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SPECTRAL PROPERTIES OF SOMBOR MATRIX OF THRESHOLD GRAPHS

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Abstract. We investigate the Sombor spectral properties of threshold graphs, a formula for the Sombor index is presented, the Sombor eigenvalues are given, graphs with simple Sombor eigenvalues are characterized, bounds on the smallest/largest Sombor eigenvalues are presented, the multiplicities of the Sombor eigenvalues are discussed, formulae for trace and determinant of the associated quotient matrix are given, the Sombor spread bound and the bounds on the Sombor energy along with the characterization of extremal graphs. At the end, the conclusion states that all our results are valid for adjacency matrix and other adjacency type matrices.

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1. Introduction

Consider a simple graph G with vertex set $V(G) = \{u_1, u_2, \dots, u_n\}$, edge set $E(G)$ and degree sequence $\{d_1, d_2, \dots, d_n\}$ (or d_{u_i} of vertex u_i). The number $|E(G)|$ is the *size* m and $|V(G)|$ is the *order* n of G. If d_{u_i} is same for each i, then G is called a *regular* graph. The star graph (complete bipartite) with partite sets 1 and n_1 is symbolized by K_{1,n_1}, other notations can be seen in [7].

The adjacency $A(G)$ matrix of G is a square matrix, where (i, j) element is 1, if u_i is adjacent to u_j and 0, otherwise. The matrix $A(G)$ is real and symmetric, so its eigenvalues are real and can be listed in non-increasing order as: $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$. The largest eigenvalue λ_1 is known as the spectral radius of G. It is known that λ_1 is simple by Perron–Frobenius theorem. Also $|\lambda_i| \leq \lambda_1$ and the associated eigenvector say X of λ_1 has positive entries. In theoretical chemistry, the trace norm of real symmetric matrix $|\lambda_1| + |\lambda_2| + \dots + |\lambda_n|$ of $A(G)$ is known as the graph energy (see [13]), denoted by $\mathsf{E}(A(G))$, and its source goes back to theoretical chemistry, where it used in calculating the π-electron energy of hydrocarbons. There is a vast literature available related to $\mathsf{E}(A(G))$, and the other spectral invariants of $A(G)$, see [12, 22, 25].

In the theory of chemical/mathematical literature, several degree/distance based graph invariants exists which are useful in chemical studies of alkanes, they are usually known as topological indices. They play main role in

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QSAR (quantitative structure-activity relationship) and QSPR (quantitative structure-property relationship) related analysis, see [19] and the references cited there in.

Gutman [14] put forward a Topological index called the *Sombor index*, denoted by $\mathsf{SO}(G)$ of G, and is given as

$$\mathsf{SO}(G) = \sum_{u_iu_j \in E(G)} \sqrt{d_i^2 + d_j^2}.$$

The Sombor index is of great importance, since it is not just a vertex degree based invariants, rather it has am interesting geometry related meaning, see [14,16]. The sharp inequalities for $\mathsf{SO}(G)$ and its connection with the Zagreb indices of G can be seen in [9]. It is proved that the cycle C_n has the minimum Sombor index among the class of unicyclic graphs [32]. Other chemical applicability and molecular type of properties of $\mathsf{SO}(G)$ can be found in [8,10,31]. The Sombor index is closely related to graph energy, see [20,34,35].

The *Sombor matrix* $S(G)$ of G is an n-square matrix given as

$$S(G) = (s_{ij}) = \begin{cases} \sqrt{d_i^2 + d_j^2}, & \text{if } u_i \text{ is adjacent to } u_j \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that $S(G) = r\sqrt{2}A(G)$, provided G is r-regular graph. The Sombor matrix $S(G)$ is a real symmetric matrix. We label the eigenvalues of $S(G)$ by ξ_i's and put them in increasing order as: $\xi_1 \geq \xi_2 \geq \cdots \geq \xi_n$. The set of the eigenvalues of $S(G)$ with repetitions is called the *Sombor spectrum* of $S(G)$. The largest eigenvalue of $S(G)$ is called as the *Sombor spectral radius* of G, and it is precisely the *spectral norm* of the real symmetric matrix $S(G)$. By Perron–Frobenius theorem, ξ_1 is unique, $|\xi_i| \leq \xi_1$ and its corresponding eigenvector Y have positive entries. The Sombor energy (trace norm) $\mathsf{E}(S(G))$ of $S(G)$, is defined by

$$\mathsf{E}(S(G)) = \sum_{i=1}^{n} |\xi_i|.$$

The Sombor matrix $S(G)$ is related to adjacency matrix and because of this reason, large numbers of articles were published in the mean time. Several paper on properties of $S(G)$ can be found in the literature, like the Sombor energy, the Sombor spectral norm, the Sombor Estrada index, relation of $\mathsf{E}(A(G))$ with $S(G)$, $\mathsf{SO}(G)$ and others spectral invariants can be seen in [15,17,23,26,29,30,34]. Inequalities for the Sombor index and energy of $S(G)$ can be seen in [36]. The Sombor spread of $S(G)$ is give by $s(S(G)) = \xi_1 - \xi_n$. If the Sombor eigenvalue μ is repeated k-times, then we represent it as $\mu^{[k]}$. Also, J_n denotes the matrix of ones and I_n is the identity matrix.

A threshold graph is constructed/defined in many ways, we generate it by a binary sequences as given in [1]. A *threshold graph* $T(b)$ is constructed from its binary sequence $b = (a_1a_2 \ldots a_n)$ in as given below:

(i) for $i = 1$, we take $T_1 = T(a_1) \cong K_1$, an isolated vertex;
(ii) for $i \geq 2$, we already have $T_{i-1} = T(a_1a_2 \ldots a_{i-1})$, and $T_i = T(a_1a_2 \ldots a_{i-1}a_i)$ is obtained by inserting an isolated vertex to T_{i-1} if $a_i = 0$ (that is, a vertex not adjacent to any vertex in T_{i-1}) or by introducing a dominating vertex (adjacent to all vertices in T_{i-1}) to T_{i-1} provided $a_i = 1$.

It is evident that $T(b) = T_n$. We denote the addition of an isolated vertex by 0 and a dominating vertex by 1, then we see that $G \cong T(b)$ is constructed by a binary code $b = 0^{n_1}1^{m_1}0^{n_2}1^{m_2} \ldots 0^{n_h}1^{m_h}$, where the powers denotes the repetition of binary number in the code. A threshold graph is a $\{P_4, C_4, 2K_2\}$-free graph, that is, it does not have induced subgraphs isomorphic to the path P_4, the cycle C_4 or the pair of non-adjacent edges $2K_2$. A threshold graph is presented in Figure 1, its vertex set is partitioned into sets (cells) L_i and U_i, $(i = 1, 2, \ldots, h)$. The vertices in $U_1 \cup \cdots \cup U_h$ induce a clique, while the vertices in $L_1 \cup \cdots \cup L_h$ induce a co-clique (a non empty totally disconnected graph), that is, a non empty graph without any edge. There are cross edges from

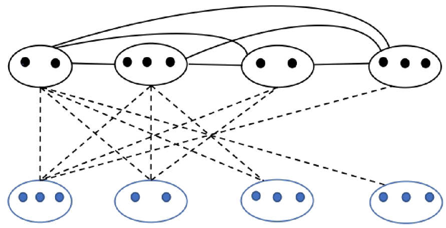

FIGURE 1. Schematic representation of a threshold graph $T(2,3,2,3;3,2,3,3)$.

each vertex of $v_{ij} \in L_i$, $(i = 1, 2, \ldots, h, j = 1, 2, \ldots, m_i)$ to every vertex $u_{ik} \in \bigcup_{j=1}^{h-(i-1)} U_j$, $(k = 1, 2, \ldots, n_i)$ and $j = 1, 2, \ldots, h$. From onwards, we denote a threshold graph by $T(b) = T(0^{n_1}1^{m_1}0^{n_2}1^{m_2}\ldots 0^{n_h}1^{m_h})$ by $T(m_1, m_2, \ldots, m_h; n_1, n_2, \ldots, n_h)$ with order $n = \sum_{i=1}^{h} m_i + \sum_{i=1}^{t} n_i$. We note that authors used different notations for threshold graphs NSG (nested split graph) [33], $G(b)$ [4]. For $h = 1$, we get the complete split graph $T(m_1; n_1)$, a graph with clique K_{m_1} and and independent set of cardinality n_1 such that each vertex of K_{m_1} is adjacent to the every vertex of independent set of size n_1. Further for $h = 1$ and $m_1 = 1$, we obtain a star graph $T(1; n_1)$ of order $n_1 + 1$. Figure 1 shows a threshold graph $T(2,3,2,3;3,2,3,3)$ of order 21, where black dots are U_i's and blue dots denote L_i's. The lines (dashed) between two sets L_i and U_j means that each vertex of L_i is adjacent to every vertex of U_j, while the solid lines between U_i's represent each vertex is adjacent themselves and they form a clique.

Jacobs *et al.* [21] obtained the bounds for the energy and the Laplacian energy of threshold graphs. Milica *et al.* [4] studied the eccentricity eigenvalue distribution of threshold graphs and characterized graphs with free extended intervals, also eigenvalue-free intervals of distance based matrices of threshold and chain graphs can be seen in [2]. Dahl [11] related Laplacian energy with majorization of graphs. Qiu and Tang [27] studied the eccentricity spectra of threshold graphs. Motivated by this, we study the spectral properties of the $S(G)$ of graphs, we investigate the eigenvalue distribution, eigenvalue multiplicities, spread, and present the sharp bounds for the $\mathsf{E}(S(G))$. With some parallel computations, our theory is equally valid for the adjacency matrix, ABC (atom-bond connectivity) matrix, geometric-arithmetic matrix and other adjacency weighted type of matrices.

In Section 2, we give the Sombor index of threshold graphs and discuss some consequences. Section 3 introduces the Sombor eigenvalue of thresholds graphs, the Sombor spectral invariants like the determinant and the trace of quotient matrix, Sombor eigenvalues distribution, inequalities on the largest, smallest Sombor eigenvalues, Sombor spread and graphs with simple Sombor eigenvalues. Section 4 presents the sharp upper/lower bounds for the Sombor energy $\mathsf{E}(S(G))$ along with their extremal characterization. Section 5 gives conclusion and says that the theory is equally valid for adjacency matrix, geometric-arithmetic matrix, ABC matrix and other adjacency type matrices.

2. Sombor index of threshold graphs

The very first result gives the closed formulae for the Sombor index of threshold graph families.

