RAIRO-Oper. Res. 58 (2024) 3521–3524
https://doi.org/10.1051/ro/2024134

RAIRO Operations Research
www.rairo-ro.org

A NOTE ON THE DISLOCATION HYPERBOLIC TRANSFORMATION FUNCTION

Lennin Mallma Ramirez[1,*] and Nelson Maculan[2]

Abstract. In work of Roman A. Polyak [R.A. Polyak, *Math. Program.* **92** (2002) 197–235.], the Modified Chen-Harker-Kanzow-Smale (CHKS) function was studied to relate a multiplier method and an Interior Prox method with the second order distance function. Independently, the dislocated hyperbolic penalty function (DHPF) was proposed by A.E. Xavier (1992). DHPF was rewritten and studied in [L. Mallma Ramirez, N. Maculan, A.E. Xavier and V.L. Xavier, *RAIRO:RO* **57** (2023) 2941–2950.] and [L. Mallma Ramirez, N. Maculan, A.E. Xavier and V.L. Xavier, *J. Convex Anal.* To appear (2024).]. Thus, this function was called the dislocated hyperbolic function (DHF). In this work, we note that DHF is a particular case of CHKS function. Then we will call the DHF function as the dislocation hyperbolic transformation function.

Mathematics Subject Classification. 90C30.

Received March 25, 2024. Accepted June 26, 2024.

1. Introduction

In nonlinear optimization, the following problem is of interest to solve

$$(P) \quad \min\{f(x) \mid x \in \Omega\}, \tag{1.1}$$

with,

$$\Omega = \{x \in I\!R^n \mid c_i(x) \geq 0, \ \ i = 1, \dots, q\},$$

where $f : I\!R^n \to I\!R$ is convex and $c_i : I\!R^n \to I\!R, \ i = 1, \dots, q$ are concave are continuously differentiable.

The problem (1.1) can be solved by a wide variety of algorithms, some of them are: the penalty and augmented Lagrangian algorithms.

In 1967, Zangwill [10] studied the exact penalty function (EPF) in his penalty algorithm, this function is as follows:

$$\beta \in I\!R \quad \mapsto \quad (\beta)_+ = -\min\{\beta, 0\}. \tag{1.2}$$

Keywords. Nonlinear programming, transformation function, Kernel function.

[1] Federal University of Rio de Janeiro, Systems Engineering and Computer Science Program (COPPE), Rio de Janeiro, Brazil.
[2] Federal University of Rio de Janeiro, Systems Engineering and Computer Science Program-Applied Mathematics (COPPE/IM), Rio de Janeiro, Brazil.
*Corresponding author: lenninmr@gmail.com

Later, in the work of Xavir [6], he proposes a smoothing of that exact function, as follows

$$P(y, \lambda, \tau) = -\lambda y + \sqrt{(\lambda y)^2 + \tau^2}, \tag{1.3}$$

where $P : I\!R \times I\!R_{++} \times I\!R_{++} \to I\!R$. This function was called the hyperbolic penalty function (HPF), this function was important to propose the Hyperbolic Penalty Algorithm [6] and the Hyperbolic Smoothing Clustering Method [9]. In [7], the dislocation hyperbolic penalty function is proposed and defined as:

$$p(y, \lambda, \tau) = -\lambda y + \sqrt{(\lambda y)^2 + \tau^2} - \tau, \tag{1.4}$$

where $p : I\!R \times I\!R_{++} \times I\!R_{++} \to I\!R$.

On the other hand, in the works of Polyak, see [3, 4] and [5], the Modified Chen-Harker-Kanzow-Smale function is studied and defined as follows $\hat{\psi}_5 : I\!R \to I\!R$,

$$\hat{\psi}_5(t) = t - \sqrt{t^2 + 4\eta} + 2\sqrt{\eta}, \quad \eta > 0. \tag{1.5}$$

2. Previous work

In this work, we are going to maintain the same mathematical notation considered in the work [5].

2.1. Roman A. Polyak, 2015, [5]

In [5], the author considers the class Ψ of twice continuous differentiable functions $\psi : I\!R \to I\!R$. This class of functions satisfies the following properties:

1. $\psi(0) = 0$.
2.
   (a) $\psi'(t) > 0$.
   (b) $\psi'(0) = 1$ and $\psi'(t) \leq \frac{a}{t},\ a > 0,\ t > 0$.
3. $-\frac{1}{b} \leq \psi''(t) < 0,\ \forall t \in]-\infty, \infty[$.
4. $\psi''(t) \leq \frac{-1}{M},\ \forall t \in]-\infty, 0[,\ 0 < b < M < \infty$.

