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REINSURANCE CONTRACTS UNDER STACKELBERG GAME AND MARKET EQUILIBRIUM

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Abstract. In this paper, we investigate the robust reinsurance contracts under Stackelberg game and market equilibrium. Each reinsurance contract contains two decision makers, one insurer and one reinsurer. The insurer is ambiguity-neutral and adopts a loss-dependent premium principle to collect premium. The reinsurer is ambiguity-averse and is a Bayesian learner. By using the stochastic dynamic programming method and the inverse method, the analytical expressions of the optimal risk allocation proportion and reinsurance price are derived for the two types of reinsurance contracts. We shows that the loss-dependent premium principle has the penalty-reward nature. Both the reinsurance price and demand decrease as the extrapolative intensity increases. Learning has important significance and always puts down the reinsurance price and puts up the reinsurance demand. On the contrary, the reinsurer's ambiguity aversion raises the reinsurance price and decreases the reinsurance demand. Finally, numerical analysis reveals that the reinsurance price is greater under the Stackelberg game than that under the market equilibrium.

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1. Introduction

In recent years, natural disasters, diseases and financial risks occur frequently all over the world. Insurance has always been an effective mean to transfer catastrophe risks and compensate for disaster losses. The key to a mature catastrophe insurance system is to effectively spread risks, expand underwriting capacity and stabilize operating results through proper reinsurance arrangements, so that catastrophe risks become insurable and daring to be insured. Therefore, how to set the risk allocation ratio or amount between insurance companies and reinsurance companies, as well as the price of reinsurance has received extensive attention from scholars (see *e.g.*, [5–7, 22, 31–33, 40]).

Reinsurance usually covers catastrophe risk events and limited available data makes it difficult for decision makers and actuaries to accurately estimate the specific probability measure of insurance claims (see *e.g.*, [10, 15]). This means that claim models have model ambiguity (model uncertainty, model misspecification) (see

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e.g., [13]). Therefore, many scholars are keen to explore robust reinsurance strategies under the background of model ambiguity. Yi *et al.* [36, 37] investigated the robust proportional reinsurance and investment strategies for the ambiguity-averse insurer with constant absolute risk aversion utility and mean-variance criteria. Li *et al.* [25] allowed the insurer to have different levels of ambiguity aversion by revising the α-maxmin mean-variance criterion, and examined the ambiguity-averse insurer's reinsurance-investment strategy. Chen and Yang [9] considered the robust optimal reinsurance-investment selection problem with price jumps and correlated claims for the ambiguity-averse insurer. More recently, Yuan *et al.* [39] explored the robust optimal reinsurance in minimizing the penalized expected time to reach a goal. Although there have been many similar studies, most of them only considered the interests of the insurer.

Hu *et al.* [19, 20] pointed out that an effective reinsurance contract should take into account the interests of both the insurer and the reinsurer, and in practical applications, reinsurance price should not be constant. They first introduced the relative safety loading of reinsurance to characterize reinsurance price and built a principal-agent framework to examine the robust reinsurance demand and price for the proportional and excess-loss reinsurance contracts. In the principal-agent framework, Wang and Siu [34] introduced risk constraint and studied the robust reinsurance contract, and Bai *et al.* [2] investigated a hybrid stochastic differential reinsurance and investment game with bounded memory. More related studies can be seen in Gu *et al.* [17], Cheung *et al.* [11], Cao *et al.* [8] and so on. However, the decision makers with ambiguity aversion did not modify claim models with the update of claim information and all these studies ignored the learning ability of the insurer and the reinsurer. Moreover, in reality, many insurance companies use the experience rating rather than the expected premium principle and the variance premium principle to charge premiums (see *e.g.*, [12, 14, 23, 30]).

In view of the above situation, we investigate the robust reinsurance contract design with a loss-dependent premium principle, model ambiguity and the reinsurer's unilateral Bayesian learning. Given that the number of reinsurers is much smaller than that of insurers and not every insurance company needs to buy reinsurance, we build not only a Stackelberg game model, but also a market equilibrium model in this paper. In the Stackelberg game model, the reinsurer is the leader and determines the optimal reinsurance price, and the insurer is the follower and determines the optimal risk retention proportion based on the reinsurance price. In the market equilibrium model, the optimal reinsurance price and risk retention proportion are determined by supply-demand equilibrium. The closed-form expressions of the robust reinsurance contracts and the corresponding value functions are derived by using the inverse method and the stochastic dynamic programming method. Then, we determine the appropriate size of ambiguity by adopting the detection-error probability and combine numerical experiments to explore the effects of the loss-dependent premium principle, ambiguity aversion and learning on the two types of reinsurance contracts.

The research in this paper is a supplement and extension of the existing research. The main academic contributions are mainly reflected in the following three aspects. Firstly, there are few articles exploring reinsurance demand and price issues from the perspective of market equilibrium. In this paper, we investigate reinsurance demand and price under Stackelberg game and market equilibrium. Secondly, we explore the impact of the loss-dependent premium principle (experience premium principle) on the equilibrium demand and price of reinsurance. Thirdly, most scholars explore reinsurance without claim information updating, and this paper introduces the incompleteness of information and Bayesian learning.

Four main findings emerge. First, we emphasize that the loss-dependent premium principle has the nature of reward and punishment for the initial policyholder and plays a certain role in insurance for the insurer. The insurer with higher extrapolation intensity will reduce demand for reinsurance although the reinsurance price decreases as the extrapolation intensity increases. Second, we show that the ambiguity-averse reinsurer is pessimistic and he sets higher reinsurance price relative to the ambiguity-neutral reinsurer, which leads to higher risk retention proportion. Third, through the reinsurer's learning, claims under his subjective beliefs become more stable, which reduces the price of reinsurance and the insurer's risk retention proportion. Finally, we find that the optimal reinsurance price under the Stackelberg game is always greater than that under the market equilibrium, but the risk retention proportion under the market equilibrium is more sensitive to the change of parameters than that under the Stackelberg game.

The remainder of this paper is organized as follows. The basic model setting is presented in Section 2. In Section 3, we introduce unilateral Bayesian learning and ambiguity aversion of the reinsurer, and study the robust risk retention proportion and reinsurance price in the Stackelberg game model. In Section 4, we are devoted to investigating the robust reinsurance contract in the market equilibrium model. Section 5 illustrates our results with numerical examples. Section 6 summarizes this paper. The main proofs are given in Appendixes A and B.

2. Model setup

Suppose that there are so many policyholders that insurance claims arrive regularly. In this case, the insurer uses a diffusion process with drift to characterize the cumulative claims. Specifically,

$$\mathrm{d}L(t) = \mu\,\mathrm{d}t - \sigma\,\mathrm{d}W^{\Gamma}(t), \tag{1}$$

where μ is the claim rate or is called as expected growth rate under the insurer's subjective beliefs, $\sigma > 0$ is a constant volatility parameter and $W^{\Gamma}(t)$ is a standard Brownian motion under probability measure Γ[1].

2.1. The insurer's wealth dynamic

In reality, in order to reduce the moral hazard of policyholders and guard against heavy losses, the insurer usually adopts a premium principle of reward and punishment when the current premium is collected. That is, when realized losses exceed expectations, the insurer puts up the premium as a penalty to policyholders, and when realized losses are less than expectations, the insurer puts down the premium as a reward to policyholders. This type of premium principle is called experience rate in some studies, such as Watt and Vazquez [35], Jean-Baptiste and Santomero [23] and Meyer [28]. Motivated by Barberis *et al.* [4], Hu and Wang [18] and Yuan *et al.* [38], we first define an exponential weighted average of past claims as follows

$$m(t) = \beta \int_0^t e^{-\beta(t-s)}\,\mathrm{d}L(s-\,\mathrm{d}t), \qquad \beta > 0, \tag{2}$$

where $\mathrm{d}L(s-\,\mathrm{d}t) = L(s) - L(s-\,\mathrm{d}t)$ represents the claim occurred in time interval $[s-\,\mathrm{d}t, s]$ and $\beta e^{-\beta(t-s)}$ is it's weight. This kind of weighted average contains all historical claims occurred in time interval $[0, t]$. Given parameter β, weight $\beta e^{-\beta(t-s)}$ increases as s approaches t. This means that the insurer pays more attention to recent claims and relatively less attention to early claims. This weight design method is in line with memory characteristics. Parameter β is important in $m(t)$. When it is high, the weighted average is mainly determined by the most recent claims, and otherwise, the claims occurred in the distant past may also play important roles on the weighted average. We call β extrapolation intensity. Obviously, $m(t)$ is also a stochastic process and evolves as[2].

$$\mathrm{d}m(t) = -\beta m(t)\,\mathrm{d}t + \beta\,\mathrm{d}L(t). \tag{3}$$

Substituting equation (1) into this expression, we have

$$\mathrm{d}m(t) = \beta(\mu - m(t))\,\mathrm{d}t - \beta\sigma\,\mathrm{d}sW^{\Gamma}(t). \tag{4}$$

When charging premium, the insurer not only considers future claims, but also pays attention to history claims. According to this idea, we define a loss-dependent premium principle as

$$c(t) = (1+\theta)\big(m(t) + e^{-\beta t}\mu\big), \tag{5}$$

[1] This diffusion model is a good approximate model of the compound Possion process $L(t) = \sum_{i=1}^{N(t)} Z_i$ with claim rate $\mu = \lambda E[Z_i]$, volatility $\sigma = \sqrt{\lambda E[(Z_i)^2]}$, where $\lambda > 0$ is the jump intensity (see *e.g.*, [16, 21]).

[2] According to Eq. (2), the differential of $m(t)$ is $\mathrm{d}m(t) = \beta \int_0^t e^{-\beta(t-s)}\,\mathrm{d}L(s-\,\mathrm{d}t)(-\beta\,\mathrm{d}t) + \beta\,\mathrm{d}L(t-\,\mathrm{d}t)$, By the definition of differential, $dL(t) = L(t) - L(t-dt)$, which is same with the expression of $\mathrm{d}L(t-\,\mathrm{d}t)$. Therefore, we can derive equation (3).

where $\theta > 0$ is the relative safety loading of insurance and $e^{-\beta t}$ can be seen as the weight assigned to claim rate μ. This weight just makes the total weight $\int_0^t \beta e^{-\beta(t-s)}\,\mathrm{d}s + e^{-\beta t}$ equal to 1. Obviously, the larger historical claim $m(t)$, the larger premium is charged by the insurer and otherwise, the opposite result holds. Therefore, this premium principle has a penalty-reward nature. This nature can prevent adverse selection and moral hazard to some extent. Moreover, when extrapolation intensity $\beta = 0$, $m(t) = 0$ and the loss-dependent premium principal degenerates into the traditional expected premium principal. Therefore, the loss-dependent premium principle includes the expected premium principle as a special case.

