T_2^h\} \\ \Rightarrow\ & t_{ij} + T_2^L \geq T_1^r + T_2^r > \bar{T}_1 + \bar{T}_2 \ (\text{using } (29)) \\ \Rightarrow\ & t_{ij} > \bar{T}_1 + \bar{T}_2 - T_2^L \\ \Rightarrow\ & t_{ij} > \bar{T}_1 \quad \left(\text{as } T_2^L \leq \bar{T}_2\right). \end{aligned}$$

Thus, $t_{ij}^{**} = t_{ij}$, $\forall (i,j) \in I \times J$ for which $t_{ij} \leq \bar{T}_1$.

This proves that $\bar{Z}$ is an M-feasible solution of the problem $P_1^*(T_2^r)$ which is not possible as this problem is not M-feasible.

So $\bar{T}_2 \geq T_2^r$ and $\bar{T}_1 + \bar{T}_2 < T_1^r + T_2^r$, we have

$$\bar{T}_1 < T_1^r. \tag{31}$$

For $r = 0$, (31) contradicts the optimality of T_1^0.

For $r \geq 1$, consider the problem $P_1^*(T_2^{r-1})$ whose optimal objective function value is T_1^r.

Here, P_1^* and $P_1^*(T_2^{r-1})$ have difference of partial sum constraints as defined below

$$\sum_{i \in J_i^{r-1}} z_{ij} + z_{i,n+1} \leq \sum_{i \in J_i^{r-1}} u_{ij}, \qquad \forall i \in I^{r-1}$$

and cell blockage by setting $t_{ij} = M$ for all those (i,j) for which

$$\begin{aligned} & t_{ij} + T_2^L \geq \min_{h=0,1,2,\dots,r-1}\{T_1^h + T_2^h\} \\ \Rightarrow\ & t_{ij} + T_2^L \geq T_1^{r-1} + T_2^{r-1} \\ \Rightarrow\ & t_{ij} \geq T_1^{r-1} + \left(T_2^{r-1} - T_2^L\right) \\ \Rightarrow\ & t_{ij} > T_1^{r-1} \\ \text{and } & t_{i,n+1}^{**} = \begin{cases} M & \text{if } \min_{i \in I}\{t_{ij}\} \geq T_1^{r-1} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

There are different possibilities depending upon the feasibility of $\bar{Z}$.

Case 1. If $\bar{Z}$ is a feasible solution of problem $P_1^*(T_2^{r-1})$, $r \geq 1$.

It means that $\bar{Z}$ satisfies all the partial sum constraints of $P_1^*(T_2^{r-1})$. All other constraints of problem $P_1^*(T_2^{r-1})$ are same as of problem P_1^* and given that $\bar{Z}$ is a feasible solution of P_1^*.

Sub-case 1.1. If $\bar{Z}$ is an M-feasible solution of the problem $P_1^*(T_2^{r-1})$.

Then $\bar{Z}$ provides $\bar{T}_1$ as the optimal value of objective function of problem $P_1^*(T_2^{r-1})$ such that $\bar{T}_1 < T_1^r$, which contradicts the optimality of T_1^r.

Sub-case 1.2. If $\bar{Z}$ is not an M-feasible solution of the problem $P_1^*(T_2^{r-1})$.

In this case $\bar{T}_1 < T_1^r$, therefore non-M-feasibility may arise only corresponding to the allocation in the $(n+1)$th column.

$$\begin{aligned} &i.e., \ \bar{z}_{i,n+1} > 0, \qquad\qquad\qquad\qquad \forall i \in I \backslash I^{r-1} \\ &\Rightarrow \bar{T}_2 \geq T_2^{r-1} \text{ (due to Restriction (i))} \\ &\text{But } \bar{T}_1 + \bar{T}_2 < T_1^{r-1} + T_2^{r-1} \\ &\Rightarrow \bar{T}_1 < T_1^{r-1} + (T_2^{r-1} - \bar{T}_2) \\ &\Rightarrow \bar{T}_1 < T_1^{r-1} \ \text{ (as } T_2^{r-1} \leq \bar{T}_2). \end{aligned}$$

Now, consider the problem $P_1^*(T_2^{r-2})$ and proceed as discussed below.

Sub-case 1.2.1. If $\bar{Z}$ is an M-feasible solution of the problem $P_1^*(T_2^{r-2})$, then $\bar{Z}$ provides $\bar{T}_1$ as the optimal value of objective function of problem $P_1^*(T_2^{r-2})$ such that $\bar{T}_1 < T_1^{r-1}$, contradicting the optimality of T_1^{r-1}.

