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ON GRAPH-BASED NETWORK PARAMETERS AND COMPONENT FACTORS IN NETWORKS

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Abstract. Many physical structures can conveniently be simulated by networks. To study the properties of the network, we use a graph to simulate the network. A graph H is called an $\mathcal{F}$-factor of a graph G, if H is a spanning subgraph of G and every connected component of H is isomorphic to a graph from the graph set $\mathcal{F}$. An $\mathcal{F}$-factor is also referred as a component factor. The graph-based network parameter *degree sum* of G is defined by

$$\sigma_k(G) = \min_{X \subseteq V(G)} \Big\{ \sum_{x \in X} d_G(x) : X \text{ is an independent set of } k \text{ vertices} \Big\}.$$

In this article, we give the precise degree sum condition for a graph to have $\{P_2, C_3, P_5, \mathcal{T}(3)\}$-factor and $\{K_{1,1}, K_{1,2}, \ldots, K_{1,k}, \mathcal{T}(2k+1)\}$-factor. We also obtain similar results for $\{P_2, C_3, P_5, \mathcal{T}(3)\}$-factor avoidable graph and $\{K_{1,1}, K_{1,2}, \ldots, K_{1,k}, \mathcal{T}(2k+1)\}$-factor avoidable graph, respectively.

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1. Introduction

Enterprises face risks caused by unpredictable emergencies in the supply chain system. The system needs to increase supply chain resilience, globally identify supply chain patterns, monitor weak links, and minimize risk impact. The network data model has the ability of intelligent analysis and reasoning, which can quickly conduct deep chain queries and reasoning, making it easy to predict the impact of demand changes, price changes, supply problems, warehousing problems, etc. on upstream and downstream in the supply chain. It connects data with business decisions, giving enterprises sufficient time to adjust pricing or marketing strategies, and avoiding risks. In response to the supply chain risk model, we use vertices of the network stand for suppliers and components, and edges of the network act as links between components and suppliers, thereby constructing an end-to-end graph of the supply chain. When there are cost fluctuations, component shortages, and other problems in the

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supply chain system, we can use graph algorithms to trace the source upwards and identify the scope of impact downwards, providing data support for enterprise business decision-making.

Some graph-based network parameters can directly measure the vulnerability of the network, and are directly related to the existence of the fractional factor and the $\mathcal{F}$-factor. This study models the network as a graph structure, and studies the relationship between graphic parameters (for instance, *degree sum*) and the stability of the network structure and data transmission from a theoretical point of view.

Let $G = (V(G), E(G))$ be a graph representing a network, where $V(G)$ is the vertex set and $E(G)$ is the edge set of G, respectively. A subgraph H of G is called a spanning subgraph of G if $V(H) = V(G)$ and $E(H) \subseteq E(G)$. Given a vertex $v \in V(G)$, let $N_G(v)$ be the set of vertices adjacent to v in G and $d_G(v) = |N_G(v)|$ be the degree of v in G. We say a vertex v is an *isolated vertex* if $d_G(v) = 0$ in G, and use $i(G)$ to denote the number of isolated vertices in G. A subgraph H of G is called an induced subgraph of G if every pair of vertices in H which are adjacent in G are also adjacent in H. For any subset $S \subseteq V(G)$, let $G[S]$ denote the subgraph of G induced by S, and $G - S := G[V(G) \setminus S]$ is the resulting graph after deleting the vertices of S from G. For any $M \subseteq E(G)$, we use $G - M$ to denote the subgraph obtained from G by deleting M. Especially, we write $G - x = G - \{x\}$ for $S = \{x\}$ and $G - e = G - \{e\}$ for $M = \{e\}$. We use K_n, P_n and C_n to denote the complete graph, the path and the cycle of order n, respectively. $K_{i,j}$ denotes the complete bipartite graph with the bipartition (X, Y), where $|X| = i, |Y| = j$. More notations and terminologies of graph theory used in this article, but not listed specifically, can be referred to Bondy and Murty [4].

