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REALIZABILITY PROBLEM OF DISTANCE-EDGE-MONITORING NUMBERS

Zhen Ji[1], Yaping Mao[2,*], Eddie Cheng[3] and Xiaoyan Zhang[4]

Abstract. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For a set M of vertices and an edge e of a graph G, let $P(M,e)$ be the set of pairs (x,y) with a vertex x of M and a vertex y of $V(G)$ such that $d_G(x,y) \neq d_{G-e}(x,y)$. Given a vertex x, an edge e is said to be monitored by x if there exists a vertex v in G such that $(x,v) \in P(\{x\},e)$, and the collection of such edges is $\mathrm{EM}(x)$. A set M of vertices of a graph G is distance-edge-monitoring (DEM for short) set if every edge e of G is monitored by some vertex of M, that is, the set $P(M,e)$ is nonempty. The DEM number $\mathrm{dem}(G)$ of a graph G is defined as the smallest size of such a set in G. The vertices of M represent distance probes in a network modeled by G; when the edge e fails, the distance from x to y increases, and thus we are able to detect the failure. In this paper, we first give some bounds or exact values of line graphs of trees, grids, complete bipartite graphs, and obtain the exact values of DEM numbers for some graphs and their line graphs, including the friendship and wheel graphs. Next, for each $n,m>1$, we obtain that there exists a graph $G_{n,m}$ such that $\mathrm{dem}(G_{n,m}) = n$ and $\mathrm{dem}(L(G_{n,m})) = 4$ or $2n+t$, for each integer $t \geq 0$. In the end, the DEM number for the line graph of a small-world network (DURT) is given.

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1. Introduction

All graphs considered in this paper are undirected, finite and simple. We refer to the book [8] for graph theoretical notation and terminology not described here. Let G be a graph with the vertex set $V(G)$ and the edge set $E(G)$. For a vertex $v \in V(G)$, we use $d_G(v)$ to express the degree of v in G. Let $L(G)$ be the line graph of a graph G; see Figure 1. In this paper, we denote by $d_G(x,y)$ the length of a shortest path between two vertices x and y in a graph G. If there are no paths between x and y in G, then $d_G(x,y) = \infty$. For an edge set Y of G, we denote by $G - Y$ the graph obtained by deleting all edges in Y from G. If $Y = \{e\}$, we simply write $G - e$ for $G - Y$. For a vertex set X of G, we denote by $G \setminus X$ the graph obtained by deleting all vertices in X from G. If $X = \{v\}$, we simply write $G \setminus v$ for $G \setminus X$. We use $X \setminus S$ to denote the vertex subset

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[1] School of Computer, Qinghai Normal University, Xining, Qinghai 810008, P.R. China.

[2] Academy of Plateau Science and Sustainability, and School of Mathematics and Statistis, Qinghai Normal University, Xining, Qinghai 810008, P.R. China.

[3] Department of Mathematics and Statistics, Oakland University, Rochester, MI 48309, USA.

[4] School of Mathematical Science & Institute of Mathematics, Nanjing Normal University, Nanjing 210023, P.R. China.

*Corresponding author: maoyaping@ymail.com

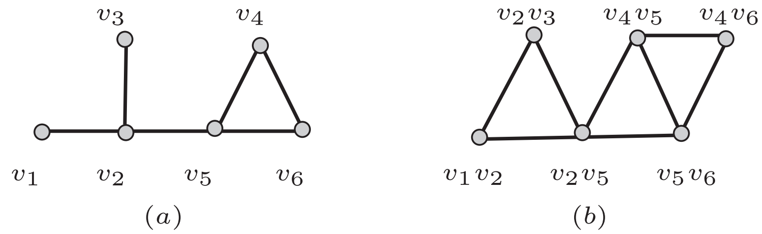

FIGURE 1. (a) A graph G; (b) its line graph $L(G)$.

of X obtained by removing all the vertices of S from X and $Y - W$ to denote the edge subset of Y obtained by removing all the edges of W from Y. If $S = \{v\}$, we simply write $X \setminus v$ for $X \setminus S$. The *connectivity* $\kappa(G)$ of a graph G is the smallest number of vertices whose removal from G results in either a disconnected graph or a trivial graph. The *Cartesian product* G of two graphs G_1 and G_2, commonly denoted by $G_1 \Box G_2$, has vertex set $V(G) = V(G_1) \times V(G_2)$, where two distinct vertices (u, v) and (x, y) of $G_1 \Box G_2$ are adjacent if either $u = x$ and $vy \in E(G_2)$, or $v = y$ and $ux \in E(G_1)$. A vertex v in a graph G is called a *cut vertex* if deleting v from G increases the number of components of G.

Distance-edge-monitoring (DEM for short), introduced by Foucaud *et al.* [15], means network monitoring using distance probes. Networks are naturally modeled by finite undirected simple connected graphs, whose vertices represent computers and whose edges represent connections between them. When a connection (an edge) fails in the network, we can detect this failure, and thus achieve the purpose of monitoring the network. Probes are made up of vertices we choose in the network. At any given moment, a probe of the network can measure its graph distance to every other vertex of the network. Whenever an edge of the network fails, one of the measured distances changes, so the probes are able to detect the failure of this edge. Probes that measure distances in graphs are present in real-life networks. It is useful in the fundamental task of routing [13,16] and is also frequently used for problems concerning network verification [2,4,7]. Given that this concept is important, one of the next steps is to study the computational complexity of the corresponding parameter, $\text{dem}(G)$ for the graph G. Indeed Foucaud *et al.* [15] showed that the decision problem for the DEM number is NP-complete (even for apex graphs). Of course, for special classes of graphs, it may still be possible to theoretically determine the corresponding DEM number as shown in [15] for grids, hypercubes and complete bipartite graphs.

The line graphs are important as it helps connect many areas of graph theory. Ma *et al.* [26] studied a series of specific line graphs of bipartite lattices, and it can be applied to 2D crystalline materials and metamaterials. See [31] for a survey and [1] for a more up-to-date study. In particular, line graphs have been used in realizability problems for graph parameters. In [11], Chartrand and Stewart showed that the line graph of an n-connected graph is itself n-connected if $n \geq 2$. Capobianco and Molluzzo [9] showed that the difference between the connectivity of a graph and its line graph can be arbitrarily large, and they proposed an open problem: For any two integers $n, m (1 < n < m)$, whether there exists a graph G such that $\kappa(G) = n$ and $\kappa(L(G)) = m$? In 1979, Bauer and Tindell [3] answered this problem. In this paper, our goal is to show that there is a graph $G_{n,m}$ such that $\text{dem}(G_{n,m}) = n$ and $\text{dem}(L(G_{n,m})) = m$ for each $n, m > 1$, and we show that this goal is attained except for "small cases".

A small-world network refers to an ensemble of networks in which the mean geodesic (*i.e.*, shortest-path) distance between nodes increases sufficiently slowly as a function of the number of nodes in the network. Small-world properties can be found in many real-world networks, such as road maps, food chains, metabolite processing networks, networks of brain neurons, and social influence networks; see [12, 29, 37] for more details. The first property of small-world networks is that their average path length (APL) grows proportionally to the logarithm of the number of nodes in the network. The second typical property of small-world networks is that they have a high clustering coefficient. In 2008, Zhang *et al.* [38] obtained a detailed analysis of the deterministic uniform recursive tree (DURT) from the viewpoint of complex networks. They first derived topological characteristics of

DURT, such as degree distribution, average path length, betweenness distribution, and degree correlations. The DURT's average path length shows a logarithmic scaling with the size of the network; however, its clustering coefficient is zero. By adding some edges in each iteration with a simple rule, a small-world network can be derived from the DURT.