In addition to charging higher premium to cover early excess claims, the insurer also can buy reinsurance from a reinsurer to spread the risk of claims. Assume that the reinsurer provides proportional reinsurance. The insurer retains risk proportion $p(t)$ for each claim and the rest risk proportion $1 - p(t)$ is transferred to the reinsurer. Both the insurance and reinsurance surpluses are invested in a fixed income asset with interest rate r. After purchasing reinsurance, the insurer's wealth dynamic is expressed as

$$\begin{aligned}\mathrm{d}X(t) &= rX(t)\,\mathrm{d}t + \left[(1+\theta)\big(m(t) + e^{-\beta t}\mu\big) - (1+\eta(t))(1-p(t))\mu\right]\mathrm{d}t - p(t)(\mu\,\mathrm{d}t - \sigma\,\mathrm{d}W^{\Gamma}(t)) \\ &= \left[rX(t) + (1+\theta)\big(m(t) + e^{-\beta t}\mu\big) - (1+\eta(t))\mu + p(t)\eta(t)\mu\right]\mathrm{d}t + \sigma p(t)\,\mathrm{d}W^{\Gamma}(t),\end{aligned} \tag{6}$$

where $\eta(t) > 0$ is the relative safety loading of reinsurance and is used to characterize reinsurance price. This means that the reinsurance premium is calculated by the expected premium principle. However, unlike the traditional expected premium principle, in which $\eta(t)$ is a fixed constant, $\eta(t)$ will be adjusted according to reinsurance demand and be determined by game equilibrium or market equilibrium.

2.2. The reinsurer's wealth dynamic

The insurer and the reinsurer commonly observe the path of claim $L(t)$. Claim rate μ used by the insurer is only an approximation to the real claim rate. The insurer believes that he is perfectly informed and there is no need to revise the estimate of claim[3]. The reinsurer knows the insurer's estimate of claim, but does not agree with this estimate. The reinsurer learns about the state variable by observing realized claims and using Bayesian learning. Assume that the priors about the claim rate at time 0 are Gaussian with mean μ_0 and variance α_0. According to Kalman-Bucy filter theory, the prior mean and variance evolve as

$$\mathrm{d}\hat{\mu}(t) = -\frac{\alpha(t)}{\sigma}(\mathrm{d}L(t) - \mu\,\mathrm{d}t) = -\frac{\alpha(t)}{\sigma}\,\mathrm{d}W^P(t), \tag{7}$$

$$\mathrm{d}\alpha(t) = -\frac{\alpha(t)^2}{\sigma^2}\,\mathrm{d}t. \tag{8}$$

Combining the initial condition $\alpha(0) = \alpha_0$, there is

$$\alpha(t) = \frac{\alpha_0\sigma^2}{\sigma^2 + \alpha_0 t}\cdot \tag{9}$$

Obviously, the variance of $\hat{\mu}$ is decreasing as learning time t increases. This indicates that claim rate $\hat{\mu}$ becomes more stable as information is updated. The innovation process can be expressed as

$$\mathrm{d}W^P(t) = \frac{\hat{\mu}\,\mathrm{d}t - \mathrm{d}L(t)}{\sigma} = \frac{\hat{\mu} - \mu}{\sigma}\,\mathrm{d}t + \mathrm{d}W^{\Gamma}(t), \tag{10}$$

where $W^P(t)$ is a standard Brownian motion under probability measure P. Through learning, the claim process evolves as

$$\mathrm{d}L(t) = \hat{\mu}\,\mathrm{d}t - \sigma\,\mathrm{d}W^P(t). \tag{11}$$

[3]The insurer usually has more information about the claim than the reinsurer, or because of differences in capacity, so we assume that the insurer believes it is fully informed.

Even though the filtered model above is the best estimate available to the reinsurer given current claim information, he is still sceptical about the current claim model and allows for some ambiguity due to finite data. The reinsurer treats the filtered model (11) as the reference model and takes into account some alternative models which are statistically difficult to be distinguished from the reference model. Similar to Anderson *et al.* [1], we assume that all the alternative models are absolutely continuous with respect to the reference model. Then, each alternative model can be characterized by a real process $\varphi(t)$ and an associated probability measure Q^{φ}, such that the Radon–Nikodym derivative of Q^{φ} with respect to P can be expressed as

$$\upsilon^{\varphi}(t) = \frac{\mathrm{d}Q^{\varphi}}{\mathrm{d}P} = \exp\left[\int_0^t \varphi(s)\,\mathrm{d}W^p(s) - \frac{1}{2}\int_0^t \varphi(s)^2\,\mathrm{d}s\right]. \tag{12}$$

As specified, the process $\upsilon^{\varphi}(t)$ is a P-martingale. $\varphi(t)$ is called as density operator, which is endogenous and depends on the reinsurer's degree of ambiguity aversion. Moreover, $\varphi(t)$ satisfies the Novikov's condition, that is

$$E_t^P\left[\exp\left(\frac{1}{2}\int_0^t \|\varphi(s)^2\|\,\mathrm{d}s\right)\right] < \infty,$$

for all $t \in [0,T]$. Here, $T > 0$ is a finite constant representing the insurance period. According to Girsanov's theorem (see [24]), there is a standard Brownian motion $W^{Q^{\varphi}}(t)$ under probability measure Q^{φ}, which satisfies

$$\mathrm{d}W^{Q^{\varphi}}(t) = \mathrm{d}W^P(t) - \varphi(t)\,\mathrm{d}t. \tag{13}$$

Then, under the reinsurer's subjective beliefs, the claim rate can be expressed as

$$\mathrm{d}\hat{\mu}(t) = -\frac{\alpha(t)}{\sigma}\varphi(t)\,\mathrm{d}t - \frac{\alpha(t)}{\sigma}\,\mathrm{d}W^{Q^{\varphi}}, \tag{14}$$

and the distorted stochastic claim process (1) can be written as

$$\mathrm{d}L(t) = \hat{\mu}(t)\,\mathrm{d}t - \sigma\,\mathrm{d}W^P(t) = (\hat{\mu}(t) - \sigma\varphi(t))\,\mathrm{d}t - \sigma\,\mathrm{d}W^{Q^{\varphi}}(t). \tag{15}$$

Obviously, when $\varphi(t)$ is zero, the probability measures P and Q^{ϕ} agree and the claim model (15) is reverted to the reference model (11).

In our model setup, the insurer and the reinsurer can agree to disagree, so that they still do not agree on the claim's evolution even after signing the reinsurance contract. The insurer only wants to pay premium based on his own subjective beliefs. To prevent the misspecification of claim model, the reinsurer can adjust reinsurance price based on reinsurance demand and his subjective beliefs on claims. Therefore, the reinsurer's wealth dynamic can be given by

$$\begin{aligned}\mathrm{d}Y(t) &= rY(t)\,\mathrm{d}t + (1-p(t))(1+\eta(t))\mu\,\mathrm{d}t - (1-p(t))\Big[(\hat{\mu}(t) - \sigma\varphi(t))\,\mathrm{d}t - \sigma\,\mathrm{d}W^{Q^{\varphi}}(t)\Big]\\ &= [rY(t) + (1-p(t))(\mu - \hat{\mu}(t)) + (1-p(t))(\mu\eta(t) + \sigma\varphi(t))]\,\mathrm{d}t + \sigma(1-p(t))\,\mathrm{d}W^{Q^{\varphi}}(t).\end{aligned} \tag{16}$$

3. Stackelberg reinsurance game

In this section, we assume that the risk retention proportion and the reinsurance price are determined by a Stackelberg game. In the dynamic game, the reinsurer is a leader and the insurer is a follower[4]. Namely, the reinsurer first designs the reinsurance price, and then the insurer determines the level of reinsurance coverage. Using the inverse method of dynamic game, we first solve the response function of the insurer for given reinsurance price and then solve the reinsurer's optimization problem.

[4]In reinsurance market, the number of insurers is much larger than the number of reinsurers. Therefore, it is reasonable to give more decision-making flexibility to the reinsurer.

3.1. The insurer's optimization problem

The insurer has utility preference of constant absolute risk aversion (CARA). For given reinsurance price $\eta(t)$, the insurer's goal is to maximize the utility with respect to terminal wealth $X(T)$ by controlling risk retention proportion $p(t)$. Then the insurer's optimization problem can be expressed as

$$J(t,m,X) = \sup_{p(t)} E_t^{\Gamma}\left[-\frac{1}{\gamma}\exp(-\gamma X(T))\right], \tag{17}$$

subject to wealth dynamic (6) and final condition $J(T,m,X) = -\frac{1}{\gamma}e^{-\gamma X}$. $\gamma > 0$ is the coefficient of absolute risk aversion. By using the stochastic dynamic programming method, we derive the insurer's optimal response function.

Proposition 1. *For given reinsurance price $\eta(t)$, the insurer's optimal risk retention proportion is*

$$p^*(t) = \left[\frac{\mu\eta(t)}{\gamma\sigma^2} + \beta b(t)\right]e^{-r(T-t)}, \tag{18}$$

where function $b(t)$ is given by

$$b(t) = \frac{1+\theta}{r+\beta}\left[e^{r(T-t)} - e^{-\beta(T-t)}\right]. \tag{19}$$

The insurer's value function is given by

$$J(t,m,X) = -\frac{1}{\gamma}\exp\left\{-\gamma[Xe^{r(T-t)} + f(t) + g(t,m)]\right\}, \tag{20}$$

where functions $f(t)$ and $g(t,m)$ are presented in equations (A.5) and (A.6), respectively.

Proof. The proof of this proposition is presented in Appendix A.

