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A SPECTRAL CONDITION FOR COMPONENT FACTORS IN GRAPHS

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Abstract. Let G be a graph. A $\{K_{1,2}, K_{1,3}, K_5\}$-factor of G is a spanning subgraph of G, in which every component is isomorphic to a member of $\{K_{1,2}, K_{1,3}, K_5\}$. In this paper, we establish a lower bound on the spectral radius of G to ensure that G contains a $\{K_{1,2}, K_{1,3}, K_5\}$-factor.

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1. Introduction

In this article, we only deal with finite and undirected graphs which possess neither loops nor multiple edges. Let $G = (V(G), E(G))$ be a graph, where $V(G)$ denotes its vertex set and $E(G)$ denotes its edge set. The order of a graph G is the number $n = |V(G)|$ of its vertices. For a vertex subset S of G, we denote by $G[S]$ the subgraph of G induced by S, and by $G - S$ the subgraph obtained from G by deleting the vertices in S and the edges incident to vertices in S. For two distinct graphs G_1 and G_2, we denote by $G_1 \vee G_2$ the join of G_1 and G_2, and by $G_1 \cup G_2$ the union of G_1 and G_2. For a graph G and an integer $k \geq 2$, we use kG to denote the disjoint union of k copies of G. As usual, a complete graph, a path and a star of order n are denoted by K_n, P_n and $K_{1,n-1}$, respectively. Let r be a real number. Recall that $\lceil r \rceil$ is the smallest integer such that $\lceil r \rceil \geq r$.

Suppose that the vertex set of G is $V(G) = \{v_1, v_2, \ldots, v_n\}$. The adjacency matrix $A(G) = (a_{ij})_{n\times n}$ of G is a $(0,1)$-matrix in which the entry $a_{ij} = 1$ if and only if v_i and v_j are adjacent. Note that $A(G)$ is a real nonnegative symmetric matrix. Let $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$ be the eigenvalues of $A(G)$. In particular, the largest eigenvalue $\lambda_1(G)$ is called the adjacency spectral radius (or spectral radius, for short) of G.

A subgraph of a graph G is spanning if the subgraph covers all vertices of G. Let $\mathcal{H}$ denote a set of connected graphs. Then an $\mathcal{H}$-factor of G is a spanning subgraph of G, in which every component is isomorphic to a member of $\mathcal{H}$. An $\mathcal{H}$-factor is also referred as a component factor. Let k be an integer with $k \geq 2$. A $P_{\geq k}$-factor of G is its spanning subgraph each of whose components is a path of order at least k. A $\{K_{1,j} : 1 \leq j \leq k\}$-factor is also called a star-factor of G.

In mathematical literature, the study on component factors attracted much attention. Kaneko [4] gave a necessary and sufficient condition for a graph having a $P_{\geq 3}$-factor. Zhou *et al.* [26] provided some sufficient conditions for graphs to have $P_{\geq 3}$-factors with given properties. Wang and Zhang [16,17] obtained some results on the existence of $P_{\geq 3}$-factors in graphs. Wu [20] showed two degree conditions for graphs to have $P_{\geq 3}$-factors.

Keywords. Graph, spectral radius, component factor.

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Zhou *et al.* [23,24,27,30,31] claimed some sufficient conditions for graphs to contain $P_{\geq 3}$-factors. Amahashi and Kano [1], and Las Vergnas [7] independently presented a characterization for a graph with a $\{K_{1,j} : 1 \leq j \leq k\}$-factor, where $k \geq 2$ is an integer. Kano *et al.* [6] verified that a graph G contains a $\{K_{1,2}, K_{1,3}, K_5\}$-factor if $i(G-S) \leq \frac{|S|}{2}$ for any $S \subseteq V(G)$. Kano and Saito [5] provided a sufficient condition for a graph to contain a $\{K_{1,j} : m \leq j \leq 2m\}$-factor, where $m \geq 2$ is an integer. Zhou *et al.* [28,34] got some results on the existence of star-factors in graphs. Wang and Zhang [18] derived some results on the existence of $\{K_{1,j} : 1 \leq j \leq k\}$-factors in graphs. Some other results on graph factors were found in [11,19,22,29,32].