The function $\hat{\psi}_5$ does not satisfy the properties 4 ($M = \infty$). In the work of Polyak [5], a slight modification is made to the CHKS function, so that it can belong to the class of Ψ. The author considers the following: Let $-1 < \tau < 0$, and $\hat{\psi}_5$, be defined as follows $\psi_5 = \hat{\psi}_5(t),\ \infty > t \geq \tau,\ \psi_5 = q_5(t),\ \ -\infty < t \leq \tau$, where $q_5(t) = a_5 t^2 + b_5 t + c_5$ and $a_5 = 0.5\hat{\psi}_5''(\tau)$, $b_5 = \hat{\psi}_5'(\tau) - \tau\hat{\psi}''(\tau)$, $c_5 = \hat{\psi}_5'(\tau) - \tau\hat{\psi}_5'(\tau) + 0.5\tau^2\hat{\psi}_5''(\tau)$. So, $\psi_5 \in \Psi$, that is, the properties 1,2,3 and 4 hold. The kernel function of the function ψ_5 is obtained in [5] as follows, and it is called as the CHKS Kernel

$$\hat{\varphi}_5(s) = -2\sqrt{\eta}(\sqrt{(2-s)s} - 1), \quad \hat{\varphi}_5(0) = 2\sqrt{\eta},$$

is infinitely differentiable on $]0, 1 + |\tau|\, M^{-1}[$.

Definition 2.1. *([5]) A kernel $\varphi \in \Phi$ is well defined iff $\varphi(0) < \infty$.*

The properties of kernel $\varphi \in \Phi$ are induced by the properties 1-4 of transformation $\psi \in \Psi$ and are given by the following.

Lemma 2.1. *The kernel $\varphi \in \Phi$ is strongly convex on $I\!R^q_+$, twice continuously differentiable and possess the following properties*

(1) if $\varphi(s) \geq 0,\ \forall s \in]0, \infty[$ and $\min_{s \geq 0} \varphi(s) = \varphi(1) = 0$.
(2) (a) $\lim_{s \to 0^+} \varphi'(s) = -\infty$,
(b) $\varphi'(s)$ is monotone increasing.
(c) $\varphi'(1) = 0$
(3) (a) $\varphi''(s) \geq b > 0,\ \forall s \in]0, \infty[$.
(b) $\varphi''(s) \leq M < \infty,\ \forall s \in [1, \infty[$.

2.2. Dislocation hyperbolic transformation function

The function (1.4) was rewritten into [1], as follows:

$$p\left(g_i(x), \lambda_i, \tau\right) = \lambda_i g_i(x) - \sqrt{\left(\lambda_i g_i(x)\right)^2 + \tau^2} + \tau$$

$$= \tau \left(\frac{\lambda_i g_i(x)}{\tau} - \sqrt{\left(\frac{\lambda_i g_i(x)}{\tau}\right)^2 + 1} + 1 \right) = \tau\, h\left(\frac{\lambda_i g_i(x)}{\tau}\right), \tag{2.1}$$

where the function $h : I\!R \to I\!R$, will be called by the dislocation hyperbolic transformation function (DHTF), and is defined as

$$h(t) = t - \sqrt{t^2 + 1} + 1.$$

The conjugate function of the DHTF function was obtained in [2], as follows:

$$h^*(s) = \inf\{st - h(t) \;:\; t \in I\!R\} = \sqrt{1 - (1-s)^2} - 1, \quad \forall s \in (0, 2). \tag{2.2}$$

We define the function $\varphi : [0, 2] \to I\!R$ as

$$\varphi(s) = -h^*(s) = 1 - \sqrt{1 - (1-s)^2}, \tag{2.3}$$

3. The modified Chen-Harker-Kanzow-Smale and dislocation hyperbolic transformation functions

This section is based on the works of [1,5] and [2]. Taking $\eta = \frac{1}{4}$ and substituting in

$$\hat{\psi}_5(t) = t - \sqrt{t^2 + 4\eta} + 2\sqrt{\eta}, \quad \eta > 0,$$

we have that

$$\hat{\psi}_5(t) = t - \sqrt{t^2 + 1} + 1.$$

The kernel function of $\hat{\varphi}_5$ is

$$\hat{\varphi}_5(s) = -2\sqrt{\eta}(\sqrt{(2-s)s} - 1), \quad \hat{\varphi}_5(0) = 2\sqrt{\eta}.$$

Let $\eta = \frac{1}{4}$ in the function $\hat{\varphi}_5$, so we have

$$\hat{\varphi}_5(s) = 1 - \sqrt{(2-s)s} = 1 - \sqrt{1 - (1-s)^2}, \quad \hat{\varphi}_5(0) = 1.$$

This way, we can observe that when we consider the value of $\eta = \frac{1}{4}$ in the CHKS function, we obtain the dislocation hyperbolic transformation function.

In [1], the function h is used to construct an augmented Lagrangian type algorithm. Convergence towards a KKT point is assured under the assumptions of nonconvexity, for that augmented Lagrangian Algorithm. On the other hand, Polyak assures the convergence of an augmented Lagrangian type algorithm considering the CHKS function, under the assumptions of convexity. Now, considering $\eta = \frac{1}{4}$ in the CHKS function, we can say that we are extending the convergence result for the algorithm proposed by Polyak [5].

Acknowledgements

The first author was supported by *Fundação de Amparo à Pesquisa do Estado do Rio de Janeiro* ($E - 26/205.684/2022$ - PÓS-DOUTORADO NOTA 10/FAPERJ) and the second author was supported by COPPETEC Foundation, Brazil and CNPq (grants 302435/2019-0).

Conflict of Interest

No potential conflict of interest was reported by the author(s).

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