SPECTRAL PROPERTIES OF SOMBOR MATRIX OF THRESHOLD GRAPHS

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Abstract. We investigate the Sombor spectral properties of threshold graphs, a formula for the Sombor index is presented, the Sombor eigenvalues are given, graphs with simple Sombor eigenvalues are characterized, bounds on the smallest/largest Sombor eigenvalues are presented, the multiplicities of the Sombor eigenvalues are discussed, formulae for trace and determinant of the associated quotient matrix are given, the Sombor spread bound and the bounds on the Sombor energy along with the characterization of extremal graphs. At the end, the conclusion states that all our results are valid for adjacency matrix and other adjacency type matrices.

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1. Introduction

Consider a simple graph $G$ with vertex set $V(G) = \{u_1, u_2, \ldots, u_n\}$, edge set $E(G)$ and degree sequence $\{d_1, d_2, \ldots, d_n\}$ (or $d_{u_i}$ of vertex $u_i$). The number $|E(G)|$ is the size $m$ and $|V(G)|$ is the order $n$ of $G$. If $d_{u_i}$ is same for each $i$, then $G$ is called a regular graph. The star graph (complete bipartite) with partite sets $1$ and $n_1$ is symbolized by $K_{1,n_1}$, other notations can be seen in [7].

The adjacency $A(G)$ matrix of $G$ is a square matrix, where $(i,j)$ element is 1, if $u_i$ is adjacent to $u_j$ and 0, otherwise. The matrix $A(G)$ is real and symmetric, so its eigenvalues are real and can be listed in non-increasing order as: $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. The largest eigenvalue $\lambda_1$ is known as the spectral radius of $G$. It is known that $\lambda_1$ is simple by Perron–Frobenius theorem. Also $|\lambda_i| \leq \lambda_1$ and the associated eigenvector say $X$ of $\lambda_1$ has positive entries. In theoretical chemistry, the trace norm of real symmetric matrix $|\lambda_1| + |\lambda_2| + \cdots + |\lambda_n|$ of $A(G)$ is known as the graph energy (see [13]), denoted by $E(A(G))$, and its source goes back to theoretical chemistry, where it used in calculating the $\pi$-electron energy of hydrocarbons. There is a vast literature available related to $E(A(G))$, and the other spectral invariants of $A(G)$, see [12,22,25].

In the theory of chemical/mathematical literature, several degree/distance based graph invariants exists which are useful in chemical studies of alkanes, they are usually known as topological indices. They play main role in
QSAR (quantitative structure-activity relationship) and QSPR (quantitative structure-property relationship) related analysis, see [19] and the references cited there in.

Gutman [14] put forward a Topological index called the Sombor index, denoted by \( SO(G) \) of \( G \), and is given as

\[
SO(G) = \sum_{u, u \in E(G)} \sqrt{d_i^2 + d_j^2}.
\]

The Sombor index is of great importance, since it is not just a vertex degree based invariants, rather it has an interesting geometry related meaning, see [14,16]. The sharp inequalities for \( SO(G) \) and its connection with the Zagreb indices of \( G \) can be seen in [9]. It is proved that the cycle \( C_n \) has the minimum Sombor index among the class of unicyclic graphs [32]. Other chemical applicability and molecular type of properties of \( SO(G) \) can be found in [8,10,31]. The Sombor index is closely related to graph energy, see [20,34,35].

The Sombor matrix \( S(G) \) of \( G \) is an \( n \times n \) square matrix given as

\[
S(G) = (s_{ij}) = \begin{cases} \sqrt{d_i^2 + d_j^2}, & \text{if } u_i \text{ is adjacent to } u_j \\ 0, & \text{otherwise.} \end{cases}
\]

It is clear that \( S(G) = r\sqrt{2}A(G) \), provided \( G \) is a \( r \)-regular graph. The Sombor matrix \( S(G) \) is a real symmetric matrix. We label the eigenvalues of \( S(G) \) by \( \xi_i \)’s and put them in increasing order as: \( \xi_1 \geq \xi_2 \geq \cdots \geq \xi_n \). The set of the eigenvalues of \( S(G) \) with repetitions is called the Sombor spectrum of \( S(G) \). The largest eigenvalue of \( S(G) \) is called as the Sombor spectral radius of \( G \), and it is precisely the spectral norm of the real symmetric matrix \( S(G) \). By Perron–Frobenius theorem, \( \xi_1 \) is unique, \( |\xi_i| \leq \xi_1 \) and its corresponding eigenvector \( Y \) have positive entries. The Sombor energy (trace norm) \( E(S(G)) \) of \( S(G) \), is defined by

\[
E(S(G)) = \sum_{i=1}^{n} |\xi_i|.
\]

The Sombor matrix \( S(G) \) is related to adjacency matrix and because of this reason, large numbers of articles were published in the mean time. Several paper on properties of \( S(G) \) can be found in the literature, like the Sombor energy, the Sombor spectral norm, the Sombor Estrada index, relation of \( E(A(G)) \) with \( S(G) \), \( SO(G) \) and others spectral invariants can be seen in [15,17,23,26,29,30,34]. Inequalities for the Sombor index and energy of \( S(G) \) can be seen in [36]. The Sombor spread of \( S(G) \) is given by \( s(S(G)) = \xi_1 - \xi_n \). If the Sombor eigenvalue \( \mu \) is repeated \( k \)-times, then we represent it as \( \mu^{[k]} \). Also, \( J_n \) denotes the matrix of ones and \( I_n \) is the identity matrix.

A threshold graph is constructed/defined in many ways, we generate it by a binary sequences as given in [1].

A threshold graph \( T(b) \) is constructed from its binary sequence \( b = (a_1a_2 \ldots a_n) \) in as given below:

(i) for \( i = 1 \), we take \( T_1 = T(a_1) \equiv K_1 \), an isolated vertex;
(ii) for \( i \geq 2 \), we already have \( T_{i-1} = T(a_1a_2 \ldots a_{i-1}) \), and \( T_i = T(a_1a_2 \ldots a_{i-1}a_i) \) is obtained by inserting an isolated vertex to \( T_{i-1} \) if \( a_i = 0 \) (that is, a vertex not adjacent to any vertex in \( T_{i-1} \)) or by introducing a dominating vertex (adjacent to all vertices in \( T_{i-1} \)).

It is evident that \( T(b) = T_n \). We denote the addition of an isolated vertex by 0 and a dominating vertex by 1, then we see that \( G \cong T(b) \) is constructed by a binary code \( b = 0^{n_1}1^{m_1}0^{n_2}1^{m_2} \ldots 0^{n_k}1^{m_k} \), where the powers denotes the binary repetition of number in the code. A threshold graph is a \( \{P_1,C_4,2K_2\} \)-free graph, that is, it does not have induced subgraphs isomorphic to the path \( P_1 \), the cycle \( C_4 \) or the pair of non-adjacent edges \( 2K_2 \). A threshold graph is presented in Figure 1, its vertex set is partitioned into sets (cells) \( L_i \) and \( U_i \), \( (i = 1,2,\ldots,h) \). The vertices in \( U_1 \cup \cdots \cup U_h \) induce a clique, while the vertices in \( L_1 \cup \cdots \cup L_h \) induce a co-clique (a non empty totally disconnected graph), that is, a non empty graph without any edge. There are cross edges from
each vertex of \( v_{ij} \in L_i \), \( i = 1, 2, \ldots, h \), \( j = 1, 2, \ldots, m_i \) to every vertex \( u_{ik} \in \bigcup_{j=1}^{h-(i-1)} U_j \), \( k = 1, 2, \ldots, n_i \) and \( j = 1, 2, \ldots, h \). From onwards, we denote a threshold graph by \( T(b) = T(0^{m_1}1^{n_1}0^{m_2}1^{n_2} \ldots 0^{m_h}1^{n_h}) \) by \( T(m_1, m_2, \ldots, m_h; n_1, n_2, \ldots, n_h) \) with order \( n = \sum_{i=1}^{h} n_i + \sum_{i=1}^{h} m_i \). We note that authors used different notations for threshold graphs NSG (nested split graph) [33], \( G(b) \) [4]. For \( h = 1 \), we get the complete split graph \( T(m_1; n_1) \), a graph with clique \( K_{m_1} \) and an independent set of cardinality \( n_1 \) such that each vertex of \( K_{m_1} \) is adjacent to the every vertex of independent set of size \( n_1 \). Further for \( h = 1 \) and \( m_1 = 1 \), we obtain a star graph \( T(1; n_1) \) of order \( n_1 + 1 \). Figure 1 shows a threshold graph \( T(2, 3, 2; 3, 2, 3, 3) \) of order 21, where black dots are \( U_i \)’s and blue dots denote \( L_i \)’s. The lines (dashed) between two sets \( L_i \) and \( U_j \) means that each vertex of \( L_i \) is adjacent to every vertex of \( U_j \), while the solid lines between \( U_i \)’s represent each vertex is adjacent themselves and they form a clique.