Let $\mathcal{F}$ be the set of connected graphs. A $\mathcal{F}$-factor of graph G is a spanning subgraph such that each component is isomorphic to a member of $\mathcal{F}$. An $\mathcal{F}$-factor is also referred as a component factor. If $G-e$ admits an $\mathcal{F}$-factor for any $e \in E(G)$, then we say that G is an $\mathcal{F}$-factor avoidable graph.

The first attempt on the study of factors was made by Danish mathematician Petersen in 1891, who proved that every graph of even degrees can be decomposed into the union of edge-disjoint 2-factors. This was motivated from the study of an algebraic factorization problem. He also showed that every 2-connected cubic graph has a 1-factor. These two results can be viewed as a forerunner of modern graph factor theory. Since Tutte gave a characterization (*i.e.*, so-called Tutte's 1-Factor Theorem) for the existence of perfect matchings in arbitrary graphs [22] in 1947, it has become a cornerstone of factor theory. Till now, this elegant theorem is still one of the most fundamental results in factor theory. Subsequently, Tutte [22] extended the techniques in the proof of 1-Fador Theorem to obtain a sufficient and necessary condition for a graph to have an f-factor. For the more comprehensive account of study on matching theory and graph factors, readers can refer to [1, 24] for more information and related references.

As early as 1953, Tutte [23] obtained a necessary and sufficient condition for a graph to have a $\{P_2, C_n : n \geq 3\}$-factor.

Theorem 1.1. (Tutte [23]) *A graph G admits a $\{P_2, C_n : n \geq 3\}$-factor if and only if $i(G - S) \leq |S|$ for every $S \subset V(G)$, where $i(G - S)$ denotes the number of isolated vertices in $G - S$.*

Klopp and Steffen [19] posed some properties for the existence of $\{K_{1,1}, K_{1,2}, C_m : m \geq 3\}$-factors in graphs. Kano, Lee and Suzuki [16] showed two results for graphs to admit path and cycle factors. Kano, Lu and Yu [17] derived a result for a graph having a $\{K_{1,2}, K_{1,3}, K_5\}$-factor.

Theorem 1.2. (Kano, Lu, Yu [17]) *A graph G admits a $\{K_{1,2}, K_{1,3}, K_5\}$-factor if $i(G - S) \leq \frac{|S|}{2}$ for every $S \subset V(G)$.*

Amahashi and Kano [3] obtained a criterion for a graph with a $\{K_{1,j} : 1 \leq j \leq k\}$-factor. Kano and Saito [14] used isolated k-toughness to ensure the existence of $\{K_{1,j} : k \leq j \leq 2k\}$-factor.

Theorem 1.3. (Amahashi, Kano [3]) *A graph G admits a $\{K_{1,j} : 1 \leq j \leq k\}$-factor if $i(G - S) \leq k|S|$ for every $S \subset V(G)$.*

Akiyama *et al.* [2] characterized those graphs admitting $P_{\geq 2}$-factor, which is stated as follows.

Theorem 1.4. *(Akiyama et al.[2]) A graph G has a $P_{\geq 2}$-factor if and only if $i(G-S) \leq 2|S|$ for all $S \subseteq V(G)$.*

Using the concept of sun components, Kaneko [13] presented a good characterization for a graph admitting $P_{\geq 3}$-factors. Kano, Katona and Király [15] gave a simpler proof of Kaneko's result. Kano, Lu and Yu [17] demonstrated that a graph G has a $P_{\geq 3}$-factor if $i(G-S) \leq \frac{2}{3}|S|$ for all $S \subseteq V(G)$. Dai *et al.* [5,8,10], Liu [20], Gao [12], Zhou *et al.* [25,26,26,27] posed some sufficient conditions for graphs having $P_{\geq 3}$-factors with given properties.