Many researchers [3,10,13,15,25,33] studied some interesting spectral properties of $A(G)$. Suil [14] obtained a spectral radius condition for a graph to have a $\{K_2\}$-factor. Li and Miao [9], Zhou *et al.* [35] presented some spectral conditions for the existence of $P_{\geq 2}$-factors in graphs. Miao and Li [12] showed two spectral conditions for a graph with a $\{K_{1,j} : 1 \leq j \leq k\}$-factor, where $k \geq 2$ is an integer. In this paper, we investigate the existence of a $\{K_{1,2}, K_{1,3}, K_5\}$-factor in a graph, and put forward a new sufficient condition for a graph to contain a $\{K_{1,2}, K_{1,3}, K_5\}$-factor by using spectral radius.

Theorem 1.1. *Let G be a connected graph of order n.*

(1) *For $n \geq 3$ and $n \notin \{4,5,6,7,8,11\}$, if $\lambda_1(G) \geq \theta(n)$, then G contains a $\{K_{1,2}, K_{1,3}, K_5\}$-factor unless $G = K_1 \vee (K_{n-2} \cup K_1)$, where $\theta(n) = \lambda_1(K_1 \vee (K_{n-2} \cup K_1))$ is the largest root of $x^3 - (n-3)x^2 - (n-1)x + n - 3 = 0$.*
(2) *For $n = 4$, if $\lambda_1(G) \geq \frac{1+\sqrt{17}}{2}$, then G contains a $\{K_{1,2}, K_{1,3}, K_5\}$-factor.*
(3) *For $n = 5$, if $\lambda_1(G) \geq 1 + \sqrt{7}$, then G contains a $\{K_{1,2}, K_{1,3}, K_5\}$-factor unless $G = K_3 \vee 2K_1$.*
(4) *For $n = 6$, if $\lambda_1(G) \geq 1 + \sqrt{10}$, then G contains a $\{K_{1,2}, K_{1,3}, K_5\}$-factor.*
(5) *For $n = 7$, if $\lambda_1(G) \geq \frac{3+\sqrt{57}}{2}$, then G contains a $\{K_{1,2}, K_{1,3}, K_5\}$-factor.*
(6) *For $n = 8$, if $\lambda_1(G) \geq 2 + \sqrt{19}$, then G contains a $\{K_{1,2}, K_{1,3}, K_5\}$-factor.*
(7) *For $n = 11$, if $\lambda_1(G) \geq 3 + \sqrt{37}$, then G contains a $\{K_{1,2}, K_{1,3}, K_5\}$-factor.*

2. Preliminaries

In this section, we provide some necessary preliminary lemmas, which will be used in the proofs of our main results.

Lemma 2.1 ([8])**.** *If G is a connected graph, and H is a proper subgraph of G, then*

$$\lambda_1(G) > \lambda_1(H).$$

Let M be a real matrix whose rows and columns are indexed by $V = \{1, 2, \ldots, n\}$. Assume that M, with respect to the partition $\pi : V = V_1 \cup V_2 \cup \cdots \cup V_m$, can be written as

$$M = \begin{pmatrix} M_{11} & M_{12} & \cdots & M_{1m} \\ M_{21} & M_{22} & \cdots & M_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ M_{m1} & M_{m2} & \cdots & M_{mm} \end{pmatrix},$$

where M_{ij} denotes the submatrix (block) of M formed by rows in V_i and columns in V_j. Let q_{ij} denote the average row sum of M_{ij}, namely, q_{ij} is the sum of all entries in M_{ij} divided by the number of rows. Then matrix $M_\pi = (q_{ij})$ is called the quotient matrix of M. If the row sum of every block M_{ij} is a constant, then the partition is equitable.