Jacobs et al. [21] obtained the bounds for the energy and the Laplacian energy of threshold graphs. Milica et al. [4] studied the eccentricity eigenvalue distribution of threshold graphs and characterized graphs with free extended intervals, also eigenvalue-free intervals of distance based matrices of threshold and chain graphs can be seen in [2]. Dahl [11] related Laplacian energy with majorization of graphs. Qiu and Tang [27] studied the eccentricity spectra of threshold graphs. Motivated by this, we study the spectral properties of the \( S(G) \) of graphs, we investigate the eigenvalue distribution, eigenvalue multiplicities, spread, and present the sharp bounds for the \( E(S(G)) \). With some parallel computations, our theory is equally valid for the adjacency matrix, \( ABC \) (atom-bond connectivity) matrix, geometric-arithmetic matrix and other adjacency weighted type of matrices.

In Section 2, we give the Sombor index of threshold graphs and discuss some consequences. Section 3 introduces the Sombor eigenvalue of thresholds graphs, the Sombor spectral invariants like the determinant and the trace of quotient matrix, Sombor eigenvalues distribution, inequalities on the largest, smallest Sombor eigenvalues, Sombor spread and graphs with simple Sombor eigenvalues. Section 4 presents the sharp upper/lower bounds for the Sombor energy \( E(S(G)) \) along with their extremal characterization. Section 5 gives conclusion and says that the theory is equally valid for adjacency matrix, geometric-arithmetic matrix, \( ABC \) matrix and other adjacency type matrices.

2. SOMBOR INDEX OF THRESHOLD GRAPHS

The very first result gives the closed formulae for the Sombor index of threshold graph families.

**Theorem 2.1.** Let \( G \) be a threshold graph of order \( n \). Then

\[
SO(G) = \sum_{i=1}^{h} \frac{m_i(m_i - 1)}{\sqrt{2}} d_{u_i} \sum_{i=1}^{h} \sum_{j=i+1}^{h} m_i m_j \sqrt{d_{u_i}^2 + d_{u_j}^2} + \sum_{i=1}^{h} \sum_{j=1}^{h-(i-1)} m_i n_j \sqrt{d_{u_i}^2 + d_{v_j}^2},
\]

where \( d_{u_i} = \sum_{j=1}^{h} m_j - 1 + \sum_{j=1}^{h-(i-1)} n_j \) and \( d_{v_i} = \sum_{j=1}^{h-(i-1)} m_j \), for \( i = 1, 2, \ldots, h \).
Proof. We label the vertices of $G$ from $L_i$’s to $U_i$’s for $i = 1, \ldots, h$. Let $v_{ij}, i = 1, \ldots, h$ and $j = 1, \ldots, m_i$ be the vertices of $L_i$ and $u_{ij}, i = 1, 2, \ldots, h$ and $j = 1, 2, \ldots n_i$ be the vertices of $U_i$. Note that

$$d(u_{11}) = d(u_{12}) = \cdots = d(u_{1m_1}) = \sum_{i=1}^{h} m_i + \sum_{j=1}^{h-i} n_j - 1,$$

as $u_i$’s form a clique in $G$. We represent this common degree by $d_u$. Similarly, $d_v$ is the common degree of the vertices of $U_i, i = 2, 3, \ldots, h$ and in general, we have

$$d_{u_i} = \sum_{i=1}^{h} m_i - 1 + \sum_{j=1}^{h-i} n_j, \quad \text{for } i = 1, 2, \ldots, h.$$

Similarly, let $d_v$ be the common degree of each $L_i, i = 1, 2, \ldots, n_h$. As with above idea, the degrees of $v_i$’s are

$$d_{v_i} = \sum_{j=1}^{h-i-1} m_j, \quad \text{for } i = 1, 2, \ldots, h.$$

The number of edges inside each $U_i$ are $m_i(m_i - 1)/2$, the number of edges from each vertex of $U_i$ to every vertex of $L_j$’s are $m_in_j$ and the total number of edges from each vertex of $U_i$ to each vertex of $L_j$ are $\sum_{i=1}^{h} \sum_{j=1}^{h-i-1} m_i n_j$.

Therefore with above information, the Sombor index of $G$ is

$$\text{SO}(G) = \sum_{i=1}^{h} \frac{m_i(m_i - 1)}{\sqrt{2}}d_{u_i} + \sum_{i=1}^{h} \sum_{j=i+1}^{h} m_i m_j \sqrt{d_{u_i}^2 + d_{u_j}^2} + \sum_{i=1}^{h} \sum_{j=1}^{h-i} m_i n_j \sqrt{d_{u_i}^2 + d_{v_j}^2},$$

where $d_{u_i} = \sum_{j=1}^{h} m_j - 1 + \sum_{j=1}^{h-i-1} n_j$ and $d_{v_i} = \sum_{j=1}^{h-i-1} m_j$, for $i = 1, 2, \ldots, h$. \hfill \Box

The following is an immediate consequences of Theorem 2.1.

Corollary 2.2. The following hold for $G \cong T(m_1, m_2, \ldots, m_h; n_1, n_2, \ldots, n_h)$.

(i) For $h = 1$, $G \cong T(m_1; n_2)$ (complete split graph) and its Sombor index is

$$\text{SO}(G) = \frac{m_1(m_1 - 1)}{\sqrt{2}}d_{u_1} + m_1 n_1 \sqrt{d_{u_1}^2 + d_{v_1}^2},$$

where $d_{u_1} = m_1 - 1 + n_2$ and $d_{v_1} = n_1$.

(ii) For $h = 1$, with $m_1 = 1$, $G \cong T(1; n_2)$ (star graph) and its Sombor index is

$$\text{SO}(G) = n_1 \sqrt{d_{u_1}^2 + d_{v_1}^2},$$

where $d_{u_1} = n_2$ and $d_{v_1} = 1$.

Theorem 2.1 gives a closed expression for the Sombor index of $T$, but when order increases, the computations significantly increases, so search for better bounds can be useful in such situations.
3. Sombor spectral properties of threshold graphs

The real sequence \( a'_1 \geq a'_2 \geq \cdots \geq a'_m \) is said to interlace the another real sequence \( a_1 \geq a_2 \geq \cdots \geq a_n \) \((m < n)\) if \( a_i \geq a'_i \geq a_{n-m+i} \), for \( i = 1, 2, \ldots, m \) and the interlacing is said to be tight if there exists a positive integer \( k \in [0, m] \) such that
\[
a_i = a'_i \quad \text{for} \quad i = 1, 2, \ldots, k \quad \text{and} \quad a_{n-m+i} = a'_i \quad \text{for} \quad k + 1 \leq i \leq m.
\]

Consider a square matrix \( M \) of order \( n \) in block form
\[
M = \begin{pmatrix}
A_{1,1} & A_{1,2} & \cdots & A_{1,s-1} & A_{1,s} \\
A_{2,1} & A_{2,2} & \cdots & A_{2,s-1} & A_{2,s} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
A_{s-1,1} & A_{s-1,2} & \cdots & A_{s-1,s-1} & A_{s-1,s} \\
A_{s,1} & A_{s,2} & \cdots & A_{s,s-1} & A_{ss}
\end{pmatrix},
\]
such that its rows/columns are indexed on the basis of a partition \( \pi = \{\ell_1, \ell_2, \ldots, \ell_s\} \) of the index set \( I = \{1, \ldots, n\} \). The quotient matrix \( Q = (q_{ij})_{s \times s} \) (see [7]) is a square matrix of order \( s \), with its \((i, j)\)-th entry equal to the average row sum of the block \( A_{ij} \) of \( M \). If each block \( A_{ij} \) of \( M \) has row sum equal to some constant, then partition \( \pi \) is equitable (or regular), and \( Q \) is called an equitable quotient matrix.

The relation between the eigenvalues of \( M \) and \( Q \) are given below.

**Theorem 3.1 ([7]).** Let \( M_{n \times n} \) be a real symmetric matrix and \( Q \) be its quotient matrix of order \( m, (n > m) \). Then

(i) if \( \pi \) is not equitable, then interlacing holds
\[
\lambda_i(M) \geq \lambda_i(Q) \geq \lambda_{i+n-m}(M), \quad \text{for} \quad i = 1, 2, \ldots, m,
\]

(ii) if \( \pi \) is equitable, then the spectrum of \( Q \) as a set is contained in the spectrum of \( M \).

For the matrix \( S(G) \), the Frobenius norm is \( \xi_1^2 + \xi_2^2 + \cdots + \xi_n^2 = \| S(G) \|_F^2 \), and energy is \( E(S(G)) = |\xi_1| + |\xi_2| + \cdots + |\xi_n| \). We recall that \( \| S(G) \|_F^2 \) and \( E(S(G)) \) are norms for a normal matrix \( (S(G)) \) [25, 30].

We have the following known results, which will be used later.

**Lemma 3.2 ([26]).** Let \( G \) be a connected graph with \( n \) vertices and \( \omega = \{u_1, u_2, \ldots, u_{\omega}\} \) be a clique in \( G \) such that \( N(u_i) - \omega = N(u_j) - \omega, 1 \leq i, j \leq \omega \). Then \(-d\sqrt{2}\) is the eigenvalue of \( S(G) \) with multiplicity greater or equal to \( \omega - 1 \), where \( d = d_{u_i} \), for \( i = 1, 2, \ldots, \omega \).