Theorem 2.1. *Let G be a threshold graph of order n. Then*

$$\mathsf{SO}(G) = \sum_{i=1}^{h} \frac{m_i(m_i-1)}{\sqrt{2}} d_{u_i} + \sum_{i=1}^{h}\sum_{j=i+1}^{h} m_i m_j \sqrt{d_{u_i}^2 + d_{u_j}^2} + \sum_{i=1}^{h}\sum_{j=1}^{h-(i-1)} m_i n_j \sqrt{d_{u_i}^2 + d_{v_j}^2},$$

where $d_{u_i} = \sum_{j=1}^{h} m_j - 1 + \sum_{j=1}^{h-(i-1)} n_j$ and $d_{v_i} = \sum_{j=1}^{h-(i-1)} m_j$, for $i = 1, 2, \ldots, h$.

Proof. We label the vertices of G from L_i's to U_i's for $i = 1, \dots, h$. Let $v_{ij}, i = 1, \dots, h$ and $j = 1, \dots, m_i$ be the vertices of L_i and $u_{ij}, i = 1, 2, \dots, h$ and $j = 1, 2, \dots n_i$ be the vertices of U_i. Note that

$$d(u_{11}) = d(u_{12}) = \cdots = d(u_{1m_1}) = \sum_{i=1}^{h} m_i + \sum_{i=1}^{h} n_j - 1,$$

as u_i's form a clique in G. We represent this common degree by d_{u_1}. Similarly, d_{u_i} is the common degree of the vertices of $U_i, i = 2, 3, \dots, h$ and in general, we have

$$d_{u_i} = \sum_{i=1}^{h} m_i - 1 + \sum_{j=1}^{h-(i-1)} n_i, \quad \text{for} \quad i = 1, 2, \dots, h.$$

Similarly, let d_{v_i} be the common degree of each $L_i, i = 1, 2, \dots, n_h$. As with above idea, the degrees of v_i's are

$$d_{v_i} = \sum_{j=1}^{h-(i-1)} m_j, \quad \text{for} \quad i = 1, 2, \dots, h.$$

The number of edges inside each U_i are $m_i(m_i - 1)/2$, the number of edges from each vertex of U_i to every vertex of L_j's are $m_i n_j$ and the total number of edges from each vertex of U_i to each vertex of L_j are $\sum_{i=1}^{h} \sum_{j=1}^{h-(i-1)} m_j n_j$.

Therefore with above information, the Sombor index of G is

$$\mathsf{SO}(G) = \sum_{i=1}^{h} \frac{m_i(m_i - 1)}{\sqrt{2}} d_{u_i} + \sum_{i=1}^{h} \sum_{j=i+1}^{h} m_i m_j \sqrt{d_{u_i}^2 + d_{u_j}^2} + \sum_{i=1}^{h} \sum_{j=1}^{h-(i-1)} m_i n_j \sqrt{d_{u_i}^2 + d_{v_i}^2},$$

where $d_{u_i} = \sum_{j=1}^{h} m_j - 1 + \sum_{j=1}^{h-(i-1)} n_j$ and $d_{v_i} = \sum_{j=1}^{h-(i-1)} m_j$, for $i = 1, 2, \dots, h$. □

The following is an immediate consequences of Theorem 2.1.

Corollary 2.2. *The following hold for* $G \cong T(m_1, m_2, \dots, m_h; n_1, n_2, \dots, n_h)$.

(i) For $h = 1$, $G \cong T(m_1; n_2)$ *(complete split graph) and its Sombor index is*

$$\mathsf{SO}(G) = \frac{m_1(m_1 - 1)}{\sqrt{2}} d_{u_1} + m_1 n_1 \sqrt{d_{u_1}^2 + d_{v_1}^2},$$

where $d_{u_1} = m_1 - 1 + n_2$ *and* $d_{v_1} = n_1$.

(ii) For $h = 1$, *with* $m_1 = 1$, $G \cong T(1; n_2)$ *(star graph) and its Sombor index is*

$$\mathsf{SO}(G) = n_1 \sqrt{d_{u_1}^2 + d_{v_1}^2},$$

where $d_{u_1} = n_2$ *and* $d_{v_1} = 1$.

Theorem 2.1 gives a closed expression for the Sombor index of T, but when order increases, the computations significantly increases, so search for better bounds can be useful in such situations.

3. Sombor spectral properties of threshold graphs

The real sequence $a_1' \geq a_2' \geq \cdots \geq a_m'$ is said to interlace the another real sequence $a_1 \geq a_2 \geq \cdots \geq a_n$ $(m < n)$ if $a_i \geq a_i' \geq a_{n-m+i}$, for $i = 1, 2, \ldots, m$ and the interlacing is said to be tight if there exists a positive integer $k \in [0, m]$ such that

$$a_i = a_i' \quad \text{for} \quad i = 1, 2, \ldots, k \quad \text{and} \quad a_{n-m+i} = a_i' \quad \text{for} \quad k+1 \leq i \leq m.$$

Consider a square matrix M of order n in block form

$$M = \begin{pmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,s-1} & A_{1,s} \\ A_{2,1} & A_{2,2} & \cdots & A_{2,s-1} & A_{2,s} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A_{s-1,1} & A_{s-1,2} & \cdots & A_{s-1,s-1} & A_{s-1,s} \\ A_{s,1} & A_{s,2} & \cdots & A_{s,s-1} & A_{s,s} \end{pmatrix},$$

such that its rows/columns are indexed on the basis of a partition $\pi = \{\ell_1, \ell_2, \ldots, \ell_s\}$ of the index set $I = \{1, \ldots, n\}$. The quotient matrix $Q = (q_{ij})_{s\times s}$ (see [7]) is a square matrix of order s, with its (i,j)-th entry equal to the average row sum of the block A_{ij} of M. If each block $A_{i,j}$ of M has row sum equal to some constant, then partition π is *equitable* (or regular), and Q is called an *equitable quotient* matrix.

The relation between the eigenvalues of M and Q are given below.

Theorem 3.1 ([7]). *Let $M_{n\times n}$ be a real symmetric matrix and Q be its quotient matrix of order $m, (n > m)$. Then*

(i) if π is not equitable, then interlacing holds

$$\lambda_i(M) \geq \lambda_i(Q) \geq \lambda_{i+n-m}(M), \quad \text{for} \quad i = 1, 2, \ldots, m,$$

(ii) if π is equitable, then the spectrum of Q as a set is contained in the spectrum of M.

For the matrix $S(G)$, the Frobenius norm is $\xi_1^2 + \xi_2^2 + \cdots + \xi_n^2 = \|S(G)\|_F^2$, and energy is $\mathsf{E}(S(G)) = |\xi_1| + |\xi_2| + \cdots + |\xi_n|$. We recall that $\|S(G)\|_F^2$ and $\mathsf{E}(S(G))$ are norms for a normal matrix $(S(G))$ [25, 30].

We have the following known results, which will be used later.

Lemma 3.2 ([26]). *Let G be a connected graph with n vertices and $\omega = \{u_1, u_2, \ldots, u_\omega\}$ be a clique in G such that $N(u_i) - \omega = N(u_j) - \omega, 1 \leq i, j \leq \omega$. Then $-d\sqrt{2}$ is the eigenvalue of $S(G)$ with multiplicity greater or equal to $\omega - 1$, where $d = d_{u_i}$, for $i = 1, 2, \ldots, \omega$.*

Lemma 3.3 ([26]). *For a graph connected G with independent set $\{u_1, \ldots, u_\alpha\}$ such that $N(u_i) = N(u_j)$ for all $1 \leq i, j \leq \alpha$. Then 0 is the Sombor eigenvalue of G with multiplicity greater or equal to $\alpha - 1$.*

We sketch the proof of Lemmas 3.2 and 3.3 is similar.

We first index the vertex of the independent set $I = \{u_1, \ldots, u_\alpha\}$ of $S(G)$ such that each vertex share the common neighbourhood. So that the Sombor matrix of G can be written as

$$S(G) = \begin{pmatrix} 0_\alpha & D_{\alpha\times(n-\alpha)} \\ D_{(n-\alpha)\times\alpha} & S'(G(V \setminus I))_{n-\alpha} \end{pmatrix},$$

where $S'(G(V \setminus I))_{n-\alpha}$ is the part of $S(G)$ on the vertex set $V(G) \setminus I$. Now, it is easy to see that

$$X_{j-1} = (-1, x_{2j}, x_{3j}, \ldots, x_{\alpha j}, \underbrace{0, 0, \ldots, 0}_{n-\alpha})^T, \quad \text{for} \quad j = 2, 3, \ldots, \alpha,$$

where x_{ij} is the Kronecker delta $\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$. Clearly, 0 is the eigenvalues of $S(G)$ with eigenvectors $X_1, \dots, X_{\alpha-1}$.

With d_{u_i} given in Theorem 2.1, in the next result, we discuss the Sombor spectrum of threshold graphs.

Theorem 3.4. *Let G be a threshold graph. Then the Sombor spectrum of G consists of the eigenvalue $-d_{u_i}\sqrt{2}$ with multiplicity $m_i - 1$, the eigenvalue 0 with multiplicity $\sum_{i=1}^{h} n_i - h$, and the remaining $2h$ Sombor eigenvalues of G are the eigenvalues of the matrix given in (3.1).*

Proof. Indexing the vertices from U_i's to V_j's, the Sombor matrix of G is

$$\begin{pmatrix} A_{m_1+m_2+\dots+m_h} & B_{(m_1+m_2+\dots+m_h)\times(n_1+n_2+\dots+n_h)} \\ B^T & \mathbf{0}_{n_1+n_2+\dots+n_h} \end{pmatrix}, \tag{3.1}$$

where $\mathbf{0}$ is a zero matrix and A and B are given below:

$$A = \begin{pmatrix} d_{u_1}\sqrt{2}(J_{m_1} - I_{m_1}) & \sqrt{d_{u_1}^2 + d_{u_2}^2} J_{m_1\times m_2} & \dots & \sqrt{d_{u_1}^2 + d_{u_{h-1}}^2} J_{m_1\times m_{h-1}} & \sqrt{d_{u_1}^2 + d_{u_h}^2} J_{m_1\times m_h} \\ \sqrt{d_{u_1}^2 + d_{u_2}^2} J_{m_2\times m_1} & d_{u_2}\sqrt{2}(J_{m_2} - I_{m_2}) & \dots & \sqrt{d_{u_2}^2 + d_{u_{h-1}}^2} J_{m_2\times m_{h-1}} & \sqrt{d_{u_2}^2 + d_{u_h}^2} J_{m_2\times m_h} \\ \vdots & \vdots & \ddots & \vdots & \\ \sqrt{d_{u_1}^2 + d_{u_{h-1}}^2} J_{m_{h-1}\times m_1} & \sqrt{d_{u_2}^2 + d_{u_{h-1}}^2} J_{m_{h-1}\times m_2} & \dots & d_{u_{h-1}}\sqrt{2}(J_{m_{h-1}} - I_{m_{h-1}}) & \sqrt{d_{u_h}^2 + d_{u_{h-1}}^2} J_{m_{h-1}\times m_h} \\ \sqrt{d_{u_1}^2 + d_{u_h}^2} J_{m_h\times m_1} & \sqrt{d_{u_2}^2 + d_{u_h}^2} J_{m_h\times m_2} & \dots & \sqrt{d_{u_{h-1}}^2 + d_{u_h}^2} J_{m_h\times m_{h-1}} & d_{u_h}\sqrt{2}(J_{m_h} - I_{m_h}) \end{pmatrix}$$