Lemma 2.2 ([2,21])**.** *Let M be a real matrix with an equitable partition π, and let M_π be the corresponding quotient matrix. Then each eigenvalue of M_π is an eigenvalue of M. Furthermore, if M is nonnegative, then the spectral radiuses of M and M_π are equal.*

Let $I(G)$ and $i(G)$ denote the set of isolated vertices and the number of isolated vertices of G, respectively. Kano *et al.* [6] provided a sufficient condition for a graph to contain a $\{K_{1,2}, K_{1,3}, K_5\}$-factor.

Lemma 2.3 ([6])**.** *If a graph G satisfies*

$$i(G-S) \leq \frac{|S|}{2}$$

for any vertex subset S of G, then G contains a $\{K_{1,2}, K_{1,3}, K_5\}$-factor.

3. The proof of Theorem 1.1

Proof of Theorem 1.1. Suppose that G contains no $\{K_{1,2}, K_{1,3}, K_5\}$-factor. Then by Lemma 2.3, there exists some nonempty subset S of $V(G)$ such that

$$i(G-S) \geq \frac{|S|+1}{2}.$$

By the integrity of $i(G-S)$, we obtain

$$i(G-S) \geq \left\lceil \frac{|S|+1}{2} \right\rceil.$$

Let $|S| = s$. Then G is a spanning subgraph of $G_1 = K_s \vee (K_{n_1} \cup \lceil \frac{s+1}{2} \rceil K_1)$, where n_1 is a nonnegative integer with $n_1 = n - s - \lceil \frac{s+1}{2} \rceil$. By means of Lemma 2.1, we conclude

$$\lambda_1(G) \leq \lambda_1(G_1), \tag{3.1}$$

where the equality holds if and only if $G = G_1$. In what follows, we shall consider three cases by the value of n_1.

Case 1. $n_1 \geq 2$.

In this case, $G_1 = K_s \vee (K_{n-s-\lceil \frac{s+1}{2} \rceil} \cup \lceil \frac{s+1}{2} \rceil K_1)$ and $n \geq s + \lceil \frac{s+1}{2} \rceil + 2$. In view of the equitable partition $V(G_1) = V(K_s) \cup V(K_{n-s-\lceil \frac{s+1}{2} \rceil}) \cup V(\lceil \frac{s+1}{2} \rceil K_1)$, the quotient matrix of $A(G_1)$ equals

$$M_1 = \begin{pmatrix} s-1 & n-s-\lceil \frac{s+1}{2} \rceil & \lceil \frac{s+1}{2} \rceil \\ s & n-s-\lceil \frac{s+1}{2} \rceil - 1 & 0 \\ s & 0 & 0 \end{pmatrix}.$$

So we derive its characteristic polynomial as

$$\begin{aligned} f_{M_1}(x) = x^3 &- \left(n - \left\lceil \frac{s+1}{2} \right\rceil - 2\right) x^2 - \left(n + s\left\lceil \frac{s+1}{2} \right\rceil - \left\lceil \frac{s+1}{2} \right\rceil - 1\right) x \\ &+ s\left\lceil \frac{s+1}{2} \right\rceil \left(n - s - \left\lceil \frac{s+1}{2} \right\rceil - 1\right). \end{aligned}$$

In view of Lemma 2.2, the largest root, say θ_1, of $f_{M_1}(x) = 0$ satisfies $\theta_1 = \lambda_1(G_1)$. Note that K_{s+2} is a proper subgraph of G_1. Using Lemma 2.1, we get

$$\theta_1 = \lambda_1(G_1) > \lambda_1(K_{s+2}) = s+1. \tag{3.2}$$

Write $G_2 = K_1 \vee (K_{n-2} \cup H)$, where $H = K_1$. Consider the equitable partition $V(G_2) = V(K_1) \cup V(K_{n-2}) \cup V(H)$. Then the quotient matrix of $A(G_2)$ equals

$$M = \begin{pmatrix} 0 & n-2 & 1 \\ 1 & n-3 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