**Lemma 3.3 ([26]).** For a graph connected \( G \) with independent set \( \{u_1, \ldots, u_\alpha\} \) such that \( N(u_i) = N(u_j) \) for all \( 1 \leq i, j \leq \alpha \). Then 0 is the Sombor eigenvalue of \( G \) with multiplicity greater or equal to \( \alpha - 1 \).

We sketch the proof of Lemmas 3.2 and 3.3 is similar.

We first index the vertex of the independent set \( I = \{u_1, \ldots, u_\alpha\} \) of \( S(G) \) such that each vertex share the common neighbourhood. So that the Sombor matrix of \( G \) can be written as
\[
S(G) = \begin{pmatrix}
0_\alpha \\
D_{(n-\alpha) \times \alpha} & S'(G(V \setminus I))_{n-\alpha}
\end{pmatrix},
\]
where \( S'(G(V \setminus I))_{n-\alpha} \) is the part of \( S(G) \) on the vertex set \( V(G) \setminus I \). Now, it is easy to see that
\[
X_{j-1} = (-1, x_{2j}, x_{3j}, \ldots, x_{\alpha j}, 0, 0, \ldots, 0)^T, \quad \text{for} \quad j = 2, 3, \ldots, \alpha,
\]
where \( x_{ij} \) is the Kronecker delta \( \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases} \). Clearly, 0 is the eigenvalues of \( S(G) \) with eigenvectors \( X_1, \ldots, X_{n-1} \).

With \( d_{u_i} \) given in Theorem 2.1, in the next result, we discuss the Sombor spectrum of threshold graphs.

**Theorem 3.4.** Let \( G \) be a threshold graph. Then the Sombor spectrum of \( G \) consists of the eigenvalue \(-d_{u_i}\sqrt{2}\) with multiplicity \( m_i - 1 \), the eigenvalue 0 with multiplicity \( \sum_{i=1}^{h} n_i - h \), and the remaining 2h Sombor eigenvalues of \( G \) are the eigenvalues of the matrix given in (3.1).

**Proof.** Indexing the vertices from \( U_i \)'s to \( V_j \)'s, the Sombor matrix of \( G \) is

\[
\begin{pmatrix}
A_{m_1+m_2+\cdots+m_h} & B_{(m_1+m_2+\cdots+m_h) \times (n_1+n_2+\cdots+n_h)} \\
0 & 0
\end{pmatrix},
\]

where 0 is a zero matrix and \( A \) and \( B \) are given below:

\[
A = \\
\begin{pmatrix}
\sqrt{d_{u_1}}(J_{m_1} - I_{m_1}) & \sqrt{d_{u_1} + d_{u_2}} J_{m_1 \times m_2} & \cdots & \sqrt{d_{u_1} + d_{u_{h-1}} - d_{u_h}} J_{m_1 \times m_{h-1}} & \sqrt{d_{u_1} + d_{u_h}} J_{m_1 \times m_h} \\
\sqrt{d_{u_2}} J_{m_2 \times m_1} & \sqrt{d_{u_2} + d_{u_3}} J_{m_2 \times m_3} & \cdots & \sqrt{d_{u_2} + d_{u_{h-1}} - d_{u_h}} J_{m_2 \times m_{h-1}} & \sqrt{d_{u_2} + d_{u_h}} J_{m_2 \times m_h} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\sqrt{d_{u_{h-1}} + d_{u_1}} J_{m_{h-1} \times m_1} & \sqrt{d_{u_{h-1}} + d_{u_2}} J_{m_{h-1} \times m_2} & \cdots & \sqrt{d_{u_{h-1}} + d_{u_{h-1}} - d_{u_h}} J_{m_{h-1} \times m_{h-1}} & \sqrt{d_{u_{h-1}} + d_{u_h}} J_{m_{h-1} \times m_h} \\
\sqrt{d_{u_h}} J_{m_h \times m_1} & \sqrt{d_{u_h} + d_{u_3}} J_{m_h \times m_3} & \cdots & \sqrt{d_{u_h} + d_{u_{h-1}} - d_{u_{h-1}}} J_{m_h \times m_{h-1}} & \sqrt{d_{u_h} + d_{u_{h-1}}} J_{m_h \times m_{h-1}} \\
\end{pmatrix}
\]

\[
B = \\
\begin{pmatrix}
\sqrt{d_{u_1}} J_{m_1 \times m_1} & \sqrt{d_{u_1} + d_{u_2}} J_{m_1 \times m_2} & \cdots & \sqrt{d_{u_1} + d_{u_{h-1}} - d_{u_h}} J_{m_1 \times m_{h-1}} & \sqrt{d_{u_1} + d_{u_h}} J_{m_1 \times m_h} \\
\sqrt{d_{u_2}} J_{m_2 \times m_1} & \sqrt{d_{u_2} + d_{u_3}} J_{m_2 \times m_3} & \cdots & \sqrt{d_{u_2} + d_{u_{h-1}} - d_{u_h}} J_{m_2 \times m_{h-1}} & \sqrt{d_{u_2} + d_{u_h}} J_{m_2 \times m_h} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\sqrt{d_{u_{h-1}} + d_{u_1}} J_{m_{h-1} \times m_1} & \sqrt{d_{u_{h-1}} + d_{u_2}} J_{m_{h-1} \times m_2} & \cdots & \sqrt{d_{u_{h-1}} + d_{u_{h-1}} - d_{u_h}} J_{m_{h-1} \times m_{h-1}} & \sqrt{d_{u_{h-1}} + d_{u_h}} J_{m_{h-1} \times m_h} \\
\sqrt{d_{u_h}} J_{m_h \times m_1} & \sqrt{d_{u_h} + d_{u_3}} J_{m_h \times m_3} & \cdots & \sqrt{d_{u_h} + d_{u_{h-1}} - d_{u_{h-1}}} J_{m_h \times m_{h-1}} & \sqrt{d_{u_h} + d_{u_{h-1}}} J_{m_h \times m_{h-1}} \\
\end{pmatrix}
\]

As each \( U_i \), \( i = 1, 2, \ldots, h \) is a clique and each vertex of such cliques share the same neighbourhood, so by Lemma 3.2, \( G \) has the Sombor eigenvalue \(-d_{u_i}\sqrt{2}\) with multiplicity \( m_i - 1 \), where \( d_{u_i} \) represents the common degree of clique induced by \( U_i \). Also, each \( L_i \), \( i = 1, 2, \ldots, h \) is an independent set and each vertex of \( L_i \) share the same common neighbourhood. So, from Lemma 3.3, it follows that 0 is the eigenvalue of \( S(G) \) with multiplicity \( \sum_{i=1}^{h} n_i - h \). The remaining Sombor eigenvalues of \( G \) are the eigenvalues of the following quotient matrix with partitions \( \{ |U_1|, |U_2|, \ldots, |U_h|, |L_1|, |L_2|, \ldots, |L_h| \} \)

\[
Q = \begin{pmatrix} \mathbf{A}_h & \mathbf{B}_h \\ \mathbf{C}_h & \mathbf{0}_h \end{pmatrix},
\]

where

\[
A = \\
\begin{pmatrix}
(m_1 - 1)d_{u_1} \sqrt{2} & m_2 \sqrt{d_{u_1} + d_{u_2}} & \cdots & m_{h-1} \sqrt{d_{u_1} + d_{u_{h-1}}} & m_h \sqrt{d_{u_1} + d_{u_h}} \\
m_1 \sqrt{d_{u_1} + d_{u_2}} & (m_2 - 1)d_{u_2} \sqrt{2} & \cdots & m_{h-1} \sqrt{d_{u_2} + d_{u_{h-1}}} & m_h \sqrt{d_{u_2} + d_{u_h}} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
m_1 \sqrt{d_{u_{h-1}} + d_{u_1}} & m_2 \sqrt{d_{u_{h-1}} + d_{u_2}} & \cdots & (m_{h-1} - 1)d_{u_{h-1}} \sqrt{2} & m_h \sqrt{d_{u_{h-1}} + d_{u_h}} \\
m_1 \sqrt{d_{u_h}} & m_2 \sqrt{d_{u_h}} & \cdots & m_{h-1} \sqrt{d_{u_h}} & (m_h - 1)d_{u_h} \sqrt{2}
\end{pmatrix}
\]
The trace of $\mathcal{Q}$ given in (3.2) by

$$
\begin{pmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9
\end{pmatrix}
$$

We denote matrix $\det(Q)$ evaluate its determinant and trace of $Q^2$, which are later used in our results.

Next, we find the determinant of $Q$ and the trace of matrix $Q^2$.