$$B = \begin{pmatrix} \sqrt{d_{u_1}^2 + d_{v_1}^2} J_{m_1\times n_1} & \sqrt{d_{u_1}^2 + d_{v_2}^2} J_{m_1\times n_2} & \dots & \sqrt{d_{u_1}^2 + d_{v_{h-1}}^2} J_{m_1\times n_{h-1}} & \sqrt{d_{u_1}^2 + d_{v_h}^2} J_{m_1\times n_h} \\ \sqrt{d_{u_2}^2 + d_{v_1}^2} J_{m_2\times n_1} & \sqrt{d_{u_2}^2 + d_{v_2}^2} J_{m_2\times n_2} & \dots & \sqrt{d_{u_2}^2 + d_{v_{h-1}}^2} J_{m_2\times n_{h-1}} & \mathbf{0}_{m_2\times n_h} \\ \sqrt{d_{u_3}^2 + d_{uv_1}^2} J_{m_3\times n_1} & \sqrt{d_{u_3}^2 + d_{v_2}^2} J_{m_3\times n_2} & \dots & \mathbf{0}_{m_3\times n_{h-1}} & \mathbf{0}_{m_3\times n_h} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \sqrt{d_{u_{h-1}}^2 + d_{v_1}^2} J_{m_{h-1}\times n_1} & \sqrt{d_{u_{h-1}}^2 + d_{v_2}^2} J_{m_{h-1}\times n_2} & \dots & \mathbf{0}_{m_{h-1}\times n_{h-1}} & \mathbf{0}_{m_{h-1}\times n_h} \\ \sqrt{d_{u_h}^2 + d_{v_1}^2} J_{m_h\times n_1} & \mathbf{0}_{m_h\times n_2} & \dots & \mathbf{0}_{m_h\times n_{h-1}} & \mathbf{0}_{m_h\times n_h} \end{pmatrix}.$$

As each $U_i, i = 1, 2, \dots, h$ is a clique and each vertex of such cliques share the same neighbourhood, so by Lemma 3.2, G has the Sombor eigenvalue $-d_{u_i}\sqrt{2}$ with multiplicity $m_i - 1$, where d_{u_i} represents the common degree of clique induced by U_i. Also, each $L_i, i = 1, 2, \dots, h$ is an independent set and each vertex of L_i share the same common neighbourhood. So, from Lemma 3.3, it follows that 0 is the eigenvalue of $S(G)$ with multiplicity $\sum_{i=1}^{h} n_i - h$. The remaining Sombor eigenvalues of G are the eigenvalues of the following quotient matrix with partitions $\{|U_1|, |U_2|, \dots, |U_h|, |L_1|, |L_2|, \dots, |L_h|\}$

$$Q = \begin{pmatrix} \mathcal{A}_h & \mathcal{B}_h \\ \mathcal{C}_h & \mathbf{0}_h \end{pmatrix}, \tag{3.2}$$

where

$$\mathcal{A} = \begin{pmatrix} (m_1 - 1)d_{u_1}\sqrt{2} & m_2\sqrt{d_{u_1}^2 + d_{u_2}^2} & \dots & m_{h-1}\sqrt{d_{u_1}^2 + d_{u_{h-1}}^2} & m_h\sqrt{d_{u_1}^2 + d_{u_h}^2} \\ m_1\sqrt{d_{u_1}^2 + d_{u_2}^2} & (m_2 - 1)d_{u_2}\sqrt{2} & \dots & m_{h-1}\sqrt{d_{u_{h-1}}^2 + d_{u_2}^2} & m_h\sqrt{d_{u_h}^2 + d_{u_2}^2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ m_1\sqrt{d_{u_1}^2 + d_{u_{h-1}}^2} & m_2\sqrt{d_{u_2}^2 + d_{u_{h-1}}^2} & \dots & (m_{h-1} - 1)d_{u_{h-1}}\sqrt{2} & m_h\sqrt{d_{u_h}^2 + d_{u_{h-1}}^2} \\ m_1\sqrt{d_{u_1}^2 + d_{u_h}^2} & m_2\sqrt{d_{u_2}^2 + d_{u_h}^2} & \dots & m_{h-1}\sqrt{d_{u_{h-1}}^2 + d_{u_h}^2} & (m_h - 1)d_{u_h}\sqrt{2} \end{pmatrix}$$

$$\mathcal{B}=\begin{pmatrix} n_1\sqrt{d_{u_1}^2+d_{v_1}^2} & n_2\sqrt{d_{u_1}^2+d_{v_2}^2} & \dots & n_{h-1}\sqrt{d_{u_1}^2+d_{v_{h-1}}^2} & n_h\sqrt{d_{u_1}^2+d_{v_h}^2}\\ n_1\sqrt{d_{u_2}^2+d_{v_1}^2} & n_2\sqrt{d_{u_2}^2+d_{v_2}^2} & \dots & n_{h-1}\sqrt{d_{u_2}^2+d_{v_{h-1}}^2} & 0\\ n_1\sqrt{d_{u_3}^2+d_{v_1}^2} & n_2\sqrt{d_{u_3}^2+d_{v_2}^2} & \dots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ n_1\sqrt{d_{u_{h-1}}^2+d_{v_1}^2} & n_2\sqrt{d_{u_{h-1}}^2+d_{v_2}^2} & \dots & 0 & 0\\ n_1\sqrt{d_{u_h}^2+d_{v_1}^2} & 0 & \dots & 0 & 0 \end{pmatrix}$$

$$\mathcal{C}=\begin{pmatrix} m_1\sqrt{d_{u_1}^2+d_{v_1}^2} & m_2\sqrt{d_{u_2}^2+d_{v_1}^2} & \dots & m_{h-1}\sqrt{d_{u_{h-1}}^2+d_{v_1}^2} & m_h\sqrt{d_{u_h}^2+d_{v_1}^2}\\ m_1\sqrt{d_{u_1}^2+d_{v_2}^2} & m_2\sqrt{d_{u_2}^2+d_{v_2}^2} & \dots & m_{h-1}\sqrt{d_{u_{h-1}}^2+d_{v_2}^2} & 0\\ m_1\sqrt{d_{u_1}^2+d_{v_3}^2} & m_2\sqrt{d_{u_2}^2+d_{v_3}^2} & \dots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ m_1\sqrt{d_{u_1}^2+d_{v_h}^2} & m_2\sqrt{d_{u_2}^2+d_{v_{h-1}}^2} & \dots & 0 & 0\\ m_1\sqrt{d_{u_1}^2+d_{v_h}^2} & 0 & \dots & 0 & 0 \end{pmatrix}.$$

□

It is not possible to find all the eigenvalues of the matrix given in (3.1) explicitly, since by the definition of the quotient matrix, all its eigenvalues are simple. We will establish bounds for the largest and the smallest eigenvalues of matrix Q, evaluate its determinant and trace of Q^2, which are later used in our results.

Next, we find the determinant of Q and the trace of matrix Q^2.

Corollary 3.5. *The following hold for the matrix given in* (3.2)*.*

(i) The determinant of matrix Q is

$$\det(Q)=(-1)^h\prod_{i=1}^{h}m_in_i\Big(d_{u_i}^2+d_{v_{h-(i-1)}}^2\Big)=(-1)^h\prod_{i=1}^{h}m_in_i\Big(d_{u_{h-(i-1)}}^2+d_{v_i}^2\Big).$$

(ii) The trace of Q^2 is

$$\operatorname{tr}\big(Q^2\big)=2\sum_{i=1}^{h}d_{u_i}^2(m_i-1)^2+\sum_{j=1}^{h}\sum_{\substack{i=1\\ i\neq j}}^{h}m_im_j\Big(d_{u_i}^2+d_{u_j}^2\Big)+2\sum_{j=1}^{h}\sum_{i=1}^{h-(j-1)}m_jn_i\Big(d_{u_j}^2+d_{v_i}^2\Big).$$

Proof. (i) We denote matrix Q given in (3.2) by $Q(m_1,m_2,\dots,m_h,n_1,n_2,\dots,n_h)$. We expand $\det(Q(m_1,m_2,\dots,m_h,n_1,n_2,\dots,n_h))$ by $2h$-th row, there is only one non-zero cofactor $\det(Q(m_1,m_2,\dots,m_h;n_1,n_2,\dots,n_{h-2}))$ multiplied by $m_1\sqrt{d_{u_1}^2+d_{u_h}^2}$, as the other entries of $2h$-th row are zeros. Similarly, for $(2h-1)$-th cofactor expansion there is one non-zero cofactor of size $2h-2$ (that is, $\det(Q(m_1,m_2,\dots,m_h;n_1,n_2,\dots,n_{h-2})))$ multiplied by $m_2\sqrt{d_{u_2}^2+d_{u_{h-1}}^2}$ and so on. The process is illustrated below

$$\begin{aligned}&\det(Q(m_1,m_2,\dots,m_h,n_1,n_2,\dots,n_h))\\ &\quad=(-1)m_1\sqrt{d_{u_1}^2+d_{u_h}^2}\,m_2\sqrt{d_{u_2}^2+d_{u_{h-1}}^2}\det(Q(m_1,m_2,\dots,m_h,n_1,n_2,\dots,n_{h-2}))\\ &\quad=(-1)^2\prod_{i=1}^{4}m_i\sqrt{d_{u_i}^2+d_{u_{h-(i-1)}}^2}\det(Q(m_1,m_2,\dots,m_h;n_1,n_2,\dots,n_{h-4}))\end{aligned}$$

$$\begin{aligned}
&\vdots\\
&=(-1)^{\frac{h}{2}}\prod_{i=1}^{h} m_i\sqrt{d_{u_i}^2+d_{u_{h-(i-1)}}^2}\det(Q(m_1,m_2,\ldots,m_h))\\
&=(-1)^{\frac{h}{2}+1}n_1m_hn_2m_{h-1}\left(d_{u_h}^2+d_{u_1}^2\right)\left(d_{u_{h-1}}^2+d_{u_2}^2\right)\cdot\prod_{i=1}^{h-2} m_i\sqrt{d_{u_i}^2+d_{u_{h-(i-1)}}^2}\det(Q(m_1,m_2,\ldots,m_{h-1}))\\
&\vdots\\
&=(-1)^{\frac{h}{2}+\frac{h-1}{2}}\prod_{i=1}^{h-2} m_{h-(i-1)}n_i\left(d_{u_{h-(i-1)}}^2+d_{u_i}^2\right)\cdot\prod_{i=h-1}^{h} m_{h-(i-1)}\sqrt{d_{u_{h-(i-1)}}^2+d_{u_i}^2}\det(Q(m_1,m_2))\\
&=(-1)^{h}\prod_{i=1}^{h} m_in_i\left(d_{u_i}^2+d_{u_{h-(i-1)}}^2\right)=(-1)^{h}\prod_{i=1}^{h} m_in_i\left(d_{u_{h-(i-1)}}^2+d_{u_i}^2\right).
\end{aligned}$$