The orderly configuration of a communication network's components (links, nodes, etc.) is known as its topology. The arrangement of different telecommunication network types, such as industrial field buses, command and control radio networks, and other key networks, is known as network topology. Regarding tree topology, it is a topology that consists of a tree structure, with all computers connected to one another like the branches of the tree. A bus and star network topology combined to form a tree topology is known in computer networks [14, 28]. In 2015, Chapman [10] gave an analysis framework for a class of dynamic composite networks, and these networks are formed from smaller factor networks *via* graph Cartesian products. He provided a composition method for extending the controllability and observability of the factor networks to that of the composite network, then delved into the effectiveness of designing control and estimation algorithms for the composite network *via* symmetry in the network. And he provided examples and applications to demonstrate the results, including distributed output feedback stabilizers and social network applications. In [34], being able to map Internet topologies exhibiting the two layers of connection would open new perspectives in Internet modeling and topology generation. Indeed, one could then model the Internet topology as a bipartite graph, *i.e.*, a graph in which vertices can be divided into two disjoint sets, A (*e.g.*, Ethernet switches) and B (*e.g.*, routers), such that every edge connects a vertex in A to one in B. The friendship graph is nothing but an undirected graph where edges are bi-directional. One of the popular social networking sites is Facebook, where we could observe these bi-directional communications [6]. Martín *et al.* [27] proposed a simulation of an arbitrary network of evolutionary processors by a network having a special underlying graph, namely a wheel graph. This work continues a series of results devoted to simulations between networks of evolutionary processors with various topologies.

1.1. Distance-edge-monitoring numbers

Foucaud *et al.* [15] gave the following definitions.

Definition 1.1. For a set M of vertices and an edge e of a graph G, let $P(M, e)$ be the set of pairs (x, y) with x a vertex of M and y a vertex of $V(G)$ such that $d_G(x, y) \neq d_{G-e}(x, y)$. In other words, e belongs to all shortest paths between x and y in G.

Definition 1.2. For a vertex x, let $\mathrm{EM}(x)$ be the set of edges e such that there exists a vertex v in G with $(x, v) \in P(\{x\}, e)$. If $e \in \mathrm{EM}(x)$, we say that e is *monitored* by x.

Definition 1.3. A set M of vertices of a graph G is *DEM set* if every edge e of G is monitored by some vertex of M, that is, for every edge e, the set $P(M, e)$ is nonempty. Equivalently, $\bigcup_{x\in M} \mathrm{EM}(x) = E(G)$.

Definition 1.4. The *DEM number* $\mathrm{dem}(G)$ of a graph G is defined as the smallest size of DEM sets of G.

Note that $V(G)$ is always a DEM set of G, so $\mathrm{dem}(G)$ is always well-defined. We start with the following proposition.

Proposition 1.5 ([15])**.** *Let M be a DEM set of a graph G. Then, for any two distinct edges e and e' in G, we have $P(M, e) \neq P(M, e')$.*

Thus, assume that we have placed probes on a DEM set M of a network G and initially computed all the sets $P(M, e)$. In the case a unique edge of the network has failed, Proposition 1.5 shows that by measuring the set of pairs (x, y) with $x \in M$ and $y \in V(G)$ whose distance has changed, we know exactly which is the edge that has failed.

Given a graph G, one would want to build a smallest possible DEM set of G. Bampas *et al.* [2] and Beerliova *et al.* [4] studied a weaker model as a network discovery problem, that is, where one seeks a set S of vertices such that for each edge e, there exists a vertex x of S and a vertex y of G such that e belongs to some shortest path from x to y. In [5], Bejeranoa *et al.* studied a different and weaker model as the Link Monitoring problem: given a graph G, for a tree T_v and any vertex $v \in V(G)$, find the smallest subset $S \subseteq V(G)$ such that $\cup_{v\in S} E(T_v) = E(G)$, where T_v defines the routing paths from the vertex v to all the other vertices in $V(G)$. One seeks to monitor the edges of a graph network by selecting vertices to act as probes, and it is essentially required that each edge of the graph belongs to one of the trees. So for a vertex v, v monitors a tree in Link Monitoring, but v monitors a forest in DEM.

1.2. Recent progress and our results

In 1979, Bauer and Tindell [3] studied graphs with prescribed connectivity and line graph connectivity.

Theorem 1.6 ([3])**.** *For each n, m, $1 < n < m$, there is a graph $G_{n,m}$ such that $\kappa(G_{n,m}) = n$ and $\kappa(L(G_{n,m})) = m$.*

Li and Mao [23] investigated this problem for generalized connectivity.

In this paper, in order to study the relation between DEM numbers of graph and its line graph, we consider the following problem.

Problem 1.7. For integers $s, t > 1$, is there a graph $G_{s,t}$ such that $\text{dem}(G_{s,t}) = s$ and $\text{dem}(L(G_{s,t})) = t$?

For a graph G, a set X of vertices is called a vertex cover of G if every edge of G has one of its endpoints in X. The smallest size of a vertex cover of G is denoted by $vc(G)$.

Foucaud *et al.* [15] derived the following result for complete graphs.

Theorem 1.8 ([15])**.** *In any graph G of order n, any vertex cover is a DEM set, and hence $\text{dem}(G) \le vc(G) \le n-1$. Moreover, $\text{dem}(G) = n - 1$ if and only if G is the complete graph of order n.*

Given a vertex x of a graph G and an integer i, we let $r_i(x)$ denote the set of vertices at distance i of x in G.

Lemma 1.9 ([15])**.** *Let x be a vertex of a connected graph G. Then, an edge uv belongs to $\text{EM}(x)$ if and only if for some integer i, $u \in r_i(x)$ and v is the only neighbor of u in $r_{i-1}(x)$.*

Yang *et al.* [35] gave upper and lower bounds of $P(M, e), \text{EM}(x), \text{dem}(G)$, respectively, and extremal graphs attaining the bounds were characterized. Also, they characterized the graphs with $\text{dem}(G) = 3$. Li *et al.* [24] obtained the results of DEM numbers about cartesian product of graphs. Ji *et al.* [19] studied the Erdös-Gallai-type problems for DEM numbers. For more results on the DEM sets and numbers, we refer the readers to the papers [17, 21, 22, 33, 36].

In Section 2, we give bounds or exact values for DEM numbers of line graphs of trees, grids, and complete bipartite graphs. In Section 3, we obtain some bounds for DEM numbers of some graphs and their line graphs, including the friendship and wheel graphs. For each $n, m > 1$, we obtain that there is a graph $G_{n,m}$ such that $\text{dem}(G_{n,m}) = n$ and $\text{dem}(L(G_{n,m})) = 4$ or $2n + t$, for each integer $t \ge 0$. In Section 4, we study the DEM number of the line graph of a small-world network (DURT).

2. Results for line graphs

In this section, we give the bounds of line graphs of trees and exact values of grids and complete bipartite graphs.

2.1. Bounds

For a connected graph G, Ji *et al.* [19] given the lower bound of G as follows.

Lemma 2.1 ([19]). *For a connected graph G, if G contains a complete subgraph with order $r \geq 2$, then $\text{dem}(G) \geq r-1$.*

An edge e in a graph G is a *bridge* if $G-e$ has more connected components than G.

Lemma 2.2 ([3]). *Let G be a connected graph and let e be a bridge of G. For any vertex x of G, we have $e \in \text{EM}(x)$.*

For a tree T, let $\text{Leaf}(T)$ be the set of leaves in T and $\Delta(T)$ be the maximum degree of T. Now, we can derive upper and lower bounds for $\text{dem}(L(T))$.

Theorem 2.3. *Let T be a tree of order n ($n \geq 3$). Then*

$$\Delta(T) - 1 \leq \text{dem}(L(T)) \leq |\,\text{Leaf}(T)| - 1.$$

Moreover, the bounds are sharp.