**Corollary 3.5.** The following hold for the matrix given in (3.2).

(i) The determinant of matrix $Q$ is

$$
\det(Q) = (-1)^h \prod_{i=1}^h m_i n_i \left( d_{u_i}^2 + d_{v_i}^2 \right) = (-1)^h \prod_{i=1}^h m_i n_i \left( d_{u_{h-i}}^2 + d_{v_i}^2 \right).
$$

(ii) The trace of $Q^2$ is

$$
\text{tr}(Q^2) = 2 \sum_{i=1}^h d_{u_i}^2 (m_i - 1)^2 + 2 \sum_{j=1}^h \sum_{i=1}^h m_i m_j \left( d_{u_i}^2 + d_{u_j}^2 \right) + 2 \sum_{j=1}^h \sum_{i=1}^{h-j+1} m_j n_i \left( d_{u_j}^2 + d_{v_i}^2 \right).
$$

**Proof.** (i) We denote matrix $Q$ given in (3.2) by $Q(m_1, m_2, \ldots, m_h, n_1, n_2, \ldots, n_h)$. We expand $\det(Q(m_1, m_2, \ldots, m_h, n_1, n_2, \ldots, n_h))$ by 2h-th row, there is only one non-zero cofactor $\det(Q(m_1, m_2, \ldots, m_h, n_1, n_2, \ldots, n_h))$ multiplied by $m_1 \sqrt{d_{u_1}^2 + d_{u_1}^2}$, as the other entries of 2h-th row are zeros. Similarly, for $(2h - 1)$-th cofactor expansion there is one non-zero cofactor of size $2h - 2$ (that is, $\det(Q(m_1, m_2, \ldots, m_h, n_1, n_2, \ldots, n_h))$ multiplied by $m_2 \sqrt{d_{u_2}^2 + d_{u_{h-1}}^2}$ and so on. The process is illustrated below

$$
\det(Q(m_1, m_2, \ldots, m_h, n_1, n_2, \ldots, n_h)) = (-1)^h m_1 \sqrt{d_{u_1}^2 + d_{u_1}^2} m_2 \sqrt{d_{u_2}^2 + d_{u_{h-1}}^2} \det(Q(m_1, m_2, \ldots, m_h, n_1, n_2, \ldots, n_{h-2}))
$$

$$
= (-1)^2 \prod_{i=1}^4 m_i \sqrt{d_{u_i}^2 + d_{u_{h-i-1}}^2} \det(Q(m_1, m_2, \ldots, m_h, n_1, n_2, \ldots, n_{h-4}))
$$

□
(ii) By computing $Q$ of cardinality $\text{graph}$, it is clear that

\[ (-1)^{\frac{h}{2} + 1} n_1 m_h n_2 m_{h-1} (d_{u_1}^2 + d_{u_1}^2) \left( d_{u_{h-1}}^2 + d_{u_{h-1}}^2 \right) \cdot \prod_{i=1}^{h-2} m_i \sqrt{d_{u_i}^2 + d_{u_{h-i}}^2} \det(Q(m_1, m_2, \ldots, m_h)) \]

\[ (-1)^{\frac{h}{2} + \frac{h-1}{2}} \prod_{i=1}^{h-2} m_{h-(i-1)} n_i \left( d_{u_{h-(i-1)}}^2 + d_{u_i}^2 \right) \cdot \prod_{i=h-1}^{h} m_{h-(i-1)} \sqrt{d_{u_{h-(i-1)}}^2 + d_{u_i}^2} \det(Q(m_1, m_2)) \]

\[ (-1)^h \prod_{i=1}^{h} m_i n_i \left( d_{u_i}^2 + d_{u_{h-(i-1)}}^2 \right) = (-1)^h \prod_{i=1}^{h} m_i n_i \left( d_{u_{h-(i-1)}}^2 + d_{u_i}^2 \right). \]

Summing all these expressions, we have

\[ \text{tr}(Q^2) = 2 \sum_{i=1}^{h} d_{u_i}^2 (m_i - 1)^2 + \sum_{i=1}^{h} m_i m_j \left( d_{u_i}^2 + d_{u_j}^2 \right) + 2 \sum_{j=1}^{h} \sum_{i \neq j} m_j n_i \left( d_{u_j}^2 + d_{u_i}^2 \right). \]

The eigenvalues of $Q$ are simple, that is, equivalently saying that the Sombor eigenvalues of $G(1, 1, \ldots, 1; 1, 1, \ldots, 1)$ are simple, since Lemmas 3.2 and 3.3 have no roles. We ask the following question:

What happens if at least one $m_i \geq 2$ (or respectively, $n_i \geq 2$)? In this case, we have the following consequence from Theorem 3.4.

**Corollary 3.6.** Let $G \cong T(m_1, m_2, \ldots, m_{h-1}, m_h; n_1, n_2, \ldots, n_h)$ be the threshold graph of order $n$ with $m_h = 2$ and $n_1 = 1$. Then $-d\sqrt{2}$ is the eigenvalue of $S(G)$ with multiplicity 2, where $d = m_1 + m_2 + \cdots + m_{h-1} + 2$.

**Proof.** Let $\{u_1, u_2\}$ and $\{u_3\}$ be the vertices of $U_h$ and $L_1$, respectively. From the structure of the threshold graph, it is clear that $\{u_1, u_2, u_3\}$ induces a clique $K_3$, and each $u_i$ share the same neighbourhood $\bigcup_{i=1}^{h-1} U_i$ of cardinality $\sum_{i=1}^{h-1} m_i$. Therefore, Lemma 3.2 implies that $\sqrt{2}(\sum_{i=1}^{h-1} m_i + 2)$ is the eigenvalue of $S(G)$ with multiplicity 2. \(\square\)
Next, we put restriction on other \( m_i \)'s (respectively \( n_j \)'s) for \( i = 1, 2, \ldots, h - 1 \) and \( j = 1, 2, \ldots, h \).

**Corollary 3.7.** Let \( G \cong T(m_1, m_2, \ldots, m_{h-1}, m_h; n_1, n_2, \ldots, n_h) \) be the threshold graph. Then the Sombor eigenvalues of \( S(G) \) are simple iff (if and only if) \( m_i \leq 2 \), for all \( i \) and \( n_i \leq 2 \) for exactly one \( i \) with exception that \( m_h \neq 2 \) and \( n_1 \neq 1 \) simultaneously.

**Proof.** The exception case \( G \cong T(m_1, m_2, \ldots, m_{h-1}, 2; 1, n_2, \ldots, n_h) \) is done in Corollary 3.6. As the eigenvalue of matrix (3.2) are simple. We only need to check for the cardinalities of \( m_i \)'s and \( n_i \)'s. Also with \( \det(Q) \neq 0 \), Lemma 3.3 implies that 0 is the simple eigenvalue of \( S(G) \) iff the cardinality of exactly one \( L_i \) is 2. Also, we note that \( d_u \) is distinct for each cell \( U_i \), so by Lemma 3.2, \(-d_u, \sqrt{2}\) is the eigenvalue of \( S(G) \) with multiplicity one iff the cardinality of each \( U_i = 2 \), for all \( i = 1, 2, \ldots, h \) except with choice \( m_h = 2 \) and \( n_1 = 1 \). \( \square \)

Corollaries 3.6 and 3.7 give us an algorithm, we need to discard those graphs where \( m_h = 2 \) and \( n_1 = 1 \) for the candidate graphs without repeated Sombor eigenvalues. A similar type of algorithm for graphs with the simple Laplacian eigenvalues can be seen in [3].

**Algorithm 1.** Chain graphs with \( 2h \) cells and without repeated SE.

**Input:** Threshold graph \( G \cong T(m_1, m_2, \ldots, m_{h-1}, m_h; n_1, n_2, \ldots, n_h) \) with \( m_i \in \{1, 2\} \) and exactly one \( n_i = 2 \), for some \( i \).

**Output:** \( G \) without repeated SE.

**Algorithm find** (\( G \) with simple SE.)

1. \( I := \{1, 2, \ldots, h\} \)
2. \( F := \{m_1, m_2, \ldots, m_{h-1}, m_h; n_1, n_2, \ldots, n_h\} \), where \( m_i \in \{1, 2\} \) for all \( i \in I \) and exactly \( n_i = 2 \)
3. Test
4. for \( m_i \in \{1, 2\} \)
5. if \( m_h = 1 \) return \( G \) is without repeated eigenvalues
6. else if \( m_h = 2 \) and \( n_1 = 2 \) return \( G \) is without repeated eigenvalues
7. else
8. \( m_h = 2 \) and \( n_1 = 1 \) return \( G \) is with repeated eigenvalues
9. end loop

For \( h = 3 \), and \( m_i \leq 2 \ (n_i \leq 2) \), there are 32 non-isomorphic graphs eligible for simple Sombor. These graphs are listed below:

\[
\begin{align*}
T(1, 1, 1; 1, 1, 1), & \ T(2, 1, 1; 1, 1, 1), \ T(1, 2, 1; 1, 1, 1), \ T(1, 1, 2; 1, 1, 1), \ T(2, 2, 1; 1, 1, 1), \\
T(2, 1, 2; 1, 1, 1), & \ T(1, 2, 2; 1, 1, 1), \ T(2, 2, 2; 1, 1, 1), \ T(1, 1, 1; 2, 1, 1), \ T(1, 1, 1; 1, 2, 1), \\
T(1, 1, 1; 1, 2), & \ T(2, 1, 1; 2, 1, 1), \ T(2, 1, 1; 1, 2), \ T(2, 1, 1; 1, 2), \ T(1, 2, 1; 2, 1, 1), \\
T(1, 2, 1; 1, 2), & \ T(1, 2, 1; 1, 2), \ T(1, 1, 2; 2, 1, 1), \ T(1, 1, 2; 1, 2, 1), \ T(1, 1, 2; 1, 1, 2), \\
T(2, 2, 1; 2, 1, 1), & \ T(2, 2, 1; 1, 2, 1), \ T(2, 2, 1; 1, 1, 2), \ T(2, 1, 2; 2, 1, 1), \ T(2, 1, 2; 1, 2, 1), \\
T(2, 1, 2; 1, 1, 2), & \ T(1, 2, 2; 2, 1, 1), \ T(1, 2, 2; 1, 2, 1), \ T(1, 2, 2; 1, 1, 2), \ T(2, 2, 2; 1, 1, 2), \\
T(2, 2, 2; 1, 1, 2), & \ T(2, 2, 2; 1, 1, 2).
\end{align*}
\]

Based on Corollaries 3.6 and 3.7 and Algorithm 1, there are exactly 20 graphs with simple Sombor eigenvalues and are listed in Table 1 along with their Sombor spectrum.