Sub-case 1.2.2. If $\bar{Z}$ is not an M-feasible solution of the problem $P_1^*(T_2^{r-2})$.

Then repeat Sub-case 1.2 with $\bar{T}_1 < T_1^{r-2}$.

Continuing like this, we reach at a stage with $\bar{T}_2 \geq T_2^0$ which gives $\bar{T}_1 < T_1^0$, a contradiction to the optimality of T_1^0.

Case 2. If $\bar{Z}$ is not a feasible solution of the problem $P_1^*(T_2^{r-1})$. This implies that $\bar{Z}$ does not satisfy partial sum constraints.

This implies that there exists some $i \in \bar{I}^{r-1}$ such that

$$\sum_{j \in J_i^{r-1}} \bar{z}_{ij} + \bar{z}_{i,n+1} > \sum_{j \in J_i^{r-1}} u_{ij}.$$

This implies that $\bar{Z}$ yields Stage-II time $\bar{T}_2 \geq T_2^{r-1}$ (using Algorithm-B). But

$$\begin{aligned} \bar{T}_1 + \bar{T}_2 &< T_1^{r-1} + T_2^{r-1} \\ \Rightarrow \ \bar{T}_1 &< T_1^{r-1}. \end{aligned}$$

Now, consider the problem $P_1^*(T_2^{r-2})$. Continuing like the same way as discussed in Case 1, either we reach a contradiction or a situation with

$$\begin{aligned} \bar{T}_2 &\geq T_2^0 \\ \Rightarrow \ \bar{T}_1 &< T_1^0 \end{aligned}$$

contradicting the optimality of T_1^0.

Therefore, our supposition is wrong, and hence, $\tilde{T}_1 + \tilde{T}_2$ is the optimal value of the objective function of problem P_3. □

5. Numerical examples

Example 1. Consider 4×4 transportation problem Ex-1 with 4 sources and 4 destinations whose time of transportation is given in Table 7 and capacities along various links are given in Table 8. Given that $\sum_{i=1}^{4} a_i = 1200$, Stage-I flow is $F_1 = 900$ and, flow in Stage-II is $F_2 = 100$.

Using Remark 4.12, $T_2^L = 9$.

Table 7. Time of transportation along various source-destination links Ex-1.

	D_1	D_2	D_3	D_4	a_i
S_1	10	11	30	43	300
S_2	9	12	31	10	300
S_3	34	35	32	44	300
S_4	10	43	44	45	300
b_j	200	200	200	100	

Table 8. Capacities along various source-destination links Ex-1.

	$j=1$	$j=2$	$j=3$	$j=4$
$i=1$	200	200	200	100
$i=2$	200	200	200	100
$i=3$	200	200	200	100
$i=4$	100	100	100	100

Table 9. Problem P_1^* corresponding to Ex-1.

	D_1	D_2	D_3	D_4	D_5	D_6	a_i^*
S_1	200[a]	200	200	100	∞	∞	
	10[b]	11	30	43	0	0	300
S_2	200	200	200	100	∞	∞	
	9	12	31	10	0	0	300
S_3	200	200	200	100	∞	∞	
	34	35	32	44	0	0	300
S_4	100	100	100	100	∞	∞	
	10	43	44	45	0	0	300
S_5	500	500	500	300	∞	∞	
	0	0	0	0	M	M	1600
b_j^*	700	700	700	400	100	200	

Notes. [a]Entry at this position along each transportation link represents the maximum capacity along this link.
[b]Entry at this position along each transportation link represents the transportation time along this link.

Iteration 1

Step 1. Formulate the problem P_1^* as given in Table 9 along with capacities along various links. Solve this problem and find its optimal solution Z^0 which is given in Table 10. It is an M-feasible solution, so go to Step 2.

Step 2. This optimal M-feasible solution Z^0 yields Stage-I and Stage-II solutions X^0 and Y^0 respectively as

$$X^0: \; x_{12}=200, x_{13}=100, x_{21}=200, x_{24}=100, x_{33}=200, x_{41}=100, x_{ij}=0, \text{ otherwise;}$$
$$Y^0: \; y_{31}=100, y_{ij}=0 \text{ otherwise.}$$

Note down $T_1^0 = 32$, $T_2^0 = 34$ and go to Step 3(b).

TABLE 10. An optimal feasible solution Z^0 of problem P_1^* for Ex-1.