(ii) By computing Q^2, the diagonal entries are

$$\begin{aligned}
d_{11}&=2d_{u_1}^2(m_1-1)^2+\sum_{\substack{i=1\\i\neq 1}}^{h} m_1m_i\left(d_{u_1}^2+d_{u_i}^2\right)+m_1\sum_{i=1}^{h} n_i\left(d_{u_1^2}+d_{v_i}^2\right),\\
d_{22}&=2d_{u_2}^2(m_2-1)^2+\sum_{\substack{i=1\\i\neq 2}}^{h} m_2m_i\left(d_{u_2}^2+d_{u_i}^2\right)+m_2\sum_{i=1}^{h-1} n_i\left(d_{u_2^2}+d_{v_i}^2\right),\\
&\vdots\\
d_{(2h-1)(2h-1)}&=n_{h-1}\sum_{i=1}^{2} m_i\left(d_{u_i^2}+d_{v_{h-1}}^2\right), d_{(2h)(2h)}=n_hm_1\left(d_{u_1^2}+d_{v_h}^2\right).
\end{aligned}$$

Summing all these expressions, we have

$$\operatorname{tr}\left(Q^2\right)=2\sum_{i=1}^{h} d_{u_i}^2(m_i-1)^2+\sum_{j=1}^{h}\sum_{\substack{i=1\\i\neq j}}^{h} m_im_j\left(d_{u_i}^2+d_{u_j}^2\right)+2\sum_{j=1}^{h}\sum_{i=1}^{h-(j-1)} m_jn_i\left(d_{u_j}^2+d_{v_i}^2\right).$$

□

The eigenvalues of Q are simple, that is, equivalently saying that the Sombor eigenvalues of $G(\underbrace{1,1,\ldots,1}_{h};\underbrace{1,1,\ldots,1}_{h})$ are simple, since Lemmas 3.2 and 3.3 have no roles. We ask the following question:

What happens if at least one $m_i \geq 2$ (or respectively, $n_i \geq 2$)? In this case, we have the following consequence from Theorem 3.4.

Corollary 3.6. *Let $G \cong T(m_1, m_2, \ldots, m_{h-1}, m_h; n_1, n_2, \ldots, n_h)$ be the threshold graph of order n with $m_h = 2$ and $n_1 = 1$. Then $-d\sqrt{2}$ is the eigenvalue of $S(G)$ with multiplicity 2, where $d = m_1 + m_2 + \cdots + m_{h-1} + 2$.*

Proof. Let $\{u_1, u_2\}$ and $\{u_3\}$ be the vertices of U_h and L_1, respectively. From the structure of the threshold graph, it is clear that $\{u_1, u_2, u_3\}$ induces a clique K_3, and each u_i share the same neighbourhood $\bigcup_{i=1}^{h-1} U_i$ of cardinality $\sum_{i=1}^{h-1} m_i$. Therefore, Lemma 3.2 implies that $\sqrt{2}(\sum_{i=1}^{h-1} m_i + 2)$ is the eigenvalue of $S(G)$ with multiplicity 2. □

Next, we put restriction on other m_i's (respectively n_j's) for $i = 1, 2, \dots, h-1$ and $j = 1, 2, \dots, h$.

Corollary 3.7. *Let $G \cong T(m_1, m_2, \dots, m_{h-1}, m_h; n_1, n_2, \dots, n_h)$ be the threshold graph. Then the Sombor eigenvalues of $S(G)$ are simple iff (if and only if) $m_i \le 2$, for all i and $n_i \le 2$ for exactly one i with exception that $m_h \ne 2$ and $n_1 \ne 1$ simultaneously.*

Proof. The exception case $G \cong T(m_1, m_2, \dots, m_{h-1}, 2; 1, n_2, \dots, n_h)$ is done in Corollary 3.6. As the eigenvalue of matrix (3.2) are simple. We only need to check for the cardinalities of m_i's and n_i's. Also with $\det(Q) \ne 0$, Lemma 3.3 implies that 0 is the simple eigenvalue of $S(G)$ iff the cardinality of exactly one L_i is 2. Also, we note that d_{u_i} is distinct for each cell U_i, so by Lemma 3.2, $-d_{u_i}\sqrt{2}$ is the eigenvalue of $S(G)$ with multiplicity one iff the cardinality of each $U_i = 2$, for all $i = 1, 2, \dots, h$ except with choice $m_h = 2$ and $n_1 = 1$. □

Corollaries 3.6 and 3.7 give us an algorithm, we need to discard those graphs where $m_h = 2$ and $n_1 = 1$ for the candidate graphs without repeated Sombor eigenvalues. A similar type of algorithm for graphs with the simple Laplacian eigenvalues can be seen in [3].

Algorithm 1. Chain graphs with 2h cells and without repeated SE.

```
Input: Threshold graph G ≅ T(m_1, m_2, ..., m_{h-1}, m_h; n_1, n_2, ..., n_h) with m_i ∈ {1,2} and exactly one n_i = 2,
for some i.
Output: G without repeated SE.
Algorithm find (G with simple SE.)
𝕀 := {1, 2, ..., h}
𝔽 := {m_1, m_2, ..., m_{h-1}, m_h; n_1, n_2, ..., n_h}, where m_i ∈ {1,2} for all i ∈ 𝕀 and exactly n_i = 2
→ Test
for m_i ∈ {1,2}
    if m_h = 1 return G is without repeated eigenvalues
    else if m_h = 2 and n_1 = 2 return G is without repeated eigenvalues
        else
            m_h = 2 and n_1 = 1 return G is with repeated eigenvalues
end loop
```

For $h = 3$, and $m_i \le 2$ ($n_i \le 2$), there are 32 non-isomorphic graphs eligible for simple Sombor. These graphs are listed below:

$$T(1,1,1;1,1,1),\ T(2,1,1;1,1,1),\ T(1,2,1;1,1,1),\ T(1,1,2;1,1,1),\ T(2,2,1;1,1,1),$$
$$T(2,1,2;1,1,1),\ T(1,2,2;1,1,1),\ T(2,2,2;1,1,1),\ T(1,1,1;2,1,1),\ T(1,1,1;1,2,1),$$
$$T(1,1,1;1,1,2),\ T(2,1,1;2,1,1),\ T(2,1,1;1,2,1),\ T(2,1,1;1,1,2),\ T(1,2,1;2,1,1),$$
$$T(1,2,1;1,2,1),\ T(1,2,1;1,1,2),\ T(1,1,2;2,1,1),\ T(1,1,2;1,2,1),\ T(1,1,2;1,1,2),$$
$$T(2,2,1;2,1,1),\ T(2,2,1;1,2,1),\ T(2,2,1;1,1,2),\ T(2,1,2;2,1,1),\ T(2,1,2;1,2,1),$$
$$T(2,1,2;1,1,2),\ T(1,2,2;2,1,1),\ T(1,2,2;1,2,1),\ T(1,2,2;1,1,2),\ T(2,2,2;2,1,1),$$
$$T(2,2,2;1,2,1),\ T(2,2,2;1,1,2).$$

Based on Corollaries 3.6 and 3.7 and Algorithm 1, there are exactly 20 graphs with simple Sombor eigenvalues and are listed in Table 1 along with their Sombor spectrum.

Our next result establishes bounds for the number of distinct Sombor eigenvalues for threshold graphs.

Corollary 3.8. *Let α be the number of distinct Sombor eigenvalues of the threshold graph G. Then*

$$2h \le \alpha \le 3h + 1,$$

TABLE 1. Threshold graphs with simple Sombor eigenvalue for $h = 3$.

Graph G	Sombor spectrum
$T(1,1,1;1,1,1)$	18.2555, 1.94335, 0.91183, −4.24264, −6.72151, −10.1465
$T(2,1,1;1,1,1)$	32.4487, 3.05619, 1.28781, −5.65685, −8.48528, −8.53571, −14.1148
$T(1,2,1;1,1,1)$	30.2198, 2.56269, 1.60231, −5.65685, −7.07107, −9.22891, −12.428
$T(2,2,1;1,1,1)$	47.2402, 4.11152, 2.12289, −7.07107, −8.48528, −9.89949, −11.3241, −16.6947
$T(1,1,1;2,1,1)$	25.1233, 2.49302, 1.10019,0,−6.74953, −8.68487,−13.2821
$T(1,1,1;1,2,1)$	23.0429, 2.40407, 1.35974, 0, −4.24264, −8.74963, −13.8145
$T(1,1,1;1,1,2)$	20.652, 3.05139, 1.05983, 0, −4.24264, −6.90506, −13.6155
$T(2,1,1;2,1,1)$	41.8582, 3.82506, 1.4847, 0, −8.57047, −9.89949, −10.6197, −18.0782
$T(2,1,1;1,2,1)$	39.6118, 3.53404, 1.9181, 0, −5.65685, −9.89949, −10.6681, −18.8395
$T(2,1,1;1,1,2)$	37.1512, 4.56461, 1.45073, 0, −5.65685, −8.82102, −9.89949, −18.7892
$T(1,2,1;2,1,1)$	39.5414, 3.13126, 1.89388, 0, −8.48528, −8.5736, −11.3823, −16.1253
$T(1,2,1;1,2,1)$	37.1367, 3.14266, 2.24935, 0, −5.65685, −8.48528, −11.4522, −16.9344
$T(1,2,1;1,1,2)$	32.324, 4.36698, 1.75558, 0, −5.65685, −7.07107, −9.67506, −16.0436
$T(1,1,2;2,1,1)$	37.1157, 4.07037, 1.42022, 0, −7.07107, −9.19296, −10.4904, −15.8519
$T(2,2,1;2,1,1)$	59.1275, 4.89996, 2.46477, 0, 2.46477, −10.4023, −11.3137, −13.7336, −21.1432
$T(2,2,1;1,2,1)$	56.576, 4.64244, 3.09889, 0, −7.07107, −9.89949, −11.3137, −13.8449, −22.1882
$T(2,2,1;1,1,2)$	51.5529, 6.70421, 2.24098, 0, −7.07107, −8.48528, −11.3137, −11.8687, −21.7593
$T(2,1,2;2,1,1)$	56.5648, 6.13342, 1.88402, 0, −8.48528, −11.1149, −11.3137, −12.5368, −21.1316
$T(1,2,2;2,1,1)$	54.3856, 4.9109, 2.45069, 0, −8.48528, −9.89949, −11.2519, −13.2328, −18.8777
$T(2,2,2;2,1,1)$	76.69, 7.15568, 3.33604, 0, −9.89949, −11.3137, −12.7279, −13.2511, −15.6854, −24.3041

with equality on left iff $G \cong T(1,1,\dots,1;1,1,\dots,1)$ *and equality on right iff*

$$G \cong T(m_1, m_2, \dots, m_h; n_1, n_2, \dots, n_h),$$

where $m_i \geq 2$, *for* $i = 1, 2, \dots, h$ *and* $n_i \geq 2$, *for at least one* i *with exception that* $m_h = 2$ *and* $n_1 = 1$.