Proof. Let E_v be the set of edges associated with the vertex v, where $v \in V(T)$ with $d_T(v) = \Delta(T)$. Then the graph induced by E_v is a complete subgraph of order $\Delta(T)$ in $L(T)$, by Lemma 2.1, we have $\text{dem}(L(T)) \geq \Delta(T) - 1$.

To show the upper bound, let D be the set of edges associated with leaves of T and the edge set $D' = D - e_0$, where e_0 is an edge associated with some leaf of T. For any bridge e of $L(T)$, it follows from Lemma 2.2 that e can be monitored by any vertex in D'. For any complete subgraph H with order $r \geq 3$ of $L(T)$, it implies that v is a cut vertex or $v \in D$ for any vertex $v \in V(H)$. If v is a cut vertex, then let C_1, C_2 be the two connected components of $L(T) \setminus v$, where $u \in V(C_1)$. For any vertex $w \in D' \cap V(C_2)$, we have $\text{EM}(w) \cap E(C_1) = \text{EM}(v) \cap E(C_1)$, and hence by Theorem 1.8, $E(H)$ can be monitored by vertices in D', and so $\text{dem}(L(T)) \leq |D'| = |\,\text{Leaf}(T)| - 1$.

To show the upper bound, let D be the set of edges associated with leaves of T and the edge set $D' = D - e_0$, where e_0 is an edge associated with some leaf of T. For any bridge e of $L(T)$, it follows from Lemma 2.2 that e can be monitored by any vertex in D'. For any complete subgraph H with order $r \geq 3$ of $L(T)$, it implies that v is a cut vertex or $v \in D \cap V(H)$ for any $v \in V(H)$. If v is a cut vertex, then let C_1, C_2 be the two connected components of $L(T) \setminus v$, where $H \setminus v$ is a subgraph of C_1. For any vertex $w \in D' \cap V(C_2)$, we have $\text{EM}(w) \cap E(H) = \text{EM}(v) \cap E(H)$, and hence by Theorem 1.8, $E(H)$ can be monitored by the vertex in D', and so $\text{dem}(L(T)) \leq |D'| = |\,\text{Leaf}(T)| - 1$.

To show the sharpness of the bounds, we use the following example.

Example 2.4. Let G be a star $K_{1,n-1}$. From Theorem 1.8, we have $\text{dem}(L(G)) = n-2 = \Delta(G) - 1 = |\,\text{Leaf}(G)| - 1$. Therefore the bounds are sharp.

□

2.2. Special graphs

Let $Y_{a,b}$ denote the grid of dimension $a \times b$, that is, $Y_{a,b} = P_a \square P_b$ with $V(Y_{a,b}) = \{(i,j) \,|\, 1 \leq i \leq a,\ 1 \leq j \leq b\}$ and $E(Y_{a,b}) = \{(i,j)(\ell,k) \mid |i-\ell| = 1 \textit{ and } |j-k| = 0 \text{ or } |i-\ell| = 0 \textit{ and } |j-k| = 1, (i,j),(\ell,k) \in V(Y_{a,b})\}$; see [32].

Theorem 2.5 ([15]). *For any integers $a, b \geq 2$, we have $dem(Y_{a,b}) = \max\{a,b\}$.*

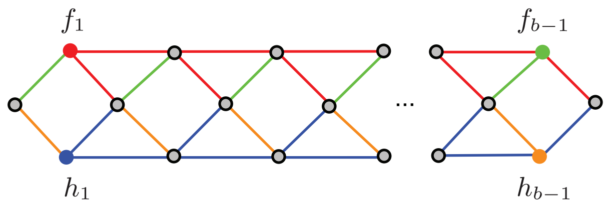

FIGURE 2. The line graph $L(Y(2, b))$: The red, green, blue and yellow edges represent the edges in the sets A, B, C and D, respectively.

For the line graph of $Y_{2,b}$, $b \geq 5$, let

$$V(L(Y_{2,b})) = \{f_i, g_j, h_k \mid 1 \leq i, k \leq b-1, 1 \leq j \leq b\}$$

and

$$E(L(Y_{2,b})) = \{f_i f_{i+1}, h_i h_{i+1} \mid 1 \leq i \leq b-2\} \cup \{f_{i-1} g_i, f_i g_i, h_{i-1} g_i, h_i g_i \mid 2 \leq i \leq b-1\}$$
$$\cup \{g_1 f_1, g_1 h_1, g_b f_{b-1}, g_b h_{b-1}\}.$$

Theorem 2.6. *For any integer $b \geq 5$, we have*

$$\mathrm{dem}(L(Y_{2,b})) = \begin{cases} 4 & \text{if } b \geq 6, \\ 3 & \text{if } b = 5. \end{cases}$$

Proof. Let $G = Y_{2,b}$ with $b \geq 6$, and let the edge set

$$A = \{f_i f_{i+1} \mid 1 \leq i \leq b-2\} \cup \{f_i g_{i+1} \mid 1 \leq i \leq b-1\}.$$

By Lemma 1.9, $A \subseteq \mathrm{EM}(f_1)$.

By the symmetry of $L(G)$, we let $B = \{f_i g_i \mid 1 \leq i \leq b-1\}$, $C = \{h_i h_{i+1} \mid 1 \leq i \leq b-2\} \cup \{h_i g_{i+1} \mid 1 \leq i \leq b-1\}$ and $D = \{h_i g_i \mid 1 \leq i \leq b-1\}$. Then $B \subseteq \mathrm{EM}(f_{b-1})$, $C \subseteq \mathrm{EM}(h_1)$ and $D \subseteq \mathrm{EM}(h_{b-1})$; see Figure 2. Since $A \cup B \cup C \cup D = E(L(G))$, it follows that $\mathrm{dem}(L(G)) \leq 4$.

To show $\mathrm{dem}(L(G)) \geq 4$, we suppose that $\mathrm{dem}(L(G)) \leq 3$. Let $E' = \{g_i h_{i-1}, g_i h_i \mid 2 \leq i \leq b-1\} \cup \{g_1 h_1, g_b h_{b-1}\}$, $E_i = \{g_i h_{i-1}, g_{i+1} h_{i+1}\}$ for $1 \leq i \leq b-1$, and $E'_j = \{g_i h_{i-1}, g_i h_i\}$ for $1 \leq j \leq b$. Note that the edges $g_1 h_0$ and $g_b h_b$ do not exist. Let the edge set $D_j = \{g_i h_{i-1} \mid j+1 \leq i \leq b\} \cup \{g_i h_i \mid 1 \leq i \leq j\}$, where $1 \leq j \leq b-1$. Now, we give the following claim.

Claim 1. For $b \geq 5$, we have $|\mathrm{EM}(f_i) \cap E'| \leq 2$, $|\mathrm{EM}(h_i) \cap E'| = b$ for $1 \leq i \leq b-1$ and $|\mathrm{EM}(g_i) \cap E'| \leq 2$ for $1 \leq i \leq b$.

Proof of Claim 1. By Definition 1.2, $\mathrm{EM}(f_1) \cap E' = \{g_2 h_2\}$, $\mathrm{EM}(f_{b-1}) \cap E' = \{g_{b-1} h_{b-2}\}$ and $\mathrm{EM}(f_i) \cap E' = E_i$ for $2 \leq i \leq b-2$, and hence $|\mathrm{EM}(f_i) \cap E'| \leq 2$, where $1 \leq i \leq b-1$. By Definition 1.2, $\mathrm{EM}(h_i) \cap E' = D_i$, and hence $|\mathrm{EM}(h_i) \cap E'| = |D_i| = (b-i) + i = b$, where $1 \leq i \leq b-1$. Similarly, we have $|\mathrm{EM}(g_i) \cap E'| \leq 2$ for $1 \leq i \leq b$. □

Let $M = \{x_1, x_2, x_3\}$ be a DEM set of $L(G)$. By symmetry, we just consider the following three cases; see Table 1.