and the characteristic polynomial of M is

$$f_M(x) = x^3 - (n-3)x^2 - (n-1)x + n - 3.$$

In terms of Lemma 2.2, the largest root $\theta(n)$ of $f_M(x) = 0$ satisfies $\theta(n) = \lambda_1(G_2)$.
Note that $f_{M_1}(\theta_1) = 0$. By a direct computation, we have

$$\begin{aligned} f_M(\theta_1) &= f_M(\theta_1) - f_{M_1}(\theta_1) \\ &= -\left(\left\lceil \frac{s+1}{2} \right\rceil - 1\right)\theta_1^2 + \left\lceil \frac{s+1}{2} \right\rceil (s-1)\theta_1 - s\left\lceil \frac{s+1}{2} \right\rceil \left(n - s - \left\lceil \frac{s+1}{2} \right\rceil - 1\right) + n - 3. \end{aligned} \tag{3.3}$$

Subcase 1.1. s is odd.

Obviously, $\lceil \frac{s+1}{2} \rceil = \frac{s+1}{2}$. Combining this with (3.2), (3.3), $s \geq 1$ and $n \geq s + \lceil \frac{s+1}{2} \rceil + 2 = \frac{3s+5}{2}$, we obtain

$$\begin{aligned} f_M(\theta_1) &= -\left(\left\lceil \frac{s+1}{2} \right\rceil - 1\right)\theta_1^2 + \left\lceil \frac{s+1}{2} \right\rceil (s-1)\theta_1 - s\left\lceil \frac{s+1}{2} \right\rceil \left(n - s - \left\lceil \frac{s+1}{2} \right\rceil - 1\right) + n - 3 \\ &= \frac{s-1}{2}\left(-\theta_1^2 + (s+1)\theta_1 - s\left(n - \frac{3(s+1)}{2}\right) - 2n + 3s + 6\right) \\ &\leq \frac{s-1}{2}\left(-\theta_1^2 + (s+1)\theta_1 - s\left(\frac{3s+5}{2} - \frac{3(s+1)}{2}\right) - (3s+5) + 3s + 6\right) \\ &= \frac{s-1}{2}\left(-\theta_1^2 + (s+1)\theta_1 - s + 1\right) \leq \frac{s-1}{2}\left(-(s+1)\theta_1 + (s+1)\theta_1 - s + 1\right) \\ &= -\frac{(s-1)^2}{2} \leq 0, \end{aligned}$$

which leads to $\lambda_1(G_1) = \theta_1 \leq \theta(n) = \lambda_1(K_1 \vee (K_{n-2} \cup K_1))$ with equality if and only if $G_1 = K_1 \vee (K_{n-2} \cup K_1)$. Combining this with (3.1), we conclude

$$\lambda_1(G) \leq \lambda_1\left(K_1 \vee (K_{n-2} \cup K_1)\right)$$

with equality if and only if $G = K_1 \vee (K_{n-2} \cup K_1)$, a contradiction.

Subcase 1.2. s is even.

Clearly, $\lceil \frac{s+1}{2} \rceil = \frac{s+2}{2}$. Together with (3.2), (3.3), $s \geq 2$ and $n \geq s + \lceil \frac{s+1}{2} \rceil + 2 = \frac{3s+6}{2}$, we have

$$\begin{aligned} f_M(\theta_1) &= -\left(\left\lceil \frac{s+1}{2} \right\rceil - 1\right)\theta_1^2 + \left\lceil \frac{s+1}{2} \right\rceil (s-1)\theta_1 - s\left\lceil \frac{s+1}{2} \right\rceil \left(n - s - \left\lceil \frac{s+1}{2} \right\rceil - 1\right) + n - 3 \\ &= -\frac{s}{2}\theta_1^2 + \frac{s^2+s-2}{2}\theta_1 - \frac{s^2+2s}{2}\left(n - \frac{3s+4}{2}\right) + n - 3 \\ &< -\frac{s(s+1)}{2}\theta_1 + \frac{s^2+s-2}{2}\theta_1 - \frac{s^2+2s}{2}\left(n - \frac{3s+4}{2}\right) + n - 3 \\ &= -\theta_1 - \frac{s^2+2s}{2}\left(n - \frac{3s+4}{2}\right) + n - 3 \\ &< -(s+1) - \frac{s^2+2s}{2}\left(\frac{3s+6}{2} - \frac{3s+4}{2}\right) + \frac{3s+6}{2} - 3 \\ &= -\frac{s^2+s+2}{2} < 0, \end{aligned}$$