Intuitively, if a graph is dense enough, it will have a $P_{\geq 3}$-factor. Recall that a graph G is *Hamiltonian* if it has a spanning cycle (a Hamiltonian cycle), that is a cycle containing all the vertices of G. Again, intuitively, if a graph is dense enough, it has a Hamiltonian cycle. Suppose G is of order $n \geq 3$. Then clearly if G is Hamiltonian, G has a $P_{\geq 3}$-factor by deleting one edge from a Hamiltonian cycle. Recall the two classic results in the study of Hamiltonian graphs with precise definitions of denseness. The first one is due to Dirac [11]: Let G be a graph of order $n \geq 3$, then G is Hamiltonian if $d_G(v) \geq n/2$ for each vertex v of G. Here the denseness condition is $d_G(v) \geq n/2$ for each vertex v of G. Although this condition is sharp, the denseness condition can be refined giving Ore's Theorem [21]: Let G be a graph of order $n \geq 3$, then G is Hamiltonian if $d_G(u) + d_G(v) \geq n$ for each pair of nonadjacent vertices of G. Our goal is to replace "$d_G(u) + d_G(v) \geq n$ for every pair of nonadjacent vertices of G" by a weaker condition of the same flavor. Here, if G is a graph containing at least k independent vertices, then we use the graphic parameter *degree sum* defined as

$$\sigma_k(G) = \min_{X \subseteq V(G)} \Big\{ \sum_{x \in X} d_G(x) : \text{the set} X \text{ is independent and contains } k \text{ vertices} \Big\}.$$

Note that when $k = 2$, this corresponds to taking the minimum of $d_G(u) + d_G(v)$ over every pair of nonadjacent vertices of G, part of the crux of the statement of Ore's Theorem [21]. On the study of the existence of component factors using the simple Ore-type sufficient condition refer to [6,7].

In this paper, in terms of degree sum (that is, a denseness condition), we investigate the existence of component factors with given properties in graphs, which are shown in Sections 2 and 3.

2. Graph with a $\{P_2, C_3, P_5, \mathcal{T}(3)\}$-factor

For a tree T, the set of leaves is denoted by $Leaf(T)$. An edge of T incident with a leaf is called a pendant edge. In particular, the number of leaves of T is equal to that of pendant edges of T.

Recently, using fractional factors as a tool, Kano, Lu and Yu [18] construct a new component factor called $\{P_2, C_3, P_5, \mathcal{T}(3)\}$-factor, where $\mathcal{T}(3)$ is defined as follows: for any $\{1,3\}$-tree R (*i.e.*, a tree such that $d_R(x) \in \{1,3\}, \forall x \in V(R)$), a new tree T_R is obtained from R by inserting a new vertex of degree 2 into every edge of R, and by adding a new pendant edge to each leaf of R. Then the tree T_R is a $\{1,2,3\}$-tree admitting $|E(R)| + |Leaf(R)|$ vertices of degree 2 and possesses the same number of leaves as R (*e.g.*, see Fig. 1). The collection of such $\{1,2,3\}$-trees T_R generated from all $\{1,3\}$-trees R is denoted by $\mathcal{T}(3)$.

Kano, Lu and Yu [18] derived a characterization for a graph with $\{P_2, C_3, P_5, \mathcal{T}(3)\}$-factors.

Theorem 2.1. (Kano, Lu, Yu [18]) *A graph G admits a $\{P_2, C_3, P_5, \mathcal{T}(3)\}$-factor if and only if*

$$i(G-S) \leq \frac{3}{2}|S|$$

for every $S \subset V(G)$.

Using Theorem 2.1, we shall verify the following two results on the graph admits $\{P_2, C_3, P_5, \mathcal{T}(3)\}$-factors.