Corollary 3.9. *Let* $G \cong T(m_1, m_2, \dots, m_{h-1}, 2; 1, n_2, \dots, n_h), m_i \geq 2, i = 1, 2, \dots, h-1$ *and* $n_i = 2$ *for some* i. *Then* G *has* $3h$ *distinct SE.*

The following corollary gives the multiplicity of the Sombor eigenvalue 0, the lower bound for the number of the positive Sombor eigenvalue of $T(m_1, m_2 \dots, m_h; n_1, n_2, \dots, n_h)$.

Corollary 3.10. *For* $G \cong T(m_1, m_2, \dots, m_h; n_1, n_2, \dots, n_h), m_i, n_i \geq 2$, *the multiplicity of the Sombor eigenvalue* 0 *is exactly* $\sum_{i=1}^{h} n_i - h$, *the number of the negative Sombor eigenvalues of* G *is at least* $\sum_{i=1}^{h} m_i - h$ *and the number of positive Sombor eigenvalues is at most* h.

One of the interesting problem posed in [5] is looking for conditions on parent matrix M, so that the spectrum of equitable quotient matrix contains all the different eigenvalues. The problem was studied in [28], it was actually about the smallest equitable partitions, and restated as: Characterize those matrices M such that with the smallest possible equitable partition its quotient matrix Q contains all the distinct eigenvalues of M? We consider this problem for $G \cong T(m_1, m_2, \dots, m_h; n_1, n_2, \dots, n_h), m_i, n_i \geq 2$, and look for the smallest possible equitable partition so that quotient matrix contains all the distinct eigenvalues of $S(G)$. Since G has at most $3h+1$, so the order of the equitable quotient matrix can be at most $3h+1$. One such partition is

$$\{\{1, 2, \dots, m_1 - 1\}, \{m_1\}, \{1, 2, \dots, m_2 - 1\}, \{m_2\}, \dots, \{1, 2, \dots, m_h - 1\}, \{m_h\}\},$$

and any partition cell among $\{1, 2, \dots, n_1\}, \{1, 2, \dots, n_2\}, \dots, \{1, 2, \dots, n_h\}$ can be repartitioned, say $\{1, 2, \dots, n_1\} = \{1, 2, \dots, n_1 - 1\} \cup \{n_1\}$. The corresponding equitable quotient matrix is

$$Q_1 = \begin{pmatrix} \mathcal{A}_{2h} & \mathcal{B}_{2h \times (h+1)} \\ \mathcal{C}_{(h+1) \times 2h} & \mathbf{0}_{h+1} \end{pmatrix},$$

where

$$\mathcal{A}=\begin{pmatrix}
(m_1-2)d_{u_1}\sqrt{2} & d_{u_1}\sqrt{2} & \dots & (m_h-1)\sqrt{d_{u_1}^2+d_{u_h}^2} & \sqrt{d_{u_1}^2+d_{u_h}^2}\\
(m_1-1)d_{u_1}\sqrt{2} & 0 & \dots & (m_h-1)\sqrt{d_{u_1}^2+d_{u_h}^2} & \sqrt{d_{u_1}^2+d_{u_h}^2}\\
(m_1-1)\sqrt{d_{u_1}^2+d_{u_2}^2} & \sqrt{d_{u_1}^2+d_{u_2}^2} & \dots & (m_h-1)\sqrt{d_{u_h}^2+d_{u_2}^2} & \sqrt{d_{u_h}^2+d_{u_2}^2}\\
\vdots & \vdots & \ddots & \vdots & \vdots\\
(m_1-1)\sqrt{d_{u_1}^2+d_{u_{h-1}}^2} & \sqrt{d_{u_1}^2+d_{u_{h-1}}^2} & \dots & (m_h-1)\sqrt{d_{u_h}^2+d_{u_{h-1}}^2} & \sqrt{d_{u_h}^2+d_{u_{h-1}}^2}\\
(m_1-1)\sqrt{d_{u_1}^2+d_{u_h}^2} & \sqrt{d_{u_1}^2+d_{u_h}^2} & \dots & (m_h-2)d_{u_h}\sqrt{2} & d_{u_h}\sqrt{2}\\
(m_1-1)\sqrt{d_{u_1}^2+d_{u_h}^2} & \sqrt{d_{u_1}^2+d_{u_h}^2} & \dots & (m_h-1)d_{u_h}\sqrt{2} & 0
\end{pmatrix},$$

$$\mathcal{B}=\begin{pmatrix}
(n_1-1)\sqrt{d_{u_1}^2+d_{u_1}^2} & \sqrt{d_{u_1}^2+d_{u_1}^2} & n_2\sqrt{d_{u_1}^2+d_{u_2}^2} & \dots & n_{h-1}\sqrt{d_{u_1}^2+d_{u_{h-1}}^2} & n_h\sqrt{d_{u_1}^2+d_{u_h}^2}\\
(n_1-1)\sqrt{d_{u_1}^2+d_{u_1}^2} & \sqrt{d_{u_1}^2+d_{u_1}^2} & n_2\sqrt{d_{u_1}^2+d_{u_2}^2} & \dots & n_{h-1}\sqrt{d_{u_1}^2+d_{u_{h-1}}^2} & n_h\sqrt{d_{u_1}^2+d_{u_h}^2}\\
(n_1-1)\sqrt{d_{u_2}^2+d_{u_1}^2} & \sqrt{d_{u_2}^2+d_{u_1}^2} & n_2\sqrt{d_{u_2}^2+d_{u_2}^2} & \dots & n_{h-1}\sqrt{d_{u_2}^2+d_{u_{h-1}}^2} & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
(n_1-1)\sqrt{d_{u_{h-1}}^2+d_{u_1}^2} & \sqrt{d_{u_{h-1}}^2+d_{u_1}^2} & n_2\sqrt{d_{u_{h-1}}^2+d_{u_2}^2} & \dots & 0 & 0\\
(n_1-1)\sqrt{d_{u_h}^2+d_{u_1}^2} & \sqrt{d_{u_h}^2+d_{u_1}^2} & 0 & \dots & 0 & 0\\
(n_1-1)\sqrt{d_{u_h}^2+d_{u_1}^2} & \sqrt{d_{u_h}^2+d_{u_1}^2} & 0 & \dots & 0 & 0
\end{pmatrix},$$

$$\mathcal{C}=\begin{pmatrix}
(m_1-1)\sqrt{d_{u_1}^2+d_{u_1}^2} & \sqrt{d_{u_1}^2+d_{u_1}^2} & \dots & (m_h-1)\sqrt{d_{u_h}^2+d_{u_1}^2} & \sqrt{d_{u_h}^2+d_{u_1}^2}\\
(m_1-1)\sqrt{d_{u_1}^2+d_{u_1}^2} & \sqrt{d_{u_1}^2+d_{u_1}^2} & \dots & (m_h-1)\sqrt{d_{u_h}^2+d_{u_1}^2} & \sqrt{d_{u_h}^2+d_{u_1}^2}\\
(m_1-1)\sqrt{d_{u_1}^2+d_{u_2}^2} & \sqrt{d_{u_1}^2+d_{u_2}^2} & \dots & 0 & 0\\
\vdots & \vdots & \ddots & \vdots & \vdots\\
(m_1-1)\sqrt{d_{u_1}^2+d_{u_h}^2} & \sqrt{d_{u_1}^2+d_{u_h}^2} & \dots & 0 & 0\\
(m_1-1)\sqrt{d_{u_1}^2+d_{u_h}^2} & \sqrt{d_{u_1}^2+d_{u_h}^2} & \dots & 0 & 0
\end{pmatrix}.$$

The matrix Q_1 contains all the $3h+1$ distinct Sombor eigenvalues of G. We note that there exists many such types of partitions , since each U_i can be partitioned into $\{|U_i-1|\cup\{x\}\}, x\notin U_i\}$ in $\binom{|U_i|}{1}=|U_i|$ choices, for $i=1,2,\dots,h$. It is easy to verify that Q_1 contains the repeated Sombor eigenvalues $d_{u_1}\sqrt{2}, d_{u_2}\sqrt{2},\dots,d_{u_h}\sqrt{2}$, and 0 with associated eigenvectors

$$X_1=\Big(\tfrac{1}{m_1-1},-1,\underbrace{0,0,\dots,0}_{2h-2},\underbrace{0,0,\dots,0}_{h+1}\Big)^T,\quad X_2=\Big(0,0,\tfrac{1}{m_2-1},-1,\underbrace{0,0,\dots,0}_{2h-4},\underbrace{0,0,\dots,0}_{h+1}\Big)^T,$$

$$\vdots$$

$$X_h=\Big(\underbrace{0,0,\dots,0}_{2h-2},\tfrac{1}{m_h-1},-1,\underbrace{0,0,\dots,0}_{h+1}\Big)^T,\quad X_{h+1}=\Big(\underbrace{0,0,\dots,0}_{2h},\tfrac{1}{n_1-1},-1,\underbrace{0,0,\dots,0}_{h-1}\Big)^T.$$

A similar type of partition

$$\Big\{\{1,2,\dots,m_1-1\},\{m_1\},\{1,2,\dots,m_2-1\},\{m_2\},\dots,\{1,2,\dots,m_{h-1}-1\},\{m_{h-1}\},$$
$$\{1,2\},\{1\},\{1,2,\dots,n_2-1\},\{n_2\},\{1,2,\dots,n_3\},\dots,\{1,2,\dots,n_h\}\Big\},$$

can be considered for $T(m_1,m_2,\dots,m_{h-1},2;1,n_2,\dots,n_h)$, so that its associated quotient matrix contains all the $3h$ distinct Sombor eigenvalues. Similarly, depending on the number of distinct Sombor eigenvalues of $T(m_1,m_2,\dots,m_h,n_1,\dots,n_h)$, the partitions can be modified, so that its corresponding quotient matrix contains all the distinct eigenvalue of G. We illustrate the first case with the help of the following example.