which yields $\lambda_1(G_1) = \theta_1 < \theta(n) = \lambda_1(K_1 \vee (K_{n-2} \cup K_1))$. Together with (3.1), we obtain

$$\lambda_1(G) < \lambda_1\left(K_1 \vee (K_{n-2} \cup K_1)\right),$$

which contradicts $\lambda_1(G) \geq \lambda_1(K_1 \vee (K_{n-2} \cup K_1))$.

Case 2. $n_1 = 1$.

In this case, $G_1 = K_s \vee (\lceil \frac{s+1}{2} \rceil + 1)K_1$ and $n = s + \lceil \frac{s+1}{2} \rceil + 1$. The quotient matrix of $A(G_1)$ with respect to the equitable partition $V(G_1) = V(K_s) \cup V((\lceil \frac{s+1}{2} \rceil + 1)K_1)$ is

$$M_2 = \begin{pmatrix} s-1 & \lceil \frac{s+1}{2} \rceil + 1 \\ s & 0 \end{pmatrix},$$

and the characteristic polynomial of M_2 is

$$f_{M_2}(x) = x^2 - (s-1)x - s\left(\left\lceil \frac{s+1}{2} \right\rceil + 1\right).$$

According to Lemma 2.2, the largest root, say θ_2, of $f_{M_2}(x) = 0$ satisfies

$$\theta_2 = \lambda_1(G_1) = \frac{s-1+\sqrt{(s-1)^2 + 4s\left(\lceil \frac{s+1}{2} \rceil + 1\right)}}{2}.$$

Note that $f_{M_2}(\theta_2) = 0$ and $n = s + \lceil \frac{s+1}{2} \rceil + 1$. By a direct computation, we have

$$\begin{aligned} f_M(\theta_2) &= f_M(\theta_2) - \theta_2 f_{M_2}(\theta_2) = -(n-s-2)\theta_2^2 + \left(-n + s\left(\left\lceil \frac{s+1}{2} \right\rceil + 1\right) + 1\right)\theta_2 + n - 3 \\ &= -\left(\left\lceil \frac{s+1}{2} \right\rceil - 1\right)\theta_2^2 + \left\lceil \frac{s+1}{2} \right\rceil (s-1)\theta_2 + s + \left\lceil \frac{s+1}{2} \right\rceil - 2. \end{aligned} \tag{3.4}$$

