ON MAXIMAL ROMAN DOMINATION IN GRAPHS: COMPLEXITY AND ALGORITHMS

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Abstract. For a simple undirected connected graph $G = (V, E)$, a maximal Roman dominating function (MRDF) of $G$ is a function $f : V(G) \rightarrow \{0, 1, 2\}$ with the following properties: (i) For every vertex $v \in \{v \in V | f(v) = 1\}$, there exists a vertex $u \in N(v)$ such that $f(u) = 2$. (ii) The set \{v \in V | f(v) = 0\} is not a dominating set of $G$. In other words, there exists a vertex $v \in \{v \in V | f(v) \neq 0\}$ such that $N(v) \cap \{u \in V | f(u) = 0\} = \emptyset$. The weight of an MRDF of $G$ is the sum of its function values over all vertices, denoted as $f(G) = \sum_{v \in V(G)} f(v)$, and the maximal Roman domination number of $G$, denoted by $\gamma_{mR}(G)$, is the minimum weight of an MRDF of $G$. In this paper, we establish some bounds of the maximal Roman domination number of graphs. Additionally, we develop an integer linear programming formulation to compute the maximal Roman domination number of any graph. Furthermore, we prove that maximal Roman domination problem (MRD) is NP-complete even restricted to star convex bipartite graphs and chordal bipartite graphs. Lastly, we show the maximal Roman domination number of threshold graphs, trees, and block graphs can be computed in linear time.

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1. Introduction

In this paper, all graphs are simple undirected graphs with no loops or multiple edges. Let $G = (V, E)$ be a graph with the vertex set $V$ and the edge set $E$. For a vertex $u \in V$, the open neighborhood $N(u)$ of $u$ is the set of all the vertices adjacent to $u$, and the close neighborhood $N[u]$ of $u$ is the set $N(u) \cup \{u\}$. The degree of a vertex $v \in V$ is denoted by $\deg_G(v) = |N(v)|$, abbr $\deg(v)$. The maximum degree of $G$ is $\Delta(G) = \max\{\deg(v) | v \in V\}$, abbr $\Delta$. The minimum degree of $G$ is $\delta(G) = \min\{\deg(v) | v \in V\}$, abbr $\delta$. The complement graph of $G$ is denoted as $\overline{G}$. For a vertex set $V_1 \subseteq V$, $G[V_1]$ is the induced subgraph of $G$ based on $V_1$. In particular, we denote $G - v$ as the induced subgraph $G[V - \{v\}]$ where $v \in V$.

A path with $n$ vertices is denoted as $P_n$, and a cycle with $n$ vertices is denoted as $C_n$. A bipartite graph $G = (V, E)$ is a graph whose vertices set $V$ can be divided into two sets $X$ and $Y$ such that $X \cap Y = \emptyset$ and $X \cup Y = V$, and each edge in $E$ has one endpoint in $X$ and one endpoint in $Y$. Hence a bipartite graph can

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also be noted as $G = (X, Y, E)$, A bipartite graph $G = (X, Y, E)$ is a star convex bipartite graph if there exists a star graph $S = (X, E')$ based on $X$ such that $S[N_G(y)]$ is a star graph for all $y \in Y$. A chord of a cycle is an edge that connects two non-adjacent vertices of the cycle. A chordal graph is a graph such that every cycle with length at least four has a chord. A chordal bipartite graph is a bipartite graph such that every cycle of length at least six has a chord. For other terminology and notations not mentioned here, please refer to [1].

A vertex set $S \subseteq V$ is a dominating set in $G = (V, E)$ if every $v \in V$ has $N[v] \cap S \not= \emptyset$. A vertex $v \in S$ dominates the vertices of its closed neighborhood. If a vertex $v$ is dominated by at least one vertex from the set $D$, we simply say $v$ is dominated by $D$.

In this paper, let $f$ be a function of the graph $G = (V, E)$ such that $f : V(G) \rightarrow \{0, 1, 2, \ldots, k\}$. The vertex set $V$ can be partitioned into distinct subsets based on the function values, yielding the representation $f = (V_0^f, V_1^f, V_2^f, \ldots, V_k^f)$, where $V_i^f = \{v \in V|f(v) = i\}$ and $V = V_0^f \cup V_1^f \cup \cdots \cup V_k^f$. Besides, we can use the shorthand $V_i$ to represent $V_i^f$ when convenient, allowing us to abbr the function as $f = (V_0, V_1, V_2, \ldots, V_k)$. The weight of the function $f$ of $G$ is the sum of the function value over all vertices in $V(G)$, which denoted by $f(G) = \sum_{v \in V} f(v)$.

A Roman dominating function (RDF) of a graph $G = (V, E)$ is a function $f : V(G) \rightarrow \{0, 1, 2\}$ (or $f = (V_0^f, V_1^f, V_2^f)$) with the property that if $f(v) = 0$ for $v \in V$, there are some $u \in N(v)$ such that $f(u) = 2$. The Roman domination number denoted by $\gamma_R(G)$ is the minimum weight of an RDF of $G$. An RDF of $G$ is called a $\gamma_R$-function if its weight $f(G) = \gamma_R(G)$. For an RDF $f = (V_0^f, V_1^f, V_2^f)$ of $G$, we define set $M^f = \{v \in V_1^f \cup V_2^f | N(v) \cap V_0^f = \emptyset\}$, i.e. $M^f$ is the set of all vertices of $V(G)$ not dominated by $V_0^f$. A maximal Roman dominating function (MRDF) of a graph $G = (V, E)$ is a Roman dominating function $f : V(G) \rightarrow \{0, 1, 2\}$ (or $f = (V_0^f, V_1^f, V_2^f)$) having the property that $V_0^f$ is not a dominating set of $G$, i.e. $M^f \not= \emptyset$. The maximal Roman domination number denoted by $\gamma_{mR}(G)$ is the minimum weight of an MRDF of $G$. An MRDF of $G$ is called a $\gamma_{mR}$-function if its weight $f(G) = \gamma_{mR}(G)$.

Inspired by the pioneering works of Stewart [2] and ReVelle et al. [3], Cockayne et al. initiated the study of Roman domination. Subsequently, a plethora of variations emerged and were investigated, which include total Roman domination [4], independent Roman domination [5], outer independent Roman domination [6], perfect Roman domination [7], global Roman domination [8], and maximal Roman domination [9]. Two comprehensive review literature about the varieties of Roman domination can be seen at [10, 11]. Among these varieties, our particular interest lies in maximal Roman domination, which is introduced by Ahangar et al. [9] in 2017. Further studies of maximal Roman domination can be found in [12-14].

Maximal type domination problems have been widely studied. In addition to maximal Roman domination, Kulli et al. introduced maximal domination [15] in 1997; Ahangar et al. introduced maximal 2-rainbow domination [16] in 2016; and Ahangar et al. introduced maximal double Roman domination [17] in 2022. In practical terms, maximal type domination strategies are useful in optimizing models with confidentiality requirements, such as in the design of enterprise networks. Let $f = (V_0, V_1, \ldots, V_k)$ be a function representing the network topology, where the vertex assigned label 0 can be regarded as external service recipients lacking trust and the vertices assigned value equal or greater than 1 are deemed trustful and capable of providing service to vertices in $V_0$. Maximal type domination strategies allows us to deploy confidentiality items at the vertices which is not dominated by $V_0$ while designing a network to provide service covering all the vertices with minimum cost. Specifically, in the maximal Roman domination strategy, three vertex sets $V_0, V_1$, and $V_2$ are defined, and only the vertices belonging to $V_2$ can provide service to those in $V_0$.

In this paper, we explore some bounds of the maximal Roman domination number of graphs in Section 2. Then we develop an integer linear programming formulation with $3|V|$ binary variables and $3|V|$ constraints to compute the maximal Roman domination number of a graph $G = (V, E)$ in Section 3. After that, we prove that the maximal Roman domination problem (MRD) is NP-complete even restricted to star convex bipartite graphs and chordal bipartite graphs in Section 4. At last we show the maximal Roman domination number of threshold graphs, trees, and block graphs can be computed in linear time in Section 5.
2. Bounds of maximal Roman domination number

In this section, we will establish a lower bound of maximal Roman domination number by discharging approach, which was first studied on domination-type problems by Shao et al. [19]. Subsequently, we will employ a probabilistic method similar to Alon et al. [19], in order to refine the upper bound of maximal Roman domination number. For a graph \( G = (V, E) \) and a vertex \( v \in V \), we also establish the bounds of \( \gamma_{mR}(G-v) \).

Following this, we will derive bounds on \( \gamma_{mR}(G) + \gamma_{mR}(\overline{G}) \) and determine the exact value of \( \gamma_{mR}(P_n) + \gamma_{mR}(\overline{P_n}) \) where \( P_n \) is a path. To commence, there are some useful results we need to introduce.

Observation 2.1 ([9]). For any graph \( G = (V, E) \), \( \gamma_R(G) \leq \gamma_{mR}(G) \leq |V| \).

Lemma 2.2 ([20]). For any graph \( G = (V, E) \) with maximum degree \( \Delta \geq 1 \), \( \gamma_{mR}(G) \geq \frac{2n}{\Delta+1} \).

Lemma 2.3 ([9]). For \( k \geq 1 \), \( \gamma_{mR}(P_{3k}) = 2k + 1 \), \( \gamma_{mR}(P_{3k+1}) = 2k + 2 \) and \( \gamma_{mR}(P_{3k+2}) = 2k + 2 \).

Lemma 2.4 ([12]). Let \( f = (V_0, V_1, V_2) \) be a \( \gamma_{mR} \)-function on a graph \( G = (V, E) \), every vertex of \( V \) not dominated by \( V_0 \) belongs to \( V_1 \).

From Observation 2.1 and Lemma 2.2, the following corollary holds. Next we will give a lower bound slightly improving this corollary by discharging approach.

Corollary 2.5. For any graph \( G = (V, E) \) with maximum degree \( \Delta \geq 1 \), let \( n = |V| \), then \( \gamma_{mR}(G) \geq \frac{2n}{\Delta+1} \).

Theorem 2.6. For any graph \( G = (V, E) \) with maximum degree \( \Delta \geq 1 \), let \( n = |V| \), then

\[
\gamma_{mR}(G) \geq \min \left\{ \frac{2n + \Delta}{\Delta + 1}, \frac{2n + 2\Delta - 2}{\Delta + 1} \right\}.
\]

Proof. Let the function \( f = (V_0, V_1, V_2) \) be a \( \gamma_{mR} \)-function on the graph \( G \). Suppose the initial charge function is denoted as \( s(u) = f(u) \) for all vertices \( u \in V \). Next, we proceed with a discharging procedure based on following rules:

Rule 1. For a vertex \( u \) with \( s(u) = 2 \), \( u \) distributes charge \( \frac{2}{\Delta+1} \) to each of its adjacent vertex.

Rule 2. For a vertex \( u \) with \( s(u) = 1 \) or 0, \( u \) sends no charge to its adjacent vertex.