The second term $\beta b(t)e^{-r(T-t)}$ in equation (18) comes from the loss-dependent premium principle. If extrapolation intensity $\beta = 0$, this part disappears. Moreover, $\beta b(t)e^{-r(T-t)} > 0$ always holds. This means that the loss-dependent premium principle increases the insurer's risk retention proportion. The reason is that the loss-dependent premium principle can play an insurance role for the insurer as discussed in the above section. From equation (18), we can obtain

$$\frac{\partial p^*(t)}{\partial \eta} = \frac{\mu}{\gamma\sigma^2}e^{-r(T-t)} > 0, \quad \frac{\partial p^*(t)}{\partial \mu} = \frac{\eta(t)}{\gamma\sigma^2}e^{-r(T-t)} > 0, \quad \frac{\partial p^*(t)}{\partial \sigma} = -\frac{2\mu\eta(t)}{\gamma\sigma^3}e^{-r(T-t)} < 0,$$
$$\frac{\partial p^*(t)}{\partial \theta} = \frac{\beta e^{-r(T-t)}}{r+\beta}\left[e^{r(T-t)} - e^{-\beta(T-t)}\right] > 0, \quad \frac{\partial p^*(t)}{\partial \gamma} = -\frac{\mu\eta(t)}{\gamma^2\sigma^2}e^{-r(T-t)} < 0.$$

Therefore, the insurer's optimal risk retention proportion is strictly and positively correlated with reinsurance price $\eta(t)$, claim rate μ and insurance price θ, and strictly negative correlation with claim volatility σ and his own risk aversion level γ. It is worth noting that if the traditional expected premium principle is adopted by the insurer, the optimal risk retention proportion will be independent of insurance price θ. The high insurance price means that the insurer has a higher repayment ability, which reduces the demand for reinsurance. From this perspective, the loss-dependent premium principle is more in line with the actual situation. □

3.2. The reinsurer's optimization problem

The reinsurer is ambiguity-averse and he designs a reinsurance contract based on the worst-case probability measure and thus, the worst-case density operator $\varphi(t)$. This robust reinsurance contract not only works well when the reference model governing the claim state is the true model, but also performs well when the true claim is governed by the alternative model. In order to introduce robustness into our model, we use the penalty-based multiple-priors objective function in continuous-time proposed by Anderson *et al.* [1] and the homothetic robustness adopted by Maenhout [26] to characterize the reinsurer's robustness preference. Thereinto, the penalty size is described by the distant between probability measures P and Q^{φ}, which is captured by relative entropy. Then, the reinsurer's objective function can be expressed as

$$V(t,\hat{\mu},Y)=\sup_{\eta(t)}\inf_{\varphi(t)}E_t^{Q^{\varphi}}\left[-\frac{1}{k}\exp(-kY(T))+\int_t^T\frac{1}{2\phi(s)}\varphi(s)^2\,\mathrm{d}s\right]. \tag{21}$$

subject to

$$p^*(t)=\left[\frac{\mu\eta(t)}{\gamma\sigma^2}+\beta b(t)\right]e^{-r(T-t)},$$

wealth dynamic (16) in which $p(t)$ is replaced by $p^*(t)$, and terminal condition $V(T,\hat{\mu},Y)=-\frac{1}{k}e^{-kY}$. Here, the expected utility is calculated under the alternative probability measure Q^{φ}. The second term in the indirect utility is the relative entropy penalty scaled by function $\phi(t)$, which penalizes any derivations. Therefore, the penalty term exists if and only if the alternative model Q^{φ} deviates from the reference model P. Function $\phi(t)$ is nonnegative and captures the reinsurer's ambiguity aversion and parameter $k>0$ is the reinsurer's risk aversion coefficient.

The homothetic robustness proposed by Maenhout [26] assumes that

$$\phi(t)=-\frac{\varepsilon}{kV(t,\hat{\mu},Y)}, \tag{22}$$

where $\varepsilon>0$ is a constant and represents the reinsurer's ambiguity aversion coefficient. $\frac{1}{V(t,\hat{\mu},Y)}$ in $\phi(t)$ can be seen as a normalization factor, which makes sure that the relative entropy has the same unit with that of the value function. By solving the reinsurer's optimization problem, we obtain the following results.

Theorem 1. *Under the Stackelberg game, the reinsurer's robust reinsurance price is given by*

$$\eta^*(t)=\frac{\gamma\sigma^2(\gamma+k+\varepsilon)}{\mu(2\gamma+k+\varepsilon)}\left[e^{r(T-t)}-\beta b(t)\right]-\frac{\gamma(k+\varepsilon)}{\mu(2\gamma+k+\varepsilon)}\alpha(t)(B(t)+2D(t)\hat{\mu})-\frac{\gamma(\mu-\hat{\mu})}{\mu(2\gamma+k+\varepsilon)}, \tag{23}$$

the worst-case density operator is given by

$$\varphi^*(t)=-\frac{\varepsilon(\mu-\hat{\mu})}{\sigma(2\gamma+k+\varepsilon)}-\frac{\gamma\varepsilon\sigma}{2\gamma+k+\varepsilon}\left[e^{r(T-t)}-\beta b(t)\right]+\frac{2\gamma\varepsilon}{\sigma(2\gamma+k+\varepsilon)}\alpha(t)[B(t)+2D(t)\hat{\mu}]. \tag{24}$$

Given this price, the insurer's optimal risk retention proportion is given by

$$\begin{aligned}p^*(t)=&\frac{\gamma+k+\varepsilon}{2\gamma+k+\varepsilon}+\frac{\gamma\beta}{2\gamma+k+\varepsilon}b(t)e^{-r(T-t)}-\frac{k+\varepsilon}{\sigma^2(2\gamma+k+\varepsilon)}\alpha(t)e^{-r(T-t)}[B(t)+2D(t)\hat{\mu}]\\&-\frac{\mu-\hat{\mu}}{\sigma^2(2\gamma+k+\varepsilon)}e^{-r(T-t)}.\end{aligned} \tag{25}$$

Under this robust reinsurance contract, the reinsurer's value function is given by

$$V(t,\hat{\mu},Y)=-\frac{1}{k}\exp\left\{-k\left[Ye^{r(T-t)}+A(t)+B(t)\hat{\mu}+D(t)\hat{\mu}^2\right]\right\}, \tag{26}$$

where functions $A(t), B(t)$ and $D(t)$ are presented in equations (B.22), (B.21) *and* (B.17)*, respectively; substituting $\eta^*(t)$ into functions $f(t)$ and $g(t,m)$, the insurer's value function $J(t,m,X)$ can be derived.*

Proof. The proof of this theorem is presented in Appendix B. □

4. Market equilibrium

In the above analysis, we only consider one insurer and one reinsurer. If the reinsurance market only exists one insurance company and one reinsurance company, the risk retention proportion and the reinsurance price may be determined by market equilibrium (market clearing condition) rather than the Stackelberg reinsurance game. In this section, we explore the robust reinsurance contract under the market equilibrium.

In the market equilibrium model, the insurer's goal is to choose the optimal reinsurance demand for maximizing the expected utility of terminal wealth, which is same with that presented in equation (17), and the reinsurer's goal is to determine how much coverage to provide rather than the reinsurance price for maximizing the penalty-based multiple-priors objective function. The equilibrium reinsurance price is determined by the equal reinsurance demand and reinsurance supply. Different from the Stackelberg game, in this model setup, the insurer and the reinsurer make decisions simultaneously. Obviously, for the insurer, the optimal risk retention proportion is given by

$$p_I^*(t) = \left[\frac{\mu\hat{\eta}(t)}{\gamma\sigma^2} + \beta b(t)\right]e^{-r(T-t)}, \tag{27}$$

and for the reinsurer, the optimal risk retention proportion is determined by

$$\hat{V}(t, \hat{\mu}, Y) = \sup_{p_R(t)} \inf_{\hat{\varphi}(t)} E_t^{Q^{\hat{\varphi}}}\left[-\frac{1}{k}\exp(-kY(T)) + \int_t^T \frac{1}{2\phi(s)}\hat{\varphi}(s)^2\,\mathrm{d}s\right], \tag{28}$$

subject to wealth dynamic (16) and terminal condition $\hat{V}(T, \hat{\mu}, Y) = -\frac{1}{k}e^{-kY}$.

By solving the corresponding Hamilton–Jacobi–Bellmann (HJB) equation of optimization problem (28), we have

$$\hat{\varphi}^*(t) = \frac{\varepsilon\alpha(t)}{\sigma}\Big(\hat{B}(t) + 2\hat{D}(t)\hat{\mu}\Big) - \varepsilon\sigma e^{r(T-t)}(1 - p_R(t)), \tag{29}$$

$$p_R^*(t) = 1 - \frac{\alpha(t)}{\sigma^2}e^{-r(T-t)}\Big(\hat{B}(t) + 2\hat{D}(t)\hat{\mu}\Big) - \frac{\mu - \hat{\mu}}{\sigma^2(k+\varepsilon)}e^{-r(T-t)} - \frac{\mu\hat{\eta}(t)}{\sigma^2(k+\varepsilon)}e^{-r(T-t)}, \tag{30}$$

where $\hat{B}(t)$ and $\hat{D}(t)$ are presented in Theorem 2.

Remark 1. In this section, $p_I^*(t)$ is the proportion of claims that the insurer is willing to bear, and $p_R^*(t)$ is the proportion of claims that the reinsurer expects the insurer to bear. Without loss of generality, $1 - p_I^*(t)$ can be understood as reinsurance demand and $1 - p_R^*(t)$ can be understood as reinsurance supply. Then, the market equilibrium can be expressed as $1 - p_I^*(t) = 1 - p_R^*(t)$, that is $p_I^*(t) = p_R^*(t)$.

Combining the corresponding HJB equation and the equal reinsurance demand and reinsurance supply ($p_I^*(t) = p_R^*(t)$) in equilibrium, we can obtain the following results.

Theorem 2. *From the perspective of market equilibrium, the robust risk retention proportion is*

$$\begin{aligned}\hat{p}^*(t) = {}& \frac{k+\varepsilon}{\gamma+k+\varepsilon} + \frac{\gamma\beta}{\gamma+k+\varepsilon}b(t)e^{-r(T-t)} - \frac{k+\varepsilon}{\sigma^2(\gamma+k+\varepsilon)}\alpha(t)e^{-r(T-t)}\Big(\hat{B}(t) + 2\hat{D}(t)\hat{\mu}\Big) \\ & - \frac{\mu-\hat{\mu}}{\sigma^2(\gamma+k+\varepsilon)}e^{-r(T-t)},\end{aligned} \tag{31}$$

the robust reinsurance price is

$$\hat{\eta}^*(t) = \frac{\gamma\sigma^2(k+\varepsilon)}{\mu(\gamma+k+\varepsilon)}\Big[e^{r(T-t)} - \beta b(t)\Big] - \frac{\gamma(k+\varepsilon)}{\mu(\gamma+k+\varepsilon)}\alpha(t)\Big(\hat{B}(t) + 2\hat{D}(t)\hat{\mu}\Big) - \frac{\gamma(\mu-\hat{\mu})}{\mu(\gamma+k+\varepsilon)}, \tag{32}$$

and under the reinsurer's beliefs, the worst-case density operator is

$$\hat{\varphi}^*(t) = -\frac{\varepsilon(\mu-\hat{\mu})}{\sigma(\gamma+k+\varepsilon)} - \frac{\gamma\varepsilon\sigma}{\gamma+k+\varepsilon}\Big[e^{r(T-t)} - \beta b(t)\Big] + \frac{\gamma\varepsilon}{\sigma(\gamma+k+\varepsilon)}\alpha(t)\Big(\hat{B}(t) + 2\hat{D}(t)\hat{\mu}\Big). \tag{33}$$