Example 3.11. Let $G \cong T(3,3,3,2,2,3)$ be the threshold graph of order 16. Then the quotient matrix contains all the distinct eigenvalues of $S(G)$ with any equitable partition given below:

$$\big\{\{u_1,u_2\},\{u_3\},\{u_4,u_5\},\{u_6\},\{u_7,u_8\},\{u_9\},\{u_{10}\},\{u_{11}\},\{u_{12},u_{13}\},\{u_{14},u_{15},u_{16}\}\big\}.$$

Corollary 3.8 and guarantees that G has 10 distinct eigenvalues. The Sombor matrix of G is

$$S(G)=\begin{pmatrix} 15\sqrt{2}(J_3-I_3) & \sqrt{369}J_{3\times3} & \sqrt{325}J_{3\times3} & \sqrt{306}J_{3\times2} & \sqrt{261}J_{3\times2} & \sqrt{234}J_{3\times3}\\ \sqrt{369}J_{3\times3} & 12\sqrt{2}(J_3-I_3) & \sqrt{244}J_{3\times3} & \sqrt{225}J_{3\times2} & \sqrt{180}J_{3\times2} & \mathbf{0}_{3\times3}\\ \sqrt{325}J_{3\times3} & \sqrt{244}J_{3\times3} & 10\sqrt{2}(J_3-I_3) & \sqrt{181}J_{3\times2} & \mathbf{0}_{3\times2} & \mathbf{0}_{3\times3}\\ \sqrt{306}J_{2\times3} & \sqrt{225}J_{2\times3} & \sqrt{181}J_{2\times3} & \mathbf{0}_2 & \mathbf{0}_{2\times2} & \mathbf{0}_{2\times3}\\ \sqrt{261}J_{2\times3} & \sqrt{180}J_{2\times3} & \mathbf{0}_{2\times3} & \mathbf{0}_{2\times2} & \mathbf{0}_2 & \mathbf{0}_{2\times3}\\ \sqrt{234}J_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times2} & \mathbf{0}_{3\times2} & \mathbf{0}_3 \end{pmatrix}. \tag{3.3}$$

By computer calculations, the eigenvalues of $S(G)$ are

$$\Big\{180.602, 26.1356, 12.5139, 0^{[4]}, (-14.1421)^{[2]}, (-16.9706)^{[2]}, -21.1872, (-21.2132)^{[2]}, -32.2938, -61.1186\Big\}.$$

The quotient matrix of $S(G)$ is

$$Q(S(G))=\begin{pmatrix} 2\cdot15\sqrt{2} & 3\sqrt{369} & 3\sqrt{325} & 2\sqrt{306} & 2\sqrt{261} & 3\sqrt{234}\\ 3\sqrt{369} & 2\cdot12\sqrt{2} & 3\sqrt{244} & 2\sqrt{225} & 2\sqrt{180} & 0\\ 3\sqrt{325} & 3\sqrt{244} & 2\cdot10\sqrt{2} & 2\sqrt{181} & 0 & 0\\ 3\sqrt{306} & 3\sqrt{225} & 3\sqrt{181} & 0 & 0 & 0\\ 3\sqrt{261} & 3\sqrt{180} & 0 & 0 & 0 & 0\\ 3\sqrt{234} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The eigenvalues of $Q(S(G))$ are

$$\{180.602, -61.1186, -32.2938, 26.1356, -21.1872, 12.5139\}.$$

The equitable quotient matrix with the modified partition

$$\{\{u_1,u_2\},\{u_3\},\{u_4,u_5\},\{u_6\},\{u_7,u_8\},\{u_9\},\{u_{10}\},\{u_{11}\},\{u_{12},u_{13}\},\{u_{14},u_{15},u_{16}\}\}$$

is given below

$$Q'(S(G))=\begin{pmatrix} 15\sqrt{2} & 15\sqrt{2} & 2\sqrt{369} & \sqrt{369} & 2\sqrt{325} & \sqrt{325} & \sqrt{306} & \sqrt{306} & 2\sqrt{261} & 3\sqrt{234}\\ 2\cdot15\sqrt{2} & 0 & 2\sqrt{369} & \sqrt{369} & 2\sqrt{325} & \sqrt{325} & \sqrt{306} & \sqrt{306} & 2\sqrt{261} & 3\sqrt{234}\\ 2\sqrt{369} & \sqrt{369} & 12\sqrt{2} & 12\sqrt{2} & 2\sqrt{244} & \sqrt{244} & \sqrt{225} & \sqrt{225} & 2\sqrt{180} & 0\\ 2\sqrt{369} & \sqrt{369} & 2\cdot12\sqrt{2} & 0 & 2\sqrt{244} & \sqrt{244} & \sqrt{225} & \sqrt{225} & 2\sqrt{180} & 0\\ 2\sqrt{325} & \sqrt{325} & 2\sqrt{244} & \sqrt{244} & 10\sqrt{2} & 10\sqrt{2} & \sqrt{181} & \sqrt{181} & 0 & 0\\ 2\sqrt{325} & \sqrt{325} & 2\sqrt{244} & \sqrt{244} & 2\cdot10\sqrt{2} & 0 & \sqrt{181} & \sqrt{181} & 0 & 0\\ 2\sqrt{306} & \sqrt{306} & 2\sqrt{225} & \sqrt{225} & 2\sqrt{181} & \sqrt{181} & 0 & 0 & 0 & 0\\ 2\sqrt{306} & \sqrt{306} & 2\sqrt{225} & \sqrt{225} & 2\sqrt{181} & \sqrt{181} & 0 & 0 & 0 & 0\\ 2\sqrt{261} & \sqrt{261} & 2\sqrt{180} & \sqrt{180} & 0 & 0 & 0 & 0 & 0 & 0\\ 2\sqrt{234} & \sqrt{234} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The eigenvalues of $Q'(S(G))$ are

$$\{180.602, -61.1186, -32.2938, 26.1356, -21.2132, -21.1872, -16.9706, -14.1421, 12.5139, 0\}.$$

Therefore, the quotient matrix $Q'(S(G))$ with the partition

$$\{\{u_1,u_2\},\{u_3\},\{u_4,u_5\},\{u_6\},\{u_7,u_8\},\{u_9\},\{u_{10}\},\{u_{11}\},\{u_{12},u_{13}\},\{u_{14},u_{15},u_{16}\}\},$$

contains all the distinct eigenvalues of $S(G)$.

As it is hard to explicitly find the largest/smallest eigenvalue of Q, next in the following result, we present their sharp bounds.

Corollary 3.12. *Let $G \cong T(m_1,m_2,\ldots,m_h;n_1,n_2,\ldots,n_h)$ be the threshold graph of order n. Then*

$$\xi_1(G) \geq \frac{1}{2h}\Big(R+\sqrt{R^2+4R'R''}\Big) \quad \text{and} \quad \xi_n(G) \leq \frac{1}{2h}\Big(R-\sqrt{R^2+4R'R''}\Big),$$

where

$$R=\sum_{i=1}^{h} l_i=\sum_{i=1}^{h}\left((m_i-1)d_{u_i}\sqrt{2}+\sum_{\substack{j=1\\ j\neq i}}^{h} m_j\sqrt{d_{u_i}^2+d_{v_j}^2}\right), \quad R'=\sum_{i=1}^{h} l'_i=\sum_{i=1}^{h}\sum_{j=1}^{h-(i-1)} n_j\sqrt{d_{u_i}^2+d_{v_j}^2},$$

and $R''=\sum_{i=1}^{h} l''_i=\sum_{i=1}^{h}\sum_{j=1}^{h-(i-1)} m_j\sqrt{d_{u_j}^2+d_{v_i}^2}$, where l_i, l'_i, and l''_i, are the i-th row sum of the matrices $\mathcal{A},\mathcal{B}$ and $\mathcal{C}$ defined in (3.2). Equalities hold iff $h=1$, that is, G is the complete split graph graph.

Proof. The quotient matrix of (3.2) with $\pi=\{\{1,2,\ldots,h\},\{1,2,\ldots,h\}\}$ is

$$Q^*=\frac{1}{h}\begin{pmatrix} R & R' \\ R'' & 0 \end{pmatrix}, \tag{3.4}$$

where

$$R=\sum_{i=1}^{h} l_i=\sum_{i=1}^{h}\left((m_i-1)d_{u_i}\sqrt{2}+\sum_{\substack{j=1\\ j\neq i}}^{h} m_j\sqrt{d_{u_i}^2+d_{v_j}^2}\right), \quad R'=\sum_{i=1}^{h} l'_i=\sum_{i=1}^{h}\sum_{j=1}^{h-(i-1)} n_j\sqrt{d_{u_i}^2+d_{v_j}^2},$$

and $R''=\sum_{i=1}^{h} l''_i=\sum_{i=1}^{h}\sum_{j=1}^{h-(i-1)} m_j\sqrt{d_{u_j}^2+d_{v_i}^2}$, where l_i, l'_i, and l''_i, are the i-th row sum of the matrices $\mathcal{A},\mathcal{B}$ and $\mathcal{C}$ (R, R', and R'', respectively).

The eigenvalues of (3.4) are

$$\chi_1(Q^*)=\frac{1}{2h}\Big(R+\sqrt{R^2+4R'R''}\Big) \quad \text{and} \quad \chi_2(Q^*)=\frac{1}{2h}\Big(R-\sqrt{R^2+4R'R''}\Big).$$

From Theorem 3.1, we have

$$\xi_1(Q)\geq\chi_1(Q^*)\geq\xi_2(Q)\geq\chi_2(Q^*)\geq\xi_3(Q)\geq\xi_4(Q)\geq\cdots\geq\xi_{2h}(Q),$$

which further implies that

$$\xi_1(G) \geq \frac{1}{2h}\Big(R+\sqrt{R^2+4R'R''}\Big) \quad \text{and} \quad \xi_n(G) \leq \frac{1}{2h}\Big(R-\sqrt{R^2+4R'R''}\Big),$$

with equalities if and only if the matrix Q^* is equitable quotient matrix of (3.2) with partition $\big\{\{1,2,\ldots,h\},\{1,2,\ldots,h\}\big\}$, that is, same as saying the $\mathcal{A}$ has constant row sum $l_1=l_2=\cdots=l_h$, $\mathcal{B}$ has constant row sum $l'_1=l'_2=\cdots=l'_h$ and $\mathcal{C}$ has constant row sum $l''_1=l''_2=\cdots=l''_h$. In this case, the partition

$\{\{1,2,\ldots,h\},\{1,2,\ldots,h\}\}$ of (3.2) is an equitable partition and Q^* is an equitable quotient matrix of Q and each eigenvalue of Q is the eigenvalue of $S(G)$, including the spectral radius for non negative matrix and the smallest eigenvalue for such an equitable partition.