After applying the discharging procedure, we examine the remaining charge of each vertex. We denote the remaining charge function as \( s'(u) \). For \( u \in V_0 \), as \( N(u) \cap V_0 = \emptyset \), \( u \) can get at least \( \frac{2}{\Delta+1} \) charge. So \( s'(u) \geq \frac{2}{\Delta+1} \). For \( u \in V_1 \), \( u \) doesn’t send charge to its neighbors, therefore \( s'(u) = s(u) = 1 \). For \( u \in V_2 \), the remaining charge of \( u \) depends on its degree \( \deg(u) \), yielding \( s'(u) = s(u) - \deg(u) \cdot \frac{2}{\Delta+1} \geq \frac{2}{\Delta+1} \).

From Lemma 2.4, we infer that the vertices not dominated by \( V_0 \) must belong to \( V_1 \), implying \( |V_1| \geq 1 \). Let \( x = |V_1| \). Since the discharging procedure preserves the sum of charge on \( G \), there are two cases.

Case 1. \( N(u) \cap V_2 = \emptyset \) for all \( u \in M_f \).

We get \( x \geq 2 \), and

\[
\sum_{u \in V} s(u) = \sum_{u \in V} s'(u) \geq \frac{2}{\Delta+1} \cdot (|V_0| + |V_2|) + 1 \cdot |V_1| = \frac{2n + (\Delta - 1)x}{\Delta + 1} \geq \frac{2n + 2\Delta - 2}{\Delta + 1}.
\]

Case 2. \( N(u) \cap V_2 \neq \emptyset \) for some \( u \in M_f \).

The vertex \( u \) receives at least \( \frac{1}{\Delta+1} \) charge from its adjacent vertices assigned 2. So we have

\[
\sum_{u \in V} s(u) = \sum_{u \in V} s'(u) \geq \frac{2}{\Delta+1} \cdot (|V_0| + |V_2|) + 1 \cdot |V_1| + \frac{1}{\Delta + 1} = \frac{2n + (\Delta - 1)x + 1}{\Delta + 1} \geq \frac{2n + \Delta}{\Delta + 1}.
\]

In conclusion, \( f(G) = \sum_{u \in V} s(u) \geq \min\{\frac{2n+\Delta}{\Delta+1}, \frac{2n+2\Delta-2}{\Delta+1}\} \), i.e. \( \gamma_{mR}(G) \geq \min\{\frac{2n+\Delta}{\Delta+1}, \frac{2n+2\Delta-2}{\Delta+1}\} \).

\[\square\]
Theorem 2.7. For a graph $G = (V, E)$ with $n = |V|$ and minimum degree $\delta$, 
\[ \gamma_{mR}(G) \leq \frac{2 + 2\ln(1 + \delta)}{1 + \delta} n. \]

Proof. For any vertices in $V$, we randomly and independently assigned 2 with a probability $p$. Let $A$ be the selected vertices set. We can get the size of $A$ is $np$. Let $A^*$ be the vertex set of all vertices $u \in A$ such that $N(u) \cap A = \emptyset$. Note $B$ as the set of all the vertices assigned 1 such that none of their adjacent vertices is assigned 2. Lastly, we define $B^*$ as the set of all the vertices assigned 1 such that all of their adjacent vertices are in $A^* \cup B$. Based on the definition above, next we will discuss the probabilities, noted $Pr(v)$, of a vertex $v$ belonging to these sets and the size of these sets respectively.

For any vertices $v \in A$, we can get the size of $\gamma_v = (V \setminus \{v\}, E \setminus \{v\})$.

Case 2. $\mathcal{H}_2 = (V_2^{h_G}, V_3^{h_G})$ is an MRDF on $G$ such that $\mathcal{H}_2 \setminus N[\mathcal{H}_2]$.

Firstly, if $v \in V_1^{h_G}$, then the function $h_G(v') = h_G(u)$ for $u \in V(G) \setminus N[v]$ is an MRDF on $G - v$ of weight $\gamma_{mR}(G) - 1$. Secondly, if $v \in V_2^{h_G}$, then $v$ dominates its neighbor. Let $h_G(v') = h_G(u)$ for $u \in V(G) \setminus N[v]$, then we can get an MRDF $h_G$ on $G - v$. We have $h_G(G - v) = \gamma_{mR}(G) - h_G(v) + \sum_{u \in N(v)} f(u) \leq \gamma_{mR}(G) - 2 + \Delta.
Subcase 2.2. \( N(v) \cap V_0^{h_G} = \emptyset \).

Based on Lemma 2.4, we get \( v \) must belong to \( V_1^{h_G} \). We pick a vertex \( w \in N(v) \) and assign 1 to all \( u \in N(w) \). Please note that \( \deg(w) \leq \Delta - 1 \) after we remove \( v \). Hence \( h_{G-v}(G-v) = h_G(G)-1+(\Delta-1) = \gamma_{mR}(G) + \Delta - 2 \).

Considering the upper bound and the lower bound above, the proof is complete. \( \Box \)

**Theorem 2.9.** For any graph \( G = (V, E) \) with maximum degree \( \Delta(G) \) and minimum degree \( \delta(G) \), if \( \Delta(G) \geq \delta(G) + 1 \), then \( \gamma_{mR}(G) + \gamma_{mR}(G) \leq 2n - 2\Delta(G) + 2\delta(G) + 2 \).

**Proof.** Apparently, \( f = (V_0^f = \emptyset, V_1^f = V(G), V_2^f = \emptyset) \) is an MRDF on \( G \). We choose a vertex \( u \) with the maximum degree and a vertex \( v \) with the minimum degree. Firstly, we need to define a function \( g = (V_0^g, V_1^g, V_2^g) \) on \( G \). Let \( g(u) = 2 \) and \( g(w) = 0 \) for \( w \in N_G(u) \), otherwise inherit from \( f \). Clearly, the function \( g \) is an RDF on \( G \). Then we need to maintain at least one vertex not dominated by \( V_0^g \), and we choose \( v \). Please note that \( v \) could be \( u \), or adjacent to \( u \), or adjacent to the neighbor of \( u \). We reassign \( h(v) = 1 \), and \( h(w) = 1 \) for \( w \in N_G(v) \), otherwise inherit from \( g \). Clearly, \( h \) is an MRDF on \( G \) and \( h(G) \leq n - \Delta(G) + \delta(G) + 1 \). Hence \( \gamma_{mR}(G) \leq n - \Delta(G) + \delta(G) + 1 \). Further more, if \( \Delta(G) > \delta(G) + 1 \), \( \gamma_{mR}(G) \leq n - \Delta(G) + \delta(G) + 1 < n \). Based on this, we deduce

\[
\gamma_{mR}(G) + \gamma_{mR}(G) \leq (n - \Delta(G) + \delta(G) + 1) + (n - \Delta(G) + \delta(G) + 1) \\
= 2n - 2\Delta(G) + 2\delta(G) + 2.
\]

Further more, if \( \Delta(G) > \delta(G) + 1 \), \( \gamma_{mR}(G) + \gamma_{mR}(G) \leq 2n - 2\Delta(G) + 2\delta(G) + 2 < 2n \). \( \Box \)

**Lemma 2.10.** For a path \( P_n \) with \( n \geq 4 \), \( \gamma_{mR}(P_n) = n - 1 \).

**Proof.** Consider a path \( P_n \) with vertex set \( V(P_n) = \{x_1, x_2, \ldots, x_n\} \) and edge set \( E(P_n) = \{x_ix_{i+1} | 1 \leq i \leq n-1\} \), where \( n \geq 4 \). For the upper bound, we can observe that \( x_1 \) is adjacent to all \( u \in V(P_n) \setminus \{x_2\} \), and \( x_3 \) is adjacent to all \( u \in \{x_1, x_2\} \) in \( P_n \). So \( f = (V_0^f = V(P_n) \setminus N_{P_n}[x_3], V_1^f = N_{P_n}[x_3] \setminus \{x_1\}, V_2^f = \{x_1\}) \) is an MRDF on \( P_n \) of weight \( f(P_n) \leq 2 + \delta(P_n) = 2 + (n - 1 - \Delta(P_n)) = n - 1 \). So \( \gamma_{mR}(P_n) \leq n - 1 \).

For the lower bound, let \( g = (V_0^g, V_1^g, V_2^g) \) be a \( \gamma_{mR} \)-function on \( P_n \). Let \( v \) be a vertex not dominated by \( V_0^g \). Then \( \sum_{u \in V(N_{P_n}[v])} g(u) \geq \delta(P_n) + 1 = n - 2 \) where equality holds if \( \deg_{P_n}(v) = \delta(P_n) \) and all \( u \in V(N_{P_n}[v]) \) are assigned 1. Additionally, there must exist some \( w \in V(P_n) \) such that \( uv \notin E(P_n) \), which means \( g(w) = 0 \) while \( g(P_n-w) \geq n - 1 \), \( g(w) \geq 1 \) while \( g(P_n-w) \geq n - 2 \). Hence \( \gamma_{mR}(P_n) \geq g(P_n) \geq g(P_n-w) + g(w) \geq n - 1 \).

Considering the derived upper and lower bounds, we have \( \gamma_{mR}(P_n) = n - 1 \). \( \Box \)

**Theorem 2.11.** For a path \( P_n \) with \( n \geq 4 \) and \( k \geq 1 \), \( \gamma_{mR}(P_n) + \gamma_{mR}(P_n) = \begin{cases} 5k, & \text{if } n = 3k, \\ 5k + 2, & \text{if } n = 3k + 1, \\ 5k + 3, & \text{if } n = 3k + 2. \end{cases} \)

The correctness of Theorem 2.11 follows by Lemmas 2.3 and 2.10, hence omitted.

3. AN INTEGER LINEAR PROGRAMMING MODEL

In this section, we leverage the definition of the maximal Roman domination number in a graph \( G = (V, E) \) to naturally formulate an integer linear programming model, denoted as ILP-I. This model involves \( 3|V| \) binary variables and \( 3|V| \) constraints.