Our next result establishes bounds for the number of distinct Sombor eigenvalues for threshold graphs.

**Corollary 3.8.** Let \( \alpha \) be the number of distinct Sombor eigenvalues of the threshold graph \( G \). Then

\[
2h \leq \alpha \leq 3h + 1,
\]
with equality on left iff $G \cong T(1,1,\ldots,1;1,1,\ldots,1)$ and equality on right iff

$$G \cong T(m_1, m_2, \ldots, m_h; n_1, n_2, \ldots, n_h),$$

where $m_i \geq 2$, for $i = 1, 2, \ldots, h$ and $n_i \geq 2$, for at least one $i$ with exception that $m_h = 2$ and $n_1 = 1$.

**Corollary 3.9.** Let $G \cong T(m_1, m_2, \ldots, m_{h-1}, 2; n_2, \ldots, n_h), m_i \geq 2, i = 1, 2, \ldots, h-1$ and $n_i = 2$ for some $i$. Then $G$ has $3h$ distinct SE.

The following corollary gives the multiplicity of the Sombor eigenvalue 0, the lower bound for the number of the positive Sombor eigenvalue of $T(m_1, m_2, \ldots, m_h; n_1, n_2, \ldots, n_h)$.

**Corollary 3.10.** For $G \cong T(m_1, m_2, \ldots, m_h; n_1, \ldots, n_h), m_i, n_i \geq 2$, the multiplicity of the Sombor eigenvalue 0 is exactly $\sum_{i=1}^{h} n_i - h$, the number of the negative Sombor eigenvalues of $G$ is at least $\sum_{i=1}^{h} m_i - h$ and the number of positive Sombor eigenvalues is at most $h$.

One of the interesting problem posed in [5] is looking for conditions on parent matrix $M$, so that the spectrum of equitable quotient matrix contains all the different eigenvalues. The problem was studied in [28], it was actually about the smallest equitable partitions, and restated as: Characterize those matrices $M$ such that with the smallest possible equitable partition its quotient matrix $Q_1$ contains all the distinct eigenvalues of $M$? We consider this problem for $G \cong T(m_1, m_2, \ldots, m_h; n_1, \ldots, n_h), m_i, n_i \geq 2$, and look for the smallest possible equitable partition so that quotient matrix contains all the distinct eigenvalues of $S(G)$. Since $G$ has at most $3h+1$, so the order of the equitable quotient matrix can be at most $3h+1$. One such partition is

$$\{\{1,2,\ldots,m_1-1\},\{m_1\},\{1,2,\ldots,m_2-1\},\{m_2\},\ldots,\{1,2,\ldots,m_h-1\},\{m_h\}\},$$

and any partition cell among $\{1,2,\ldots,n_1\},\{1,2,\ldots,n_2\},\ldots,\{1,2,\ldots,n_h\}$ can be repartitioned, say $\{1,2,\ldots,n_1\}=\{1,2,\ldots,n_1-1\}\cup\{n_1\}$. The corresponding equitable quotient matrix is

$$Q_1 = \begin{pmatrix} A_{2h} & B_{2h \times (h+1)} \\ C_{(h+1) \times 2h} & 0_{h+1} \end{pmatrix},$$
where

\[
\begin{pmatrix}
(m_1 - 2)d_{u_1} \sqrt{2} & d_{u_1} \sqrt{2} & \ldots & (m_h - 1)\sqrt{d_{u_1}^2 + d_{u_2}^2} & \sqrt{d_{u_1}^2 + d_{u_h}^2} \\
(m_1 - 1)d_{u_1} \sqrt{2} & 0 & \ldots & (m_h - 1)\sqrt{d_{u_1}^2 + d_{u_h}^2} & \sqrt{d_{u_1}^2 + d_{u_h}^2} \\
(m_1 - 1)\sqrt{d_{u_1}^2 + d_{u_2}^2} & \sqrt{d_{u_1}^2 + d_{u_2}^2} & \ldots & (m_h - 1)\sqrt{d_{u_2}^2 + d_{u_h}^2} & \sqrt{d_{u_2}^2 + d_{u_h}^2} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
(m_1 - 1)\sqrt{d_{u_{h-1}}^2 + d_{u_1}^2} & \sqrt{d_{u_{h-1}}^2 + d_{u_1}^2} & \ldots & (m_h - 1)\sqrt{d_{u_{h-1}}^2 + d_{u_h}^2} & \sqrt{d_{u_{h-1}}^2 + d_{u_h}^2} \\
(m_1 - 1)\sqrt{d_{u_{h-1}}^2 + d_{u_h}^2} & \sqrt{d_{u_{h-1}}^2 + d_{u_h}^2} & \ldots & (m_h - 1)d_{u_h} \sqrt{2} & d_{u_h} \sqrt{2}
\end{pmatrix},
\]

\[
\begin{pmatrix}
(n_1 - 1)\sqrt{d_{u_1}^2 + d_{u_1}^2} & \sqrt{d_{u_1}^2 + d_{u_1}^2} & n_2\sqrt{d_{u_1}^2 + d_{u_2}^2} & \ldots & n_{h-1}\sqrt{d_{u_{h-1}}^2 + d_{u_1}^2} & n_h\sqrt{d_{u_1}^2 + d_{u_h}^2} \\
(n_1 - 1)\sqrt{d_{u_1}^2 + d_{u_1}^2} & \sqrt{d_{u_1}^2 + d_{u_1}^2} & n_2\sqrt{d_{u_1}^2 + d_{u_2}^2} & \ldots & n_{h-1}\sqrt{d_{u_{h-1}}^2 + d_{u_1}^2} & n_h\sqrt{d_{u_1}^2 + d_{u_h}^2} \\
(n_1 - 1)\sqrt{d_{u_1}^2 + d_{u_1}^2} & \sqrt{d_{u_1}^2 + d_{u_1}^2} & n_2\sqrt{d_{u_1}^2 + d_{u_2}^2} & \ldots & n_{h-1}\sqrt{d_{u_{h-1}}^2 + d_{u_1}^2} & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
(n_1 - 1)\sqrt{d_{u_{h-1}}^2 + d_{u_1}^2} & \sqrt{d_{u_{h-1}}^2 + d_{u_1}^2} & n_2\sqrt{d_{u_{h-1}}^2 + d_{u_2}^2} & \ldots & 0 & 0 \\
(n_1 - 1)\sqrt{d_{u_{h-1}}^2 + d_{u_1}^2} & \sqrt{d_{u_{h-1}}^2 + d_{u_1}^2} & 0 & \ldots & 0 & 0 \\
(n_1 - 1)\sqrt{d_{u_{h-1}}^2 + d_{u_1}^2} & 0 & \ldots & 0 & 0 & 0 \\
\end{pmatrix},
\]

\[
\begin{pmatrix}
(m_1 - 1)\sqrt{d_{u_1}^2 + d_{u_1}^2} & \sqrt{d_{u_1}^2 + d_{u_1}^2} & \ldots & (m_h - 1)\sqrt{d_{u_1}^2 + d_{u_2}^2} & \sqrt{d_{u_1}^2 + d_{u_h}^2} \\
(m_1 - 1)\sqrt{d_{u_1}^2 + d_{u_1}^2} & \sqrt{d_{u_1}^2 + d_{u_1}^2} & \ldots & (m_h - 1)\sqrt{d_{u_2}^2 + d_{u_h}^2} & \sqrt{d_{u_2}^2 + d_{u_h}^2} \\
(m_1 - 1)\sqrt{d_{u_1}^2 + d_{u_1}^2} & \sqrt{d_{u_1}^2 + d_{u_1}^2} & \ldots & 0 & \sqrt{d_{u_1}^2 + d_{u_2}^2} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
(m_1 - 1)\sqrt{d_{u_{h-1}}^2 + d_{u_1}^2} & \sqrt{d_{u_{h-1}}^2 + d_{u_1}^2} & \ldots & 0 & \sqrt{d_{u_{h-1}}^2 + d_{u_2}^2} \\
(m_1 - 1)\sqrt{d_{u_{h-1}}^2 + d_{u_1}^2} & \sqrt{d_{u_{h-1}}^2 + d_{u_1}^2} & \ldots & 0 & 0 \\
\end{pmatrix}.
\]