In this case, the insurer's value function is given by

$$\hat{J}(t,m,X) = -\frac{1}{\gamma}\exp\Big\{-\gamma\Big[Xe^{r(T-t)} + \hat{f}(t) + b(t)m + \hat{d}(t)\Big]\Big\}, \tag{34}$$

where $b(t)$ is presented in equation (19),

$$\hat{f}(t) = \frac{\mu(1+\theta)}{\beta+r}\Big[e^{rT-(\beta+r)t} - e^{-\beta T}\Big] - \mu\int_t^T (1+\hat{\eta}^*(s))e^{r(T-s)}ds + \frac{\mu^2}{2\gamma\sigma^2}\int_t^T \hat{\eta}^*(s)^2\,\mathrm{d}s, \tag{35}$$

$$\hat{\mathrm{d}}(t) = \beta\mu\int_t^T (1+\hat{\eta}^*(s))b(s)\,\mathrm{d}s, \tag{36}$$

and the reinsurer's value function is given by

$$\hat{V}(t,\hat{\mu},Y) = -\frac{1}{k}\exp\Big\{-k\Big[Ye^{r(T-t)} + \hat{A}(t) + \hat{B}(t)\hat{\mu} + \hat{D}(t)\hat{\mu}^2\Big]\Big\}, \tag{37}$$

where

$$\begin{aligned}\hat{A}(t) = {}& \frac{\mu^2(k+\varepsilon)}{2\sigma^2(\gamma+k+\varepsilon)^2}(T-t) + \frac{\gamma^2\sigma^2(k+\varepsilon)}{2(\gamma+k+\varepsilon)^2}\int_t^T\Big[e^{r(T-s)} - \beta b(s)\Big]^2\mathrm{d}s \\ &+ \frac{\mu(k+\varepsilon)^2}{\sigma^2(\gamma+k+\varepsilon)^2}\int_t^T\alpha(s)\hat{B}(s)\,\mathrm{d}s + \frac{\gamma(k+\varepsilon)}{(\gamma+k+\varepsilon)^2}\int_t^T\alpha(s)\hat{B}(s)\Big[e^{r(T-s)} - \beta b(s)\Big]\mathrm{d}s \\ &- \frac{\gamma(k+\varepsilon)(\gamma+2k+2\varepsilon)}{2\sigma^2(\gamma+k+\varepsilon)^2}\int_t^T\alpha(s)^2\hat{B}(s)^2\,\mathrm{d}s + \frac{1}{\sigma^2}\int_t^t\alpha(s)^2\hat{D}(s)\,\mathrm{d}s,\end{aligned} \tag{38}$$

$$\hat{B}(t) = e^{-\int_t^T \hat{M}(s)\,\mathrm{d}s}\int_t^T \hat{N}(s)e^{\int_s^T \hat{M}(u)\,\mathrm{d}u}\,\mathrm{d}s, \tag{39}$$

$$\hat{D}(t) = \frac{\big(\sigma^2+\alpha_0 t\big)\Big(1 - \frac{\sigma^2+\alpha_0 t}{\sigma^2+\alpha_0 T}\Big)}{\alpha_0\sigma^2\Big[2(k+\varepsilon) + \frac{2\gamma(\gamma+2k+2\varepsilon)(\sigma^2+\alpha_0 t)}{(k+\varepsilon)(\sigma^2+\alpha_0 T)}\Big]}, \tag{40}$$

$$\hat{M}(t) = \frac{(k+\varepsilon)^2}{\sigma^2(\gamma+k+\varepsilon)^2}\alpha(t) + \frac{2\gamma(k+\varepsilon)(\gamma+2k+2\varepsilon)}{\sigma^2(\gamma+k+\varepsilon)^2}\alpha(t)^2\hat{D}(t), \tag{41}$$

$$\hat{N}(t) = \frac{\gamma(k+\varepsilon)}{(\gamma+k+\varepsilon)^2}\Big[e^{r(T-t)} - \beta b(t)\Big]\Big(2\alpha(t)\hat{D}(t) - 1\Big) + \frac{\mu(k+\varepsilon)}{\sigma^2(\gamma+k+\varepsilon)^2}\Big[2(k+\varepsilon)\alpha(t)\hat{D}(t) - 1\Big]. \tag{42}$$

Proof. The proof of this theorem is similar to that of Theorem 1 and we omit it here. □

By observing and comparing Proposition 1, Theorems 1 and 2, we note that the reinsurer's ambiguity aversion coefficient ε and risk aversion coefficient k have the same effects on the risk retention proportion, the reinsurance price and the value functions of the insurer and the reinsurer. Therefore, in our model setup, introducing the reinsurer's preference for the robust reinsurance contract is essentially equivalent to increasing his effective risk aversion level. It is not difficult to test that if the insurer is ambiguity-averse, this conclusion is still valid.

Proposition 2. *When the diffusion claim model is uncertain, an ambiguity-averse decision maker with a constant absolute risk aversion utility function $u(Y) = -\frac{1}{k}e^{-kY}$ and homothetic robustness preference $\phi(t) = -\frac{\varepsilon}{kV(t,Y)}$ is observationally equivalent to an ambiguity-neutral decision maker under a constant absolute risk aversion utility function with risk aversion coefficient $k+\varepsilon$, that is $u(Y) = -\frac{1}{k+\varepsilon}e^{-(k+\varepsilon)Y}$.*

Proposition 2 reveals that the reinsurer's ambiguity aversion is equivalent to increasing his risk aversion by the same amount. The conclusion in Proposition 2 is only valid when the claim follows a diffusion process and is no longer valid when the claim follows a Cramér-Lundberg process or a Lévy process (see *e.g.*, [8, 19]). Moreover, it is worth noting that ambiguity aversion coefficient and risk aversion coefficient have different effects on the deviation degree between the worst-case model and the reference model. When the reinsurer is ambiguity-neutral, that is $\varepsilon = 0$, the worst-case density operators $\varphi^*(t)$ and $\hat{\varphi}^*(t)$ are equal to zero and model misspecifications do not generate effect on the risk retention proportion and the reinsurance price. However, when the reinsurer is risk-neutral, that is $k = 0$, this result does not hold. Therefore, whether model ambiguity affects decision makers' behaviors depends on whether there exists ambiguity-averse.

Remark 2. In this section, the demand and supply for reinsurance depend on the reinsurance price. The equilibrium reinsurance price is just a special case that satisfies this condition, which allows the reinsurance market to clear. Baltas *et al.* [3] point out that the equilibrium reinsurance price is "fair"[5]. The reason is that at this price, reinsurance can be used optimally by the insurer and the reinsurer to transfer risk and maximize utility. Therefore, when the insurer and the reinsurer are allowed to negotiate the reinsurance price, the equilibrium reinsurance price can be used as a benchmark.

5. Comparative analysis

The above analysis shows that the two robust reinsurance contracts are different under the Stackelberg game and the market equilibrium. This section contains three contents. We first analyze the effects of the reinsurer's risk aversion and ambiguity aversion on the deviation degree between the worst-case model and the reference model, and then select the rational size of ambiguity by using the detection-error probability. At last, we are devoted to comparing the two types of reinsurance contracts and analyzing how the loss-dependent premium principle, model ambiguity and Bayesian learning influence the risk allocation and reinsurance price strategies combining numerical analysis. In the following numerical examples, unless otherwise stated, the basic model parameter values are given in Table 1.

5.1. Degree of deviation

From equation (15), we note that the degree of deviation between the worst-case model and the reference model is $-\sigma\phi^*(t)$. Without loss of generality, we use the worst-case density operators $\phi^*(t)$ and $\hat{\phi}^*(t)$ to represent the degrees of deviation under the Stackelberg game and the market equilibrium, respectively. In addition, the expressions of equations (24) and (33) indicate that the effects of the reinsurer's ambiguity aversion and risk aversion on the worst-case density operator are not equivalent. We now analyze how the reinsurer's ambiguity aversion and risk aversion influence the worst-case density operators $\phi^*(t)$ and $\hat{\phi}^*(t)$.

Looking at Panels A and B in Figure 1, we find that the reinsurer's ambiguity aversion and risk aversion generate the exact opposite effects on the degree of deviation between the two worst-case models and the reference model. Panel A shows that the worst-case density operators are subtraction functions of ambiguity aversion level and are always negative for the other parameters given. Therefore, the degrees of deviation $-\sigma\phi^*(t)$ and $-\sigma\hat{\phi}^*(t)$ are positive and increasing functions of ambiguity aversion level. This means that the reinsurer with ambiguity aversion pessimistically believes that the worst-case claim amount will be greater than the reference claim amount. Moreover, the higher the degree of ambiguity aversion, the more the worst-case model deviates from the reference model. Panel B reveals that the worst-case density operator is increasing with respect to the reinsurer's risk aversion. Therefore, higher risk aversion reduces the gap between the worst-case model and the reference model. Figure 1 also indicates that this gap under the market equilibrium is always greater than that under the Stackelberg game. This is mainly because, compared with the Stackelberg game, the reinsurer loses the priority of pricing decision under the market equilibrium, which makes him more pessimistic to believe that the claim will deviate greatly.

[5]The research focus of Baltas *et al.* [3] is the optimal investment and reinsurance policies with inside information. They simply solve and mention the equilibrium reinsurance price without in-depth analysis.

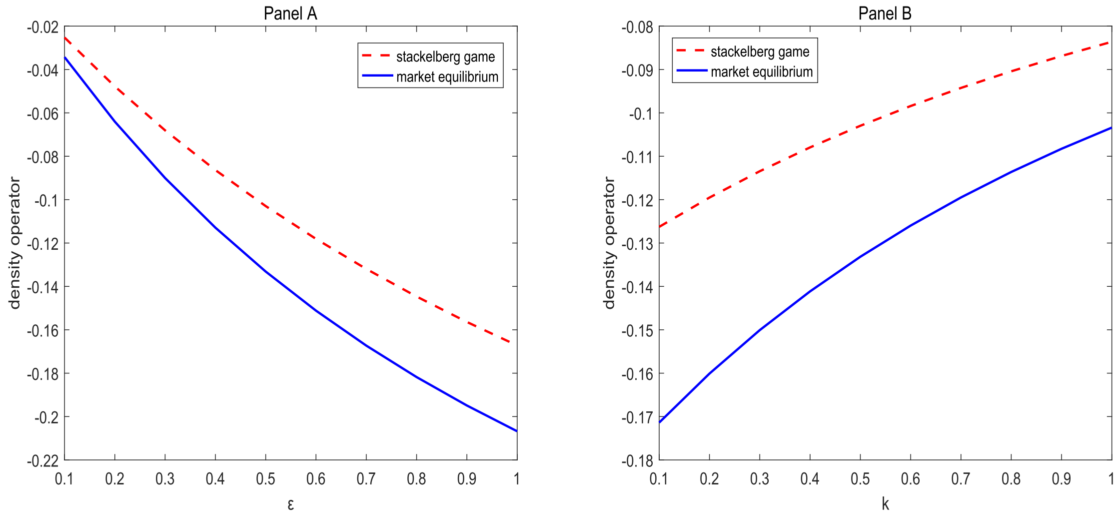

FIGURE 1. Effects of ambiguity aversion and risk aversion on the worst-case density operator.