Let \( f : V(G) \to \{0, 1, 2\} \) be an MRDF of a graph \( G = (V, E) \). Please note that \( V_i^f = \{v \in V|f(v) = i\} \) and \( M^f \) is the set of all vertices not dominated by \( V_0^f \). Assume \( M \) is a sufficiently large const integer, then we define the following variables and integer linear programming formulations for the graph \( G \):

\[
x_{v,1} = \begin{cases} 1, & \text{if } f(v) = 1, \\ 0, & \text{otherwise}, \end{cases} 
x_{v,2} = \begin{cases} 1, & \text{if } f(v) = 2, \\ 0, & \text{otherwise}, \end{cases} 
z_v = \begin{cases} 0, & \text{if } v \in M^f, \\ 1, & \text{otherwise}, \end{cases}
\]
ILP-I:

\[
\text{minimum: } \sum_{v \in V} x_{v,1} + 2 \sum_{v \in V} x_{v,2}, \quad \text{for } v \in V \tag{1}
\]

subject to: \(x_{v,1} + x_{v,2} + \sum_{u \in N(v)} x_{u,2} \geq 1, \quad \text{for } v \in V \tag{2}\)

\[
(x_{v,1} + x_{v,2}) \deg(v) \leq \sum_{u \in N(v)} x_{u,1} + \sum_{u \in N(v)} x_{u,2} + z_v M, \quad \text{for } v \in V \tag{3}
\]

\[
\sum_{v \in V} z_v \leq n - 1
\]

\[
x_{v,1}, x_{v,2}, z_v \in \{0,1\}, \quad \text{for } v \in V. \tag{4}
\]

The objective function is defined by (1). The formulations (4) and (5) can maintain \(M^f \neq \emptyset\). The formulation (2) can ensure that for \(v \in V\), \(f(v) \geq 1\), or \(f(v) = 0\) such that \(N(v) \cap V_2 \neq \emptyset\). However, in ILP-I, it’s possible for both \(x_{v,1}\) and \(x_{v,2}\) to equal 1 simultaneously. To address this, we introduce the following lemma to establish that in an optimal solution of ILP-I, \(x_{v,1} + x_{v,2} \leq 1\) for \(v \in V\). After that, we can deduce that in an optimal solution of ILP-I, \(f\) is a \(\gamma_{mR}\)-function on \(G\).

**Lemma 3.1.** In an optimal solution of ILP-I on the graph \(G = (V, E)\), we have \(x_{v,1} + x_{v,2} \leq 1\) for \(v \in V\).

**Proof.** Let the optimal solution for ILP-I be represented by an \(n \times 3\) matrix \((X_{v,1}, X_{v,2}, Z_v)\) in which \(X_{v,i}\) is an \(n \times 1\) vector \((x_{v,1,i}, x_{v,2,i}, \ldots, x_{v,n,i})\) for \(i \in \{1,2\}\) and \(Z_v\) is an \(n \times 1\) vector \((z_1, z_2, \ldots, z_n)\). We will now prove by reduction to absurdity.

Assume there exists some \(v_j \in V\) such that \(x_{v_j,1} = x_{v_j,2} = 1\) in \((X_{v,1}, X_{v,2}, Z_v)\). Define \(V_3 = \{v_j \in V : x_{v_j,1} = x_{v_j,2} = 1\}\). Then let \(x'_{v_j,1} = \begin{cases} 0, & \text{if } v_k \in V_3, \\ x_{v_k,1}, & \text{otherwise}, \end{cases}\) and \(X'_{v,1}\) be an \(n \times 1\) vector \((x'_{v,1,1}, x'_{v,2,1}, \ldots, x'_{v,n,1})\). Next we will prove \((X'_{v,1}, X_{v,2}, Z_v)\) is a feasible solution for ILP-I with less objective function value than the solution \((X_{v,1}, X_{v,2}, Z_v)\).

Clearly, when \(v_k \in V_3\), we have \(x'_{v_k,1} = 0, x_{v_k,2} = 1\) and \(x'_{v_k,1} + x_{v_k,2} < x_{v_k,1} + x_{v_k,2}\). It can be observed that (2)–(5) will be satisfied with \(x'_{v,1}, x_{v,2}\) and \(z_k\). Therefore, \((X'_{v,1}, X_{v,2}, Z_v)\) is a feasible solution for ILP-I. Since \(\sum_{v_k \in V} x'_{v_k,1} < \sum_{v_k \in V} x_{v_k,1}\), we can deduce that \((X'_{v,1}, X_{v,2}, Z_v)\) has less objective function value than the solution \((X_{v,1}, X_{v,2}, Z_v)\). A contradiction to the assumption. Therefore, we can conclude that \(x_{v,1} + x_{v,2} \leq 1\) for \(v_j \in V\) in an optimal solution of ILP-I on the graph \(G\). \(\square\)

### 4. Complexity results

Ahangar *et al.* [12] had found the Maximal Roman Domination Problem (MRD) is NP-complete even restricted to bipartite graphs and planar graphs. In this section, we will further investigate the NP-completeness of MRD in star convex bipartite graphs and chordal bipartite graphs. The definitions of the problems we needed are as follows.

**Maximal Roman Domination Problem (MRD)**

**Instance:** A graph \(G = (V, E)\) and a positive integer \(k \leq |V|\).

**Question:** Does \(G\) have an MRDF \(f\) of weight \(f(G) \leq k\)?
Exact-3-Cover (X3C)

Instance: A set $X$ with $3q$ elements and a collection $C$ of 3-element subsets of $X$.

Question: Does there exist a subcollection $C' \subseteq C$ with each element of $X$ occurring exactly once in $C'$?

Dominating Set Problem (DS)

Instance: A graph $G = (V, E)$ and a positive integer $k$.

Question: Does there exist a dominating set $D \subseteq V$ with $|D| \leq k$?

Roman Domination Problem (RDP)

Instance: A graph $G = (V, E)$ and a positive integer $k \leq |V|$.

Question: Does $G$ have an RDF $f$ of weight $f(G) \leq k$?

4.1. MRD in star convex bipartite graphs

In this subsection, we will establish that MRD is NP-complete for star convex bipartite graphs by using a polynomial time reduction from the well-known NP-complete problem Exact-3-Cover (X3C) [21]. To begin with, we will give a polynomial time construction to generate an instance $< G, k >$ of MRD from an arbitrary instance $< X, C >$ of X3C, as is follow.

Given an arbitrary instance $< X, C >$ of X3C with $X = \{x_1, x_2, x_3, \ldots, x_{3q}\}$ and $C = \{C_1, C_2, C_3, \ldots, C_t\}$, we construct a vertex $x_i$ for each element $x_i \in X$ and a vertex $y_j$ for each element $C_j \in C$. Besides, for each element $x_i \in X$ if $x_i \in C_j$ in $< X, C >$, we add an edge $x_iy_j$ connecting the vertex $x_i$ to $y_j$. Next, for every vertex vertices $x_i$ with $i \in \{1, 2, \ldots, 3q\}$, we add two additional vertices $x_i^1$ and $x_i^2$, along with edges $x_ix_i^1, x_ix_i^2$. After that we add a star $S_{1,4} = \{x, a_1, a_2, a_3, a_4\}$ central at $x$ and we add edges to connect each $x_i^1, x_i^2$, and $y_j$ to $x$, respectively, where $i \in \{1, 2, \ldots, 3q\}$ and $j \in \{1, 2, \ldots, t\}$. Let $G_X = (V, E)$ be the graph generated from an arbitrary instance $< X, C >$ of X3C by using the construction above, and it’s obvious that $G_X$ is a star convex bipartite graph. For illustration we denote that the vertex set $X' = \{x_1, x_2, \ldots, x_{3q}\}$ and $Y = \{y_1, y_2, \ldots, y_t\}$. We also denote $H = \{a_1, a_2, a_3, a_4\}$ and $X_{ii} = \{x_i^1 \mid 1 \leq i \leq 3q\} \cup \{x_i^2 \in V \mid 1 \leq i \leq 3q\}$. An example of a generated graph $G_X$ is provided in Figure 1.

Lemma 4.1. Let $f$ be an MRDF of a graph $G_X = (V, E)$ generated from an arbitrary instance of X3C with $f(G_X) \leq 2q + 3$, it follows that $f(x) = 2$.

Proof. Suppose $f(x) = 0$ or 1 when $f(G_X) \leq 2q + 3$. Since $f$ is an MRDF, we have $f(v) \neq 2$ for all $v \in X'$, or $f(v) \geq 2$ for some $v \in X'$ such that $f(u) \geq 1$ for $u \in N(v) \cap X_{ii}$. Hence $f(x_i) + f(x_i^1) + f(x_i^2) \geq 2$ for $i \in \{1, 2, \ldots, 3q\}$. Then we can deduce that $f(G_X)$ is at least $6q$, which is a contradiction to $f(G_X) \leq 2q + 3$ as $q \geq 1$. Therefore, we can conclude that if $f(G_X) \leq 2q + 3$, then $f(x) = 2$. \qed

Theorem 4.2. MRD is NP-complete for star convex bipartite graphs.

Proof. The check for whether a function $f$ is an MRDF in a graph $G$ of weight $f(G) \leq k$ can be performed in polynomial time, hence MRD is a member of NP. Now let’s discuss the hardness of MRD.

If there exists an instance $< X, C >$ of X3C with a subcollection $C' \subseteq C$ such that each element of $X$ occurring exactly once in $C'$, a graph $G_X = (V, E)$ can be generated from this instance. Note $Y' \subseteq Y$ is the corresponding vertex set from $C'$. We can define a function $g = (V^0, V^1, V^2)$ on $G_X$ as follow:

$$g(v) = \begin{cases} 2, & \text{if } v \in \{x\} \cup \{y_j \mid y_j \in Y'\}, \\ 1, & \text{if } v = a_1, \\ 0, & \text{otherwise.} \end{cases}$$
Clearly, $g$ is an RDF of $G_X$ and the vertex $a_1$ is not dominated by $V_0^g$, so $g$ is an MRDF of $G_X$ of weight $g(G_X) = 3 + 2q$. Therefore, if there exists an instance $⟨X, C⟩$ of X3C with a subcollection $C' \subseteq C$ such that each element of $X$ occurs exactly once in $C'$, then $G_X$ have an MRDF $g$ with weight $g(G_X) \leq k = 3 + 2q$.

Conversely, assume there exists an MRDF $f = (V_0^f, V_1^f, V_2^f)$ of weight $f(G_X) \leq k = 2q + 3$ on the graph $G_X = (V, E)$ generated from an arbitrary instance of X3C. Let $r$ be the number of the vertex $x_i \in X'$ such that $f(x_i) = 0$, $s$ be the number of $y_j \in Y$ such that $f(y_j) = 2$, and $t$ be the number of $u \in X_{ii}$ such that $f(u) = 2$. We denote $f(X') = \sum_{v \in X'} f(v)$, $f(Y) = \sum_{v \in Y} f(v)$, $f(X_{ii}) = \sum_{v \in X_{ii}} f(v)$, and $f(H) = \sum_{v \in H} f(v)$. Then $f(G_X) = f(X') + f(Y) + f(X_{ii}) + f(x) + f(H)$. Please note that $M^f$ is the set of all vertices not dominated by $V_0^f$. The vertex $x$ must be dominated by $V_0^f$, otherwise $f(G_X) \geq 6q + 6 > 3 + 2q$, hence we consider the following four cases.

**Case 1.** $M^f \cap H \neq \emptyset$.

In this case we have $f(G_X) = f(X') + f(Y) + f(X_{ii}) + f(x) + f(H) \geq (3q - r) + 2s + 2t + 2 + 1$ where $f(G_X) \leq 2q + 3$ and $r \leq 3q$. Since the vertex assigned 0 in $X'$ is dominated by vertices from its neighbor, we have $3s + t \geq r$. There are two subcases.