	D_1	D_2	D_3	D_4	D_5	D_6	a_i^*
S_1	10	11 [200]	30 [100]	43	0	0	300
S_2	9 [200]	12	31	10 [100]	0	0	300
S_3	34	35	32 [200]	44	0 [100]	0	300
S_4	10 [100]	43	44	45	0	0 [200]	300
S_5	0 [400]	0 [500]	0 [400]	0 [300]	M	M	1600
b_j^*	700	700	700	400	100	200	

Notes. Entries in rectangular boxes along various links represent the amount of commodity transported along these links.

TABLE 11. Problem $P_1^*(34)$ and its optimal feasible solution Z^1 for Ex-1.

	D_1	D_2	D_3	D_4	D_5	D_6	a_i^*
S_1	10	11 [200]	30	43	0 [100]	0	300
S_2	9 [200]	12	31	10	0	0 [100]	300
S_3	34	35	32 [200]	44 [100]	0	0	300
S_4	10	43 [100]	44 [100]	45	0	0 [100]	300
S_5	0 [500]	0 [400]	0 [400]	0 [300]	M	M	1600
b_j^*	700	700	700	400	100	200	

Step 3. (b) Formulate the problem $P_1^*(T_2^0)$, *i.e.*, $P_1^*(34)$ and find its optimal feasible solution as given in Table 11. Using Restriction (iii), we impose the following partial sum constraints

$$z_{31} + z_{35} \leq 200, \qquad z_{41} + z_{45} \leq 100. \tag{32}$$

It may be noted that there is no additional blockage in the transportation matrix except partial sum constraints defined by equation (32).

The solution Z^1 is an M-feasible solution. Go to Step 2.

Iteration 2

Step 2. This optimal M-feasible solution Z^1 yields Stage-I and Stage-II solutions X^1 and Y^1 respectively as

$$X^1: \; x_{12} = 200, x_{21} = 200, x_{33} = 200, x_{34} = 100, x_{42} = 100, x_{43} = 100, x_{ij} = 0 \quad \text{otherwise;}$$

TABLE 12. Problem $P_1^*(10)$ and its optimal feasible solution Z^2 for Ex-1.

	D_1	D_2	D_3	D_4	D_5	D_6	a_i^*
S_1	10 [100]	11 [100]	30	43	M	0 [100]	300
S_2	9 [100]	12	31	10 [100]	0 [100]	0	300
S_3	34	35 [100]	32 [100]	44	M	0 [100]	300
S_4	10 [100]	43 [100]	44 [100]	M	M	0	300
S_5	0 [400]	0 [400]	0 [500]	0 [300]	M	M	1600
b_j^*	700	700	700	400	100	200	

$$Y^1: \; y_{11} = 100, y_{ij} = 0 \text{ otherwise.}$$

Note down $T_1^1 = 44$, $T_2^0 = 10$ and go to Step 3(b).

Step 3. (b) Formulate the problem $P_1^*(T_2^1)$, *i.e.*, $P_1^*(10)$ and find its optimal feasible solution as given in Table 12. Using Restriction (iv), we impose the following partial sum constraint

$$z_{21} + z_{25} \leq 200. \tag{33}$$

It may be noted that links $(1,5)$, $(3,5)$, $(4,5)$ are blocked due to Restriction (i), and $(4,4)$ is blocked due to restriction (ii) in the transportation matrix.

The solution Z^2 is an M-feasible solution. Go to Step 2.

Iteration 3

Step 2. This optimal M-feasible solution Z^2 yields Stage-I and Stage-II solutions X^2 and Y^2 respectively as

$$\begin{aligned} X^2: \; & x_{12} = 100, x_{12} = 100, x_{21} = 100, x_{24} = 100, x_{32} = 100, x_{33} = 100, \\ & x_{41} = 100, x_{42} = 100, x_{43} = 100, x_{ij} = 0 \text{ otherwise;} \\ Y^2: \; & y_{21} = 100, y_{ij} = 0 \text{ otherwise.} \end{aligned}$$

Note down $T_1^1 = 44$, $T_2^0 = 9$ and go to Step 3(a).

Step 3. (a) As $T_2^2 = 9 = T_2^L$, so go to Step 4(b).

Step 4. (b) The optimal value of the objective function of Ex-1 is

$$\min\{32 + 34, 44 + 10, 44 + 9\} = 53.$$

Algorithm-C terminates.

Example 2. Consider 3×2 transportation problem Ex-2 with 3 sources and 2 destinations whose time of transportation is given in Table 13 and capacities along various links are $u_{ij} = 200 \; \forall i = 1, 2, 3; j = 1, 2$. Given that $\sum_{i=1}^{3} a_i = 900$, Stage-I flow is $F_1 = 500$ and flow in Stage-II is $F_2 = 100$.