5.2. The size of ambiguity

The gap between the worst-case model and the reference model is endogenous and indexed by the reinsurer's ambiguity aversion parameter ε. Although we use the relative entropy penalty in the reinsurer's objective function to limit model deviation, this does not limit parameter ε. Anderson *et al.* [1] argue that the detection-error probability should not be lower than 10% in order to ensure that the ambiguity-averse reinsurer is statistically difficult to differentiate the worst-case model from the reference model. In order to strengthen the practical application of this research, we need to establish a reasonable range for the reinsurer's ambiguity aversion level by using the detection-error probability.

Denote $\xi_{1,t}$ and $\xi_{2,t}$ the logs of the Radon–Nikodym derivatives $\frac{\mathrm{d}Q^{\varphi^*}}{\mathrm{d}P}$ and $\frac{\mathrm{d}Q^{\hat{\varphi}^*}}{\mathrm{d}P}$ as

$$\xi_{1,t} = \log\left[\frac{\mathrm{d}Q^{\varphi^*}}{\mathrm{d}P}\right] = \int_0^t \varphi^*(s)\mathrm{d}W^P(s) - \frac{1}{2}\int_0^t \varphi^*(s)^2\,\mathrm{d}s, \tag{43}$$

$$\xi_{2,t} = \log\left[\frac{\mathrm{d}Q^{\hat{\varphi}^*}}{\mathrm{d}P}\right] = \int_0^t \hat{\varphi}^*(s)\mathrm{d}W^P(s) - \frac{1}{2}\int_0^t \hat{\varphi}^*(s)^2\,\mathrm{d}s. \tag{44}$$

According to Anderson *et al.* [1] and Maenhout [27], the detection-error probabilities of the reinsurer under the Stackelberg game and the market equilibrium can be defined as

$$\zeta(\varepsilon) = \frac{1}{2}Pr(\xi_{1,N} > 0|P, \mathcal{F}_0) + \frac{1}{2}Pr(\xi_{1,N} < 0|Q^{\varphi^*}, \mathcal{F}_0), \tag{45}$$

$$\hat{\zeta}(\varepsilon) = \frac{1}{2}Pr(\xi_{2,N} > 0|P, \mathcal{F}_0) + \frac{1}{2}Pr(\xi_{2,N} < 0|Q^{\hat{\varphi}^*}, \mathcal{F}_0), \tag{46}$$

where $\{\mathcal{F}_t : t \geq 0\}$ describes the information available to the reinsurer at time t. $\xi_{i,N} > 0$ $(i = 1, 2)$ indicates that the reinsurer may mistakenly choose the worst-case models Q^{φ^*} and $Q^{\hat{\varphi}*}$ based on a finite sample with size N when the reference model P is correct. Conversely, if $\xi_{i,N} < 0$ $(i = 1, 2)$, the reinsurer may erroneously reject the worst-case models Q^{φ^*} and $Q^{\hat{\varphi}*}$ in favor of the reference model P. Therefore, a high detection-error probability means that the ambiguity-averse reinsurer is likely to select an incorrect model between the worst-case model and the reference model. Using the same methodology with that of Munk and Rubtsov [29] and Hu *et al.* [18] ,

TABLE 1. Parameter values.

Parameter	β	γ	k	m	r	μ	$\hat{\mu}$	σ	α_0	T	t
Value	0.05	0.8	0.5	0.5	0.02	4	4	1	0.05	5	0

TABLE 2. Detection-error probability under Stackelberg game.

ε	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
Panel A: sample of size $N = 5$ years										
$k = 0.3$	0.4832	0.4681	0.4546	0.4424	0.4313	0.4212	0.4120	0.4036	0.3958	0.3887
$k = 0.5$	0.4848	0.4711	0.4587	0.4473	0.4369	0.4274	0.4186	0.4105	0.4030	0.3961
$k = 0.7$	0.4862	0.4736	0.4620	0.4514	0.4417	0.4326	0.4243	0.4165	0.4093	0.4025
Panel B: sample of size $N = 10$ years										
$k = 0.3$	0.4526	0.4109	0.3742	0.3420	0.3138	0.2891	0.2672	0.2480	0.2309	0.2158
$k = 0.5$	0.4574	0.4194	0.3855	0.3554	0.3286	0.3048	0.2836	0.2646	0.2476	0.2324
$k = 0.7$	0.4614	0.4265	0.3950	0.3668	0.3414	0.3185	0.2979	0.2793	0.2626	0.2474

the detection-error probabilities $\zeta(\varepsilon)$ and $\hat{\zeta}(\varepsilon)$ can be reexpressed as

$$\zeta(\varepsilon) = \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \frac{1}{\omega}\exp\left(-\frac{1}{2}\omega^2\int_0^N \varphi^*(s)^2\,\mathrm{d}s\right)\sin\left(\frac{1}{2}\omega\int_0^N \varphi^*(s)^2\,\mathrm{d}s\right)\mathrm{d}\omega, \tag{47}$$

$$\hat{\zeta}(\varepsilon) = \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \frac{1}{\omega}\exp\left(-\frac{1}{2}\omega^2\int_0^N \hat{\varphi}^*(s)^2\,\mathrm{d}s\right)\sin\left(\frac{1}{2}\omega\int_0^N \hat{\varphi}^*(s)^2\,\mathrm{d}s\right)\mathrm{d}\omega. \tag{48}$$

The detailed derivation process of these two expressions is similar to Appendix D in Munk and Rubtsov [29] or Appendix 3 in Hu *et al.* [18]. We omit it here.

Given the parameter values in Table 1, Tables 2 and 3 display the detection-error probabilities for different values of risk aversion coefficient k, ambiguity aversion coefficient ε and sample size N. These two tables reveal that the probability that the ambiguity-averse reinsurer chooses the wrong model between the worst-case model and the reference model decreases with the increase of ambiguity aversion level. This is because the higher the ambiguity aversion, the larger the worst-case model deviates from the reference model as shown by Panel A of Figure 1, which makes it easier for the two models to be distinguished statistically from each other. On the contrary, the model deviation decreases with respect to the reinsurer's risk aversion level as shown by Panel B of Figure 1, and as a result, the detection-error probability increases with the increase of the reinsurer's risk aversion level. Comparing Tables 2 and 3, we find that the detection-error probability under the market equilibrium is lower than that under the Stackelberg game. The reason is that the model deviation is higher under the market equilibrium as shown by Figure 1.

Moreover, for the same conditions, Tables 2 and 3 also reveal that the larger the sample size used to estimate the claim model, the smaller the detection-error probability. For a short sample of data, $N = 5$, the detection-error probability can be above 48%. Even for a long sample of data, $N = 10$, the detection-error probability is still above 19%. This means that, for common parameter values, the detection-error probability is always greater than 10%. Therefore, in the following analysis, we assume that the reinsurer's ambiguity aversion coefficient is 0.5.

Table 3. Detection-error probability under market equilibrium.

ε	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
Panel A: sample of size $N = 5$ years										
$k = 0.3$	0.4736	0.4513	0.4323	0.4159	0.4018	0.3894	0.3785	0.3688	0.3602	0.3525
$k = 0.5$	0.4773	0.4577	0.4407	0.4257	0.4125	0.4008	0.3904	0.3810	0.3726	0.3650
$k = 0.7$	0.4802	0.4627	0.4472	0.4335	0.4212	0.4102	0.4002	0.3912	0.3830	0.3755
Panel B: sample of size $N = 10$ years										
$k = 0.3$	0.4295	0.3711	0.3231	0.2837	0.2513	0.2244	0.2021	0.1833	0.1675	0.1540
$k = 0.5$	0.4392	0.3874	0.3437	0.3068	0.2756	0.2492	0.2266	0.2073	0.1907	0.1763
$k = 0.7$	0.4467	0.4003	0.3603	0.3259	0.2961	0.2703	0.2480	0.2286	0.2116	0.1966

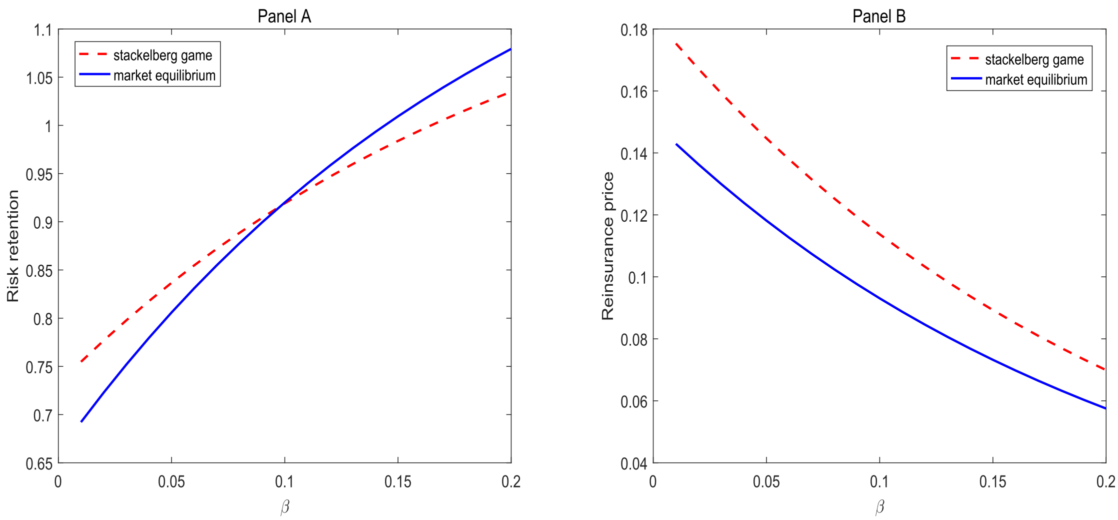

Figure 2. Effects of extrapolation intensity on reinsurance contracts. Panel A corresponds to risk retention and Panel B corresponds to price.

5.3. Robust risk retention proportion and reinsurance price

In this subsection, we investigate the qualitative and quantitative effects of the loss-dependent premium principle, learning and model ambiguity on the robust reinsurance contracts under the Stackelberg game and the market equilibrium by using some numerical examples.