**Subcase 1.1.** $3s \geq r$.

From $(3q - r) + 2s + 2t + 2 \leq 2q + 3$ and $3s \geq r$, we have $3q + 6t \leq r$. Since we also have $r \leq 3q$, we have $t \leq 0$. Apparently, we can deduce $t = 0$ and $r \geq 0$. Based on this, from $(3q - r) + 2s + 3 \leq 2q + 3$ and $3s \geq r$, we infer $\frac{r}{3} \leq s \leq \frac{q - 3q}{2}$, which also means $3q \leq r$. Since we also have $r \leq 3q$, we can get $r = 3q$.

From $2s + 3 \leq 2q + 3$ and $3s \geq r$, we deduce $s = q$. Therefore, the subcollection $C' \subseteq C$ corresponding
to the vertex set \( Y' = \{ y_j | f(y_j) = 2 \} \) is a 3-element subset of \( X \) such that each element of \( X \) occurs exactly once.

**Subcase 1.2.** \( 3s < r \).

From \((3q - r) + 2s + 2t + 3 \leq 2q + 3 \) and \( 3s + t \geq r \), we have \( r - 3s \leq t \leq \frac{r-2s-q}{2} \), i.e. \( \frac{r+q}{4} \leq s \). Furthermore, we can deduce \( 3q < s \) as \( 3s < r \), which is a contraction to \( r \leq 3q \).

**Case 2.** \( M^f \cap X_{ii} \neq \emptyset \).

Let \( u \) be a vertex in \( M^f \cap X_{ii} \). Then there are four subcases.

**Subcase 2.1.** \( f(u) = 1 \) and \( 3s \geq r \).

In this subcase we have \( f(G_X) \geq (3q - r) + 2s + (2t + 1) + 2 \) where \( f(G_X) \leq 2q + 3 \) and \( r \leq 3q - 1 \). From \((3q - r) + 2s + (2t + 1) + 2 \leq 2q + 3 \) and \( 3s \geq r \), we infer \( 3q + 6t \leq r \). Since we also have \( r \leq 3q - 1 \), we can deduce \( t \leq \frac{1}{6} \), a contradiction to \( t \geq 0 \).

**Subcase 2.2.** \( f(u) = 1 \) and \( 3s < r \).

In this subcase we also have \((3q - r) + 2s + (2t + 1) + 2 \leq 2q + 3 \) and \( r \leq 3q - 1 \). Similar to Case 1, we have \( 3s + t \geq r \). From \((3q - r) + 2s + (2t + 1) + 2 \leq 2q + 3 \) and \( 3s \geq r \), we infer \( r - 3s \leq \frac{r-2s-q}{2} \), i.e. \( \frac{r+q}{4} \leq s \). Since we also have \( 3s < r \), we can deduce \( 3q < r \), a contradiction to \( r \leq 3q - 1 \).

**Subcase 2.3.** \( f(u) = 2 \) and \( 3s \geq r \).

In this subcase we have \( f(G_X) \geq (3q - r) + 2s + 2t + 2 \) where \( f(G_X) \leq 2q + 3 \) and \( t \geq 1 \) as \( f(u) = 2 \). From \((3q - r) + 2s + 2t + 2 \leq 2q + 3 \) and \( 3s \geq r \), we infer \( 3q + 6t - 3 \leq r \). Since we also have \( r \leq 3q - 1 \), we can deduce \( t \leq \frac{1}{6} \), a contradiction to \( t \geq 1 \).

**Subcase 2.4.** \( f(u) = 2 \) and \( 3s < r \).

In this subcase because the vertex assigned 0 in \( X' \) is dominated by vertices from its neighbor and \( f(u) = 2 \), we have \( 3s + (t - 1) \geq r \) and \( t \geq 1 \). Therefore, \( r + 1 - t \leq 3s < r \), i.e. \( 1 < t \), a contradiction to \( t \geq 1 \).

**Case 3.** \( M^f \cap X' \neq \emptyset \).

In this case we need to consider that \( f(u) = 1 \) or \( 2 \) for \( u \in N(v) \cap X_{ii} \) where \( v \in M^f \cap X' \). We can have the same inequality formulas as Case 2. Hence, the proof of this case is similar to the proof of Case 2, and we can finally conclude that \( M^f \cap X' = \emptyset \).

**Case 4.** \( M^f \cap Y \neq \emptyset \).

Let \( u \) be a vertex in \( M^f \cap Y \). Then there are four subcases.

**Subcase 4.1.** \( f(u) = 1 \) and \( 3s \geq r \).

In this subcase we have \( f(G_X) \geq (3q - r) + (2s + 1) + 2t + 2 \) where \( f(G_X) \leq 2q + 3 \) and \( r \leq 3q - 3 \). Similar to Subcase 2.1, we can deduce \( t \leq \frac{1}{2} \), a contradiction to \( t \geq 0 \).

**Subcase 4.2.** \( f(u) = 1 \) and \( 3s < r \).

In this subcase we also have \((3q - r) + (2s + 1) + 2t + 2 \leq 2q + 3 \) and \( r \leq 3q - 3 \). Similar to Subcase 2.2, we can deduce \( 3q < r \), a contradiction to \( r \leq 3q - 3 \).

**Subcase 4.3.** \( f(u) = 2 \) and \( 3s \geq r \).

In this subcase we have \( f(G_X) \geq (3q - r) + 2s + 2t + 2 \) where \( f(G_X) \leq 2q + 3 \) and \( t \geq 0 \). Because the vertex assigned 0 in \( X' \) is dominated by vertices from its neighbor and \( f(u) = 2 \), we have \( 3(s - 1) + t \geq r \) and \( s \geq 1 \). Similar to Subcase 2.3, we can deduce \( t \leq 0 \). Hence \( t = 0 \). Based on this, we have \( 3q + 3 \leq r \) from \( 3(s - 1) + 0 \geq r \) and \( (3q - r) + 2s + 0 + 2 \leq 2q + 3 \), a contradiction to \( r \leq 3q - 3 \).

**Subcase 4.4.** \( f(u) = 2 \) and \( 3s < r \).

In this subcase, we also have \((3q - r) + 2s + 2t + 2 \leq 2q + 3 \) and \( 3(s - 1) + t \geq r \). Therefore, \( r - 3s + 3 \leq t \leq \frac{r+1-2s-q}{2} \), furthermore, we can deduce \( \frac{r+q}{4} \leq s \). Since we also have \( 3s < r \), we can deduce \( 3q + 15 < r \), a contradiction to \( r \leq 3q - 3 \).

Since MRD is NP-hard and NP in star convex bipartite graphs, the proof is complete. \( \square \)
4.2. MRD in chordal bipartite graphs

In this subsection, we will use a polynomial time reduction from Dominating Set Problem (DS) to Roman Domination Problem (RDP), proving that RDP is NP-complete for chordal bipartite graphs. Subsequently, we will employ a polynomial time reduction from RDP to MRD in order to prove the NP-completeness of MRD for chordal bipartite graphs. It is noteworthy that Müller et al. had already proved that DS is NP-complete for chordal bipartite graphs [22].

Figure 2. The construction of the graph $G'$ from $G$ in chordal bipartite graph.

Claim 4.3. RDP is NP-complete for chordal bipartite graphs.

Proof. Let $G = (V, E)$ be a graph with $n = |V|$. For each vertex $x_i \in V$, we add a new vertex $y_i$ and connect them with an edge $x_i y_i$. By this way we can get a graph $G'$ in polynomial time and if $G$ is a chordal bipartite graph so is $G'$. See Figure 2. We denote $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$. The verification of whether a function $f$ is an RDF in a graph $G$ of weight no more than $k$ can be accomplished in polynomial time, hence RDP is a member of NP. Now we delve into the hardness of RDP.

If there exists a dominating set $D$ on $G$ such that $|D| \leq k$, we can define a function $g = (V_0^g, V_1^g, V_2^g)$ on $G'$ as follow:

$$g(v) = \begin{cases} 2, & \text{if } v \in D, \\ 1, & \text{if } v = y_i \in Y \text{ such that } N(y_i) \cap D = \emptyset, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, $g$ is an RDF on $G'$ such that $g(G') \leq 2k + (n - k) = k + n$.

Conversely, let $f = (V_0^f, V_1^f, V_2^f)$ be an RDF on $G'$ such that $f(G') \leq k + n$. Let $y_i$ be a vertex in $Y$ and $x_i$ be the corresponding vertex in $N(y_i) \cap X$, then there are three cases we need to discuss. Firstly, if $f(y_i) = 2$, we exchange the function values between $x_i$ and $y_i$ then we can get an RDF on $G'$ with the same weight. Secondly, if $f(y_i) = 1$ and $f(x_i) = 1$, we can reassign $x_i$ and $y_i$ with 2 and 0, respectively, by which we can get an RDF on $G'$ with the same weight. Thirdly, if $f(y_i) = 1$ and $f(x_i) = 0$, there must exist some vertices $v \in N(x_i) \cap X$ such that $f(v) = 2$. Therefore, without loss of generality, we can regard $f(x_i) \neq 1$ for all $x_i \in X$ and we can deduce that $V_2^f \cap X$ is a dominating set of $G$.

Suppose there are $m$ vertices $y_i \in Y$ such that $g(y_i) \geq 1$, then there are $n - m$ vertices $y_i \in Y$ assigned 0, i.e. there are at least $n - m$ vertices $x_i \in X$ assigned 2. Let $D = V_2^f \cap X$ and $f(D) = \sum_{v \in D} g(v)$, then we have
Theorem 4.4. MRD is NP-complete for chordal bipartite graphs.

Proof. Let $G = (V,E)$ be a graph with $n = |V|$. For every vertex $x_i \in V$, we add a star $K_{1,3} = \{a_i, b_i, c_i, d_i\}$ central at $b_i$, and add an edge $x_ia_i$, then we can get a graph $G'$ from $G$, which can be constructed in polynomial time. See Figure 3. If $G$ is a chordal bipartite graph, so is $G'$. Clearly, MRD is a member of NP. Now we discuss the hardness of MRD.

If there exists an RDF $f = (V_0^f, V_1^f, V_2^f)$ on $G$ such that $f(G) \leq k$, we can define a function $f' = (V_0^{f'}, V_1^{f'}, V_2^{f'})$ of $G'$ as follow:

$$f'(v) = \begin{cases} f(v), & \text{if } v \in V(G), \\ 2, & \text{if } v = b_i \text{ with } i \in \{0 \leq i \leq n\}, \\ 1, & \text{if } v = c_2, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, $f'$ is an RDF of $G'$ and the vertex $c_2$ is not dominated by $V_0^{f'}$, so $f'$ is an MRDF on $G'$ such that $f'(G') \leq k + 2n + 1$.