Theorem 2.2. *Let G be a connected graph of order n. Then G admits a $\{P_2, C_3, P_5, \mathcal{T}(3)\}$-factor if $\sigma_3(G) \geq \lceil \frac{6}{5}n \rceil$.*

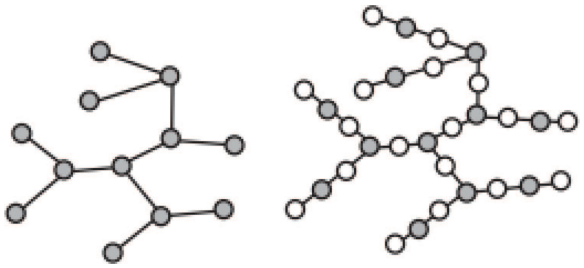

FIGURE 1. A $\{1,3\}$-tree R and the resulting $\{1,2,3\}$-tree T_R obtained from R.

Proof. Assume that there is no $\{P_2, C_3, P_5, \mathcal{T}(3)\}$-factor in G. By Theorem 2.1, there exists $S \subseteq V(G)$ satisfying that

$$i(G-S) > \frac{3}{2}|S|. \tag{1}$$

Claim 2.1. $|S| \geq 2$.

Proof. By $\sigma_3(G) \geq \frac{6}{5}n$, we can easily confirm that G is not a complete graph and $n \geq 5$. Assume that S is an empty set $\emptyset$, then $|S| = 0$, $i(G) > 0$ by (1). In terms of the integrality of $i(G)$, we obtain that $i(G) \geq 1$, that is, G has an isolated vertex. This is a contradiction to the connectivity of G. Hence, $S \neq \emptyset$ and $|S| \geq 1$. This together with (1) implies that $i(G-S) > \frac{3}{2}|S| \geq \frac{3}{2}$. Due to the integrality of $i(G-S)$, we obtain that

$$i(G-S) \geq 2. \tag{2}$$

If $|S| = 1$, then by (2), there exists a vertex $x \in V(G)$ such that $i(G-x) \geq 2$. Denote the set of isolated vertices in $G-x$ by $I(G-x)$, and choose $u, v \in I(G-x)$. Let w be a vertex in $G-\{x,u,v\}$. Then it is clear that $\{w,u,v\}$ is independent in G and $N_G(w) \subseteq V(G) \setminus \{u,v,w\}$. So, the degree $d_G(w) \leq n-3$. This together with $N_G(u) = N_G(v) = \{x\}$ implies that

$$\sigma_3(G) \leq d_G(u) + d_G(v) + d_G(w) \leq 1 + 1 + (n-3) = n-1,$$

which contradicts $\sigma_3(G) \geq \lceil \frac{6}{5}n \rceil$. So, we have that $|S| \geq 2$. ◊

It follows from Claim 2.1 that

$$i(G-S) > \frac{3}{2}|S| \geq 3. \tag{3}$$

Let $\{u_1, u_2, u_3\} \subseteq I(G-S)$, where $I(G-S)$ denotes the set of isolated vertices in $G-S$. It is clear that $\{u_1, u_2, u_3\}$ is independent in G. Then $\sum_{i=1}^{3} d_G(u_i) \geq \sigma_3(G) \geq \lceil \frac{6}{5}n \rceil$. Since $\bigcup_{i=1}^{3} N_G(u_i) \subseteq S$, we have

$$|S| \geq \max\{d_G(u_i) : i = 1,2,3\} \geq \frac{\sum_{i=1}^{3} d_G(u_i)}{3} \geq \frac{\sigma_3(G)}{3} \geq \lceil \frac{2}{5}n \rceil \geq \frac{2}{5}n.$$

Combining with (3) implies that

$$n \geq |S| + i(G-S) > |S| + \frac{3}{2}|S| = \frac{5}{2}|S| \geq n,$$

a contradiction. □

Theorem 2.3. *Let G be a 2-connected graph of order n. Then G is a $\{P_2, C_3, P_5, \mathcal{T}(3)\}$-factor avoidable graph if $\sigma_3(G) \geq \lceil \frac{6}{5}n \rceil + 1$.*