The matrix \( Q_1 \) contains all the \( 3h + 1 \) distinct Sombor eigenvalues of \( G \). We note that there exists many such types of partitions , since each \( U_i \) can be partitioned into \( \{|U_i| - 1 \cup \{x\}, x \notin U_i\} \) in \((\binom{2h}{1}) = |U_i|\) choices, for \( i = 1, 2, \ldots, h \). It is easy to verify that \( Q_1 \) contains the repeated Sombor eigenvalues \( d_{u_1} \sqrt{2}, d_{u_2} \sqrt{2}, \ldots, d_{u_h} \sqrt{2} \), and 0 with associated eigenvectors

\[
X_1 = \begin{pmatrix}
\frac{1}{m_1 - 1}, -1, 0, 0, \ldots, 0, 0, 0, \ldots, 0 \\
2h - 2 & h + 1
\end{pmatrix}^T,
X_2 = \begin{pmatrix}
0, 0, -1, 0, 0, \ldots, 0, 0, 0, \ldots, 0 \\
2h - 4 & h + 1
\end{pmatrix}^T,
\]

\[
X_h = \begin{pmatrix}
0, 0, \ldots, 0, \frac{1}{m_h - 1}, -1, 0, 0, \ldots, 0 \\
2h - 2 & h + 1
\end{pmatrix}^T,
X_{h+1} = \begin{pmatrix}
0, 0, \ldots, 0, \frac{1}{m_{h+1} - 1}, -1, 0, 0, \ldots, 0 \\
2h & h + 1
\end{pmatrix}^T.
\]

A similar type of partition

\[
\left\{\{1, 2, \ldots, m_1 - 1\}, \{m_1\}, \{1, 2, \ldots, m_2 - 1\}, \{m_2\}, \ldots, \{1, 2, \ldots, m_{h-1} - 1\}, \{m_{h-1}\}, \{1, 2\}, \{1\}, \{1, 2, \ldots, n_2 - 1\}, \{n_2\}, \{1, 2, \ldots, n_3\}, \ldots, \{1, 2, \ldots, n_h\}\right\},
\]

can be considered for \( T(m_1, m_2, \ldots, m_{h-1}, 2; 1, n_2, \ldots, n_h) \), so that its associated quotient matrix contains all the \( 3h \) distinct Sombor eigenvalues. Similarly, depending on the number of distinct Sombor eigenvalues of \( T(m_1, m_2, \ldots, m_h, n_1, \ldots, n_h) \), the partitions can be modified, so that its corresponding quotient matrix contains all the distinct eigenvalue of \( G \). We illustrate the first case with the help of the following example.
Example 3.11. Let $G \cong T(3, 3, 3, 2, 2, 3)$ be the threshold graph of order 16. Then the quotient matrix contains all the distinct eigenvalues of $S(G)$ with any equitable partition given below:

$$\{\{u_1, u_2\}, \{u_3\}, \{u_4, u_5\}, \{u_6\}, \{u_7, u_8\}, \{u_9\}, \{u_{10}\}, \{u_{11}\}, \{u_{12}, u_{13}\}, \{u_{14}, u_{15}, u_{16}\}\}.$$ 

Corollary 3.8 and guarantees that $G$ has 10 distinct eigenvalues. The Sombor matrix of $G$ is

$$S(G) = \left(\begin{array}{cccccc}
15\sqrt{2}(J_3 - I_3) & 369J_{3\times3} & 325J_{3\times3} & 306J_{3\times2} & 261J_{3\times2} & 234J_{3\times3} \\
369J_{3\times3} & 12\sqrt{2}(J_3 - I_3) & 244J_{3\times3} & 225J_{3\times2} & 180J_{3\times2} & 0_{3\times3} \\
325J_{3\times3} & 244J_{3\times3} & 10\sqrt{2}(J_3 - I_3) & 18I J_{3\times2} & 0_2 & 0_{2\times2} \\
306J_{2\times3} & 225J_{2\times3} & 18I J_{2\times3} & 0_2 & 0_{2\times2} \\
261J_{3\times3} & 180J_{2\times3} & 0_{2\times3} & 0_{2\times2} \\
234J_{3\times3} & 0_{3\times3} & 0_{3\times2} & 0_{3\times2} & 0_3
\end{array}\right). \quad (3.3)$$

By computer calculations, the eigenvalues of $S(G)$ are


The quotient matrix of $S(G)$ is

$$Q(S(G)) = \left(\begin{array}{cccccc}
2 \cdot 15\sqrt{2} & 3\sqrt{369} & 3\sqrt{325} & 2\sqrt{306} & 2\sqrt{261} & 3\sqrt{234} \\
3\sqrt{369} & 2 \cdot 12\sqrt{2} & 3\sqrt{244} & 2\sqrt{225} & 2\sqrt{180} & 0 \\
3\sqrt{325} & 3\sqrt{244} & 2 \cdot 10\sqrt{2} & 2\sqrt{181} & 0 & 0 \\
3\sqrt{306} & 3\sqrt{225} & 3\sqrt{181} & 0 & 0 & 0 \\
3\sqrt{261} & 3\sqrt{180} & 0 & 0 & 0 & 0 \\
3\sqrt{234} & 0 & 0 & 0 & 0 & 0
\end{array}\right).$$

The eigenvalues of $Q(S(G))$ are

$$\{180.602, -61.1186, -32.2938, 26.1356, -21.1872, 12.5139\}.$$ 

The equitable quotient matrix with the modified partition

$$\{\{u_1, u_2\}, \{u_3\}, \{u_4, u_5\}, \{u_6\}, \{u_7, u_8\}, \{u_9\}, \{u_{10}\}, \{u_{11}\}, \{u_{12}, u_{13}\}, \{u_{14}, u_{15}, u_{16}\}\}$$

is given below

$$Q'(S(G)) = \left(\begin{array}{cccccccccccc}
15\sqrt{2} & 15\sqrt{2} & 2\sqrt{369} & 3\sqrt{369} & 2\sqrt{325} & 3\sqrt{325} & 2\sqrt{306} & 3\sqrt{306} & 2\sqrt{261} & 3\sqrt{261} & 3\sqrt{234} \\
2 \cdot 15\sqrt{2} & 0 & 2\sqrt{369} & 3\sqrt{369} & 2\sqrt{325} & 3\sqrt{325} & 2\sqrt{306} & 3\sqrt{306} & 2\sqrt{261} & 3\sqrt{261} & 3\sqrt{234} \\
2\sqrt{369} & 3\sqrt{369} & 12\sqrt{2} & 2\sqrt{244} & 3\sqrt{244} & 2\sqrt{225} & 3\sqrt{225} & 2\sqrt{180} & 3\sqrt{180} & 0 & 0 \\
2\sqrt{369} & 3\sqrt{369} & 12\sqrt{2} & 0 & 2\sqrt{244} & 3\sqrt{244} & 2\sqrt{225} & 3\sqrt{225} & 2\sqrt{180} & 0 & 0 \\
2\sqrt{325} & 3\sqrt{325} & 2\sqrt{244} & 2\sqrt{244} & 10\sqrt{2} & 10\sqrt{2} & 18I & 18I & 0 & 0 & 0 \\
2\sqrt{325} & 3\sqrt{325} & 2\sqrt{244} & 2\sqrt{244} & 10\sqrt{2} & 0 & 18I & 18I & 0 & 0 & 0 \\
2\sqrt{306} & 3\sqrt{306} & 2\sqrt{225} & 2\sqrt{225} & 18I & 18I & 0 & 0 & 0 & 0 & 0 \\
2\sqrt{306} & 3\sqrt{306} & 2\sqrt{225} & 2\sqrt{225} & 18I & 18I & 0 & 0 & 0 & 0 & 0 \\
2\sqrt{261} & 3\sqrt{261} & 2\sqrt{180} & 2\sqrt{180} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2\sqrt{234} & 3\sqrt{234} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right).$$

The eigenvalues of $Q'(S(G))$ are

Therefore, the quotient matrix $Q'(S(G))$ with the partition
\[ \{\{u_1, u_2\}, \{u_3\}, \{u_4, u_5\}, \{u_6\}, \{u_7, u_8\}, \{u_9\}, \{u_{10}\}, \{u_{11}\}, \{u_{12}, u_{13}\}, \{u_{14}, u_{15}, u_{16}\}\}, \]
contains all the distinct eigenvalues of $S(G)$.

As it is hard to explicitly find the largest/smallest eigenvalue of $Q$, next in the following result, we present their sharp bounds.