Using Remark 4.12, $T_2^L = 18$.

TABLE 13. Time of transportation along various source-destination links for Ex-2.

	D_1	D_2	a_i
S_1	20	22	300
S_2	18	24	300
S_3	30	32	300
b_j	200	200	

TABLE 14. Problem P_1^* corresponding to Ex-2.

	D_1	D_2	D_3	D_4	a_i^*
S_1	200 20	200 22	∞ 0	∞ 0	300
S_2	200 18	200 24	∞ 0	∞ 0	300
S_3	200 30	200 32	∞ 0	∞ 0	300
S_4	400 0	400 0	∞ M	∞ M	700
b_j^*	600	600	100	300	

TABLE 15. An optimal feasible solution Z^0 of problem P_1^* for Ex-2.

	D_1	D_2	D_3	D_4	a_i^*
S_1	[100] 20	[200] 22	0	0	300
S_2	[200] 18	24	[100] 0	0	300
S_3	30	32	0	[300] 0	300
S_4	[300] 0	[400] 0	M	M	700
b_j^*	600	600	100	300	

Iteration 1

Step 1. Formulate the problem P_1^* corresponding to Ex-2 as given in Table 14 along with capacities along various links. Solve this problem and find its optimal solution Z^0 as given in Table 15. It is an M-feasible solution, so go to Step 2.

Step 2. This optimal M-feasible solution Z^0 yields Stage-I and Stage-II solutions X^0 and Y^0 respectively as $x_{11} = 100, x_{12} = 200, x_{21} = 200, x_{ij} = 0$ otherwise; $y_{32} = 100, y_{ij} = 0$ otherwise.
Note down $T_1^0 = 22$, $T_2^0 = 24$ and go to Step 3(b).

TABLE 16. Problem $P_1^*(24)$ and its optimal feasible solution Z^1 for Ex-2.

	D_1		D_2		D_3		D_4		a_i^*
S_1		100		200					
	20		22		0		0		300
S_2		100		100		100			
	18		24		0		0		300
S_3								300	
	M		M		M		0		300
S_4		400		300					
	0		0		M		M		700
b_j^*	600		600		100		300		

Step 3. (b) Formulate the problem $P_1^*(T_2^0)$, *i.e.*, $P_1^*(24)$ by imposing restrictions (i)–(iii) and find its optimal feasible solution as given in Table 16. Using Restriction (iii), we impose the following partial sum constraints

$$z_{21} + z_{23} \leq 200. \tag{34}$$

It may be noted that there are blockages in cells $(3,1)$, $(3,2)$, and $(3,3)$ in the transportation matrix along with the partial sum constraints defined by equation (34).

The solution Z^1 is an M-feasible solution. Go to Step 2.

Iteration 2

Step 2. This optimal M-feasible solution Z^1 yields Stage-I and Stage-II solutions as

$$X^1: \; x_{11} = 100, x_{12} = 200, x_{21} = 100, x_{22} = 100, x_{ij} = 0 \quad \text{otherwise;}$$
$$Y^1: \; y_{21} = 100, y_{ij} = 0 \text{ otherwise.}$$

Note down $T_1^1 = 24$, $T_2^1 = 18$ and go to Step 3(a).

Step 3. (a) As $T_2^1 = 18 = T_2^L$, then go to Step 4(b).

Step 4. (b) The optimal value of the objective function of Ex-2 is

$$\min\{22 + 24, 24 + 18\} = 42.$$

Algorithm-C terminates.

Solution of Ex 2 using Algorithm-A proposed by Kaur *et al.* [27]. Given Ex-2 formulate the problem $\bar{P}_1$ given in Table 17.

An optimal solution of problem $\bar{P}_1$ is given in Table 18.

This yields a feasible solution of Ex 2 as

Stage-I: $x_{11} = 100, x_{12} = 200, x_{21} = 200, x_{ij} = 0$ otherwise and $T_1^0 = 22$.
Stage-II: $y_{22} = 100, y_{ij} = 0$ otherwise and $T_2^0 = 24$.

Now formulate a restricted version of problem $\bar{P}_1$ as $\bar{P}_1(T_2^0)$ as given in Table 19. As per the restrictions imposed by Kaur *et al.* [27] following transportation links will be blocked in $\bar{P}_1(T_2^0)$ for which $t_{ij} + T_2^L \geq \min\{22 + 24\} = 46$, *i.e.*, $t_{ij} \geq 28 \; \forall (i,j) \in I \times J$.