Case 1. $x_1, x_2 \in \{f_i \,|\, 1 \leq i \leq b-1\}$.

TABLE 1. The three cases for the lower bound of Theorem 2.6.

Cases	$\lvert\{f_i \mid 1 \le i \le b-1\} \cap M\rvert$	$\lvert\{g_i \mid 1 \le i \le b\} \cap M\rvert$	$\lvert\{h_i \mid 1 \le i \le b-1\} \cap M\rvert$
Case 1	3	0	0
Case 1	2	1	0
Case 1	2	0	1
Case 2	1	2	0
Case 2	1	1	1
Case 2	1	0	2
Case 3	0	3	0

Suppose that $x_3 \in \{f_i \mid 1 \le i \le b-1\}$. For the edge h_1h_2 and each vertex $f_j \in \{f_i \mid 3 \le i \le b-1\}$, since $h_1 \in r_j(f_j)$ and h_2 is not the only neighbor of f_j in $r_{j-1}(f_j)$, it follows from Lemma 1.9 that the edge h_1h_2 cannot be monitored by f_j for $3 \le j \le b-1$. Since $d_{L(G)}(h_1, f_j) = d_{L(G)}(h_2, f_j)$ for $j = 1, 2$, it follows that h_1h_2 cannot be monitored by f_j, where $1 \le j \le 2$. Therefore, h_1h_2 cannot be monitored by f_j, where $1 \le j \le b-1$, and so h_1h_2 cannot be monitored by x_1, x_2, x_3, which contradicts to the fact that $\{x_1, x_2, x_3\}$ is a DEM set of $L(G)$.

Suppose that $x_3 \in \{g_i \mid 1 \le i \le b\}$. From Claim 1, $|\mathrm{EM}(x_i) \cap E'| \le 2$ for $i = 1, 2, 3$. Therefore, the set $\{x_1, x_2, x_3\}$ can monitor at most 6 edges of E', which contradicts to the fact that $|E'| \ge 10$ for $b \ge 6$.

Suppose that $x_3 \in \{h_i \mid 1 \le i \le b-1\}$. By Definition 1.2, $f_1g_2 \in \mathrm{EM}(f_1) \cap \mathrm{EM}(h_2)$ and $f_1g_2 \notin (\cup_{i=2}^{b-1}\mathrm{EM}(f_i)) \cup (\cup_{i=3}^{b-1}\mathrm{EM}(h_i)) \cup \mathrm{EM}(h_1)$, and $f_{b-1}g_{b-1} \in \mathrm{EM}(f_{b-1}) \cap \mathrm{EM}(h_{b-2})$ and $f_{b-1}g_{b-1} \notin (\cup_{i=1}^{b-2}\mathrm{EM}(f_i)) \cup (\cup_{i=1}^{b-3}\mathrm{EM}(h_i)) \cup \mathrm{EM}(h_{b-1})$. Since $|\{h_2, h_{b-2}\} \cap \{x_3\}| \le 1$, it follows that $|\{f_1, f_{b-1}\} \cap \{x_1, x_2, x_3\}| \ge 1$, otherwise $\{f_1g_2, f_{b-1}g_{b-1}\} \nsubseteq \cup_{i=1}^{3}\mathrm{EM}(x_i)$. Since $\mathrm{EM}(f_1) \cap E' = \{g_2h_2\}$, $\mathrm{EM}(f_{b-1}) \cap E' = \{g_{b-1}h_{b-2}\}$, it follows that $| \cup_{i=1}^{2} \mathrm{EM}(x_i) \cap E'| \le 3$. From Claim 1, $|\mathrm{EM}(x_3) \cap E'| = b$, and hence $| \cup_{i=1}^{3} \mathrm{EM}(x_i) \cap E'| \le b+3$, which contradicts the fact that $|E'| = 2(b-1) > b+3$ for $b \ge 6$.

Case 2. $x_1 \in \{f_i \mid 1 \le i \le b-1\}$ and $x_2, x_3 \in \{g_i \mid 1 \le i \le b\} \cup \{h_i \mid 1 \le i \le b-1\}$.

Suppose that $x_2, x_3 \in \{g_i \mid 1 \le i \le b\}$. From Claim 1, we have $|\mathrm{EM}(x_i) \cap E'| \le 2$ for $1 \le i \le 3$, and then x_1, x_2 and x_3 can monitor at most 6 edges of E', which contradicts the fact that $|E'| \ge 10$.

Suppose that $x_2 \in \{g_i \mid 1 \le i \le b\}$ and $x_3 \in \{h_i \mid 1 \le i \le b-1\}$. If $x_1 \in \{f_1, f_{b-1}\}$, then $|\mathrm{EM}(x_1) \cap E'| = 1$. From Claim 1, $|\mathrm{EM}(x_1) \cap E'| + |\mathrm{EM}(x_2) \cap E'| + |\mathrm{EM}(x_3) \cap E'| = 1 + 2 + b < 2(b-1)$, which contradicts the fact that $|E'| = 2(b-1)$ for $b \ge 6$. If $x_1 \notin \{f_1, f_{b-1}\}$, then it follows from Claim 1 that $|\mathrm{EM}(x_1) \cap E'| = 2$, and f_1g_2,$f_{b-1}g_{b-1} \notin \mathrm{EM}(x_1)$ from Definition 1.2. Moreover, $\{f_1g_2, f_{b-1}g_{b-1}\} \nsubseteq \mathrm{EM}(x_i)$, where $i = 2, 3$. Since $\{x_1, x_2, x_3\}$ is a DEM set of $L(G)$, it follows that $\{f_1g_2, f_{b-1}g_{b-1}\} \subseteq \mathrm{EM}(x_2) \cup \mathrm{EM}(x_3)$. Without loss of generality, let $f_1g_2 \in \mathrm{EM}(x_3)$ and $f_{b-1}g_{b-1} \in \mathrm{EM}(x_2)$, which implies that $x_3 = h_2$ and $x_2 = g_{b-1}$. Therefore, $\mathrm{EM}(x_3) \cap E' = \{g_ih_{i-1} \mid 3 \le i \le b\} \cup \{g_ih_i \mid 1 \le i \le 2\}$ and $\mathrm{EM}(x_2) \cap E' = \{g_{b-1}h_{b-2}, g_{b-1}h_{b-1}\}$. Clearly, $(\mathrm{EM}(x_3) \cap E') \cap (\mathrm{EM}(x_2) \cap E') = \{g_{b-1}h_{b-2}\}$. Therefore, $|\mathrm{EM}(x_1) \cap E'| + |(\mathrm{EM}(x_3) \cap E') \cup (\mathrm{EM}(x_2) \cap E')| = 2 + b + 1 < 2(b-1) = |E'|$ for $b \ge 6$, a contradiction.

Suppose that $x_2, x_3 \in \{h_i \mid 1 \le i \le b-1\}$. Similar to the case that $x_1, x_2 \in \{f_i \mid 1 \le i \le b-1\}$ and $x_3 \in \{h_i \mid 1 \le i \le b-1\}$, we can obtain that $\{x_1, x_2, x_3\}$ is not a DEM set of $L(G)$, a contradiction.

Case 3. $x_1, x_2, x_3 \in \{g_i \mid 1 \le i \le b\}$.