Proof. We first argue that $n \geq 4$. Otherwise, G is 3-cycle since G is 2-connected, and thus G is a $\{P_2, C_3, P_5, \mathcal{T}(3)\}$-factor avoidable graph. Let $H = G - e$ for every fixed edge $e \in E(G)$. To prove Theorem 2.3, it suffices to verify that H has a $\{P_2, C_3, P_5, \mathcal{T}(3)\}$-factor. On the contrary, we assume that H admits no $\{P_2, C_3, P_5, \mathcal{T}(3)\}$-factor. By Theorem 2.1, there exists $S \subseteq V(H)$ such that

$$i(H - S) > \frac{3}{2}|S|. \tag{4}$$

Claim 2.2. $|S| \geq 2$.

Proof. On the contrary, we fist assume that $S = \emptyset$. By (4), we obtain

$$i(H) = i(H - S) > 0. \tag{5}$$

On the other hand, since G is 2-connected, H is a connected graph. So we obtain $i(H) = 0$, which contradicts (5). Hence, $S \neq \emptyset$ and $|S| \geq 1$. This together with (4) implies that $i(H - S) > \frac{3}{2}|S| \geq \frac{3}{2}$. Due to the integrality of $i(H - S)$, we obtain that

$$i(G - e - S) = i(H - S) \geq 2. \tag{6}$$

If $|S| = 1$, then by (6), there exists a vertex $x \in V(G)$ such that $i(H - x) \geq 2$. Denote the set of isolated vertices in $H - x$ by $I(H - x)$, and choose $u, v \in I(H - x)$. If $e \neq uv$, then $i(G - x) \geq 1$, which contradicts that G is 2-connected. Otherwise, $e = uv$, then e is an isolated edge in $G - x$. As $n \geq 4$, then there exists some vertices in $G - \{x, u, v\}$. This together with $N_{G-e}(u) = N_{G-e}(v) = \{x\}$ implies that x is a cut vertex of G, which contradicts that G is 2-connected. ◊

It follows from Claim 2.2 that $i(H - S) > \frac{3}{2}|S| \geq 3$. Due to the integrality of $i(H - S)$, we obtain that

$$i(G - e - S) = i(H - S) \geq 4. \tag{7}$$

Let $\{u_1, u_2, u_3, u_4\} \subseteq I(H - S)$, where $I(H - S)$ denotes the set of isolated vertices in $H - S$. Let $e = uv$, then $|\{u_1, u_2, u_3, u_4\} \cap \{u, v\}| \leq 2$. Without loss of generality, we assume $\{u_1, u_2\} \cap \{u, v\} = \emptyset$. It follows that $\{u_1, u_2, u_3\}$ is independent in G. Then $\sum_{i=1}^{3} d_G(u_i) \geq \sigma_3(G) \geq \lceil \frac{6}{5}n \rceil + 1$. Since $\bigcup_{i=1}^{2} N_G(u_i) \subseteq S$ and $N_G(u_3) \subseteq S \cup \{u_4\}$, we have

$$\begin{aligned} |S| &\geq \max\{d_G(u_1), d_G(u_2), d_G(u_3) - 1\} \\ &\geq \frac{\sum_{i=1}^{3} d_G(u_i) - 1}{3} \\ &\geq \frac{\sigma_3(G) - 1}{3} \\ &\geq \frac{\lceil \frac{6}{5}n \rceil + 1 - 1}{3} \\ &\geq \lceil \frac{2}{5}n \rceil \\ &\geq \frac{2}{5}n. \end{aligned}$$