TABLE 17. Problem $\bar{P}_1$.

	D_1	D_2	D_3	a_i^*
S_1	200	200	300	
	20	22	0	300
S_2	200	200	300	
	18	24	0	300
S_3	200	200	300	
	30	32	0	300
S_4	400	400	∞	
	0	0	M	700
b_j^*	600	600	400	

TABLE 18. An optimal solution of problem $\bar{P}_1$.

	D_1	D_2	D_3	a_i^*
S_1	$\boxed{100}$	$\boxed{200}$		
	20	22	0	300
S_2	$\boxed{200}$		$\boxed{100}$	
	18	24	0	300
S_3			$\boxed{300}$	
	30	32	0	300
S_4	$\boxed{300}$	$\boxed{400}$		
	0	0	M	700
b_j^*	600	600	400	

TABLE 19. Restricted version of problem $\bar{P}_1$, *i.e.*, $\bar{P}_1(24)$.

	D_1	D_2	D_3	a_i^*
S_1				
	20	22	0	300
S_2				
	18	24	0	300
S_3				
	M	M	M	300
S_4				
	0	0	M	700
b_j^*	600	600	400	

Table 20. Time entries of various source-destination links Ex-3.

	D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8	D_9	D_{10}	D_{11}	D_{12}	D_{13}	D_{14}	D_{15}	D_{16}	D_{17}	a_i
S_1	22	38	46	15	38	49	33	27	42	24	45	25	48	19	13	12	14	12
S_2	35	24	31	10	12	7	6	14	20	26	13	37	34	25	30	8	22	10
S_3	28	31	49	23	16	26	45	32	35	36	24	22	47	21	41	15	42	18
S_4	3	21	2	39	8	35	37	29	31	15	5	35	19	38	14	7	22	13
S_5	8	4	16	14	25	21	8	39	29	29	16	31	32	23	37	29	12	11
S_6	17	14	35	44	43	28	24	48	37	35	47	21	48	35	34	41	49	35
S_7	38	1	34	10	36	30	33	20	18	27	27	33	33	16	12	6	21	19
S_8	16	12	31	8	11	1	38	20	39	14	27	19	26	27	4	16	26	19
S_9	31	5	10	19	22	20	33	20	33	29	6	2	10	10	9	26	20	51
S_{10}	29	11	32	46	41	40	42	35	36	12	27	42	40	43	16	17	30	23
S_{11}	21	33	39	21	4	34	11	38	38	30	35	24	28	39	10	8	10	29
S_{12}	12	31	33	25	45	23	13	7	35	17	9	28	21	43	8	10	19	25
S_{13}	31	22	31	2	20	36	17	4	37	32	36	46	17	42	36	14	18	40
S_{14}	8	31	24	16	31	32	36	8	43	18	38	39	28	32	18	16	13	18
S_{15}	12	19	15	25	18	21	25	7	30	15	31	4	19	4	2	9	43	37
b_j	8	24	6	27	56	35	17	9	25	28	22	12	4	16	4	26	11	

Table 21. Capacities along various source-destination links Ex-3.

	$j=1$	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$i=1$	19	30	13	19	29	23	23	30	15	31	33	34	35	40	30	11	1
2	32	32	35	36	38	5	30	33	7	8	30	39	34	33	32	30	35
3	33	34	33	23	9	35	33	39	30	34	33	23	32	21	34	35	37
4	36	34	8	21	9	35	23	12	21	34	32	36	21	21	33	35	38
5	6	10	10	10	38	32	8	6	13	15	22	34	23	34	21	21	17
6	9	15	30	20	12	20	13	12	13	15	12	17	13	13	18	19	21
7	21	12	17	16	13	10	10	15	19	39	25	20	14	13	12	14	17
8	15	16	18	18	19	19	29	19	22	32	22	12	15	18	18	10	12
9	29	1	24	27	24	17	15	27	25	28	29	26	30	23	24	25	27
10	14	27	28	28	23	27	29	24	23	23	29	28	26	23	23	28	28
11	10	10	30	30	30	30	31	33	32	40	41	40	40	30	40	30	30
12	14	18	16	17	14	18	18	14	15	17	19	18	16	16	17	18	15
13	30	30	30	30	30	30	30	30	30	30	30	30	15	12	13	10	7
14	8	3	4	16	15	13	12	15	17	18	19	18	16	14	13	10	14
15	17	18	19	20	18	19	16	17	19	20	20	20	21	22	23	24	21

Therefore in Table 19, corresponding to S_3 all source-destination links are closed and this restricted problem does not possess any special M-feasible solution. Thus, Algorithm-A terminates yielding an optimal value of Ex-2 as 46. While Algorithm-C yields an optimal value of 42 which is better than 46. Hence, Algorithm-A may not obtain the optimal value of TSTMTP-F.