From Claim 1, we have $|(\cup_{i=1}^{3}\mathrm{EM}(x_i)) \cap E'| \le 6$, which is a contradiction as $|E'| \ge 10$. From the arguments given, we have $\mathrm{dem}(L(Y_{2,b})) \ge 4$, and hence $\mathrm{dem}(L(Y_{2,b})) = 4$ for $b \ge 6$. Then, we consider the case $b = 5$. Choose the three vertices f_3, g_2 and h_3 in $L(Y_{2,5})$. From Definition 1.2 and Lemma 1.9, we have $\mathrm{EM}(f_3) \cup \mathrm{EM}(g_2) \cup \mathrm{EM}(h_3) = E(L(Y_{2,5}))$, and so $\mathrm{dem}(L(Y_{2,5})) \le 3$. To show the lower bound, we suppose that $\mathrm{dem}(L(Y_{2,5})) \le 2$. Let $\{x_1, x_2\}$ be a DEM set of $L(Y_{2,5})$. From the symmetry of $L(Y_{2,5})$, we discuss the following scenarios.

If $x_1, x_2 \in \{f_i \,|\, 1 \le i \le 4\}$, then the edge $h_1h_2 \notin \mathrm{EM}(x_1) \cup \mathrm{EM}(x_2)$, which contradicts to the fact that $\{x_1, x_2\}$ is a DEM set of $L(Y_{2,5})$. If $x_1 \in \{f_i \,|\, 1 \le i \le 4\}$ and $x_2 \in \{h_i \,|\, 1 \le i \le 4\}$, then from Claim 1, $|(\mathrm{EM}(x_1) \cup \mathrm{EM}(x_2)) \cap E'| \le 2 + 5 = 7$, which contradicts to the fact that $|E'| = 8$. If $x_1 \in \{f_i \,|\, 1 \le i \le 4\}$ and $x_2 \in \{g_i \,|\, 1 \le i \le 5\}$ or $x_1, x_2 \in \{g_i \,|\, 1 \le i \le 5\}$, then from Claim 1, $|(\mathrm{EM}(x_1) \cup \mathrm{EM}(x_2)) \cap E'| \le 4$, which contradicts to the fact that $|E'| = 8$. Therefore, we have $\mathrm{dem}(L(Y_{2,5})) \ge 3$, and hence $\mathrm{dem}(L(Y_{2,5})) = 3$. □

The following corollary is immediate from Theorems 2.5 and 2.6.

Corollary 2.7. *For an integer $s > 0$, there exists a graph G with $\mathrm{dem}(G) = s$ and $\mathrm{dem}(L(G)) = 4$.*

Now, we give the DEM numbers of $L(P_n)$, $L(K_{1,n})$ and $L(K_{a,b})$ as follows.

Theorem 2.8 ([15])**.** *Let G be a connected graph with at least one edge. Then $\mathrm{dem}(G) = 1$ if and only if G is a tree.*

Corollary 2.9. *For $n \ge 3$, we have $\mathrm{dem}(L(P_n)) = 1$ and $\mathrm{dem}(L(K_{1,n})) = n - 1$.*

Proof. From Theorem 2.8, we have $\mathrm{dem}(L(P_n)) = 1$. Since $L(K_{1,n})$ is a complete graph of order n, it follows from Theorem 1.8 that $\mathrm{dem}(L(K_{1,n})) = n - 1$. □

Lemma 2.10 ([15])**.** *Let G be a graph and $x \in V(G)$. For any edge e incident with x, $e \in \mathrm{EM}(x)$.*

Theorem 2.11 ([15])**.** *For a complete bipartite $K_{a,b}$ with part sizes a and b, $\mathrm{dem}(K_{a,b}) = \min\{a, b\}$.*

Now, we give the result of the line graph of the complete bipartite graph as follows.

Proposition 2.12. *Let $K_{a,b}$ be the complete bipartite graph with part sizes a and b ($a \ge b \ge 2$). Then $\mathrm{dem}(L(K_{a,b})) = a(b-1)$.*

Proof. Let $\{v_1, v_2, \ldots, v_a\}$ and $\{u_1, u_2, \ldots, u_b\}$ be the two parts of $K_{a,b}$. Then $V(L(K_{a,b})) = \{v_iu_j \mid 1 \le i \le a, 1 \le j \le b\}$. Let $M = V(K_{a,b}) \setminus \{v_iu_t| \; 1 \le i \le a, t = i + b - 1 \bmod (b)\}$. From the construction of $L(K_{a,b})$, the set of edges adjacent with v_iu_j is $\{(v_iu_j)(v_pu_j), (v_iu_j)(v_iu_q) \mid 1 \le i \ne p \le a, 1 \le j \ne q \le b\}$, it follows from Lemma 2.10 that $\{(v_iu_j)(v_pu_j), (v_iu_j)(v_iu_q) \mid 1 \le i \ne p \le a, 1 \le j \ne q \le b\} \subseteq \mathrm{EM}(v_iu_j)$. Therefore, $\cup_{v \in M} \mathrm{EM}(v) = E(L(K_{a,b}))$, and so $\mathrm{dem}(L(K_{a,b})) \le |M| = a(b-1)$.

Next, it suffices to show that $\mathrm{dem}(L(K_{a,b})) \ge a(b-1)$. Arbitrarily choose a vertex set $Q \subseteq V(L(K_{a,b}))$ with $|Q| = a(b-1) - 1$ as DEM set of $L(K_{a,b})$. Let the graph induced by the vertices in $\{v_iu_j \mid 1 \le j \le b\}$ be K^i, where $1 \le i \le a$. Note that K^i is a complete subgraph with order b. By the Pigeonhole Principle, there exists a complete subgraph K^r such that $|V(K^r) \cap Q| < b - 1$, where $1 \le r \le a$, then there is an edge $xy \in E(K^r)$ such that $x, y \notin Q$. Since $d(w, x) = d(w, y)$ for every vertex $w \in (V(K^r) \backslash \{x, y\}) \cap Q$, it follows from Definition 1.2 that $xy \notin \mathrm{EM}(w)$. For each vertex $z \in (V(L(K_{a,b})) \backslash V(K^r)) \cap Q$, by Lemma 1.9, we have $xy \notin \mathrm{EM}(z)$. Therefore, $\mathrm{dem}(L(K_{a,b})) \ge a(b-1)$. □

3. Results for graphs and their line graphs

In this section, we give DEM numbers of friendship graphs, the constructed graph H, wheel graphs and their line graphs.

The *friendship graph* F_n consists of n-triangles with a common vertex o; see [30]. This common vertex is called *center vertex*. We use v_i and u_i to denote the neighbors of o in the i-th triangle of F_n for $1 \le i \le n$; see Figure 3.

Proposition 3.1. *For the friendship graph F_n, we have $\mathrm{dem}(F_n) = n$.*

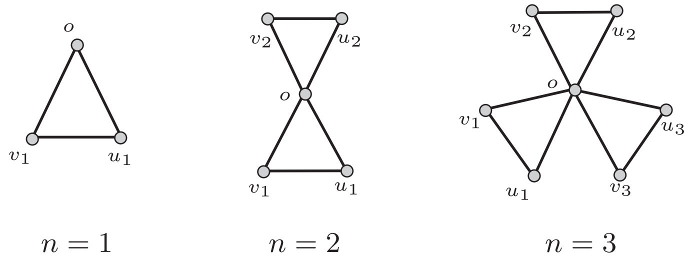

FIGURE 3. The friendship graphs with $n = 1, 2, 3$.