**Corollary 3.12.** Let $G \cong T(m_1, m_2, \ldots, m_h; n_1, n_2, \ldots, n_h)$ be the threshold graph of order $n$. Then
\[ \xi_1(G) \geq \frac{1}{2h} \left( R + \sqrt{R^2 + 4R'R''} \right) \quad \text{and} \quad \xi_n(G) \leq \frac{1}{2h} \left( R - \sqrt{R^2 + 4R'R''} \right), \]
where
\[ R = \sum_{i=1}^{h} l_i = \sum_{i=1}^{h} \left( (m_i - 1) d_u \sqrt{2} + \sum_{j=1 \atop j \neq i}^{h} m_j \sqrt{d_{w_i}^2 + d_{v_j}^2} \right), \quad R' = \sum_{i=1}^{h} l_i' = \sum_{i=1}^{h} \sum_{j=1}^{h} n_j \sqrt{d_{w_i}^2 + d_{v_j}^2}, \]
and $R'' = \sum_{i=1}^{h} l_i'' = \sum_{i=1}^{h} \sum_{j=1}^{h} m_j \sqrt{d_{w_i}^2 + d_{v_j}^2}$, where $l_i, l_i'$, and $l_i''$, are the $i$-th row sum of the matrices $A, B$ and $C$ defined in (3.2). Equalities hold iff $h = 1$, that is, $G$ is the complete split graph.

**Proof.** The quotient matrix of (3.2) with $\pi = \{\{1, 2, \ldots, h\}, \{1, 2, \ldots, h\}\}$ is
\[ Q^* = \frac{1}{h} \begin{pmatrix} R & R' \\ R'' & 0 \end{pmatrix}, \] (3.4)
where
\[ R = \sum_{i=1}^{h} l_i = \sum_{i=1}^{h} \left( (m_i - 1) d_u \sqrt{2} + \sum_{j=1 \atop j \neq i}^{h} m_j \sqrt{d_{w_i}^2 + d_{v_j}^2} \right), \quad R' = \sum_{i=1}^{h} l_i' = \sum_{i=1}^{h} \sum_{j=1}^{h} n_j \sqrt{d_{w_i}^2 + d_{v_j}^2}, \]
and $R'' = \sum_{i=1}^{h} l_i'' = \sum_{i=1}^{h} \sum_{j=1}^{h} m_j \sqrt{d_{w_i}^2 + d_{v_j}^2}$, where $l_i, l_i'$, and $l_i''$, are the $i$-th row sum of the matrices $A, B$ and $C$ ($R$, $R'$, and $R''$, respectively).

The eigenvalues of (3.4) are
\[ \chi_1(Q^*) = \frac{1}{2h} \left( R + \sqrt{R^2 + 4R'R''} \right) \quad \text{and} \quad \chi_2(Q^*) = \frac{1}{2h} \left( R - \sqrt{R^2 + 4R'R''} \right). \]

From Theorem 3.1, we have
\[ \xi_1(Q) \geq \chi_1(Q^*) \geq \chi_2(Q^*) \geq \xi_2(Q^*) \geq \xi_3(Q) \geq \cdots \geq \xi_{2h}(Q), \]
which further implies that
\[ \xi_1(G) \geq \frac{1}{2h} \left( R + \sqrt{R^2 + 4R'R''} \right) \quad \text{and} \quad \xi_n(G) \leq \frac{1}{2h} \left( R - \sqrt{R^2 + 4R'R''} \right), \]
with equalities if and only if the matrix $Q'$ is equitable quotient matrix of (3.2) with partition $\{\{1, 2, \ldots, h\}, \{1, 2, \ldots, h\}\}$, that is, same as saying the $A$ has constant row sum $l_1 = l_2 = \cdots = l_h$, $B$ has constant row sum $l_1' = l_2' = \cdots = l_h'$ and $C$ has constant row sum $l_1'' = l_2'' = \cdots = l_h''$. In this case, the partition
\text{\{1, 2, \ldots, \ell\}} \text{ of (3.2) is an equitable partition and } Q^* \text{ is an equitable quotient matrix of } Q \text{ and each eigenvalue of } Q \text{ is the eigenvalue of } S(G), \text{ including the spectral radius for non negative matrix and the smallest eigenvalue for such an equitable partition.}

Thus with } h = 1, \ G \cong T(m_1, n_1) \text{ and from [26], we have}

\[ \xi_1(G) = \frac{1}{2} \left( (m_1 - 1)(n - 1) \sqrt{2} + \sqrt{2((m_1 - 1)(n - 1))^2 + 4(m_1n_1((n - 1)^2 + m_1^2))} \right), \]

and

\[ \xi_{\ell}(G) = \frac{1}{2} \left( (m_1 - 1)(n - 1) \sqrt{2} - \sqrt{2((m_1 - 1)(n - 1))^2 + 4(m_1n_1((n - 1)^2 + m_1^2))} \right). \]

\[ \square \]

For a non-negative matrix with spectral radius \( \lambda \) and row sums \( l_1 \geq l_2 \geq \cdots \geq l_n \), it is known that \( l_n \leq \lambda \leq l_1 \). Moreover, if the matrix is irreducible, then one of the equalities holds iff the row sums of the matrix are all equal.

For \( G \cong T(m_1, m_2, \ldots, m_h; n_1, n_2, \ldots, n_h) \) with \( m_1 \geq \cdots \geq m_h \geq n_1 \geq \cdots \geq n_h \), it is clear that the maximum row sum is in the first row and the minimum row sum is the last \( 2h \)-th row. The maximum and the minimum row sum of the quotient matrix given in (3.2) is

\[ (m_1 - 1)d_{u_1} \sqrt{2} + \sum_{i=2}^{h} m_i \sqrt{d_{u_i}^2 + d_{v_i}^2} + \sum_{i=1}^{h} n_i \sqrt{d_{u_i}^2 + d_{v_i}^2} \quad \text{and} \quad m_1 \sqrt{d_{u_1}^2 + d_{v_1}^2}. \]

Thus, we have the following corollary.

**Corollary 3.13.** Let \( G \cong T(m_1, m_2, \ldots, m_h; n_1, n_2, \ldots, n_h) \) with \( m_1 \geq \cdots \geq m_h \geq n_1 \geq \cdots \geq n_h \), be the threshold graph. Then

\[ m_1 \sqrt{d_{u_1}^2 + d_{v_1}^2} \leq \xi_1(G) \leq (m_1 - 1)d_{u_1} \sqrt{2} + \sum_{i=2}^{h} m_i \sqrt{d_{u_i}^2 + d_{v_i}^2} + \sum_{i=1}^{h} n_i \sqrt{d_{u_i}^2 + d_{v_i}^2}. \]

The spread of \( M \) (real symmetric matrix) with eigenvalues \( \lambda_1(M) \geq \lambda_2(M) \geq \cdots \geq \lambda_n(M) \) is given by \( s(M) = \lambda_1(M) - \lambda_n(M) \). The spread of \( S(G) \) is \( s(S(G)) = \xi_1 - \xi_n \), see [30]. With terminology as in Corollary 3.13, we have the following result for the spread of \( S(G) \).

**Corollary 3.14.** The spread of the \( S(G) \) matrix of \( G \cong T(m_1, m_2, \ldots, m_h; n_1, n_2, \ldots, n_h) \) satisfies

\[ s(S(G)) \geq \frac{1}{h} \left( \sqrt{R^2 + 4R'R''} \right), \]

with equality holding iff \( G \cong T(m_1; n_1) \), that is, \( G \) is the complete split graph.

From Mirisky [24], the spread of \( M \) with trace \( tr(M) \) satisfies the following inequality

\[ s(M) \leq \sqrt{2\|M\|_F^2 - \frac{2}{n} (trM)^2}, \quad (3.5) \]

with equality iff \( M \) is normal such that \( n - 2 \) of its eigenvalues are equal to each other and to the arithmetic mean of the remaining two.
As trace of \( \text{tr}(S(G)) = \sum_{i=1}^{n} \xi_i = 0 \) and \( \|S(G)\|_F^2 = \sum_{i=1}^{n} \xi_i^2 \). Now,

\[
\sum_{i=1}^{n} \xi_i^2 = \sum_{i=1}^{h} 2(m_i - 1)d_{u_i} + \text{tr}(Q^2),
\]

where \( \text{tr}(Q^2) \) is the trace of (3.2).

Thus, we have the following result.

**Corollary 3.15.** Let \( T \) be threshold graph of order \( n \). Then

\[
s(S(T)) \leq \sqrt{\sum_{i=1}^{h} 2(m_i - 1)d_{u_i}^2 + \text{tr}(Q^2)},
\]

where \( \text{tr}(Q^2) \) is the trace of (3.2).

4. Sombor energy of threshold graphs

Let \( \{e_1, e_2, e_3, \ldots, e_n\} \), with \( e_i \geq 0 \) be the set of real numbers and let \( P_t \) be the average of products of \( t \)-element subset of the set \( \{e_1, e_2, e_3, \ldots, e_n\} \), that is,

\[
P_1 = \frac{e_1 + e_2 + e_3 + \cdots + e_n}{n},
\]

\[
P_2 = \frac{1}{n(n-1)}(e_1e_2 + e_1e_3 + \cdots + e_1e_n + e_2e_3 + \cdots + e_{n-1}e_n),
\]

\[\vdots\]

\[
P_n = e_1e_2 \cdots e_n.
\]

The following result relates \( P_t \)'s with themselves.

**Lemma 4.1 ([6], Maclaurin symmetric mean inequality).** For positive real numbers \( e_1, e_2, \ldots, e_n \), we have the following inequalities

\[
P_1 \geq P_2^\frac{1}{2} \geq P_3^\frac{1}{3} \geq \cdots \geq P_n^\frac{1}{n},
\]

with equalities holding iff \( e_1 = e_2 = \cdots = e_n \).