Conversely, let $g = (V_0^g, V_1^g, V_2^g)$ be an MRDF on $G'$ such that $g(G') \leq k + 2n + 1$. Apparently, for each $i \in \{1,2,\ldots,n\}$, $g(d_i) + g(c_i) + g(b_i) \geq 2$ since $d_i$ and $c_i$ need to dominate themselves or to be dominated. We also have $\sum_{i=1}^{n} g(d_i) + g(c_i) + g(b_i) \leq 3n$. Therefore, without loss of generality, we regard $g(b_i) = 2$ for all $b_i \in \{b_1, b_2, \ldots, b_n\}$ and $g(d_i) = 0$ for all $d_i \in \{d_1, d_2, \ldots, d_n\}$.

Now, we consider three cases. Firstly, if $g(a_i) = 1$, we can reassign $d_i$ and $a_i$ with 1 and 0, respectively, by which we can get an MRDF on $G'$ with the same weight. Secondly, if $g(a_i) = 2$ and $g(x_i) = 0$, we can reassign $x_i$, $a_i$ and $d_i$ with 1, 0 and 1, respectively, by which we can get an MRDF on $G'$ with the same weight. Thirdly, if $g(a_i) = 2$ and $g(x_i) = 1$, we can reassign $x_i$, $a_i$ and $d_i$ with 2, 0 and 1, respectively, by which we can also get an MRDF on $G'$ with the same weight. So, without loss of generality, we deduce that $g(a_i) = 0$ for all $a_i \in \{a_1, a_2, \ldots, a_n\}$.

\[ \frac{f(D)}{2} \geq n - m. \] Hence, if there exists an RDF on $G'$ of weight $f(G') \leq k + n$, then the set $D$ forms a dominating set of $G$ such that $|D| = f(G') - \sum_{v \in Y} g(v) - \frac{f(D)}{2} \leq f(G') - m - (n - m) \leq k$.

Since RDP is NP-hard and NP in chordal bipartite graphs, the proof is complete. \[ \square \]
As all vertices in $V(G)$ is connected to $a_i$ such that $g(a_i) = 0$, there exists at least one vertex $u \in \{c_i|0 \leq i \leq n\} \cup \{d_i|0 \leq i \leq n\}$ not dominated by $V_0$. Therefore, $g$ restricted to $G$ is an RDF on $G$ has a weight of $g(G) = g(G') - \sum_{i=0}^{n} g(x_i) - f(u) \leq k + 2n + 1 - 2n - 1 = k$.

Since MRD is NP-hard and NP in chordal bipartite graphs, the proof is complete. \hfill $\square$

5. Linear time algorithms

For any disconnected special graph $G$ with connected components $\{G_1, G_2, \ldots, G_n\}$ such that $G = G_1 \cup G_2 \cup \cdots \cup G_n$, we can compute $\gamma_{MR}(G_1)$ and $\gamma(R(G_i))$ for each component $G_i$, respectively. Then based on the definition of maximal Roman domination and Observation 2.1, we have following equation:

$$\gamma_{mR}(G) = \min_{i \in \{1, 2, \ldots, n\}} \left\{ \gamma_{mR}(G_i) + \sum_{j \in \{1, 2, \ldots, n\} \setminus \{i\}} \gamma(R(G_j)) \right\}.$$ 

Assume both $\gamma_{mR}(G_i)$ and $\gamma(R(G_i))$ can be computed in linear time. Note $A = \sum_{j \in \{1, 2, \ldots, n\}} \gamma(R(G_j))$, then $A$ can be computed in linear time first, and we have following equation, which also means $\gamma_{mR}(G)$ can be computed in linear time.

$$\gamma_{mR}(G) = \min_{i \in \{1, 2, \ldots, n\}} \left\{ \gamma_{mR}(G_i) + (A - \gamma(R(G_i))) \right\}.$$ 

Worth mentioned that there are some useful results show that the Roman domination number $\gamma(R(G))$ can be computed in linear time for threshold graphs [23], bounded tree-width graphs [23, 24] and block graphs [25]. Therefore, in this section we only consider maximal Roman domination in connected graphs.

5.1. Maximal Roman domination in threshold graphs

There are several equivalent definitions for threshold graphs, and in this context, we adopt the characterization presented in [23, 26].

**Definition 5.1.** A graph $G = (V, E)$ is a threshold graph if and only if it is a split graph having a split partition $(C, I)$ of $V$, where $C$ is a clique with a vertex order $(x_1, x_2, \ldots, x_n)$ such that $N_G[x_1] \subseteq N_G[x_2] \subseteq \cdots \subseteq N_G[x_n]$, and $I$ is an independent set with a vertex order $(y_1, y_2, \ldots, y_m)$ such that $N_G[y_1] \supseteq N_G[y_2] \supseteq \cdots \supseteq N_G[y_m]$. Therefore, we denote a threshold graph by $G = (C, I, E)$. An illustration of threshold graphs is provided in Figure 4.

**Observation 5.2.** Let $G = (C, I, E)$ be a connected threshold graph where $C$ has a vertex order $(x_1, x_2, \ldots, x_n)$ such that $N_G[x_1] \subseteq N_G[x_2] \subseteq \cdots \subseteq N_G[x_n]$, then the vertex $x_n$ is adjacent to all other vertices in $C \cup I$.

Apparantly, trivial graphs and complete graphs are a member of threshold graphs. For a threshold graph $G = (\{v\}, \emptyset)$, we have $\gamma_{mR}(G) = 1$. Now we can focus on the connected threshold graphs with more than one vertex.

**Theorem 5.3.** Let $G = (C, I, E)$ be a connected threshold graph with $|C \cup I| \geq 2$ and minimum degree $\delta$, then

$$\gamma_{mR}(G) = \begin{cases} 1 + \delta, & \text{if } G \text{ is a complete graph,} \\ 2 + \delta, & \text{if } G \text{ is not a complete graph.} \end{cases}$$

**Proof.** Suppose $C$ has a vertex order $(x_1, x_2, \ldots, x_n)$ such that $N_G[x_1] \subseteq N_G[x_2] \subseteq \cdots \subseteq N_G[x_n]$, and $I$ has a vertex order $(y_1, y_2, \ldots, y_m)$ such that $N_G[y_1] \supseteq N_G[y_2] \supseteq \cdots \supseteq N_G[y_m]$. Let $f = (V_0, V_1, V_2)$ be a $\gamma_{mR}$-function on $G$ and $V = C \cup I$. 
Case 1. $G$ is a complete graph.

If $G$ is a complete graph, i.e. $|I| = 1$, then we have $\delta = \Delta$. For the upper bound, assigning 1 to all vertices $v \in V$ yields an MRDF on $G$, so $\gamma_{mR}(G) \leq |V| = 1 + \delta$. For the lower bound, there is no vertex in $V$ assigned 0 as $G$ is a complete graph, otherwise all $v \in V$ will be dominated by $V_0$, so $\gamma_{mR}(G) \geq |V| = 1 + \delta$. Hence $\gamma_{mR}(G) = 1 + \delta$ when $G$ is a complete graph.

Case 2. $G$ is not a complete graph.

If $G$ is not a complete graph, for the upper bound, we can assign 2 to the vertex $x_n$. After that, for a vertex $v \in V$ such that $\deg(v) = \delta$, we assign 1 to all vertices $w \in N[v]$, otherwise assign 0. Obviously, $v$ is not dominated by $V_0$ and since Observation 5.2, $v$ is adjacent to $x_n$, which means we can get an MRDF of $G$. Hence, $\gamma_{mR}(G) \leq 2 + 1 + (\delta - 1) = 2 + \delta$. For the lower bound, let $u$ be one of the vertices not dominated by $V_0$, then we need to consider two cases divided by the location of $u$.

Subcase 2.1. $u \in C$.

Without loss of generality, we assume that $x_i = u$. Please note that $f(G) = f(C) + f(I)$, in which $f(C)$ ($f(I)$, respectively) is the weight of $f$ restricted to $C$ ($I$, respectively). As $C$ is a clique, $f(v) \geq 1$ for all $v \in C$, then $f(C) \geq |C|$. Firstly, we consider there exists some $v \in I$ such that $f(v) = 0$, then $f(C) \geq |C| + 1$, because at least one vertex in $C$ needs to be assigned 2 in order to dominate $v$. Besides, if $N(x_i) \cap I = \emptyset$ then we have $|C| \geq \delta + 1$, hence $f(G) = f(C) + f(I) \geq (\delta + 1) + 1 + 0 = \delta + 2$. If $N(x_i) \cap I \neq \emptyset$, we have $f(w) \geq 1$ for all $w \in N(x_i) \cap I$, hence $f(G) = f(C) + f(I) \geq |C| + 1 + 1 \geq \delta + 2$. Secondly, we consider $f(v) \geq 1$ for all $v \in I$. When $|I| = 1$, we note $I = \{y_1\}$ then we have $C \setminus N(y_1) \neq \emptyset$ as $G$ is not a complete graph. One can see the degree of the vertex $y_1$ is $\delta$ and $|C| \geq \delta + 1$. Hence $f(G) = f(C) + f(I) \geq |C| + 1 \geq (\delta + 1) + 1 = \delta + 2$. When $|I| \geq 2$, there are at least $|C| + 2$ vertices assigned equal or over 1, then $f(G) \geq |C| + 2 \geq \delta + 2$.

Subcase 2.2. $u \in I$.

Without loss of generality, we assume that $y_j = u$. Let $f(N[y_j]) = \sum_{v \in N[y_j]} f(v)$ and $f(V - N[y_j]) = \sum_{v \in V \setminus N[y_j]} f(v)$. By the definition of MRDF, we have $f(N[y_j]) \geq \delta + 1$. As $G$ is not a complete graph, then $V \setminus N[y_j] \neq \emptyset$. We have $f(G) = f(N[y_j]) + f(V - N[y_j])$. Firstly, we consider there exists some $v \in V \setminus N[y_j]$ such that $f(v) = 0$. There must exist a vertex in $V$ assigned 2. Hence $f(V - N[y_j]) \geq 2$ while $f(N[y_j]) \geq \delta + 1$, or $f(N[y_j]) \geq (\delta + 1) + 1$ while $f(V - N[y_j]) \geq 0$, which means $f(G) \geq \delta + 2$. Secondly, we consider $f(v) \geq 1$ for all $v \in V \setminus N[y_j]$. There exists at least one $w \in V \setminus N[y_j]$ such that $f(w) \geq 1$ as $V \setminus N[y_j] \neq \emptyset$, then $f(G) \geq (\delta + 1) + 1 = \delta + 2$.

Considering the upper bounds and lower bounds above, the proof is complete. \qed
According to Theorem 5.3 and Observation 5.2, we can devise a linear time algorithm based on Breadth First Search (BFS) to efficiently compute the $\gamma_{mR}(G)$ of a connected threshold graph $G = (V, E)$ and we have the following theorem.