Proof. Let the set $M = \{v_i \,|\, 1 \leq i \leq n\}$. From Definition 1.2, $\mathrm{EM}(v_i) = \{v_iu_i, v_io, v_jo, u_jo \mid 1 \leq i, j \leq n, i \neq j\}$. Then $E(F_n) = \cup_{i=1}^{n}\mathrm{EM}(v_i)$ and hence $\mathrm{dem}(F_n) \leq n$. To show $\mathrm{dem}(F_n) \geq n$, observe that for any DEM set M of F_n with $|M| \leq n-1$, then there exist $v_j, u_j \in V(F_n) \setminus o$ but $v_j, u_j \notin M$, where $1 \leq j \leq n$. Since $d_{F_n}(v_j, x) = d_{F_n}(u_j, x)$ for any vertex $x \in V(F_n)\backslash\{v_j, u_j\}$, it follows that the edge v_ju_j cannot be monitored by M, and so $\mathrm{dem}(F_n) \geq n$. □

Proposition 3.2. *For the friendship graph F_n, we have* $\mathrm{dem}(L(F_n)) = 2n$.

Proof. Let $M = \{v_io, u_io \mid 1 \leq i \leq n\}$. Note that the line graph of the graph induced by the vertices in $\{v_io, u_io \mid 1 \leq i \leq n\}$ is a complete subgraph K with order $2n$. From Theorem 1.8, we have $E(K) \subseteq \{(v_io)(u_jo) \mid 1 \leq i, j \leq n\} \cup \{(v_io)(v_jo), (u_io)(u_jo) \mid 1 \leq i, j \leq n, i \neq j\}$. Moreover, the edges $(v_iu_i)(v_io)$ and $(v_iu_i)(u_io)$ are incident with the vertices v_io and u_io, respectively. By Lemma 2.10, we have $(v_iu_i)(v_io) \in \mathrm{EM}(v_io)$ and $(v_iu_i)(u_io) \in \mathrm{EM}(u_io)$ for each $1 \leq i \leq n$. Thus we have $\cup_{i=1}^{n}(\mathrm{EM}(v_io) \cup \mathrm{EM}(u_io)) = E(F_n)$, and hence $\mathrm{dem}(L(F_n)) \leq |M| = 2n$.

To show $\mathrm{dem}(L(F_n)) \geq 2n$, suppose that the set Q with $2n-1$ vertices is a DEM set of $L(F_n)$. Let $Q' = Q \cap \{v_io, u_io \mid 1 \leq i \leq n\}$. Suppose that $|Q'| = 2n-1$. Without loss of generality, let $Q = \{v_io, u_io \mid 1 \leq i \leq n-1\} \cup \{v_no\}$. Therefore, the edge $(u_no)(u_nv_n)$ cannot be monitored by the vertex in Q. Suppose that $|Q'| < 2n-1$. For each edge e of the complete subgraph K, by Definition 1.2, e cannot be monitored by any vertex $x \in V(L(F_n)) \setminus \{v_io, u_io \mid 1 \leq i \leq n\}$. Without loss of generality, let $Q' \subseteq \{v_io, u_io \mid 1 \leq i \leq n-1\}$. Then there exists an edge $(v_no)(u_no)$ that cannot be monitored by Q in $L(F_n)$, a contradiction. □

The following corollary is immediate from Propositions 3.1 and 3.2.

Corollary 3.3. *For an integer s with $s > 1$, there exists a graph $G = F_s$ with* $\mathrm{dem}(G) = s$ *and* $\mathrm{dem}(L(G)) = 2s$.

Let F_s be the friendship graph with vertex set $V(F_s) = \{v_i, u_i| \ 1 \leq i \leq s\} \cup \{o\}$ and edge set $E(F_s) = \{v_iu_i| \ 1 \leq i \leq s\} \cup \{ov_i, ou_i| \ 1 \leq i \leq s\}$. Let P_m be a path with a vertex set $V(P_{m_1}) = \{x_i \mid 1 \leq i \leq m_1\}$ and edge set $E(P_{m_1}) = \{x_ix_{i+1} \mid 1 \leq i \leq m_1-1\}$. Let K_{1,m_2} be a path with vertex set $V(K_{1,m_2}) = \{y_i \mid 1 \leq i \leq m_2\} \cup \{y_0\}$ and edge set $E(K_{1,m_2}) = \{y_0y_i \mid 1 \leq i \leq m_2\}$.

Let H_{s,m_1,m_2} be a graph, consisting of the union of F_s, P_{m_1} and K_{1,m_2}, with $V(H_{s,m_1,m_2}) = V(F_s) \cup V(P_{m_1}) \cup V(K_{1,m_2})$ and $E(H_{s,m_1,m_2}) = E(F_s) \cup E(P_{m_1}) \cup E(K_{1,m_2})$, where $V(F_s) \cap V(P_{m_1}) = \{x_1\} = \{v_1\}$, $V(F_s) \cap V(K_{1,m_2}) = \{y_1\} = \{u_1\}$, and $V(P_{m_1}) \cap V(K_{1,m_2}) = \emptyset$; see Figure 4.

Proposition 3.4. *For the integers $s, m_1 \geq 2$ and $m_2 \geq 3$, we have* $\mathrm{dem}(H_{s,m_1,m_2}) = s$.

Proof. Let $H = H_{s,m_1,m_2}$ and a vertex set $M = \{v_i \mid 1 \leq i \leq s\}$. For each $1 \leq i \neq j \leq s$, we get $\mathrm{EM}(v_i) = \{v_iu_i, v_io, v_jo, u_jo\} \cup \{x_kx_{k+1} \mid 1 \leq k \leq m_1-1\} \cup \{y_ty_0 \mid 1 \leq t \leq m_2\}$ by Definition 1.2. Then we have $\cup_{i=1}^{s}\mathrm{EM}(v_i) = E(H)$ and hence $\mathrm{dem}(H) \leq s$. To show that $\mathrm{dem}(H) \geq s$, suppose that Q is a DEM set of H with $|Q| = s-1$, then there exist two vertices $v_j, u_j \in (V(F_n) \setminus o) \setminus Q$. Suppose $j \neq 1$. Since $d_H(v_j, x) = d_H(u_j, x)$

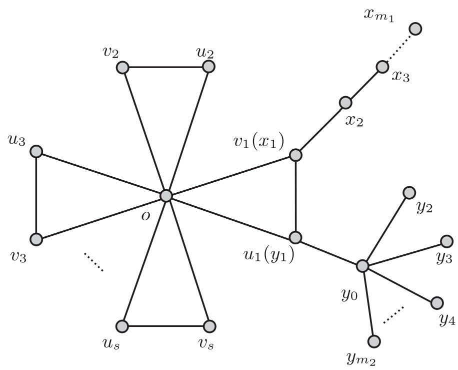

FIGURE 4. The constructed graph H_{s,m_1,m_2}.

for any vertex $x \in V(H) \setminus \{v_j, u_j\}$, it follows that the edge v_ju_j cannot be monitored by the vertex in Q. Suppose $j = 1$. If $|V(F_s) \cap Q| = s-1$, then the edge v_1u_1 cannot be monitored by the vertex in Q. If $|V(F_s) \cap Q| < s-1$, then there exist two vertices $v_t, u_t \notin Q$ for $1 < t \leq s$. Since $d_H(v_t, u) = d_H(u_t, u)$ for any vertex $u \in V(H) \setminus \{v_t, u_t\}$, where $2 \leq t \leq s$, it follows that the edge v_tu_t cannot be monitored by the vertex in Q, and hence $\mathrm{dem}(H) \geq s$. □

Theorem 3.5. *For the integers $s \geq 2$ and $m_1, m_2 \geq 3$, we have* $\mathrm{dem}(L(H_{s,m_1,m_2})) = 2s + m_2 - 2$.