If $s = 1$, then $n = 3$, $\theta_2 = \sqrt{2} = \theta(3)$ and $G_1 = G_2 = K_1 \vee (K_{n-2} \cup K_1)$. Combining this with (3.1), we conclude $\lambda_1(G) \le \theta(3)$ with equality if and only if $G = K_1 \vee (K_{n-2} \cup K_1)$, a contradiction. If $s = 2$, then $n = 5$ and $\theta_2 = 3$. In terms of (3.4), we possess $f_M(\theta_2) = -\theta_2^2 + 2\theta_2 + 2 = -1 < 0$, which implies $\lambda_1(G_1) = \theta_2 < \theta(5)$. Together with (3.1), we have $\lambda_1(G) < \theta(5) < 1+\sqrt{7}$, which contradicts $\lambda_1(G) \ge 1+\sqrt{7}$. If $s = 3$, then $n = 6$ and $\theta_2 = 1 + \sqrt{10}$. Using (3.1), we get $\lambda_1(G) \le 1 + \sqrt{10}$ with equality if and only if $G = K_3 \vee 3K_1$. If $G = K_3 \vee 3K_1$, then $\lambda_1(G) = 1+\sqrt{10}$ and G has a $\{K_{1,2}\}$-factor. Obviously, a $\{K_{1,2}\}$-factor is also a $\{K_{1,2}, K_{1,3}, K_5\}$-factor. If $G \ne K_3 \vee 3K_1$, then $\lambda_1(G) < 1 + \sqrt{10}$, a contradiction. If $s = 4$, then $n = 8$ and $\theta_2 = \frac{3+\sqrt{73}}{2}$. According to (3.4), we possess $f_M(\theta_2) = -\theta_2^2 + 9\theta_2 + 5 = -\frac{45-3\sqrt{73}}{2} < 0$, which yields $\lambda_1(G_1) = \theta_2 < \theta(8)$. Applying (3.1), we obtain $\lambda_1(G) < \theta(8) < 2 + \sqrt{19}$, which is a contradiction to $\lambda_1(G) \ge 2 + \sqrt{19}$. If $s = 5$, then $n = 9$ and $\theta_2 = 2 + 2\sqrt{6}$. According to (3.4), we derive $f_M(\theta_2) = -2\theta_2^2 + 22\theta_2 + 6 = -26 + 8\sqrt{6} < 0$, which implies $\lambda_1(G_1) = \theta_2 < \theta$. Combining this with (3.1), we conclude $\lambda_1(G) < \theta$, which contradicts $\lambda_1(G) \ge \theta$. If $s \ge 6$, then

$$\begin{aligned} \theta_2 &= \frac{s-1+\sqrt{(s-1)^2 + 4s\left(\lceil \frac{s+1}{2} \rceil + 1\right)}}{2} \ge \frac{s-1+\sqrt{(s-1)^2 + 4s\left(\frac{s+1}{2} + 1\right)}}{2} \\ &= \frac{s-1+\sqrt{3s^2+4s+1}}{2} > \frac{s-1+\sqrt{s^2+10s+25}}{2} = s+2. \end{aligned}$$

Combining this with (3.4), we get

$$\begin{aligned} f_M(\theta_2) &= -\left(\left\lceil \frac{s+1}{2} \right\rceil - 1\right)\theta_2^2 + \left\lceil \frac{s+1}{2} \right\rceil (s-1)\theta_2 + s + \left\lceil \frac{s+1}{2} \right\rceil - 2 \\ &< -\left(\left\lceil \frac{s+1}{2} \right\rceil - 1\right)(s+2)\theta_2 + \left\lceil \frac{s+1}{2} \right\rceil (s-1)\theta_2 + s + \left\lceil \frac{s+1}{2} \right\rceil - 2 \\ &= -\left(3\left\lceil \frac{s+1}{2} \right\rceil - s - 2\right)\theta_2 + s + \left\lceil \frac{s+1}{2} \right\rceil - 2 < -\left(3\left\lceil \frac{s+1}{2} \right\rceil - s - 2\right)(s+2) + s + \left\lceil \frac{s+1}{2} \right\rceil - 2 \\ &= -\left((3s+5)\left\lceil \frac{s+1}{2} \right\rceil - s^2 - 5s - 2\right) \le -\left(\frac{(3s+5)(s+1)}{2} - s^2 - 5s - 2\right) = -\frac{(s-1)^2}{2} < 0, \end{aligned}$$

which yields $\lambda_1(G_1) = \theta_2 < \theta$. Together with (3.1), we conclude $\lambda_1(G) < \theta$, which contradicts $\lambda_1(G) \geq \theta$.

Case 3. $n_1 = 0$.