Combining with (4) implies that

$$n \geq |S| + i(H - S) > |S| + \frac{3}{2}|S| = \frac{5}{2}|S| \geq n,$$

a contradiction. □

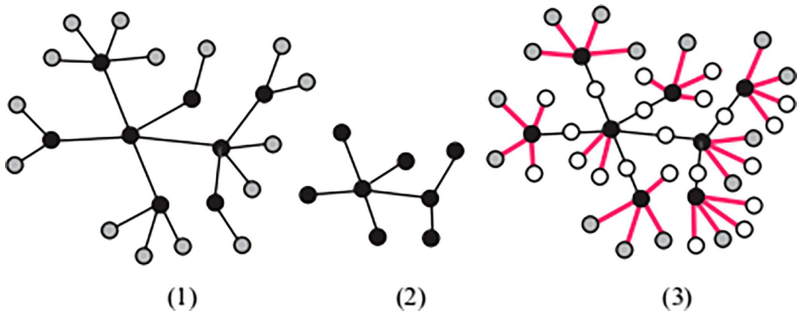

FIGURE 2. (1) A tree R that satisfies (ii) with $k = 4$; (2) The tree $R - Leaf(R)$; (3) The tree T_R obtained from R.

3. Graph with a $\{K_{1,1}, K_{1,2}, \ldots, K_{1,k}, \mathcal{T}(2k+1)\}$-factor

In this section, we we investigate the graph admits $\{K_{1,1}, K_{1,2}, \ldots, K_{1,k}, \mathcal{T}(2k+1)\}$-factor, where $k \geq 2$ is an integer and $\mathcal{T}(2k+1)(k \geq 2)$ is a more general class of trees. Let $k \geq 2$ be an integer and let R be a tree that satisfies the following conditions: for each $x \in V(R - Leaf(R))$,

(i) $d_{R-Leaf(R)}(x) \in \{1, 3, \ldots, 2k+1\}$;

(ii) 2·(the number of leaves adjacent to x in R)$+d_{R-Leaf(R)}(x) \leq 2k+1$.

For such a tree R, we derive a new tree T_R as follows:

(iii) insert a new vertex of degree 2 into each edge of $R - Leaf(R)$;

(iv) for each vertex x of $R - Leaf(R)$ with $d_{R-Leaf(R)}(x) = 2r+1 < 2k+1$, add $k - r-$(the number of leaves adjacent to x in R) pendant edges to x.

Then the set of such trees T_R for all trees R satisfying conditions (i) and (ii) is denoted by $\mathcal{T}(2k+1)$.

Recently, Kano, Lu and Yu [18] derived a characterization for graphs with $\{K_{1,1}, K_{1,2}, \ldots, K_{1,k}, \mathcal{T}(2k+1)\}$-factors.

Theorem 3.1. *Let G be a connected graph, $k \geq 2$ be an integer. Then G admits a $\{K_{1,1}, K_{1,2}, \ldots, K_{1,k}, \mathcal{T}(2k+1)\}$-factors if and only if*

$$i(G - S) \leq (k + \frac{1}{2})|S|$$

for every $S \subset V(G)$.

Using Theorem 3.1, we shall verify the following results on the graph admits $\{K_{1,1}, K_{1,2}, \ldots, K_{1,k}, \mathcal{T}(2k+1)\}$-factors.

Theorem 3.2. *Let G be a connected graph of order n, and $k \geq 2$ be an integer. Then G admits a $\{K_{1,1}, K_{1,2}, \ldots, K_{1,k}, \mathcal{T}(2k+1)\}$-factor if $\sigma_{k+2}(G) \geq \lceil \frac{2k+4}{2k+3} n \rceil$.*

Proof. Assume that G admits no $\{K_{1,1}, K_{1,2}, \ldots, K_{1,k}, \mathcal{T}(2k+1)\}$-factor in G. By Theorem 3.1, there exists $S \subseteq V(G)$ satisfying that

$$i(G - S) > (k + \frac{1}{2})|S|. \tag{8}$$

Claim 3.1. $|S| \geq 2$.