In general, Figures 2–5 show that with the change of extrapolation intensity β, initial variance α_0, ambiguity aversion coefficient ε and risk aversion coefficient γ, the change trends of the risk retention proportion and the reinsurance price under the Stackelberg game and the market equilibrium are same.

Figure 2 depicts the effects of the loss-dependent premium principle on the robust reinsurance contracts. For the Stackelberg game, the reinsurer realizes that the loss-dependent premium principle can protect the insurer from worse cases by charging higher premium at a later stage. In this case, the reinsurer needs to lower reinsurance price in order to stimulate demand for reinsurance. Therefore, the reinsurance price decreases as the extrapolation intensity increases, as shown by Panel B. The lower reinsurance price reduces the risk retention proportion (see Eq. (18)) and the higher extrapolation intensity increases the risk retention proportion. Moreover, the positive influence is superior to the negative influence, and so the proportion of risk retention increases with the increase of extrapolation intensity, as shown by Panel A. For the market equilibrium, as

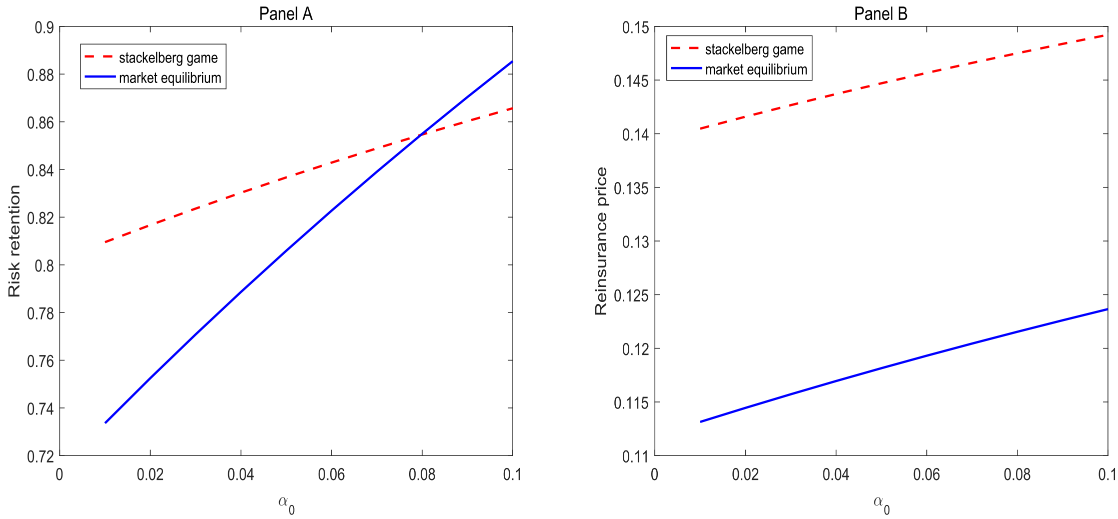

FIGURE 3. Effects of initial variance on reinsurance contracts. Panel A corresponds to risk retention and Panel B corresponds to price.

the extrapolation intensity increases, the insurer increases the risk retention proportion (Panel A), that is, he decreases the demand for reinsurance. Lower market demand leads to lower market price. Therefore, the price of reinsurance falls as the intensity of extrapolation increases (Panel B).

In the Stackelberg game, the reinsurer is the leader and has the priority to make price decision. However, in the market equilibrium, the reinsurer no longer has the priority to make price decision. In this case, both the insurer and the reinsurer decide the demand and supply for reinsurance at the same time, and the reinsurance price is determined by the equality of supply and demand. Therefore, the reinsurance price under the Stackelberg game is always greater than that under the market equilibrium, and the risk retention proportion under the market equilibrium is more sensitive to the change of parameters than that under the Stackelberg game. This result also can be seen in Panels B of Figures 3–5. Moreover, due to the higher price, the risk retention proportion under the Stackelberg game is larger than that under the market equilibrium if the extrapolation intensity is not too strong.

Figure 3 shows that the unilateral learning of the reinsurer generates great effects on the two kinds of reinsurance contract. We first analyze the Stackelberg game. As the initial variance increases, the reinsurer believes that there are considerable fluctuations in the claim rate, and thus, he charges higher reinsurance price (Panel B) to prevent the occurrence of larger claims. But in the insurer's beliefs, the claim rate is completely certain. Therefore, as the initial variance increases, the demand for reinsurance decreases and the risk retention proportion increases (Panel A). From the perspective of market equilibrium, the reinsurer, as the supplier, believes that there are large fluctuations in the claims as the initial variance increases, and thus, he reduces the supply for reinsurance. Lower supply leads to higher price and further reduces the equilibrium demand for reinsurance. Thus, the risk retention proportion increases with respect to the initial variance.

As the information is constantly updated, the prior variance evolves as $\alpha(t) = \frac{\alpha_0\sigma^2}{\sigma^2+\alpha_0 t}$, which decreases gradually. Therefore, Figure 3 can also be understood as follows: with the update of information, the initial variance decreases, that is, the claim rate becomes more stable, which leads to the reduction of reinsurance price and the increase of reinsurance demand.

Figure 4 depicts how the reinsurer's ambiguity aversion influences the robust risk retention proportion and reinsurance price. The higher the ambiguity aversion, the larger the worst-case model deviates from the reference model, which means that the worst-case model becomes worse. Therefore, as the level of ambiguity aversion

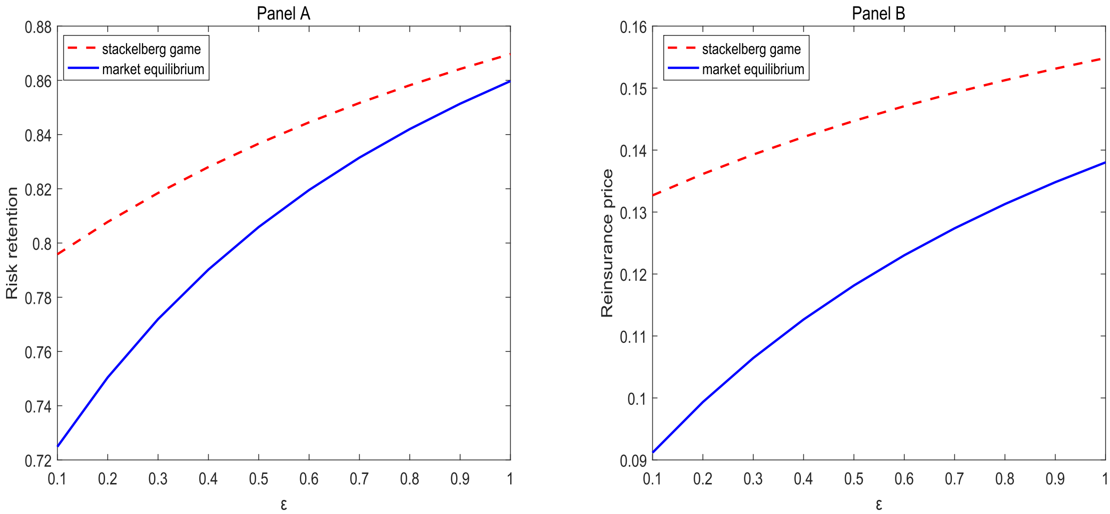

FIGURE 4. Effects of ambiguity aversion on reinsurance contracts. Panel A corresponds to risk retention and Panel B corresponds to price.

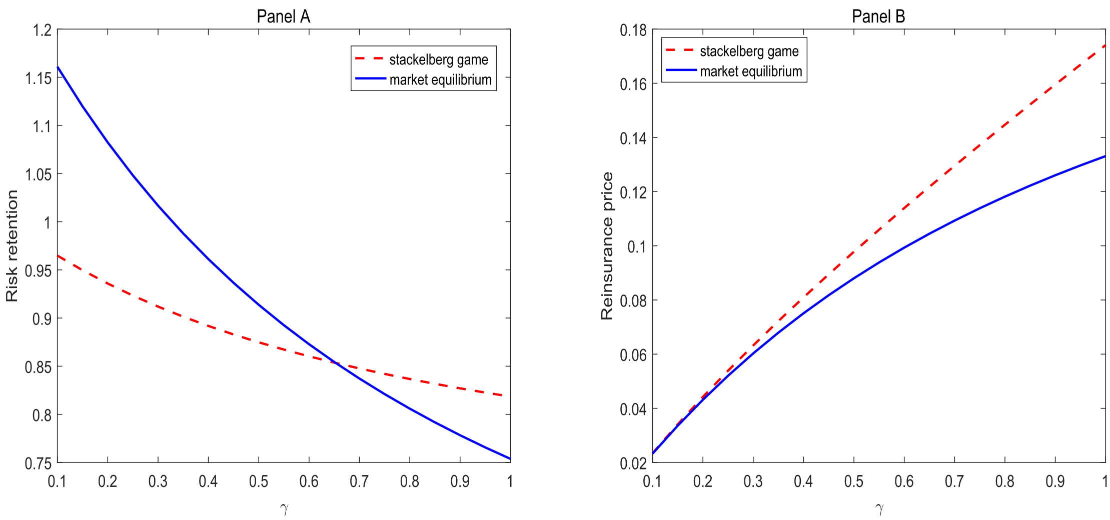

FIGURE 5. Effects of insurer's risk aversion on reinsurance contracts. Panel A corresponds to risk retention and Panel B corresponds to price.

increases, the reinsurer directly raises the price of reinsurance in the Stackelberg game, and in the market equilibrium, the reinsurer reduces the reinsurance supply. The decline of reinsurance supply leads to the rise of reinsurance price. Therefore, the reinsurance price under the market equilibrium is also increasing with respect to the reinsurer's ambiguity aversion as shown by Panel B. The insurer is ambiguity neutral, so the high price due to the reinsurer's ambiguity aversion causes the insurer to reserve higher proportion of claims as shown by Panel A. In the previous analysis, we point out that the reinsurer's risk aversion and ambiguity aversion are equivalent. Thus, Figure 4 also reveals the influence of the reinsurer's risk aversion on the reinsurance contracts.

The risk aversion of the insurer also generates great impacts on the demand and price of reinsurance as shown by Figure 5. The insurer with high risk aversion reserves less claim risk and needs to purchase more reinsurance to prevent claim risk. That is, the proportion of risk retention decreases with the insurer's risk aversion as shown by Panel A. This conclusion is intuitive. When demand goes up and supply stays the same, supply exceeds demand, which leads to the rise of equilibrium reinsurance price. Therefore, the equilibrium reinsurance price increases with respect to the insurer's risk aversion. In the Stackelberg game, when the insurer's risk aversion level increases, the reinsurer uses his first-mover advantage to grab higher price. As shown by Panel B, the reinsurance price in the Stackelberg game also increases about the insurer's risk aversion.