**Theorem 5.4.** Maximal Roman domination number $\gamma_{mR}(G)$ of threshold graphs can be computed in linear time.

### 5.2. Maximal Roman domination in block graphs

**Definition 5.5** ([27]). For a graph $G$, a cut vertex is a vertex that the connected components of $G$ will increase with whose removal, and a maximal connected subgraph of $G$ is called a block of $G$ if it has no cut vertex in it. A graph $G$ is a block graph if and only if all blocks in $G$ are cliques and the intersection of two blocks is either empty or a cut vertex.

**Definition 5.6.** Let $G = (V, E)$ be a block graph with blocks $B_G^1, B_G^2, \ldots, B_G^k$ and cut vertices $c_1, c_2, \ldots, c_s$. Denote $B = \{B_G^1, B_G^2, \ldots, B_G^k\}$ and $C = \{c_1, c_2, \ldots, c_s\}$. Define $B_i$ as a block vertex relating to the set $V(B_G^i) \setminus \{c_j \mid c_j \in V(B_G^i)\}$. The cut tree of $G$, denoted by $T_G = (B' \cup C', E')$, is a tree with the vertex set $B' \cup C'$ and the edge set $E'$ where $B' = \{B_1, B_2, \ldots, B_k\}$, $C' = \{c_1, c_2, \ldots, c_s\}$ and $E' = \{B_i c_j \mid c_j \in V(B_G^i), 1 \leq i \leq k, 1 \leq j \leq s\}$. An example is given in Figure 5.

![Figure 5](image_url)

**Figure 5.** An example of a block graph (a) and its relevant cut tree (b).

It is noteworthy that in a block graph $G$, a block can be a path, *i.e.* $P_2$. If all the blocks in the block graph $G$ are $P_2$, then $G$ is a tree. Consequently, the class of trees is a subset of block graphs, and if we can establish that the maximal Roman domination number in block graphs can be computed in linear time, the same holds true for trees. Now we propose a linear time algorithm for computing the $\gamma_{mR}(G)$ in connected block graphs based on dynamic programming, and we use its cut tree as the input. The structure of the cut tree for a block graph $G = (V, E)$ can be constructed in $O(|V| + |E|)$ time by depth first search [28]. Without loss of generality, we assume the cut trees are rooted at a specific cut vertex in this paper [29]. To start, we give the following definition:
Definition 5.7. Let $G = (V, E)$ be a graph rooted at $v$, and $S(G)$ be the set of all functions $g : V(G) \to \{0, 1, 2\}$. Suppose $f = (V_0^f, V_1^f, V_2^f) \in S(G)$ is a function on $G$, and $f_{G-v}$ be the restriction of $f$ on $G-v$. We define:

\[
\begin{align*}
\gamma^{0,0}(v, G) &= \min \{f(v) \mid f \text{ is an RDF but not an MRDF on } G \text{ such that } f(v) = 0\}, \\
\gamma^{0,2}(v, G) &= \min \{f(v) \mid f_{G-v} \text{ is an MRDF on } G-v \text{ such that } f(v) = 0 \text{ and } f(u) = 2 \text{ for some } u \in N(v)\}, \\
\gamma^{1,0}(v, G) &= \min \{f(v) \mid f \text{ is an RDF but not an MRDF on } G \text{ such that } f(v) = 1\}, \\
\gamma^{1,1}(v, G) &= \min \{f(v) \mid f_{G-v} \text{ is an RDF but not an MRDF on } G-v \text{ such that } f(v) = 1 \text{ and } f(u) \geq 1 \text{ for all } u \in N(v)\}, \\
\gamma^{1,2}(v, G) &= \min \{f(v) \mid f_{G-v} \text{ is an MRDF on } G-v \text{ such that } f(v) = 1\}, \\
\gamma^{2,0}(v, G) &= \min \{f(v) \mid f \text{ is an RDF but not an MRDF on } G \text{ such that } f(v) = 2\}, \\
\gamma^{2,1}(v, G) &= \min \{f(v) \mid f_{G-v} \text{ is an RDF but not an MRDF on } G-v \text{ such that } f(v) = 2 \text{ and } f(u) \geq 1 \text{ for all } u \in N(v)\}, \\
\gamma^{2,2}(v, G) &= \min \{f(v) \mid f_{G-v} \text{ is an MRDF on } G-v \text{ such that } f(v) = 2\}, \\
\gamma^{0,-0}(v, G) &= \min \{f(v) \mid f_{G-v} \text{ is an RDF but not an MRDF on } G-v \text{ such that } f(v) = 0 \text{ and } f(u) \leq 1 \text{ for all } u \in N(v)\}, \\
\gamma^{0,-2}(v, G) &= \min \{f(v) \mid f_{G-v} \text{ is an MRDF on } G-v \text{ such that } f(v) = 0 \text{ and } f(u) \leq 1 \text{ for all } u \in N(v)\}.
\end{align*}
\]

Let $\mathcal{A} = \{(0, 0), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 1), (2, 2), (0^-, 0), (0^-, 2)\}$ and $f = (V_0^f, V_1^f, V_2^f) \in S(G)$ be a function on a graph $G$ rooted at $v$. Apparently, for $\gamma^{i,j}(v, G)$ where $(i, j) \in \mathcal{A}$, $i$ relates to the function value of $v$, $j$ relates to the property that the set $V_0^f$ is not a dominating set of $G$. Worth mentioned that for $i = 0^-$ or $0^+$, the function value of $v$ is $f(v) = 0$. When $j = 0$, it means all $u \in V(G)$ are dominated by $V_0^f$. When $j = 1$, it means all $u \in V(G)$ are dominated by $V_0^f$ except the vertex $v$. When $j = 2$, it means there are some $u \in V(G) \setminus \{v\}$ not dominated by $V_0^f$. Besides, when $f(v) = 0$, we have $j \neq 1$. Hence, there doesn’t exist $\gamma^{0,1}(v, G)$ or $\gamma^{0,-1}(v, G)$. According to Definition 5.7, we have the following observation and claim.

Observation 5.8. Let $G$ be a connected block graph rooted at a vertex $v$ then

\[\gamma_{mR}(v, G) = \min \{\gamma^{0,2}(v, G), \gamma^{1,1}(v, G), \gamma^{1,2}(v, G), \gamma^{2,1}(v, G), \gamma^{2,2}(v, G)\}.\]

Claim 5.9. For a trivial graph $G = (\{v\}, \emptyset)$, we have:

\[
\begin{align*}
\gamma^{0,0}(v, G) &= \infty, \quad \gamma^{0,2}(v, G) = \infty, \quad \gamma^{1,0}(v, G) = \infty, \quad \gamma^{1,1}(v, G) = 1, \quad \gamma^{2,0}(v, G) = \infty, \quad \gamma^{2,1}(v, G) = 2, \\
\gamma^{2,2}(v, G) &= \infty, \quad \gamma^{0,-0}(v, G) = 0, \quad \gamma^{0,-2}(v, G) = \infty.
\end{align*}
\]

Let $G$ be a connected block graph rooted at a specific vertex $v_1$. Suppose $v_1$ belongs to a block $B^G_i$ where $V(B^G_i) = \{v_1, v_2, \ldots, v_m\}$, then the block graph $G$ can be constructed out of the disjoint union $G_1 \cup G_2 \cup \ldots \cup G_m$ by adding edges to make the vertices $v_1, v_2, \ldots, v_m$ form the block $B^G_i$, shown in Figure 6, in which $G_1, G_2, \ldots, G_m$ are the connected induced subgraphs of $G$ rooted at $v_1, v_2, \ldots, v_m$, respectively. Similarly, the nontrivial graphs in $\{G_1 \cup G_2 \cup \ldots \cup G_m\}$ are also block graphs, which also can be constructed from a sequence of their connected induced subgraphs. Converse to the construction above, we can decompose $G$ into a sequence of connected induced subgraphs recursively until all connected induced subgraphs become trivial graphs. Now let’s define these connected induced subgraphs during the recursion. Apparently, among these connected induced subgraphs only the nontrivial ones rooted at cut vertices need to be defined and we call them recursive block subgraphs.

Definition 5.10. Let $T_G = (B' \cup C', E')$ be a given cut tree rooted at a specific vertex. We define a vertex order $O_L$ for the vertices in $B' \cup C'$ with the following property: for any two vertices $u, v$ in $B' \cup C'$, if the depth
of $u$ is deeper than $v$ at $T_G$, then $O_L(u) < O_L(v)$, where $O_L(w)$ is the order for $w \in B' \cup C'$. Let $v_i \in B'$, then we can get its parent vertex $v_j$ from $T_G$. We denote $LB(v_i)$ as the vertex set $\{v \in B' \cup C' | O_L(v) \leq O_L(v_i)\}$, and we denote $TC(v_j)$ as the set of all descendants of the cut vertex $v_j$ in $T_G$. Then we can get an specific induced subtree $T_G[LB(v_i) \cap TC(v_j) \cup \{v_j\}]$ from $T_G$. Based on $O_L$, we can get a specific induced subtree order $T_L = (T_1, T_2, \ldots, T_r)$ where $r = |B'|$. We define a concept called recursive block subgraph (abbr. rbsG) that an rbsG $G_k$ is an induced subgraph of $G$ rooted at $c_v$ which is related to the specific induced subtree $T_k$ in $T_L$ and $c_v$ is the root of $T_k$. An illustration of the example in Figure 5 is provided in Figure 7.

**Lemma 5.11.** Let a block graph $G = (V, E)$ rooted at $v_1$ be constructed out of the disjoint union $H_1 \cup H_2 \cup \ldots \cup H_m$ by adding edges making the vertices $v_1, v_2, \ldots, v_m$ a clique, where $H_1, H_2, \ldots, H_m$ are the connected induced subgraphs rooted at $v_1, v_2, \ldots, v_m$ respectively. Suppose all $\gamma_{i,j}(v_k, H_k)$ for $k \in \{1, 2, \ldots, m\}$ are already determined by the graph $H_k$. Let

\[
\begin{align*}
A &= \{(0, 0), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 1), (2, 2), (0^-, 0), (0^-, 2)\}, \\
A_0 &= \{(0, 0), (0, 2)\}, \\
A_1 &= \{(1, 0), (1, 1), (1, 2)\}, \\
A_2 &= \{(2, 0), (2, 1), (2, 2)\}, \\
A_{0^-} &= \{(0^-, 0), (0^-, 2)\}, \\
B_2 &= \{(0, 2), (1, 2), (2, 2), (0^-, 2)\}, \\
m &= \{2, \ldots, m\}, \\
f(v_k, S) &= \min \{\gamma_{i,j}(v_k, H_k) : (i, j) \in S\},
\end{align*}
\]

then the following equations hold:

\[
(a_1) \quad \gamma_{0,0}^{0,0}(v_1, G) = \min_{k_1 \in [m]} \left\{ (1) \sum_{k_1 \in [m]} f(v_{k_1}, \{(0, 0), (1, 0), (1, 1)\}) + \gamma_{0,0}^{0,0}(v_1, H_1), \right. \\
(2) \quad \min_{k_2 \in [m]} \left\{ f(v_{k_2}, \{(2, 0), (2, 1)\}) + \sum_{k_3 \in [m] \setminus \{k_2\}} f(v_{k_3}, A \setminus B_2) \right\} + f(v_1, \{(0, 0), (0^-, 0)\}), \right\}
\]
Figure 7. The rbsGs $G_k$ of the example in Figure 5 where $O_L = (B_7, c_5, B_6, c_4, B_5, c_3, B_4, B_2, c_2, B, B_1, c_1)$. For example, since $LB(B_4) = \{B_7, c_5, B_6, c_4, B_5, c_3, B_4\}$ and $TC(c_2) = \{B_4, B_2, c_2\}$, we have $V(T_4) = LB(B_4) \cap TC(c_2) \cup \{c_2\} = \{B_4, c_2\}$, then we can get the rbsG $G_4$.
\[
\begin{align*}
(1) & \quad \gamma^{1,0}(v_1, H_1) + \sum_{k_1 \in [m]} f(v_{k_1}, \{(1,0), (2,0)\}), \\
(2) & \quad f(v_1, \{(1,0), (1,1)\})
\end{align*}
\]

\[
(\text{b1}) \quad \gamma^{1,0}(v_1, G) = \min \left\{ \begin{aligned}
& (2.1) \quad \min_{k_2 \in [m]} \left\{ \gamma^{0,0}(v_2, H_{k_2}) + \sum_{k_3 \in [m] \setminus \{k_2\}} f(v_{k_3}, \{(0,0), (1,0), (1,1)\}) \right\}, \\
& + \min_{k_4, k_5 \in [m]} \left\{ f(v_{k_4}, \{(2,0), (2,1)\}) + f(v_{k_5}, \{(0,0), (0^-, 0)\}) \right\}, \\
& + \sum_{k_6 \in [m] \setminus \{k_4, k_5\}} f(v_{k_6}, A \setminus B_2) \right\},
\end{aligned} \right.
\]

\[
(\text{b2}) \quad \gamma^{1,1}(v_1) = \gamma^{1,1}(v_1) + \sum_{k_1 \in [m]} f(v_{k_1}, \{(1,0), (2,0)\}),
\]

\[
(\text{b3}) \quad \gamma^{1,2}(v_1, G) = \min \left\{ \begin{aligned}
& (1.1) \quad \sum_{k_1 \in [m]} f(v_{k_1}, A_0 \cup A_1), \\
& (1.2) \quad \min_{k_2 \in [m]} \left\{ f(v_{k_2}, A_2) + \sum_{k_3 \in [m] \setminus \{k_2\}} f(v_{k_3}, A) \right\}, \\
& (2.1) \quad \min_{k_4 \in [m]} \left\{ f(v_{k_4}, \{(0,2), (1,2)\}) + \sum_{k_5 \in [m] \setminus \{k_4\}} f(v_{k_5}, A_0 \cup A_1) \right\}, \\
& (2.2) \quad \min_{k_6 \in [m]} \left\{ \gamma^{2,2}(v_6, H_{k_6}) + \sum_{k_7 \in [m] \setminus \{k_6\}} f(v_{k_7}, A) \right\}, \\
& (2.3) \quad \min_{k_8, k_9 \in [m]} \left\{ f(v_{k_8}, B_2 \setminus \{(2,2)\}) + f(v_{k_9}, \{(2,0), (2,1)\}) \right\}, \\
& + \sum_{k_{10} \in [m] \setminus \{k_8, k_9\}} f(v_{k_{10}}, A \setminus \{(2,2)\}) \right\}, \\
& (3) \quad f(v_1, \{(1,0), (1,1)\}) + \sum_{k_{11} \in [m]} f(v_{k_{11}}, \{(1,1), (2,1)\}) + \sum_{k_{12} \in [m] \setminus \{k_{11}\}} f(v_{k_{12}}, \{(1,0), (1,1), (2,0), (2,1)\}), \\
& (1.1), (2.0), (2.1))
\end{aligned} \right.
\]

\[
(1) \quad \gamma^{2,0}(v_1, H_1) + \sum_{k_1 \in [m]} f(v_{k_1}, \{(1,0), (2,0)\}),
\]

\[
(2) \quad f(v_1, \{(2,0), (2,1)\}) + \sum_{k_2 \in [m]} f(v_{k_2}, \{(0,0), (0^-, 0)\}) + \sum_{k_3 \in [m] \setminus \{k_2\}} f(v_{k_3}, A \setminus B_2),
\]

\[
(\text{c1}) \quad \gamma^{2,0}(v_1, G) = \min \left\{ \begin{aligned}
& (1) \quad \gamma^{2,0}(v_1, H_1) + \sum_{k_1 \in [m]} f(v_{k_1}, \{(1,0), (2,0)\}), \\
& (2) \quad f(v_1, \{(2,0), (2,1)\}) + \min_{k_2 \in [m]} \left\{ f(v_{k_2}, \{(0,0), (0^-, 0)\}) + \sum_{k_3 \in [m] \setminus \{k_2\}} f(v_{k_3}, A \setminus B_2) \right\},
\end{aligned} \right.
\]

\[
(\text{c2}) \quad \gamma^{2,1}(v_1, G) = \gamma^{2,1}(v_1, H_1) + \sum_{k_1 \in [m]} f(v_{k_1}, \{(1,0), (2,0)\}),
\]
When \( i = 2 \) for some \( k \), there are two cases. Depending on whether \( i_{H_k} = 0 \) or \( i_{H_k} = 1 \) for all \( k \in [m] \), there are also two subcases divided by whether \( i_{H_k} = 2 \) for some \( k \in [m] \). If not, then \( i_{H_1} = 0 \) and \( i_{H_2} = 0 \) or 1 for all \( k \in [m] \), which follows (3) of (a2). Otherwise, (4) of (a2) applies, with two subcases divided by whether \( i_{H_k} = j_{H_k} = 2 \) for some \( k \in [m] \), and if so we have (4.1), otherwise we have (4.2).

(b1).

By Definition 5.7 when \( \gamma^{i,j}(v_1, G) = \gamma^{1,0}(v_1, G) \), \( j_{H_k} \neq 2 \) for all \( k \in [1, 2, \ldots, m] \). Depending on whether \( i_{H_k} = 0 \) or \( i_{H_k} = 1 \) or \( 0 \) for some \( k \in [m] \), there are two cases. If not, then \( i_{H_k} = 1 \) or \( 2 \) for all \( k \in [m] \). Besides, \( j_{H_k} = 0 \) for all \( k \in [1, 2, \ldots, m] \) since \( i_{H_k} > 0 \), otherwise \( j_{H_k} = 2 \). Therefore, we can conclude that (1) of (b1) holds. If so, then \( (i_{H_k}, j_{H_k}) \in \{(1, 0), (1, 1)\} \) and there are two subcases divided by whether \( i_{H_k} = 2 \) for some \( k \in [m] \). When \( i_{H_k} = 2 \) for some \( k \in [m] \), then \( i_{H_k} \) can be 0 or \( 0^- \), which follows (2.2) of (b1). Otherwise, we have (2.1) of (b1).

Proof. Worth mentioned that in Algorithm 1, the \( H_k \) in \( \gamma^{i,j}(v_{c_1}, H_k) \) is either a trivial graph determined by Claim 5.9 or an rbsG determined by Definition 5.10. Let the pair \((i_{H_k}, j_{H_k})\) relate to \((i, j)\) of \( \gamma^{i,j}(v_k, H_k) \) for all \( H_k \in \{H_1, H_2, \ldots, H_m\} \).

(a1).

According to Definition 5.7 when \( \gamma^{i,j}(v_1, G) = \gamma^{0,0}(v_1, G) \), \( j_{H_k} \neq 2 \) for all \( k \in [1, 2, \ldots, m] \). Depending on whether \((i_{H_k}, j_{H_k}) \in \{(2, 0), (2, 1)\}\) for some \( k \in [m] \), there are two cases. If so, then \( i_{H_1} = 0^- \) or 0, which follows (2) of (a1). Otherwise, \( i_{H_1} = 0 \), which follows (1) of (a1).

(a2).

With Definition 5.7 when \( \gamma^{i,j}(v_1, G) = \gamma^{0,2}(v_1, G) \), two cases depend on whether \( j_{H_1} = 2 \).

Case 1. \( j_{H_1} = 2 \).

In this case we have \((i_{H_1}, j_{H_1}) \in \{(0, 2), (0^-, 2)\}\). Similar to the proof of (a1), there are two subcases divided by whether \( i_{H_k} = 2 \) for some \( k \in [m] \). If so, then \( i_{H_1} = 0^- \) or 0, following (2) of (a2). Otherwise, \( i_{H_1} = 0 \), following (1) of (a2).

Case 2. \( j_{H_1} \neq 2 \).

In this case, \((i_{H_1}, j_{H_1}) \in \{(0, 0), (0^-, 0)\}\), and there must exist \( j_{H_k} = 2 \) for some \( k \in [m] \). There are also two subcases depending on whether \( i_{H_k} = 2 \) for some \( k \in [m] \). If not, then \( i_{H_1} = 0 \) and \( i_{H_2} = 0 \) or 1 for all \( k \in [m] \), which follows (3) of (a2). Otherwise, (4) of (a2) applies, with two subcases divided by whether \( i_{H_k} = j_{H_k} = 2 \) for some \( k \in [m] \), and if so we have (4.1), otherwise we have (4.2).
(b2).

By Definition 5.7 when $\gamma^{i,j}(v_1, G) = \gamma^{1,1}(v_1, G)$, we have $(i_{H_1}, j_{H_1}) = (1,1)$ and $i_{H_k} \geq 1$ for all $k \in [m]$. Besides, $j_{H_k} = 0$ for all $k \in [m]$ since $i_{H_1} > 0$ and $j_G \neq 2$. Hence we have (b2).

(b3).

With Definition 5.7 we have $\gamma^{i,j}(v_1, G) = \gamma^{1,2}(v_1, G)$, and there are two cases.

Case 1. $j_{H_k} = 2$ for some $k \in \{1, 2, \ldots, m\}$.

Subcase 1.1. $j_{H_1} = 2$.

In this subcase $(i_{H_1}, j_{H_1}) = (1,2)$. Similar to the proof of (a1), there are two subcases divided by whether $i_{H_k} = 2$ for some $k \in [m]$. If so, then (1.2) of (b3) applies. Otherwise, (1.1) of (b3) holds.

Subcase 1.2. $j_{H_1} \neq 2$.

In this subcase, $(i_{H_1}, j_{H_1}) \in \{(1,0), (1,1)\}$ and $j_{H_k} = 2$ for some $k \in [m]$. If $i_{H_1} \neq 2$ for $k \in [m]$, we have (2.1) of (b3). Otherwise, two subcases arise. Firstly, when $(i_{H_1}, j_{H_1}) = (2,2)$ for some $k \in [m]$, we have (2.2) of (b3). Secondly, if $i_{H_1} \neq 2$ while $j_{H_k} = 2$ for all $k \in [m]$, then in order to dominate $v_k$, we have $(i_{H_1}, i_{H_k}) \in \{(2,0), (2,1)\}$ for some $k_1 \in [m] \setminus \{k\}$, hence (2.3) of (b3) holds.