Proof. By $\sigma_{k+2}(G) \geq \lceil \frac{2k+4}{2k+3} n \rceil$, we can easily confirm that G is not a complete graph and $n \geq k+4$. Assume that G is an empty set $\emptyset$, then $|S| = 0, i(G) > 0$ by (8). In terms of the integrality of $i(G)$, we obtain that $i(G) \geq 1$, that is, G has an isolated vertex. This is a contradiction to the connectivity of G. Hence, $S \neq \emptyset$ and

$|S| \geq 1$. This together with (8) implies that $i(G-S) > (k+\frac{1}{2})|S| \geq k+\frac{1}{2}$. Due to the integrality of $i(G-S)$, we obtain that

$$i(G-S) \geq k+1. \tag{9}$$

If $|S| = 1$, then by (9), there exists a vertex $x \in V(G)$ such that $i(G-x) \geq k+1$. Denote the set of isolated vertices in $G-x$ by $I(G-x)$, and choose $\{u_1, u_2, \ldots, u_k, u_{k+1}\} \subseteq I(G-x)$. Let y be a vertex in $G - \{x, u_1, u_2, \ldots, u_k, u_{k+1}\}$. Then it is clear that $\{y, u_1, u_2, \ldots, u_k, u_{k+1}\}$ is independent in G and $N_G(y) \subseteq V(G) \setminus \{y, u_1, u_2, \ldots, u_k, u_{k+1}\}$. So, the degree $d_G(y) \leq n-k-2$. Since $N_G(u_i) = \{x\}$ for any $i = 1, 2, \ldots, k+1$, we get

$$\sigma_{k+2}(G) \leq d_G(y) + \sum_{i=1}^{k+1} d_G(u_i) \leq (n-k-2) + (k+1) = n-1,$$

which contradicts $\sigma_{k+2}(G) \geq \lceil \frac{2k+4}{2k+3} n \rceil$. So, we have that $|S| \geq 2$. ◊

It follows from Claim 3.1 that

$$i(G-S) > (k+\frac{1}{2})|S| \geq 2k+1. \tag{10}$$

Let $\{v_1, v_2, \ldots, v_{k+1}, v_{k+2}\} \subseteq I(G-S)$, where $I(G-S)$ denotes the set of isolated vertices in $G-S$. It is clear that $\{v_1, v_2, \ldots, v_{k+1}, v_{k+2}\}$ is an independent subset in G. Then $\sum\limits_{i=1}^{k+2} d_G(v_i) \geq \sigma_{k+2}(G) \geq \lceil \frac{2k+4}{2k+3} n \rceil$. Since $\bigcup\limits_{i=1}^{k+2} N_G(v_i) \subseteq S$, we have

$$\begin{aligned} |S| &\geq \max\{d_G(v_i) : i = 1, 2, \ldots, k+2\} \\ &\geq \frac{\sum\limits_{i=1}^{k+2} d_G(v_i)}{k+2} \\ &\geq \frac{\sigma_{k+2}(G)}{k+2} \\ &\geq \lceil \frac{2}{2k+3} n \rceil \\ &\geq \frac{2}{2k+3} n. \end{aligned}$$

Combining with (10) implies that

$$n \geq |S| + i(G-S) > |S| + (k+\frac{1}{2})|S| = \frac{2k+3}{2}|S| \geq n,$$

a contradiction. □

Theorem 3.3. *Let G be a 2-connected graph of order n, and $k \geq 2$ be an integer. Then G is a $\{K_{1,1}, K_{1,2}, \ldots, K_{1,k}, \mathcal{T}(2k+1)\}$-factor avoidable graph if $\sigma_{k+2}(G) \geq \lceil \frac{2k+4}{2k+3} n \rceil + 1$.*

Proof. We first verify that G is not a complete graph and $n \geq k+4$ since $\sigma_{k+2}(G) \geq \frac{2k+4}{2k+3} n$. Let $H = G - e$ for every fixed edge $e \in E(G)$. To prove Theorem 3.3, it suffices to verify that H has a $\{K_{1,1}, K_{1,2}, \ldots, K_{1,k}, \mathcal{T}(2k+1)\}$-factor. On the contrary, we assume that H admits no $\{K_{1,1}, K_{1,2}, \ldots, K_{1,k}, \mathcal{T}(2k+1)\}$-factor. By Theorem 3.1, there exists $S \subseteq V(H)$ such that

$$i(H-S) > (k+\frac{1}{2})|S|. \tag{11}$$

Claim 3.2. $|S| \geq 2$.