6. Conclusions

This paper examines the robust risk retention proportion and reinsurance price under the Stackelberg game and the market equilibrium. There are two decision makers in the reinsurance contract design, one insurer and one reinsurer. The insurer has full confidence in the estimated claim model and penalizes large claims with a loss-dependent premium principle. However, the reinsurer is aware of that the estimated claim model is inaccurate. From observing realized claims, the reinsurer learns about the claim model by using Bayesian learning. Moreover, the reinsurer takes model ambiguity into account and has a preference for reinsurance contract that is robust to model misspecifications.

We find that the loss-dependent premium principle, the unilateral learning and ambiguity aversion of the reinsurer influence the structures and levels of the optimal reinsurance demand and price. Moreover, the worst-case density operator, the optimal risk retention proportion and reinsurance price are different under the Stackelberg game and the market equilibrium. Under the market equilibrium, the deviation between the worst-case model and the reference model is larger than that under the Stackelberg game. This leads to lower detection-error probability in the market equilibrium. Because the reinsurer has the first-mover advantage in the Stackelberg game, the reinsurance price in this case is always greater than that in the market equilibrium. Therefore, in most cases, the risk retention proportion under the market equilibrium is smaller than that under the Stackelberg game. For the two models, the reinsurer's ambiguity aversion always leads to higher reinsurance price and lower reinsurance demand. On the contrary, Bayesian learning reduces the reinsurance price and increases the demand for reinsurance. However, both the reinsurance price and demand increase with respect to the loss-dependent premium principle.

Appendix A. The proof of Proposition 1

Proof. We apply the dynamic programming method to solve the insurer's response function. The HJB equation of optimization problem (3) is given by

$$\begin{aligned}\sup_{p(t)}\Big\{ & J_t + \big[rX + (1+\theta)\big(e^{-\beta t}\mu + m\big) - (1+\eta(t))\mu + p(t)\eta(t)\mu\big] J_X \\ & +\beta(\mu - m)J_m + \frac{1}{2}\sigma^2 p(t)^2 J_{XX} + \frac{1}{2}\beta^2\sigma^2 J_{mm} - \beta\sigma^2 p(t) J_{Xm}\Big\} = 0.\end{aligned} \tag{A.1}$$

Conjecture that the insurer's value function has the following form,

$$J(t,m,X) = -\frac{1}{\gamma}\exp\Big\{-\gamma\Big[Xe^{r(T-t)} + f(t) + g(t,m)\Big]\Big\}, \tag{A.2}$$

subject to terminal conditions $f(T) = 0$ and $g(T,m) = 0$. Then

$$\begin{aligned} & J_t = \gamma\Big[rXe^{r(T-t)} - f_t - g_t\Big]J, \quad J_X = -\gamma e^{r(T-t)}J, \quad J_{XX} = \gamma^2 e^{2r(T-t)}J, \\ & J_m = -\gamma g_m J, \quad J_{mm} = \gamma(\gamma g_m^2 - g_{mm}), \quad J_{Xm} = \gamma^2 e^{r(T-t)} g_m J. \end{aligned}$$

Substituting these derivatives into the HJB equation, we have

$$\begin{aligned} &f_t + g_t + (1+\theta)\big(e^{-\beta t}\mu + m\big)e^{r(T-t)} - (1+\eta(t))\mu e^{r(T-t)} + \beta(\mu - m)g_m + \frac{1}{2}\beta^2\sigma^2 g_{mm} \\ &\quad - \frac{1}{2}\gamma\beta^2\sigma^2 g_m^2 + \sup_{p(t)}\left\{p(t)\eta(t)\mu e^{r(T-t)} - \frac{1}{2}\gamma\sigma^2 p(t)^2 e^{2r(T-t)} + \gamma\beta\sigma^2 p(t)e^{r(T-t)}g_m\right\} = 0. \end{aligned} \tag{A.3}$$

The second-order derivative of the left hand of the above equation about $p(t)$ is less than 0, thus the first-order condition shows that the insurer's response function is

$$p^*(t) = \left(\frac{\mu\eta(t)}{\gamma\sigma^2} + \beta g_m\right)e^{-r(T-t)}. \tag{A.4}$$

Inserting $p^*(t)$ into equation (A.3) and according to the irrelevance of function $f(t)$ about state variable m, we can obtain the following two differential equations,

$$f_t + (1+\theta)\mu e^{-\beta t}e^{r(T-t)} - (1+\eta(t))\mu e^{r(T-t)} + \frac{\mu^2\eta(t)^2}{2\gamma\sigma^2} = 0, \tag{A.5}$$

$$g_t + (1+\theta)me^{r(T-t)} + \beta(\mu - m)g_m + \frac{1}{2}\beta^2\sigma^2 g_{mm} + \beta\mu\eta(t)g_m = 0. \tag{A.6}$$

Combining equation (A.5) and the final condition $f(T) = 0$, we derive

$$f(t) = \frac{\mu(1+\theta)}{\beta + r}\Big[e^{rT-(\beta+r)t} - e^{-\beta T}\Big] - \mu\int_t^T (1+\eta(s))e^{r(T-s)}\,\mathrm{d}s + \frac{\mu^2}{2\gamma\sigma^2}\int_t^T \eta(s)^2\,\mathrm{d}s. \tag{A.7}$$

Through preliminary observation to equation (A.6), $g(t,m)$ is a quadratic function of the variable m. We conjecture that

$$g(t,m) = a(t)m^2 + b(t)m + d(t), \tag{A.8}$$

subject to terminal conditions $a(T) = 0, b(T) = 0$ and $d(T) = 0$. Substituting the corresponding derivatives into equation (A.6), the expression holds for any nonnegative state variable m, and therefore

$$a'(t) - 2\beta a(t) = 0, \tag{A.9}$$

$$b'(t) + (1+\theta)e^{r(T-t)} + 2\beta\mu a(t) - \beta b(t) + 2\beta\mu\eta(t)a(t) = 0, \tag{A.10}$$

$$d'(t) + \beta^2\sigma^2 a(t)^2 + \beta\mu(1+\eta(t))b(t) = 0. \tag{A.11}$$

Together with the terminal conditions $a(T) = 0, b(T) = 0$ and $d(T) = 0$, we obtain

$$a(t) = 0, \tag{A.12}$$

$$b(t) = \frac{1+\theta}{r+\beta}\Big[e^{r(T-t)} - e^{-\beta(T-t)}\Big], \tag{A.13}$$

$$d(t) = \beta\mu\int_t^T (1+\eta(s))b(s)\,\mathrm{d}s. \tag{A.14}$$

This completes the proof of Proposition 1. □

Appendix B. Proof of Theorem 1

Proof. We apply the dynamic programming method to solve the robust reinsurance contract. For the reinsurer's optimization problem (21), the robust HJB equation is given by

$$\begin{aligned} &\sup_{\eta(t)}\inf_{\varphi(t)}\Big\{V_t + [rY + (1-p^*(t))(\mu - \hat{\mu}) + (1-p^*(t))(\mu\eta(t) + \sigma\varphi(t))]V_Y \\ &\quad + \frac{1}{2}\sigma^2(1-p^*(t))^2 V_{YY} - \frac{\alpha(t)\varphi(t)}{\sigma}V_{\hat{\mu}} + \frac{\alpha(t)^2}{2\sigma^2}V_{\hat{\mu}\hat{\mu}} - \alpha(t)(1-p^*(t))V_{\hat{\mu}Y} - \frac{k}{2\varepsilon}\varphi(t)^2 V\Big\} = 0, \end{aligned} \tag{B.1}$$

where $p^*(t)$ is given in equation (18) and $\alpha(t) = \frac{\alpha_0\sigma^2}{\sigma^2+\alpha_0 t}$. Similar to the derivation of the response function in Appendix A, we conjecture that the reinsurer's value function has the following form,

$$V(t,\hat{\mu},Y) = -\frac{1}{k}\exp\Big\{-k\Big[Ye^{r(T-t)} + A(t) + B(t)\hat{\mu} + D(t)\hat{\mu}^2\Big]\Big\}, \tag{B.2}$$

subject to terminal conditions $A(T)=0$, $B(T)=0$ and $D(T)=0$. It is not difficult to obtain

$$\begin{aligned} V_t &= k\Big[rYe^{r(T-t)} - A'(t) - B'(t)\hat{\mu} - D'(t)\hat{\mu}^2\Big]V, & V_Y &= -ke^{r(T-t)}V,\\ V_{YY} &= k^2e^{2r(T-t)}V, & V_{\hat{\mu}} &= -k[B(t)+2D(t)\hat{\mu}]V,\\ V_{\hat{\mu}\hat{\mu}} &= -k\big[2D(t) - k(B(t)+2D(t)\hat{\mu})^2\big]V, & V_{Y\hat{\mu}} &= k^2e^{r(T-t)}[B(t)+2D(t)\hat{\mu}]V. \end{aligned}$$

Substituting these relevant derivatives of $V(t,\hat{\mu},Y)$ into HJB equation (B.1) and reorganizing this expression, we have

$$\begin{aligned} \sup_{\eta(t)}\inf_{\varphi(t)}\Big\{& A'(t) + B'(t)\hat{\mu} + D'(t)\hat{\mu}^2 + (\mu-\hat{\mu})e^{r(T-t)}(1-p^*(t)) + e^{r(T-t)}(1-p^*(t))(\mu\eta(t)+\sigma\varphi(t))\\ &- \frac{1}{2}k\sigma^2e^{2r(T-t)}(1-p^*(t))^2 - \frac{\alpha(t)\varphi(t)}{\sigma}(B(t)+2D(t)\hat{\mu}) + \frac{\alpha(t)^2}{2\sigma^2}\big[2D(t) - k(B(t)+2D(t)\hat{\mu})^2\big]\\ &+ k\alpha(t)e^{r(T-t)}(B(t)+2D(t)\hat{\mu})(1-p^*(t)) + \frac{1}{2\varepsilon}\varphi(t)^2\Big\} = 0. \end{aligned} \tag{B.3}$$

We first consider the infinimization part of the above equation. Obviously, the second derivatives of the above equation about $\varphi(t)$ is greater than 0. Solving the first condition, the worst-case density operator is given by

$$\varphi^*(t) = \frac{\varepsilon\alpha(t)}{\sigma}\Big(B(t)+2D(t)\hat{\mu}\Big) - \varepsilon\sigma e^{r(T-t)}(1-p^*(t)\Big). \tag{B.4}$$