Example 3. Consider 15×17 transportation problem Ex-3 with 15 sources and 17 destinations whose time of transportation is given in Table 20 and capacities along various links are given in Table 21. Given that $\sum_{i=1}^{15} a_i = 360$, Stage-I flow is $F_1 = 130$ and, flow in Stage-II is $F_2 = 200$.

Various pairs (T_1, T_2) generated by Algorithm-C for problem Ex-3 are listed in Table 22. The highlighted row in Table 22 shows that the optimal time of transportation of TSCTMTP-F is 4+20 which corresponds to the

TABLE 22. List of pairs generated for Ex-3.

Sr. No.	Pair generated (T_1, T_2)	Total time
1	$(4, 49)$	53
2	$(4, 48)$	52
3	$(4, 47)$	51
4	$(4, 45)$	49
5	$(4, 43)$	47
6	$(4, 42)$	46
7	$(4, 41)$	45
8	$(4, 37)$	41
9	$(4, 36)$	40
10	$(4, 35)$	39
11	$(4, 31)$	35
12	$(4, 30)$	34
13	$(4, 29)$	33
14	$(4, 28)$	32
15	$(4, 27)$	31
16	$(4, 24)$	28
17	$(4, 22)$	26
18	$(4, 21)$	25
19	$(4, 20)$	24
20	$(26, 19)$	45
21	$(25, 18)$	43
22	$(25, 17)$	42
23	$(25, 16)$	41
24	$(25, 14)$	39
25	$(36, 12)$	48
26	$(36, 11)$	47
27	$(36, 10)$	46
28	$(36, 9)$	45
29	$(36, 8)$	44
30	$(41, 7)$	48
31	$(41, 6)$	47

pair $(4, 20)$. Now, corresponding to the pair $(4, 20)$, the solution of Stage-I is given in Table 23 and the solution of Stage-II is given in Table 24. The total time taken by the CPU to solve this problem is 18.89 s

Remark 5.1. Capacitated two-stage time minimization problem discussed by Xie and Li [45] and Sharma *et al.* [37] becomes a particular case of the current problem with $F_1 = \sum_{j\in J} b_j$ and $F_2 = \sum_{i\in I} a_i - \sum_{j\in J} b_j$. Algorithm-C solves this particular case successfully with $b_{n+2} = 0$.

Here, an optimal solution of a capacitated two-stage time minimization problem of size 6×4 discussed by Xie and Li [45] and Sharma *et al.* [37] has been obtained using Algorithm-C (refer to Tabs. 25 and 26).

The corresponding problem P_1^* is given in Table 27 followed by its optimal solution in Table 28.

The optimal solution of problem P_1^* given in Table 28 yields the same optimal solution of Stage-I and Stage-II as obtained by Xie and Li [45] with optimal time as 12 units with $T_1 = 9$ and $T_2 = 3$.

6. Computational complexity of Algorithm-C

The Algorithm-C has been coded in Python and tested on a variety of randomly generated problems on a 64-bit operating system (Intel Core i5, 4 GB RAM). The technique operates well when test problems have

TABLE 23. An optimal solution of Stage-I for Ex-3.

	D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8	D_9	D_{10}	D_{11}	D_{12}	D_{13}	D_{14}	D_{15}	D_{16}	D_{17}
S_1																	
S_2																	
S_3																	
S_4			6														
S_5		8															
S_6																	
S_7																	
S_8						19											
S_9												12					
S_{10}																	
S_{11}					29												
S_{12}																	
S_{13}				27				9									
S_{14}																	
S_{15}														16	4		

TABLE 24. An optimal solution of Stage-II for Ex-3.

	D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8	D_9	D_{10}	D_{11}	D_{12}	D_{13}	D_{14}	D_{15}	D_{16}	D_{17}
S_1																11	1
S_2						4			6								
S_3					8												
S_4					7												
S_5							3										
S_6		15															
S_7									19								
S_8																	
S_9		1				12					22		4				
S_{10}										23							
S_{11}																	
S_{12}							14									11	
S_{13}					4												
S_{14}	8																10
S_{15}					8					5						4	

TABLE 25. Time of transportation along various source-destination links.