In this case, $G_1 = K_s \vee \lceil \frac{s+1}{2} \rceil K_1$ and $n = s + \lceil \frac{s+1}{2} \rceil$. The quotient matrix of $A(G_1)$ with respect to the equitable partition $V(G_1) = V(K_s) \cup V(\lceil \frac{s+1}{2} \rceil K_1)$ is

$$M_3 = \begin{pmatrix} s-1 & \lceil \frac{s+1}{2} \rceil \\ s & 0 \end{pmatrix},$$

and the characteristic polynomial of M_3 equals

$$f_{M_3}(x) = x^2 - (s-1)x - s\left\lceil \frac{s+1}{2} \right\rceil.$$

By virtue of Lemma 2.2, the largest root, say θ_3, of $f_{M_3}(x) = 0$ satisfies

$$\theta_3 = \lambda_1(G_1) = \frac{s - 1 + \sqrt{(s-1)^2 + 4s\lceil \frac{s+1}{2} \rceil}}{2}.$$

Notice that $f_{M_3}(\theta_3) = 0$ and $n = s + \lceil \frac{s+1}{2} \rceil$. By a direct calculation, we obtain

$$\begin{aligned} f_M(\theta_3) &= f_M(\theta_3) - \theta_3 f_{M_3}(\theta_3) = -(n-s-2)\theta_3^2 + \left(-n + s\left\lceil \frac{s+1}{2} \right\rceil + 1\right)\theta_3 + n - 3 \\ &= -\left(\left\lceil \frac{s+1}{2} \right\rceil - 2\right)\theta_3^2 + \left(\left\lceil \frac{s+1}{2} \right\rceil - 1\right)(s-1)\theta_3 + s + \left\lceil \frac{s+1}{2} \right\rceil - 3. \end{aligned} \tag{3.5}$$

If $s = 1$, then $n = 2$, which is a contradiction to $n \geq 3$. If $s = 2$, then $n = 4$ and $\theta_3 = \frac{1+\sqrt{17}}{2}$. According to (3.1), we conclude $\lambda_1(G) \leq \frac{1+\sqrt{17}}{2}$ with equality if and only if $G = K_2 \vee 2K_1$. If $G = K_2 \vee 2K_1$, then $\lambda_1(G) = \frac{1+\sqrt{17}}{2}$ and G has a $\{K_{1,3}\}$-factor. Obviously, a $\{K_{1,3}\}$-factor is also a $\{K_{1,2}, K_{1,3}, K_5\}$-factor. If $G \neq K_2 \vee 2K_1$, then $\lambda_1(G) < \frac{1+\sqrt{17}}{2}$, a contradiction. If $s = 3$, then $n = 5$ and $\theta_3 = 1 + \sqrt{7}$. In view of (3.1), we possess $\lambda_1(G) \leq 1 + \sqrt{7}$ with equality if and only if $G = K_3 \vee 2K_1$, a contradiction. If $s = 4$, then $n = 7$ and $\theta_3 = \frac{3+\sqrt{57}}{2}$. By (3.1), we get $\lambda_1(G) \leq \frac{3+\sqrt{57}}{2}$ with equality if and only if $G = K_4 \vee 3K_1$. If $G = K_4 \vee 3K_1$, then $\lambda_1(G) = \frac{3+\sqrt{57}}{2}$ and G has a $\{K_{1,2}, K_{1,3}\}$-factor. Obviously, a $\{K_{1,2}, K_{1,3}\}$-factor is also a $\{K_{1,2}, K_{1,3}, K_5\}$-factor. If $G \neq K_4 \vee 3K_1$, then $\lambda_1(G) < \frac{3+\sqrt{57}}{2}$, a contradiction. If $s = 5$, then $n = 8$ and $\theta_3 = 2 + \sqrt{19}$. Using (3.1), we conclude $\lambda_1(G) \leq 2 + \sqrt{19}$ with equality if and only if $G = K_5 \vee 3K_1$. If $G = K_5 \vee 3K_1$, then $\lambda_1(G) = 2 + \sqrt{19}$ and G has a $\{K_{1,3}\}$-factor. Obviously, a $\{K_{1,3}\}$-factor is also a $\{K_{1,2}, K_{1,3}, K_5\}$-factor. If $G \neq K_5 \vee 3K_1$, then $\lambda_1(G) < 2 + \sqrt{19}$, a contradiction. If $s = 6$, then $n = 10$ and $\theta_3 = 8$. Combining this with (3.5), we have $f_M(\theta_3) = -2\theta_3^2 + 15\theta_3 + 7 = -1 < 0$, which implies $\lambda_1(G_1) = \theta_3 < \theta(10)$. According to (3.1), we infer $\lambda_1(G) \leq \lambda_1(G_1) < \theta(10)$, which contradicts $\lambda_1(G) \geq \theta(10)$. If $s = 7$, then $n = 11$ and $\theta_3 = 3 + \sqrt{37}$. Using (3.1), we obtain $\lambda_1(G) \leq 3 + \sqrt{37}$ with equality if and only if $G = K_7 \vee 4K_1$. If $G = K_7 \vee 4K_1$, then $\lambda_1(G) = 3 + \sqrt{37}$ and G has a $\{K_{1,2}, K_5\}$-factor. Obviously, a $\{K_{1,2}, K_5\}$-factor is also a $\{K_{1,2}, K_{1,3}, K_5\}$-factor. If $G \neq K_7 \vee 4K_1$, then $\lambda_1(G) < 3 + \sqrt{37}$, a contradiction. If $s \geq 8$, then