Thus with $h=1$, $G\cong T(m_1,n_1)$ and from [26], and we have

$$\xi_1(G)=\frac{1}{2}\bigg((m_1-1)(n-1)\sqrt{2}+\sqrt{2\big((m_1-1)(n-1)\big)^2+4\big(m_1n_1((n-1)^2+m_1^2)\big)}\bigg),$$

and

$$\xi_n(G)=\frac{1}{2}\bigg((m_1-1)(n-1)\sqrt{2}-\sqrt{2\big((m_1-1)(n-1)\big)^2+4\big(m_1n_1((n-1)^2+m_1^2)\big)}\bigg).$$

□

For a non-negative matrix with spectral radius λ and row sums $l_1\geq l_2\geq\cdots\geq l_n$, it is known that $l_n\leq\lambda\leq l_1$. Moreover, if the matrix is irreducible, then one of the equalities holds iff the row sums of the matrix are all equal.

For $G\cong T(m_1,m_2,\ldots,m_h;n_1,n_2,\ldots,n_h)$ with $m_1\geq\cdots\geq m_h\geq n_1\geq\cdots\geq n_h$, it is clear that the maximum row sum is in the first row and the minimum row sum is the last $2h$-th row. The maximum and the minimum row sum of the quotient matrix given in (3.2) is

$$(m_1-1)d_{u_1}\sqrt{2}+\sum_{i=2}^{h}m_i\sqrt{d_{u_1}^2+d_{v_i}^2}+\sum_{i=1}^{h}n_1\sqrt{d_{u_1}^2+d_{v_i}^2}\quad\text{and}\quad m_1\sqrt{d_{u_1}^2+d_{v_1}^2}.$$

Thus, we have the following corollary.

Corollary 3.13. *Let $G\cong T(m_1,m_2,\ldots,m_h;n_1,n_2,\ldots,n_h)$ with $m_1\geq\cdots\geq m_h\geq n_1\geq\cdots\geq n_h$, be the threshold graph. Then*

$$m_1\sqrt{d_{u_1}^2+d_{v_1}^2}\leq\xi_1(G)\leq(m_1-1)d_{u_1}\sqrt{2}+\sum_{i=2}^{h}m_i\sqrt{d_{u_1}^2+d_{v_i}^2}+\sum_{i=1}^{h}n_1\sqrt{d_{u_1}^2+d_{v_i}^2}.$$

The spread of M (real symmetric matrix) with eigenvalues $\lambda_1(M)\geq\lambda_2(M)\geq\cdots\geq\lambda_n(M)$ is given by $s(M)=\lambda_1(M)-\lambda_n(M)$. The spread of $S(G)$ is $s(S(G))=\xi_1-\xi_n$, see [30]. With terminology as in Corollary 3.13, we have the following result for the spread of $S(G)$.

Corollary 3.14. *The spread of the $S(G)$ matrix of $G\cong T(m_1,m_2,\ldots,m_h;n_1,n_2,\ldots,n_h)$ satisfies*

$$s(S(G))\geq\frac{1}{h}\Big(\sqrt{R^2+4R'R''}\Big),$$

with equality holding iff $G\cong T(m_1;n_1)$, that is, G is the complete split graph.

From Mirisky [24], the spread of M with trace $\mathrm{tr}(M)$ satisfies the following inequality

$$s(M)\leq\sqrt{2\|M\|_F^2-\frac{2}{n}\big(\mathrm{tr}M\big)^2},\tag{3.5}$$

with equality iff M is normal such that $n-2$ of its eigenvalues are equal to each other and to the arithmetic mean of the remaining two.

As trace of $\mathrm{tr}(S(G)) = \sum_{i=1}^{n} \xi_i = 0$ and $\|S(G)\|_F^2 = \sum_{i=1}^{n} \xi_i^2$. Now,

$$\sum_{i=1}^{n} \xi_i^2 = \sum_{i=1}^{h} 2(m_i - 1)d_{u_i}^2 + \mathrm{tr}\big(Q^2\big),$$

where $\mathrm{tr}(Q^2)$ is the trace of (3.2).

Thus, we have the following result.

Corollary 3.15. *Let T be threshold graph of order n. Then*

$$s(S(T)) \leq \sqrt{\sum_{i=1}^{h} 2(m_i - 1)d_{u_i}^2 + \mathrm{tr}(Q^2)}.$$

4. Sombor energy of threshold graphs

Let $\{e_1, e_2, e_3, \ldots, e_n\}$, with $e_i \geq 0$ be the set of real numbers and let P_t be the average of products of t-element subset of the set $\{e_1, e_2, e_3, \ldots, e_n\}$, that is,

$$\begin{aligned}
P_1 &= \frac{e_1 + e_2 + e_3 + \cdots + e_n}{n}, \\
P_2 &= \frac{1}{\frac{n(n-1)}{2}}(e_1e_2 + e_1e_3 + \cdots + e_1e_n + e_2e_3 + \cdots + e_{n-1}e_n), \\
&\vdots \\
P_n &= e_1e_2 \ldots e_n.
\end{aligned}$$

The following result relates P_i's with themselves.

Lemma 4.1 ([6], Maclaurin symmetric mean inequality)**.** *For positive real numbers $e_1, e_2, \ldots, e_n$, we have the following inequalities*

$$P_1 \geq P_2^{\frac{1}{2}} \geq P_3^{\frac{1}{3}} \geq \cdots \geq P_n^{\frac{1}{n}},$$

with equalities holding iff $e_1 = e_2 = \cdots = e_n$.

Now, we give inequalities for the Sombor energy and characterize the extremal graphs.

Theorem 4.2. *Let $G \cong T(m_1, \ldots, m_h; n_1, \ldots, n_h)$ be a threshold graph of order n. Then, we have the following.*

(i)

$$\mathsf{E}(S(G)) \leq \sqrt{2}\sum_{i=1}^{h}(m_i - 1)d_{u_i} + \sqrt{2h \cdot \mathrm{tr}(Q^2)}, \tag{4.1}$$

with equality holding iff $G \cong T(1; n_1)$.

(ii)

$$\mathsf{E}(S(G)) \geq \sqrt{2}\sum_{i=1}^{h}(m_i - 1)d_{u_i} + \sqrt{\mathrm{tr}(Q^2) + 2h(2h-1)|\det(Q)|^{\frac{1}{h}}},$$

with equality is attained iff $G \cong T(1; n_1)$. The determinant of Q and $\mathrm{tr}(Q^2)$ are given in Corollary 3.5.

Proof. By Theorem 3.4, the Sombor spectrum of G consists of the eigenvalue $-d_{u_i}\sqrt{2}$ with multiplicity m_i-1, the eigenvalue 0 with multiplicity $\sum_{i=1}^{h} n_i - h$ and the eigenvalues of the matrix given in (3.2). Thus the Sombor energy of G is given by

$$\mathsf{E}(S(G)) = \sqrt{2}\sum_{i=1}^{h}(m_i-1)d_{u_i} + \mathsf{E}(Q), \tag{4.2}$$

where $\mathsf{E}(Q)$ is the energy of the matrix given in (3.2). Next, we calculate the bounds for $\mathsf{E}(Q)$, since its eigenvalues cannot be found explicitly. Let $\xi_1(Q) \ge \xi_2(Q) \ge \cdots \ge \xi_{2h}(Q)$ be the eigenvalues of Q. Now, we consider the following cases.

(i) Substituting $e_i = |\xi_i(Q)|$ and $n = 2h$ in Lemma 4.1, we have

$$\left(\frac{1}{2h}\sum_{i=1}^{2h}|\xi_i(Q)|\right)^2 \ge \frac{1}{\frac{2h(2h-1)}{2}}\sum_{1\le i<j\le 2h}|\xi_i(Q)||\xi_j(Q)|, \tag{4.3}$$

which is equivalent to

$$\begin{aligned}(2h-1)\left(\sum_{i=1}^{2h}|\xi_i(Q)|\right)^2 &\ge 4h\sum_{1\le i<j\le 2h}|\xi_i(Q)||\xi_j(Q)|\\ &= 2h\left(\left(\sum_{i=1}^{2h}|\xi_i(Q)|\right)^2 - \sum_{i=1}^{2h}\xi_i^2\right),\end{aligned}$$

that is,

$$(\mathsf{E}(Q))^2 = \left(\sum_{i=1}^{2h}|\xi_i(Q)|\right)^2 \le 2h\sum_{i=1}^{2h}\xi_i^2 = 2h\cdot \operatorname{tr}\big(Q^2\big),$$

that is,

$$\left(\sum_{i=1}^{2h}|\xi_i(Q)|\right)^2 \le 2h\cdot \operatorname{tr}\big(Q^2\big).$$

Thus, by equation (4.2), we have

$$\mathsf{E}(S(G)) \le \sqrt{2}\sum_{i=1}^{h}(m_i-1)d_{u_i} + \sqrt{2h\cdot \operatorname{tr}(Q^2)},$$

where $\operatorname{tr}(Q^2)$ is given in Corollary 3.5. Equality is attained in above inequality if equality holds in (4.3), that is, $|\xi_1(Q)| = |\xi_2(Q)| = \cdots = |\xi_{2h}(Q)|$. Which is true for $h=1, m_1=1$ and $n_1 \ge 2$ and in this case Q takes the form

$$Q_1 = \begin{pmatrix} 0 & n_1\sqrt{d_{u_1}^2+d_{v_1}^2}\\ \sqrt{d_{u_1}^2+d_{v_1}^2} & 0\end{pmatrix},$$

where $d_{u_1} = n_1$ and $d_{v_1} = 1$. The trace of Q_1^2 is

$$\operatorname{tr}\big(Q_1^2\big) = n_1\big(d_{u_1}^2+d_{v_1}^2\big) + n_1\big(d_{u_1}^2+d_{v_1}^2\big) = 2n_1\big(d_{u_1}^2+d_{v_1}^2\big)$$

and its eigenvalues satisfy

$$|\xi(Q_1)| = |\xi(Q_2)| = \sqrt{n_1\left(d_{u_1}^2 + d_{u_1}^2\right)}.$$

Thus, $\mathsf{E}(S(G)) = 2\sqrt{n_1\left(d_{u_1}^2 + d_{v_1}^2\right)}$ and equality holds in (4.1). Conversely, for $G \cong T(1; n_1)$, the Sombor spectrum [26] of G is

$$\left\{0^{[n_1-1]}, \pm\sqrt{n_1\left(d_{u_1}^2 + d_{v_1}^2\right)}\right\},$$

and its Sombor energy is

$$\mathsf{E}(S(G)) = 2\sqrt{n_1\left(d_{u_1}^2 + d_{v_1}^2\right)} = \sqrt{2 \cdot \mathrm{tr}(Q_1^2)},$$

where $d_{u_1} = n_1 - 1$ and $d_{v_1} = 1$.