Case 2. $j_{H_k} \neq 2$ for all $k \in \{1, 2, \ldots, m\}$.

Clearly, for a $v_k \in \{v_1, v_2, \ldots, v_m\}$ with $j_{H_k} = 1$, if all $v_k \in N[v_k]$ has $j_{H_k} \geq 1$, then $v_k$ is not dominated by $V_0^f$. When $j_{H_k} = 1$ for some $k \in [m]$, we can deduce that (3) of (b3) holds.

(d1).

By Definition 5.7 when $\gamma^{i,j}(v_1, G) = \gamma^{0^+,0}(v_1, G)$, we have $i_{H_k} \neq 2$ for all $k \in \{1, 2, \ldots, m\}$, and $(i_{H_1}, j_{H_1}) = (0^+,0)$. Hence (d1) is true.

(d2).

In this case $(i_{H_1}, j_{H_1}) \in \{(0^-,0), (0^-,2)\}$ and $i_{H_k} \neq 2$ for all $k \in [m]$. If $j_{H_1} = 2$, we have (1) of (d2). Otherwise, $j_{H_1} = 0$ and there must exist $(i_{H_k_1}, j_{H_k}) \in \{(0,2), (1,2)\}$ for some $k_1 \in [m] \setminus \{k\}$, which follows (2) of (d2).

The proofs of (c1), (c2) and (c3) are similar to the proofs of (b1), (b2) and (b3), hence omitted. In conclusion, all the proofs of Lemma 5.11 are complete. \qed

Worth mentioning, there is no need to pre-construct the recursive block subgraphs (rbsGs) $G_k$ during the computation. The purpose of defining the rbsGs is to aid in illustrating the decomposition of a block graph and validating the correctness of our algorithm. In practical terms, we construct a 10-tuple global variable of $\gamma^{i,j}(v)$ for each $v \in V(G)$, and all calculation results of $v$ are stored at this variable during the whole algorithm. As you can see in the variable of $\gamma^{i,j}(v)$ the placeholder of the graph is missing, but for the convenience of understanding we can regard that the placeholder of graph is changeable during the computation virtually. By choosing a block vertex $B_i$ based on $O_L$ reversed to the level traversal of $T_G$ in sequence, we compute the corresponding block $B_{i}^G$ at a time and store the calculation results to the specific cut vertex of $B_{i}^G$. In this way, we actually achieve the construction of the rbsG $G_k$ in order, which means we actually achieve the composition of the block graph $G_{v_n}$ and also compute the maximal Roman domination number of the block graph $G_{v_n}$. Therefore, the correctness of Algorithm 1 follows the correctness of Lemma 5.11. Now we give an example of the calculation based on Figure 5 (Tab. 1). The illustration of the rbsGs $G_k$ of the example in Figure 5 is shown in Figure 7.

Clearly, the time complexity of Algorithm 1 depends on the second for loop. For the second for loop, if we store the information of each block $B_j^G$ in a hash table with 3-tuple $(c_i, B_i$, index of $B_j^G)$, then searching $B_j^G$ and $c_i$ take $O(1)$ time at each iteration. Worth mention that the time complexity of constructing this hash table is the same as constructing a cut tree. Now, let’s discuss the computation of Lemma 5.11. For illustration, we choose (2.1) and (2.2) of (b1) as examples. The other computations of Lemma 5.11 are either similar or obvious.
Algorithm 1: MIN-MRD-BLOCK graphs.

Input: a corresponding cut tree $T_G$ rooted at $v_n$ of a connected block graph $G_v$;
Output: $\gamma_{mR}(v_n, G_v)$;

1. get a tree order $O_L$ reversed to the level traversal of $T_G$;
2. construct a 10-tuple global variable $\gamma^{i,j}(v)$ for each $v \in V(G)$;
3. for each $v_i \in V(G)$ do
   4. $\gamma^{0,0}(v_i) \leftarrow \infty$; $\gamma^{0,2}(v_i) \leftarrow \infty$; /* vertices initialize */
   5. $\gamma^{1,0}(v_i) \leftarrow \infty$; $\gamma^{1,1}(v_i) \leftarrow 1$; $\gamma^{1,2}(v_i) \leftarrow \infty$;
   6. $\gamma^{2,0}(v_i) \leftarrow \infty$; $\gamma^{2,1}(v_i) \leftarrow 2$; $\gamma^{2,2}(v_i) \leftarrow \infty$;
   7. $\gamma^{0,-2}(v_i) \leftarrow 0$; $\gamma^{0,-2}(v_i) \leftarrow \infty$;
4. end
5. for $v$ in $O_L$ do
   6. if $v$ is a block vertex then
      7. find the corresponding block $B^G_v$ of $v$;
      8. find the parent vertex $u$ of $v$ in $T_G$;
      9. suppose $V(B^G_v) = \{v_1, v_2, \ldots, v_m\}$ where $v_1 = u$, and $G_k$ is the specific rbsG corresponding to $B^G_v$;
     10. use Lemma 5.11 to compute $\gamma^{i,j}(v, G_k)$ where $\gamma^{i,j}(v, G_k)$ for all $k \in \{1, 2, \ldots, m\}$ are stored at $\gamma^{i,j}(v_k)$;
     11. update $\gamma^{i,j}(v_1) : \gamma^{i,j}(v_1) \leftarrow \gamma^{i,j}(v_1, G_k)$;
   12. else continue; /* $v$ is a cut vertex */
7. end
8. end
9. $\gamma_{mR}(v_n, G_v) = \min\{\gamma^{0,2}(v_n), \gamma^{1,1}(v_n), \gamma^{1,2}(v_n), \gamma^{2,1}(v_n), \gamma^{2,2}(v_n)\}$;
10. Result: $\gamma_{mR}(v_n, G_v)$

Table 1. An example of the calculation based on Figure 5.

<table>
<thead>
<tr>
<th>Step</th>
<th>Considered rbsG $G_k$</th>
<th>Considered block $B^G_k$</th>
<th>Cut vertex $c_k$</th>
<th>Update $\gamma^{i,j}(c_k)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$G_1 = G[V(B^G_1)]$</td>
<td>$B^G_1$</td>
<td>$c_5$</td>
<td>$\gamma^{1,3}(c_5, G_1)$</td>
</tr>
<tr>
<td>2</td>
<td>$G_2 = G[V(G_1) \cup V(B^G_2)]$</td>
<td>$B^G_2$</td>
<td>$c_4$</td>
<td>$\gamma^{1,3}(c_4, G_2)$</td>
</tr>
<tr>
<td>3</td>
<td>$G_3 = G[V(G_3) \cup V(B^G_3)]$</td>
<td>$B^G_3$</td>
<td>$c_3$</td>
<td>$\gamma^{1,3}(c_3, G_3)$</td>
</tr>
<tr>
<td>4</td>
<td>$G_4 = G[V(B^G_4)]$</td>
<td>$B^G_4$</td>
<td>$c_2$</td>
<td>$\gamma^{1,3}(c_2, G_4)$</td>
</tr>
<tr>
<td>5</td>
<td>$G_5 = G[V(G_4) \cup V(B^G_5)]$</td>
<td>$B^G_5$</td>
<td>$c_2$</td>
<td>$\gamma^{1,3}(c_2, G_5)$</td>
</tr>
<tr>
<td>6</td>
<td>$G_6 = G[V(G_6) \cup V(G_7)] \cup V(B^G_6)]$</td>
<td>$B^G_6$</td>
<td>$c_1$</td>
<td>$\gamma^{1,3}(c_1, G_6)$</td>
</tr>
<tr>
<td>7</td>
<td>$G_7 = G[V(G_6) \cup V(B^G_7)]$</td>
<td>$B^G_7$</td>
<td>$c_1$</td>
<td>$\gamma^{1,3}(c_1, G_7)$</td>
</tr>
</tbody>
</table>

To begin with, let $V(B^G_v) = \{v_1, v_2, \ldots, v_m\}$ and $[m] = \{2, \ldots, m\}$, then we reiterate the equations of (2.1) and (2.2) of $b_1$ from Lemma 5.11.

\[
(2.1) \min_{k_2 \in [m]} \left\{ \gamma^{0,0}(v_{k_2}, H_{k_2}) + \sum_{k_3 \in [m] \setminus \{k_2\}} f(v_{k_3}, \{0,0, (0,1), (1,1)\}) \right\},
\]

\[
(2.2) \min_{k_4, k_5 \in [m]} \left\{ f(v_{k_4}, \{2,0, (2,1)\}) + f(v_{k_5}, \{(0,0), (0^{-}, 0)\}) + \sum_{k_6 \in [m] \setminus \{k_4, k_5\}} f(v_{k_6}, A \setminus B_2) \right\}.
\]
Next, we define following equations:

\[ C_1 = \sum_{k_3 \in [m]} f(v_{k_3}, \{(0,0), (1,0), (1,1)\}) \]
\[ C_2 = \sum_{k_6 \in [m]} f(v_{k_6}, A \setminus B_2) \]
\[ C_3 = \min_{k_4, k_5 \in [m]} \left\{ f(v_{k_4}, \{(2,0), (2,1)\}) + f(v_{k_5}, \{(0,0), (0^{-0})\}) + \left( C_2 - \sum_{v_k \in \{v_{k_4, v_{k_5}}\}} f(v_k, A \setminus B) \right) \right\} \]

Apparently, \( C_1 \) can be computed in \( O(|V(B_G)|) \) first. Then, (2.1) of (b1) equals to \( \min_{k_2 \in [m]} \{ \gamma_{0,0}(v_{k_2}, H_{k_2}) + (C_1 - f(v_{k_2}, \{(0,0), (1,0), (1,1)\})) \} \), which can also be computed in \( O(|V(B_G^i)|) \) time. \( C_2 \) can be computed in \( O(|V(B_G^i)|) \) time first. Then, (2.2) of (b1) equals to \( C_3 \). Clearly, computing \( C_3 \) depends on the number of the pairs of \( v_{k_4}, v_{k_5} \in \{v_2, v_3, \ldots, v_m\} \), so it takes \( O(|E(B_G^i)|) \) time, i.e. the computation of (2.2) of (b1) takes \( O(|V(B_G^i)| + |E(B_G^i)|) \) time.

In conclusion, \( \gamma_{m,R}(G) \) of a connected block graph \( G = (V,E) \) can be computed in \( O(|V| + |E|) \) time. Therefore, we have the following theorem and corollary.

**Theorem 5.12.** Maximal Roman domination number \( \gamma_{m,R}(G) \) of block graphs can be computed in linear time.

**Corollary 5.13.** Maximal Roman domination number \( \gamma_{m,R}(G) \) of trees can be computed in linear time.

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**References**


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