$$\begin{aligned} \theta_3 &= \frac{s - 1 + \sqrt{(s-1)^2 + 4s\lceil \frac{s+1}{2} \rceil}}{2} \geq \frac{s - 1 + \sqrt{(s-1)^2 + 2s(s+1)}}{2} \\ &= \frac{s - 1 + \sqrt{3s^2 + 1}}{2} > \frac{\left(\sqrt{3} + 1\right)s - 1}{2}. \end{aligned}$$

Together with (3.5), we obtain

$$\begin{aligned}
f_M(\theta_3) &= -\left(\left\lceil\frac{s+1}{2}\right\rceil-2\right)\theta_3^2+\left(\left\lceil\frac{s+1}{2}\right\rceil-1\right)(s-1)\theta_3+s+\left\lceil\frac{s+1}{2}\right\rceil-3\\
&< -\left(\left\lceil\frac{s+1}{2}\right\rceil-2\right)\cdot\frac{\left(\sqrt{3}+1\right)s-1}{2}\theta_3+\left(\left\lceil\frac{s+1}{2}\right\rceil-1\right)(s-1)\theta_3+s+\left\lceil\frac{s+1}{2}\right\rceil-3\\
&= -\frac{1}{2}\left(\left(\sqrt{3}-1\right)s\left\lceil\frac{s+1}{2}\right\rceil+\left\lceil\frac{s+1}{2}\right\rceil-2\sqrt{3}s\right)\theta_3+s+\left\lceil\frac{s+1}{2}\right\rceil-3\\
&< -\frac{1}{2}\left(\left(\sqrt{3}-1\right)s\left\lceil\frac{s+1}{2}\right\rceil+\left\lceil\frac{s+1}{2}\right\rceil-2\sqrt{3}s\right)\cdot\frac{\left(\sqrt{3}+1\right)s-1}{2}+s+\left\lceil\frac{s+1}{2}\right\rceil-3\\
&= -\frac{1}{4}\left(2s^2\left\lceil\frac{s+1}{2}\right\rceil-\left(6+2\sqrt{3}\right)s^2+2s\left\lceil\frac{s+1}{2}\right\rceil+\left(2\sqrt{3}+4\right)s+3\left\lceil\frac{s+1}{2}\right\rceil-12\right)\\
&\le -\frac{1}{4}\left(s^2(s+1)-\left(6+2\sqrt{3}\right)s^2+s(s+1)+\left(2\sqrt{3}+4\right)s+\frac{3(s+1)}{2}-12\right)\\
&= -\frac{1}{8}\left(2s^3-\left(8+4\sqrt{3}\right)s^2+\left(4\sqrt{3}+13\right)s-21\right)<0,
\end{aligned}$$

which implies $\lambda_1(G_1)=\theta_3<\theta(n)$. Combining this with (3.1), we obtain $\lambda_1(G)\le\lambda_1(G_1)<\theta(n)$, which contradicts $\lambda_1(G)\ge\theta(n)$. This completes the proof of Theorem 1.1.

□

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