Substituting $\varphi^*(t)$ and $p^*(t)$ into HJB equation (B.3), we obtain

$$\begin{aligned} \sup_{\eta(t)}\Big\{& A'(t) + B'(t)\hat{\mu} + D'(t)\hat{\mu}^2 + (\mu-\hat{\mu})\Big[e^{r(T-t)} - \frac{\mu\eta(t)}{\gamma\sigma^2} - \beta b(t)\Big]\\ &+ \mu\eta(t)\Big[e^{r(T-t)} - \frac{\mu\eta(t)}{\gamma\sigma^2} - \beta b(t)\Big] - \frac{\sigma^2(k+\varepsilon)}{2}\Big[e^{r(T-t)} - \frac{\mu\eta(t)}{\gamma\sigma^2} - \beta b(t)\Big]^2\\ &+ (k+\varepsilon)\alpha(t)(B(t)+2D(t)\hat{\mu})\Big[e^{r(T-t)} - \frac{\mu\eta(t)}{\gamma\sigma^2} - \beta b(t)\Big]\\ &- \frac{k+\varepsilon}{2\sigma^2}\alpha(t)^2(B(t)+2D(t)\hat{\mu})^2 + \frac{\alpha(t)^2}{\sigma^2}D(t)\Big\} = 0. \end{aligned} \tag{B.5}$$

The first-order condition about $\eta(t)$ is

$$\begin{aligned} &(\mu-\hat{\mu})\Big(-\frac{\mu}{\gamma\sigma^2}\Big) + \mu\Big[e^{r(T-t)} - \frac{\mu\eta(t)}{\gamma\sigma^2} - \beta b(t)\Big] + \mu\eta(t)\Big(-\frac{\mu}{\gamma\sigma^2}\Big)\\ &\quad - \sigma^2(k+\varepsilon)\Big[e^{r(T-t)} - \frac{\mu\eta(t)}{\gamma\sigma^2} - \beta b(t)\Big]\Big(-\frac{\mu}{\gamma\sigma^2}\Big) + \alpha(t)(k+\varepsilon)(B(t)+2D(t)\hat{\mu})\Big(-\frac{\mu}{\gamma\sigma^2}\Big) = 0. \end{aligned} \tag{B.6}$$

Solving this first-order condition, we can obtain the robust reinsurance price that is presented in equation (23).

Substituting $\eta^*(t)$ into the insurer's optimal response function $p^*(t)$, the insurer's optimal risk retention proportion (25) can be derived. Then, substituting $p^*(t)$ into equation (B.4), we obtain the worst-case density operator as presented in equation (24).

We are now in a position to solve functions $A(t), B(t)$ and $D(t)$. Inserting the expression of $\eta^*(t)$ into equation (B.5), the corresponding equation holds for any nonnegative state variable $\hat{\mu}$, and thus, we have the following three ordinary differential equations,

$$D'(t) - \frac{4\gamma(k+\varepsilon)}{\sigma^2(2\gamma+k+\varepsilon)}\alpha(t)^2 D(t)^2 - \frac{2(k+\varepsilon)}{\sigma^2(2\gamma+k+\varepsilon)}\alpha(t)D(t) + \frac{1}{2\sigma^2(2\gamma+k+\varepsilon)} = 0, \tag{B.7}$$

$$B'(t) - \left\{\frac{2(k+\varepsilon)\alpha(t)^2}{\sigma^2}D(t) - \frac{(k+\varepsilon)\alpha(t)}{\sigma^2(2\gamma+k+\varepsilon)}[2(k+\varepsilon)\alpha(t)D(t)-1]\right\}B(t)$$
$$+ \frac{\gamma\sigma^2\left[e^{r(T-t)} - \beta b(t)\right] + \mu}{\sigma^2(2\gamma+k+\varepsilon)}[2(k+\varepsilon)\alpha(t)D(t)-1] = 0, \tag{B.8}$$

$$A'(t) - \frac{\gamma}{\sigma^2(2\gamma+k+\varepsilon)}\alpha(t)^2 B(t)^2 + \frac{k+\varepsilon}{2\gamma+k+\varepsilon}\left[\gamma(e^{r(T-t)} - \beta b(t)) + \frac{\mu}{\sigma^2}\right]\alpha(t)B(t)$$
$$+ \frac{1}{\sigma^2}\alpha(t)^2 D(t) + \frac{\gamma^2\sigma^2}{2(2\gamma+k+\varepsilon)}\left[e^{r(T-t)} - \beta b(t)\right]^2 + \frac{\mu^2}{2\sigma^2(2\gamma+k+\varepsilon)} = 0. \tag{B.9}$$

According to the characteristics of these three equations, we solve the functions $D(t), B(t)$ and $A(t)$ in order. Because $\frac{\mathrm{d}\alpha(t)}{\mathrm{d}t} = -\frac{\alpha_0^2\sigma^2}{(\sigma^2+\alpha_0 t)^2} = -\frac{1}{\sigma^2}\alpha(t)^2$, we have

$$[\alpha(t)D(t)]' = \alpha(t)D'(t) - \frac{1}{\sigma^2}\alpha(t)^2 D(t). \tag{B.10}$$

Therefore,

$$D'(t) = \frac{[\alpha(t)D(t)]' + \frac{1}{\sigma^2}\alpha(t)^2 D(t)}{\alpha(t)}. \tag{B.11}$$

Combining this expression with equation (B.7), there is

$$[\alpha(t)D(t)]' = \alpha(t)\left[\frac{4\gamma(k+\varepsilon)}{\sigma^2(2\gamma+k+\varepsilon)}\alpha(t)^2 D(t)^2 + \frac{k+\varepsilon-2\gamma}{\sigma^2(2\gamma+k+\varepsilon)}\alpha(t)D(t) - \frac{1}{2\sigma^2(2\gamma+k+\varepsilon)}\right]. \tag{B.12}$$

Denote $h(t) = \alpha(t)D(t)$, then $h(T) = \alpha(T)D(T) = 0$. The above equation can be converted to the following Riccati equation,

$$h'(t) = \alpha(t)\left[\frac{4\gamma(k+\varepsilon)}{\sigma^2(2\gamma+k+\varepsilon)}h(t)^2 + \frac{k+\varepsilon-2\gamma}{\sigma^2(2\gamma+k+\varepsilon)}h(t) - \frac{1}{2\sigma^2(2\gamma+k+\varepsilon)}\right]. \tag{B.13}$$

It is not difficult to prove that the quadratic equation,

$$\frac{4\gamma(k+\varepsilon)}{\sigma^2(2\gamma+k+\varepsilon)}h(t)^2 + \frac{k+\varepsilon-2\gamma}{\sigma^2(2\gamma+k+\varepsilon)}h(t) - \frac{1}{2\sigma^2(2\gamma+k+\varepsilon)} = 0$$

has two real roots,

$$h_1 = \frac{1}{2(k+\varepsilon)}, \qquad h_2 = -\frac{1}{4\gamma}. \tag{B.14}$$

Thus, equation (B.13) can be rewritten as

$$h'(t) = \frac{4\gamma(k+\varepsilon)}{\sigma^2(2\gamma+k+\varepsilon)}\alpha(t)(h(t)-h_1)(h(t)-h_2). \tag{B.15}$$

Combining this equation and the final condition $h(T) = 0$, we obtain

$$h(t) = \frac{h_1\left[1 - \left(\frac{\sigma^2+\alpha_0 T}{\sigma^2+\alpha_0 t}\right)^{-\frac{4\gamma(k+\varepsilon)}{2\gamma+k+\varepsilon}(h_1-h_2)}\right]}{1 - \frac{h_1}{h_2}\left(\frac{\sigma^2+\alpha_0 T}{\sigma^2+\alpha_0 t}\right)^{-\frac{4\gamma(k+\varepsilon)}{2\gamma+k+\varepsilon}(h_1-h_2)}} = \frac{h_1\left(1 - \frac{\sigma^2+\alpha_0 t}{\sigma^2+\alpha_0 T}\right)}{1 - \frac{h_1}{h_2}\frac{\sigma^2+\alpha_0 t}{\sigma^2+\alpha_0 T}}. \tag{B.16}$$

Considering $D(t) = \frac{h(t)}{\alpha(t)}$, then we have

$$D(t) = \frac{\left(\sigma^2 + \alpha_0 t\right)\left(1 - \frac{\sigma^2+\alpha_0 t}{\sigma^2+\alpha_0 T}\right)}{\alpha_0\sigma^2\left[2(k+\varepsilon) + \frac{4\gamma(\sigma^2+\alpha_0 t)}{\sigma^2+\alpha_0 T}\right]}. \tag{B.17}$$

To simplify formula (B.8), we denote

$$M(t) = \frac{2(k+\varepsilon)\alpha(t)^2}{\sigma^2}D(t) - \frac{(k+\varepsilon)\alpha(t)}{\sigma^2(2\gamma+k+\varepsilon)}[2(k+\varepsilon)\alpha(t)D(t) - 1], \tag{B.18}$$

$$N(t) = \frac{\gamma\sigma^2[e^{r(T-t)} - \beta b(t)] + \mu}{\sigma^2(2\gamma+k+\varepsilon)}[2(k+\varepsilon)\alpha(t)D(t) - 1]. \tag{B.19}$$

Then, equation (B.8) can be reexpressed as

$$B'(t) - M(t)B(t) + N(t) = 0. \tag{B.20}$$

Combining this equation and the terminal condition $B(T) = 0$, the solution of equation (B.20) can be written as

$$B(t) = e^{-\int_t^T M(s)\,\mathrm{d}s}\int_t^T N(s)e^{\int_s^T M(u)\,\mathrm{d}u}\,\mathrm{d}s. \tag{B.21}$$

At last, considering equation (B.9) and the terminal condition $A(T) = 0$, we obtain

$$\begin{aligned} A(t) = &-\frac{\gamma}{\sigma^2(2\gamma+k+\varepsilon)}\int_t^T \alpha(s)^2B(s)^2\,\mathrm{d}s + \frac{k+\varepsilon}{2\gamma+k+\varepsilon}\int_t^T\left[\gamma(e^{r(T-s)} - \beta b(s)) + \frac{\mu}{\sigma^2}\right]\alpha(s)B(s)\,\mathrm{d}s \\ &+ \frac{1}{\sigma^2}\int_t^T \alpha(s)^2D(s)\,\mathrm{d}s + \frac{\gamma^2\sigma^2}{2(2\gamma+k+\varepsilon)}\int_t^T\left[e^{r(T-s)} - \beta b(s)\right]^2\mathrm{d}s + \frac{\mu^2}{2\sigma^2(2\gamma+k+\varepsilon)}(T-t). \end{aligned} \tag{B.22}$$

Therefore, the conjecture about the reinsurer's value function is correct. Substituting the optimal reinsurance price $\eta^*(t)$ into functions $f(t)$ and $g(t,m)$, we can obtain the insurer's value function. This competes the proof of Theorem 1. □

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