Proof. Let $H = H_{s,m_1,m_2}$. To show the upper bound, we choose the vertex set $M = \{v_io, u_io \mid 2 \leq i \leq s\} \cup \{y_ky_0 \mid 2 \leq k \leq m_2 - 1\} \cup \{u_1o, v_1u_1\}$. Since the graph induced by the vertex set $\{v_io, u_io \mid 1 \leq i \leq s\}$ is a complete subgraph, denoted by G_1, we may apply Theorem 1.8 and Lemma 2.10 to conclude that the edges in $E(G_1 \cup \{(v_io)(v_iu_i), (u_io)(v_iu_i) \mid 2 \leq i \leq s\} \cup \{(u_1o)(v_1u_1), (u_1o)(y_1y_0)\}$ can be monitored by the vertices in $\{v_io, u_io \mid 2 \leq i \leq s\} \cup \{u_1o\}$, and the edges in $\{(x_ix_{i+1})(x_{i+1}x_{i+2}) \mid 1 \leq i \leq m_1 - 2\} \cup \{v_1o, x_1x_2\}$ can be monitored by the vertices in $\{v_io \,|\, 2 \leq i \leq s\}$. Also, from Lemmas 2.10 and 1.9, the edges in $\{(v_1o)(v_1u_1), (x_1x_2)(v_1u_1), (y_1y_0)(v_1u_1)\} \cup \{(y_iy_0)(y_1y_0) \mid 2 \leq i \leq m_2\}$ can be monitored by the vertex v_1u_1. Since the graph induced by the vertices in $\{y_iy_0 \mid 1 \leq i \leq m_2\}$ is a complete subgraph, denoted by G_2, it follows from Theorem 1.8 that $E(G_2) \setminus \{(y_iy_0)(y_1y_0) \mid 2 \leq i \leq m_2\}$ can be monitored by the vertices $\{y_iy_0 \,|\, 2 \leq i \leq m_2 - 1\}$. Note that $E(G_2) \setminus \{(y_iy_0)(y_1y_0) \,|\, 2 \leq i \leq m_2\} = E(G_2 - y_1y_0)$. Therefore, $\cup_{i=2}^{s}(\mathrm{EM}(v_io) \cup \mathrm{EM}(u_io)) \cup (\cup_{k=2}^{m_2-1}\mathrm{EM}(y_ky_0)) \cup \mathrm{EM}(u_1o) \cup \mathrm{EM}(v_1u_1) = E(L(H))$, and so $\mathrm{dem}(L(H)) \leq 2s + m_2 - 2$.

Now, we show this number is also a lower bound of $\mathrm{dem}(L(H))$. Choose any $2s + m_2 - 3$ vertices as a DEM set Q. Then we have

Claim 2. $|V(G_2) \cap Q| \geq m_2 - 2$.

Proof of Claim 2. Suppose that $|V(G_2) \cap Q| \leq m_2 - 3$. Let $t_1, t_2, t_3 \in V(G_2)$ but $t_1, t_2, t_3 \notin Q$. Then there are at least two vertices t_1 and t_2 such that $y_0y_1 \notin \{t_1, t_2\}$. Hence, by Definition 1.2, the edge t_1t_2 cannot be monitored by the vertex in Q. □

Let H_1 be the line graph of the graph induced by the vertices in $\{v_io, u_io, u_iv_i | \ 2 \leq i \leq s\}$.

Claim 3. $|V(H_1) \cap Q| \geq 2s - 2$.

Proof of Claim 3. Let K be the graph induced by the vertices in $\{v_io, u_io | \ 2 \leq i \leq s\}$. For each $2 \leq i \leq s$, from Definition 1.2, the edges in $E(K)$ cannot be monitored by the vertex in $V(H) \setminus V(K)$, and the edges $(v_io)(v_iu_i)$ and $(u_io)(v_iu_i)$ are monitored by its endpoints. Suppose that $|V(K) \cap Q| = 2s - 3$. Without loss of generality, let $V(K) \cap Q = \{v_io, u_io \mid 3 \leq i \leq s\} \cup \{v_2o\}$, and hence the edge $(u_2o)(v_2u_2)$ cannot be monitored by the vertex in Q. Suppose that $|V(K) \cap Q| < 2s - 3$. Since the edges in $E(K)$ cannot be monitored by the vertex in $V(H) \setminus V(K)$, it follows from Theorem 1.8 that $|Q \cap V(K)| \geq 2s - 3$, which contradicts the fact that $|V(K) \cap Q| < 2s - 3$. □

Therefore, $|(V(H) \setminus (V(H_1) \cup V(G_2))) \cap Q| \leq 1$. If $|(V(H) \setminus (V(H_1) \cup V(K_{m_2}))) \cap Q| = 0$, then, by Definition 1.2 and Lemma 1.9, the edge $(u_1 o)(v_1 o)$ cannot be monitored by the vertex in Q, a contradiction. Otherwise, $|(V(H) \setminus (V(H_1) \cup V(K_{m_2}))) \cap Q| = 1$. If $v_1 o$ or $u_1 o \in Q$, then the edge $(u_1 o)(u_1 v_1)$ or $(v_1 o)(u_1 v_1)$ cannot be monitored by the vertex in Q, respectively. If $v_1 o$ and $u_1 o \notin Q$, the edge $(u_1 o)(v_1 o)$ cannot be monitored by the vertex in Q. Therefore, $\text{dem}(L(H)) \geq 2s + m_2 - 2$. □

From Theorem 3.5 and Proposition 3.4, the following corollary holds.

Corollary 3.6. *For any two integers $s > 1$ and $m > 3$, there exists a graph G with $\text{dem}(G) = s$ and $\text{dem}(L(G)) = 2s + m - 2$.*

The *wheel graph* is defined as $W_{n+1} = C_n \vee K_1$ (K_1 is adjacent to every vertex on the cycle C_n), where $V(W_{n+1}) = \{v_i \mid 1 \leq i \leq n\} \cup \{u\}$, $E(W_{n+1}) = \{v_i v_{i+1} \mid 1 \leq i \leq n-1\} \cup \{uv_i \mid 1 \leq i \leq n\} \cup \{v_1 v_n\}$, $C_n = v_1 v_2 \cdots v_n v_1$ and $K_1 = u$. We call u the centre. Then the following results hold.

Proposition 3.7. $\text{dem}(W_4) = \text{dem}(W_5) = 3$.

Proof. Since W_4 is a complete graph K_4 with order 4, it follows from Theorem 1.8 that $\text{dem}(W_4) = 3$. Then, we prove that $\text{dem}(W_5) = 3$. Choose the three vertices u, v_1 and v_3, where u is the centre. From Lemma 2.10, we have $\{v_1 v_2, v_1 v_4\} \subseteq \text{EM}(v_1)$, $\{v_3 v_2, v_3 v_4\} \subseteq \text{EM}(v_3)$ and $\{uv_i \mid 1 \leq i \leq 4\} \subseteq \text{EM}(u)$, and hence $\text{EM}(v_1) \cup \text{EM}(v_3) \cup \text{EM}(u) = E(W_5)$, and so $\text{dem}(W_5) \leq 3$. To show the lower bound, we suppose $|M| = 2$. Let $M = \{x, y\}$ be a DEM set of W_5. If $x, y \neq u$, then there exists a vertex $z \in V(W_5) \setminus \{x, y, u\}$ such that the edge zu cannot be monitored by x and y. Otherwise, without loss of generality, let $x = u$ and $y \neq u$. Then there exist two vertices z_1 and z_2 such that $z_1 z_2 \in E(W_5 \setminus u)$ and $z_1 z_2$ cannot be monitored by the vertex in M. Therefore, $\text{dem}(W_5) \geq 3$. □

Proposition 3.8. $\text{dem}(W_6) = 4$.

Proof. Let $M = \{u, v_1, v_3, v_4\}$, where u is the centre. From Lemma 2.10, we have $\{v_1 v_2, v_1 v_5\} \subseteq \text{EM}(v_1)$, $\{v_3 v_2, v_3 v_4\} \subseteq \text{EM}(v_3)$, $\{v_4 v_5\} \subseteq \text{EM}(v_4)$ and $\{v_i u \mid 1 \leq i \leq 5\} \subseteq \text{EM}(u)$, and so $\text{EM}(v_1) \cup \text{EM}(v_3) \cup \text{EM}(v_4) \cup \{v_i u \mid 1 \leq i \leq 5\} = E(W_6)$. Therefore, $\text{dem}(W_6) \leq 4$.