	D_1	D_2	D_3	D_4	a_i
S_1	5	6	4	3	30
S_2	7	9	12	10	40
S_3	2	8	7	4	45
S_4	11	5	9	8	25
S_5	6	10	5	3	50
S_6	12	4	2	10	20
bj	40	50	30	40	

TABLE 26. Capacities along various source-destination links.

	$j=1$	$j=2$	$j=3$	$j=4$
$i=1$	10	20	15	15
$i=2$	20	20	10	15
$i=3$	15	15	15	20
$i=4$	15	10	20	20
$i=5$	20	20	20	20
$i=6$	10	20	20	15

TABLE 27. Corresponding problem P_1^*.

	D_1	D_2	D_3	D_4	D_5	D_6	a_i^*
S_1	10	20	15	15	∞	∞	
	5	6	4	3	0	0	30
S_2	20	20	10	15	∞	∞	
	7	9	12	10	0	0	40
S_3	15	15	15	20	∞	∞	
	2	8	7	4	0	0	45
S_4	15	10	20	20	∞	∞	
	11	5	9	8	0	0	25
S_5	20	20	20	20	∞	∞	
	6	10	5	3	0	0	50
S_6	10	20	20	15	∞	∞	
	12	4	2	10	0	0	20
S_7	50	55	70	65	∞	∞	
	0	0	0	0	M	M	240
b_j^*	90	105	100	105	50	0	

different input data values of t_{ij} selected from the intervals varying from $[1, 40]$ to $[1, 500]$. For each instance corresponding to a particular time interval, the CPU time (in seconds) consumed by the Python program is reported. The CPU time t is the average CPU time of 50 instances with the same specifications. From Table 29, we observe that Algorithm-C runs successfully for all these randomly generated instances.

The graphical representation of the computational results given in Table 29 is shown in Figure 1. From Figure 1, it can be concluded that the running time for a particular instance depends on the size of the problem and the variations in time entries. For a particular variation in time entries, the CPU running time increases with the increase in the size of the problem. Further, for a fixed-sized problem, the CPU running time increases with the increase in the variations in the time entries. From Figure 1a, we observe that the running time for 5×5 instance is almost the same for different variations in time entries because there can be at the most 25 different entries in the time matrix whether $t_{ij} \in [1, 40]$ or $t_{ij} \in [1, 500]$. A similar argument holds for the instances of the size 10×15 when the running time decreases from 4.43 s to 4.31 s as time entries vary from $t_{ij} \in [1, 200]$ to $t_{ij} \in [1, 500]$. Therefore, it can be concluded that the computational results are in accordance with the theoretical results mentioned in Remark 4.16.

TABLE 28. An optimal feasible solution of the problem P_1^*.

	D_1	D_2	D_3	D_4	D_5	D_6	a_i^*
S_1		15	15		0		30
S_2	20	20					40
S_3		10		20	15		45
S_4		5		20			25
S_5	20		15		15		50
S_6					20		20
S_7	50	55	70	65			240
b_j^*	90	105	100	105	50	0	

TABLE 29. CPU run time (in seconds) for randomly generated instances.

Size of the problem	Running time (in seconds)			
	$t_{ij} \in [1, 40]$	$t_{ij} \in [1, 100]$	$t_{ij} \in [1, 200]$	$t_{ij} \in [1, 500]$
5×5	0.615	0.695	0.691	0.674
5×10	0.921	1.25	1.293	1.329
10×12	2.161	3.227	3.143	4.0162
10×15	2.5	3.606	4.43	4.31
15×15	3.928	6.729	6.86	7.97
20×15	4.22	9.775	13.57	12.66
20×25	8.32	15.60	21.14	35.65
30×35	16.61	39.37	68.70	102.04
40×45	45.73	185.24	400.087	960
50×50	65.308	364.043	417.620	1200

7. CONCLUDING REMARKS

(1) An iterative algorithm "Algorithm-C" has been proposed to find an optimal solution for a two-stage capacitated time minimization transportation problem with restricted flow in both stages. This proposed algorithm overcomes all flaws of the already existing algorithm proposed by Kaur *et al.* [27] (called Algorithm-A) and generates an optimal solution of TSCTMTP-F for any given instance.