(ii) Again using $n = 2h$ and $e_i = |\xi_i(Q)|$ in Lemma 4.1 and noting that $|\det(Q)| = \prod_{i=1}^{2h} |\xi_i(Q)|$, we have

$$\frac{1}{\frac{2h(2h-1)}{2}} \sum_{1 \le i < j \le 2h} |\xi_i(Q)||\xi_j(Q)| \ge \left(\prod_{i=1}^{2k} |\xi_i(Q)|\right)^{\frac{1}{h}},$$

with equality iff $|\xi_1(Q)| = \cdots = |\xi_{2h}(Q)|$. The above expression is equivalent to

$$2 \sum_{1 \le i < j \le 2h} |\xi_i(Q)||\xi_j(Q)| \ge 2h(2h-1)|\det(Q)|^{\frac{1}{h}},$$

that is,

$$\left(\sum_{i=1}^{2h} |\xi_i(Q)|\right)^2 - \sum_{i=1}^{2h} \xi_i(Q)^2 \ge 2h(2h-1)|\det(Q)|^{\frac{1}{h}},$$

that is,

$$\mathsf{E}(Q) = \sum_{i=1}^{2h} |\xi_i(Q)| \ge \sqrt{\sum_{i=1}^{2h} \xi_i(Q)^2 + 2h(2h-1)|\det(Q)|^{\frac{1}{h}}}.$$

Therefore, from (4.2), we have

$$\mathsf{E}(S(G)) \ge \sqrt{2} \sum_{i=1}^{h} (m_i - 1) d_{u_i} + \sqrt{\mathrm{tr}(Q^2) + 2h(2h-1)|\det(Q)|^{\frac{1}{h}}},$$

where $\mathrm{tr}(Q^2)$ and $\det(Q)$ are given in Corollary 3.5. The equality is as in part (i).

□

The following example presents the numerical estimation of the above results for $G \cong T(3,3,3;2,2,3)$.

Example 4.3. Let $G \cong T(3,3,3;2,2,3)$ be the threshold graph as given in Figure 1. Then, we have the following.

(1) The Sombor index of G is $\mathsf{SO}(G) = 1223.48$.
(2) The Sombor spectrum of G is

$$\Big\{180.602, 26.1356, 12.5139, 0^{[4]}, (-14.1421)^{[2]}, (-16.9706)^{[2]}, -21.1872, (-21.2132)^{[2]}, -32.2938, -61.1186\Big\}.$$

(3) The Sombor spectral radius of G is $\xi_1(G) = 180.602$, the smallest eigenvalue is $\xi_n = -96.6512$. By Corollary 3.12, $\xi_1(G) \geq 174.719$ and $\xi_n(G) \leq -34.1197$, and by Corollary 3.13, $45.8912 \leq \xi_1(G) \leq 267.326$.
(4) The spread of Sombor matrix of G is $s(S(G)) = 241.721$, while by Corollary 3.14, the spread is $s(S(G)) \geq 208.839$ and by Corollary 3.15, $s(S(G)) \leq 389.468$.
(5) The Sombor energy of G is $\mathsf{E}(S(G)) = 438.503$. By Theorem 4.2, the lower bound for the Sombor energy is $\mathsf{E}(S(G)) \geq 306.03$ and the upper bound is $\mathsf{E}(S(G)) \leq 586.424$.

5. Conclusion

The general adjacency matrix [18] associated with a symmetric function ϕ (usually a degree base topological index) of G is a real symmetric matrix, defined by

$$A_\phi(G) = (a_\phi)_{ij} = \begin{cases} \phi_{d_u d_v} & \text{if } u \text{ is adjacency to } v \\ 0 & \text{otherwise.} \end{cases} \tag{5.1}$$

If $\phi_{d_u d_v} = 1$ when u is adjacent to v, then $A_\phi(G)$ is the adjacency $A(G)$ matrix. If $\phi_{d_u d_v} = \sqrt{d_u^2 + d_v^2}$ in (5.1), we have the $S(G)$ matrix

$$S(G) = \begin{cases} \sqrt{d_u^2 + d_v^2} & \text{if } u \text{ is adjacent to } v \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, for values of $\phi_{d_u d_v}$, we get ABC matrix, geometric-arithmetic matrix and many others. The theory of the present articles can be extended to adjacency and other matrices with similar techniques and lines along with some calculations. Further in future, the analysis of the remaining Sombor eigenvalues of the quotient matrix of threshold graphs remains an open topic and new techniques can be developed in this direction.

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Conflict of Interest

Authors state no conflict of interest.

Data availability statement

All data generated or analysed during this study are included in this article.

References

[1] A. Alazemi, M. Anđelić and S.K. Simić, Eigenvalue location for the chain graphs. *Linear Algebra Appl.* **505** (2016) 194–210.

[2] A. Alazemi, M. Anđelić, T. Koledin and Z. Stanić, Eigenvalue-free intervals of distance matrices of threshold and chain graphs. *Linear Multilinear Algebra* **69** (2021) 2959–2975.

[3] A. Alazemi, M. Anđelić, T. Koledin and Z. Stanić, Chain graphs with simple Laplacian eigenvalues and their Laplacian dynamics. *Comput. Appl. Math.* **42** (2023). DOI: `https://doi.org/10.1007/s40314-022-02141-5`.

[4] M. Anđelić, C.M. da Fonseca, T. Koledin and Z. Stanić, An extended eigenvalue-free interval for the eccentricity matrix of threshold graphs. *J. Appl. Math. Comp.* **69** (2023) 491–503.

[5] F. Atik, On equitable partition of matrices and its applications. *Linear Multilinear Algebra* **68** (2020) 2143–2156.
[6] P. Biler and A. Witkowski, Problems in Mathematical Analysis. Chapman and Hall, New York (1990).
[7] A.E. Brouwer and W.H. Haemers, Spectra of Graphs. Springer Science and Business Media, New York (2010).
[8] R. Cruz, I. Gutman and J. Rada, SI of graphs. *Appl. Math. Comp.* **399** (2021) 126018.
[9] K.C. Das, A.S. Çevik, I.N. Cangul and Y. Shang, On Sombor index. *Symmetry* **13** (2021) 140.
[10] H. Deng, Z. Tang and R. Wu, Molecular trees with extremal values of Sombor indices. *Int. J. Quantum Chem.* **121** (2021) e26622.
[11] G. Dhal, The Laplacian energy of threshold graphs and majorization. *Linear Algebra Appl.* **469** (2015) 518–530.
[12] S. Filipovski and R. Jajcay, Bounds for the energy of graphs. *Mathematics* **9** (2021) 1687.
[13] I. Gutman, The energy of a graph. *Ber. Math.-Statist. Sekt. Forsch. Graz* **103** (1978) 1–22.
[14] I. Gutman, Geometric approach to degree-based topological indices: Sombor indices. *MATCH Commun. Math. Comput. Chem.* **86** (2021) 197–220.
[15] I. Gutman, Spectrum and energy of Sombor matrix. *Mil. Tech. Cour.* **69** (2021) 551–561.
[16] I. Gutman, Sombor indices-back to geometry. *Open J. Discrete Appl. Math.* **5** (2022) 1–5.
[17] I. Gutman, I. Redžepović and J. Rada, Relating energy and Sombor energy. *Contrib. Math.* **4** (2021) 41–44.
[18] I. Gutman, J. Monsalve and J. Rada, A relation between a vertex-degree-based TI and its energy. *Linear Algebra Appl.* **636** (2022) 134–142.
[19] S.M. Hosamani, B.B. Kulkarni, R.G. Boli and V. M. Gadag, QSPR analysis of certain graph theoretical matrices and their corresponding energy. *Appl. Math. Nonlinear Sci.* **2** (2017) 131–150.
[20] M. Imran and B.A. Rather, Sombor spectral properties of chain graphs. *Heliyon* **9** (2023) E18135. DOI: https://doi.org/10.1016/j.heliyon.2023.e18135.
[21] D.P. Jacobs, V. Trevisan and F. Tura, Eigenvalues and energy in threshold graphs. *Linear Algebra Appl.* **465** (2015) 412–425.
[22] X. Li, Y. Shi and I. Gutman, Graph Energy. Springer Science and Business Media, New York (2012).
[23] Z. Lin, T. Zhou and L. Miao, On the SI, energy and Estrada index of the Sombor matrix of graphs. *Trans. Comb.* **12** (2023) 191–205.
[24] L. Mirsky, The spread of a matrix. *Mathematika* **3** (1965) 127–130.
[25] V. Nikiforov, Beyond graph energy: norms of graphs and matrices. *Linear Algebra Appl.* **506** (2016) 82–138.
[26] S. Pirzada, B.A. Rather, K.C. Das, Y. Shang and I. Gutman, On Sombor eigenvalues and Sombor energy of graphs. *Georgian Math. J.* (2024), to appear.
[27] Z. Qiu and Z. Tang, On the eccentricity spectra of threshold graphs. *Discrete Appl. Math.* **310** (2022) 75–85.
[28] B.A. Rather, A note on the distinct eigenvalues of quotient matrices. *Indian J. Pure Appl. Math.* (2023). DOI: https://doi.org/10.1007/s13226-023-00448-5.
[29] B.A. Rather and M. Imran, A note on energy and Sombor energy of graphs. *MATCH Commun. Math. Comput. Chem.* **89** (2023) 467–477.
[30] B.A. Rather, M. Imran and A. Diene, The spectral spread and trace norm of the Sombor matrix. *Indian J. Pure Appl. Math.* (2024). DOI: https://doi.org/10.1007/s13226-023-00529-5.
[31] I. Redžepović, Chemical applicability of Sombor indices. *J. Serb. Chem. Soc.* **86** (2021) 445–457.
[32] T. Réti, T. Došlić and A. Ali, On the Sombor index of graphs. *Contrib. Math.* **3** (2021) 11–18.
[33] S.K. Simić, F. Belardo, E.M. Li Marzi and D.V. Tošić, Connected graphs of fixed order and size with maximal index: some spectral bounds. *Linear Algebra Appl.* **432** (2010) 2361–2372.
[34] A. Ülker, A. Gürsoy and N.K. Gürsoy, The energy and Sombor index of graphs. *MATCH Commun. Math. Comput. Chem.* **87** (2022) 51–58.
[35] A. Ülker, A. Gürsoy, N.K. Gürsoy and I. Gutman, Relating graph energy and Sombor index. *Discrete Math. Lett.* **8** (2022) 6–9.

[36] Z. Wang, Y. Mao, I. Gutman, J. Wu and Q. Ma, Spectral radius and energy of Sombor matrix of graphs. *Filomat* **35** (2021) 5093–5100.