Now, we give inequalities for the Sombor energy and characterize the extremal graphs.

**Theorem 4.2.** Let \( G \cong T(m_1, \ldots, m_h; n_1, \ldots, n_h) \) be a threshold graph of order \( n \). Then, we have the following.

(i)

\[
E(S(G)) \leq \sqrt{2} \sum_{i=1}^{h} (m_i - 1)d_{u_i} + \sqrt{2h \cdot \text{tr}(Q^2)},
\]

with equality holding iff \( G \cong T(1; n_1) \).

(ii)

\[
E(S(G)) \geq \sqrt{2} \sum_{i=1}^{h} (m_i - 1)d_{u_i} + \sqrt{\text{tr}(Q^2) + 2h(2h - 1)|\text{det}(Q)|^\frac{1}{h}},
\]

with equality is attained iff \( G \cong T(1; n_1) \). The determinant of \( Q \) and \( \text{tr}(Q^2) \) are given in Corollary 3.5.
Proof. By Theorem 3.4, the Sombor spectrum of $G$ consists of the eigenvalue $-d_u \sqrt{2}$ with multiplicity $m_i - 1$, the eigenvalue 0 with multiplicity $\sum_{i=1}^{h} n_i - h$ and the eigenvalues of the matrix given in (3.2). Thus the Sombor energy of $G$ is given by

$$E(S(G)) = \sqrt{2} \sum_{i=1}^{h} (m_i - 1)d_u + E(Q), \quad (4.2)$$

where $E(Q)$ is the energy of the matrix given in (3.2). Next, we calculate the bounds for $E(Q)$, since its eigenvalues cannot be found explicitly. Let $\xi_1(Q) \geq \xi_2(Q) \geq \cdots \geq \xi_{2h}(Q)$ be the eigenvalues of $Q$. Now, we consider the following cases.

(i) Substituting $e_i = |\xi_i(Q)|$ and $n = 2h$ in Lemma 4.1, we have

$$\left( \frac{1}{2h} \sum_{i=1}^{2h} |\xi_i(Q)| \right)^2 \geq \frac{1}{2h(2h-1)} \sum_{1 \leq i < j \leq 2h} |\xi_i(Q)||\xi_j(Q)|, \quad (4.3)$$

which is equivalent to

$$(2h - 1)\left( \sum_{i=1}^{2h} |\xi_i(Q)| \right)^2 \geq 4h \sum_{1 \leq i < j \leq 2h} |\xi_i(Q)||\xi_j(Q)|$$

$$= 2h \left( \sum_{i=1}^{2h} |\xi_i(Q)| \right)^2 - 2h \sum_{i=1}^{2h} \xi_i^2,$$

that is,

$$\left( \sum_{i=1}^{2h} |\xi_i(Q)| \right)^2 \leq 2h \cdot \text{tr}(Q^2),$$

where $\text{tr}(Q^2)$ is given in Corollary 3.5. Equality is attained in above inequality if equality holds in (4.3), that is, $|\xi_1(Q)| = |\xi_2(Q)| = \cdots = |\xi_{2h}(Q)|$. Which is true for $h = 1$, $m_1 = 1$ and $n_1 \geq 2$ and in this case $Q$ takes the form

$$Q_1 = \begin{pmatrix} 0 & n_1 \sqrt{d_u^2 + d_v^2} \\ \sqrt{d_u^2 + d_v^2} & 0 \end{pmatrix},$$

where $d_{u_1} = n_1$ and $d_{v_1} = 1$. The trace of $Q_1^2$ is

$$\text{tr}(Q_1^2) = n_1(d_u^2 + d_v^2) + n_1(d_u^2 + d_v^2) = 2n_1(d_u^2 + d_v^2)$$
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and its eigenvalues satisfy

\[ |\xi(Q_1)| = |\xi(Q_2)| = \sqrt{n_1 \left( d_{u_1}^2 + d_{v_1}^2 \right)}. \]

Thus, \( E(S(G)) = 2\sqrt{n_1 \left( d_{u_1}^2 + d_{v_1}^2 \right)} \) and equality holds in (4.1). Conversely, for \( G \cong T(1; n_1) \), the Sombor spectrum [26] of \( G \) is

\[ \{0^{n_1-1}, \pm \sqrt{n_1 \left( d_{u_1}^2 + d_{v_1}^2 \right)}\}, \]

and its Sombor energy is

\[ E(S(G)) = 2\sqrt{n_1 \left( d_{u_1}^2 + d_{v_1}^2 \right)} = \sqrt{2 \cdot \text{tr}(Q_1^2)}, \]

where \( d_{u_1} = n_1 - 1 \) and \( d_{v_1} = 1 \).

(ii) Again using \( n = 2h \) and \( e_i = |\xi_i(Q)| \) in Lemma 4.1 and noting that \(|\text{det}(Q)| = \prod_{i=1}^{2h} |\xi_i(Q)|\), we have

\[ \frac{1}{2h(2h-1)} \sum_{1 \leq i < j \leq 2h} |\xi_i(Q)||\xi_j(Q)| \geq \left( \prod_{i=1}^{2h} |\xi_i(Q)| \right)^{\frac{1}{2h}}, \]

with equality iff \( |\xi_1(Q)| = \cdots = |\xi_{2h}(Q)| \). The above expression is equivalent to

\[ 2 \sum_{1 \leq i < j \leq 2h} |\xi_i(Q)||\xi_j(Q)| \geq 2h(2h - 1)|\text{det}(Q)|^{\frac{1}{2h}}, \]

that is,

\[ \left( \sum_{i=1}^{2h} |\xi_i(Q)| \right)^2 - \sum_{i=1}^{2h} |\xi_i(Q)|^2 \geq 2h(2h - 1)|\text{det}(Q)|^{\frac{1}{2h}}, \]

that is,

\[ E(Q) = \sum_{i=1}^{2h} |\xi_i(Q)| \geq \sqrt{\sum_{i=1}^{2h} |\xi_i(Q)|^2 + 2h(2h - 1)|\text{det}(Q)|^{\frac{1}{2h}}}. \]

Therefore, from (4.2), we have

\[ E(S(G)) \geq \sqrt{2} \sum_{i=1}^{h} (m_i - 1)d_{u_i} + \sqrt{\text{tr}(Q^2) + 2h(2h - 1)|\text{det}(Q)|^{\frac{1}{2h}}}, \]

where \( \text{tr}(Q^2) \) and \( \text{det}(Q) \) are given in Corollary 3.5. The equality is as in part (i).

The following example presents the numerical estimation of the above results for \( G \cong T(3, 3, 3; 2, 2, 3) \).

**Example 4.3.** Let \( G \cong T(3, 3, 3; 2, 2, 3) \) be the threshold graph as given in Figure 1. Then, we have the following.
1. The Sombor index of $G$ is $\text{SO}(G) = 1223.48$.

2. The Sombor spectrum of $G$ is


3. The Sombor spectral radius of $G$ is $\xi_1(G) = 180.602$, the smallest eigenvalue is $\xi_n(G) = -96.6512$. By Corollary 3.12, $\xi_1(G) \geq 174.719$ and $\xi_n(G) \leq -34.1197$, and by Corollary 3.13, $45.8912 \leq \xi_1(G) \leq 267.326$.

4. The spread of Sombor matrix of $G$ is $\delta(S(G)) = 241.721$, while by Corollary 3.14, the spread is $\delta(S(G)) \geq 208.839$ and by Corollary 3.15, $\delta(S(G)) \leq 389.468$.

5. The Sombor energy of $G$ is $E(S(G)) = 438.503$. By Theorem 4.2, the lower bound for the Sombor energy is $E(S(G)) \geq 306.03$ and the upper bound is $E(S(G)) \leq 586.424$.

5. Conclusion

The general adjacency matrix [18] associated with a symmetric function $\phi$ (usually a degree base topological index) of $G$ is a real symmetric matrix, defined by

$$A_{\phi}(G) = (a_{\phi})_{ij} = \begin{cases} \phi_{d_u,d_v} & \text{if } u \text{ is adjacency to } v \\ 0 & \text{otherwise.} \end{cases}$$ (5.1)

If $\phi_{d_u,d_v} = 1$ when $u$ is adjacent to $v$, then $A_{\phi}(G)$ is the adjacency $A(G)$ matrix. If $\phi_{d_u,d_v} = \sqrt{d_u^2 + d_v^2}$ in (5.1), we have the $S(G)$ matrix

$$S(G) = \begin{cases} \sqrt{d_u^2 + d_v^2} & \text{if } u \text{ is adjacency to } v \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, for values of $\phi_{d_u,d_v}$, we get ABC matrix, geometric-arithmetic matrix and many others. The theory of the present articles can be extended to adjacency and other matrices with similar techniques and lines along with some calculations. Further in future, the analysis of the remaining Sombor eigenvalues of the quotient matrix of threshold graphs remains an open topic and new techniques can be developed in this direction.

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Conflict of Interest

Authors state no conflict of interest.

Data availability statement

All data generated or analysed during this study are included in this article.

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