(2) Implementation of Algorithm-C is easy as compared to Algorithm-A. Firstly, the solution strategy to obtain a feasible solution of Stage-II for a given solution of Stage-I is straightforward (refer to Algorithm-B discussed in Sect. 4.1). Secondly, Algorithm-C terminates as soon as a restricted problem does not possess any M-feasible solution. On the other hand, Algorithm-A keeps on scanning for special M-feasible solutions (SMFSs) of

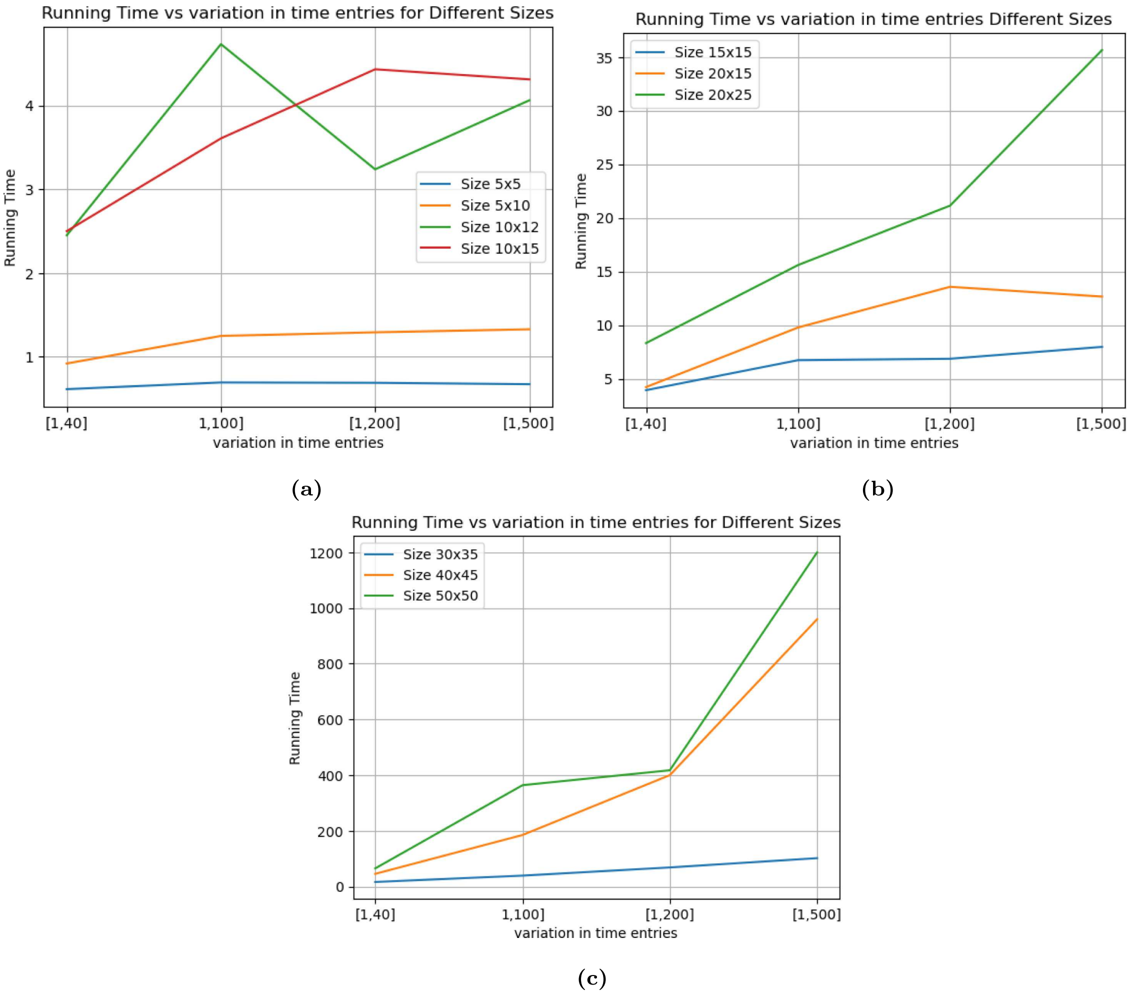

Figure 1. Graphical representation of computational results.

restricted problems and there is no definite strategy for scanning of SMFSs. This makes the implementation of Algorithm-A quite complex.

(3) Algorithm-A may not always yield an optimal solution of TSCTMTP-F. Although, Algorithm-A and Algorithm-C both yield the same objective function value for the problems discussed by Kaur *et al.* [27] and Ex 1 discussed in the current manuscript. But Algorithm-C yields a better objective function value than Algorithm-A for problem Ex 2 discussed in Section 5.
(4) Developing an alternate technique for solving TSCTMTP-F by finding optimal feasible flow using network structure properties of a transportation problem and carrying out a comparative study with the proposed technique emerges as an interesting future research problem.

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