Proof. On the contrary, we fist assume that $S = \emptyset$. By (11), we obtain

$$i(H) = i(H - S) > 0. \tag{12}$$

On the other hand, since G is 2-connected, H is a connected graph. So we obtain $i(H) = 0$, which contradicts (12). Hence, $S \neq \emptyset$ and $|S| \geq 1$. This together with (11) implies that $i(H - S) > (k + \frac{1}{2})|S| \geq k + \frac{1}{2}$. Due to the integrality of $i(H - S)$, we obtain that

$$i(G - e - S) = i(H - S) \geq k + 1. \tag{13}$$

If $|S| = 1$, then by (13), there exists a vertex $x \in V(G)$ such that $i(H - x) \geq k + 1$. After deleting an edge in a graph, the number of its isolated components increases by at most two. Hence, we obtain $i(G - x) \geq i(H - x) - 2 \geq k - 1 \geq 1$, which contradicts that G is 2-connected. ◊

It follows from Claim 3.2 that

$$i(G - e - S) = i(H - S) > (k + \frac{1}{2})|S| \geq 2k + 1 \geq k + 3. \tag{14}$$

Let $\{u_1, u_2, \ldots, u_{k+3}\} \subseteq I(H - S)$, where $I(H - S)$ denotes the set of isolated vertices in $H - S$. Let $e = uv$, then $|\{u_1, u_2, u_3, \ldots, u_{k+3}\} \cap \{u, v\}| \leq 2$. Without loss of generality, we assume $\{u_1, u_2, \ldots, u_{k+1}\} \cap \{u, v\} = \emptyset$. It follows that $\{u_1, u_2, \ldots, u_{k+2}\}$ is independent in G. Then $\sum_{i=1}^{k+2} d_G(u_i) \geq \sigma_{k+2}(G) \geq \lceil \frac{2k+4}{2k+3} n \rceil + 1$. Since $\bigcup_{i=1}^{k+1} N_G(u_i) \subseteq S$ and $N_G(u_{k+2}) \subseteq S \cup u_{k+3}$, we have

$$\begin{aligned} |S| &\geq \max\{d_G(u_1), d_G(u_2), \ldots, d_G(u_{k+2}) - 1\} \\ &\geq \frac{\sum_{i=1}^{k+2} d_G(u_i) - 1}{k+2} \\ &\geq \frac{\sigma_{k+2}(G) - 1}{k+2} \\ &\geq \frac{\lceil \frac{2k+4}{2k+3} n \rceil + 1 - 1}{k+2} \\ &\geq \frac{2}{2k+3}. \end{aligned}$$

Combining with (11) implies that

$$n \geq |S| + i(H - S) > |S| + (k + \frac{1}{2})|S| = \frac{2k+3}{2}|S| \geq n,$$

a contradiction. □

4. Open problems

As the last part of our paper, we propose the following open problems for further on-going studies.

Problem 4.1. *Does $G - V'$ have a $\{K_{1,1}, K_{1,2}, \ldots, K_{1,k}, \mathcal{T}(2k+1)\}$-factor for any $V' \subseteq V(G)$ with $|V'| = l$, or is a $(\{K_{1,1}, K_{1,2}, \ldots, K_{1,k}, \mathcal{T}(2k+1)\}, l)$-factor critical graph?*

Problem 4.2. *Does $G - E'$ have a $\{K_{1,1}, K_{1,2}, \ldots, K_{1,k}, \mathcal{T}(2k+1)\}$-factor for any $E' \subseteq E(G)$ with $|E'| = m$, or is a $(\{K_{1,1}, K_{1,2}, \ldots, K_{1,k}, \mathcal{T}(2k+1)\}, m)$-factor deleted graph?*

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