To prove that $\text{dem}(W_6) \geq 4$, suppose that Q is a DEM set of W_6 with $|Q| = 3$. If $u \notin Q$, then it follows from Definition 1.2 and Lemma 1.9 that there exists a vertex $v \in (V(W_6) \setminus u) \setminus Q$ such that uv cannot be monitored by the vertex in Q, a contradiction. If $u \in Q$, then let $Q = \{u, v_i, v_j\}$, where $v_i, v_j \in V(W_6) \setminus u$. Let F be the graph induced by the vertices in $V(W_6) \setminus Q$. For any $e \in E(F)$, by Definition 1.2 and Lemma 1.9, the edge e cannot be monitored by the vertex in Q, a contradiction. □

From Definition 1.2 and Lemma 1.9, we can easily obtain the following Lemma 3.9.

Lemma 3.9. *For the wheel graph W_{n+1} with the vertex set $V(W_{n+1}) = \{v_i \mid 1 \leq i \leq n\} \cup \{u\}$ and the edge set $E(W_{n+1}) = \{v_i v_{i+1} \mid 1 \leq i \leq n-1\} \cup \{uv_i \mid 1 \leq i \leq n\} \cup \{v_1 v_n\}$, the edge $v_i v_{i+1}$ can only be monitored by its two end-vertices, where $v_n v_{n+1} = v_n v_1$ and $1 \leq i \leq n$.*

Theorem 3.10. *For the wheel W_{n+1}, we have*

$$\text{dem}(W_{n+1}) = \begin{cases} 3 & \text{if } n = 3 \text{ or } 4, \\ 4 & \text{if } n = 5, \\ \lceil \frac{n}{2} \rceil & \text{if } n \geq 6. \end{cases}$$

Proof. By Propositions 3.7 and 3.8, the results hold for $3 \leq n \leq 5$. Now we just need to prove the case of $n \geq 6$.

To show the upper bound, we choose a vertex set $M = \{v_i \mid 1 \leq i \leq n, i \equiv 1 \bmod 2\}$ in W_{n+1}. If n is even, then let $E_i = \{v_{i-1}v_i, v_{i+1}v_i, uv_i, uv_{i+\frac{n}{2}}\}$; otherwise, let $E_i = \{v_{i-1}v_i, v_{i+1}v_i, uv_i, uv_{i+\frac{n-1}{2}}, uv_{i+\frac{n+1}{2}}\}$ for each i $(1 \leq i \leq n, i \equiv 1 \bmod 2)$, where the subscripts of v_i are in the sense of modulo n. By Lemmas 2.10 and 1.9, we have $E_i \subseteq \mathrm{EM}(v_i)$ for $1 \leq i \leq n$ and $i \equiv 1 \bmod 2$. Since $\cup_{i\in\{1\leq i\leq n, i\equiv 1 \bmod 2\}} E_i = E(W_{n+1})$, it follows that $\mathrm{dem}(W_{n+1}) \leq \lceil \frac{n}{2} \rceil$.

To prove that $\mathrm{dem}(W_{n+1}) \geq \lceil \frac{n}{2} \rceil$, suppose that Q is a DEM set of W_{n+1} with $|Q| = \lceil \frac{n}{2} \rceil - 1$. From the construction of W_{n+1} and Lemma 3.9, $|(\cup_{x\in Q}\mathrm{EM}(x)) \cap E(C_n)| \leq n-2$ if n is even; $|(\cup_{x\in Q}\mathrm{EM}(x)) \cap E(C_n)| \leq n-1$ if n is odd, which contradicts to the fact that $|E(W_{n+1} \setminus u)| = n$. Therefore, $\mathrm{dem}(W_{n+1}) = \lceil \frac{n}{2} \rceil$. ☐

For the line graph $L(W_{n+1})$ of wheel graph W_{n+1}, let $V(L(W_{n+1})) = \{v_iu, v_iv_{i+1} \mid 1 \leq i \leq n\}$ and $E(L(W_{n+1})) = E(K'_n) \cup E(C'_n) \cup \{(v_iu)(v_iv_{i+1}), (v_iu)(v_{i-1}v_i) \mid 1 \leq i \leq n\}$, where the graph induced by the vertices in $\{v_iu \mid 1 \leq i \leq n\}$ is a complete subgraph K'_n and the graph induced by the vertices in $\{v_iv_{i+1} \mid 1 \leq i \leq n\}$ is a cycle C'_n. For convenience, let $v_{n+1} = v_1$ and $v_0 = v_n$, where the subscripts of v_i are in the sense of modulo n.

Next, we will give some lemmas of the line graph of the wheel graph.

Lemma 3.11. *For the line graph $L(W_{n+1})$ of the wheel graph W_{n+1}, an edge $(v_iv_{i+1})(v_{i+1}v_{i+2})$ cannot be monitored by v_ju, $1 \leq i, j \leq n$.*

Proof. Let $G = L(W_{n+1})$. For an edge $(v_iv_{i+1})(v_{i+1}v_{i+2})$, where $1 \leq i \leq n$, if $j \neq i, i+2$, then we have $d_G(v_ju, v_iv_{i+1}) = d_G(v_ju, v_{i+1}v_{i+2})$, and hence v_ju cannot monitor the edge $(v_iv_{i+1})(v_{i+1}v_{i+2})$. If $j = i$, then we can obtain that an edge $(v_iv_{i+1})(v_{i+1}v_{i+2})$ does not belong to all shortest paths between v_iu and $v_{i+1}v_{i+2}$, such as $P = (v_iu)(v_{i+1}u)(v_{i+1}v_{i+2})$, which is a contradiction, and hence $(v_iv_{i+1})(v_{i+1}v_{i+2}) \notin \mathrm{EM}(v_iu)$. Similarly, we have $(v_iv_{i+1})(v_{i+1}v_{i+2}) \notin \mathrm{EM}(v_{i+2}u)$ if $j = i+2$. ☐

From Definition 1.2 and Lemma 1.9, the following lemma is immediate.

Lemma 3.12. *For the line graph $L(W_{n+1})$ of the wheel graph W_{n+1}, the edge $(v_iu)(v_ju) \notin \mathrm{EM}(v_kv_{k+1})$ for $1 \leq i, j, k \leq n$, $i \neq j$.*

Lemma 3.13. *For the line graph $L(W_{n+1})$ of the wheel graph W_{n+1}, the edges $(v_iu)(v_iv_{i+1})$ and $(v_iu)(v_iv_{i-1})$ can only be monitored by its two endpoints, where $1 \leq i \leq n$.*

Proof. By Lemma 2.10, we have $(v_iu)(v_iv_{i+1}) \in \mathrm{EM}(v_iu)$ or $\mathrm{EM}(v_iv_{i+1})$, and $(v_iu)(v_iv_{i-1}) \in \mathrm{EM}(v_iu)$ or $\mathrm{EM}(v_iv_{i-1})$. From Definition 1.2 and Lemma 1.9, $(v_iu)(v_iv_{i+1}), \notin \mathrm{EM}(v)$ for any $v \in V(L(W_{n+1}))\setminus\{v_iu, v_iv_{i+1}\}$, where $1 \leq i \leq n$. Therefore, $(v_iu)(v_iv_{i+1})$ can only be monitored by its two endpoints. Similarly, $(v_iu)(v_iv_{i-1})$ can only be monitored by its two endpoints. ☐

Proposition 3.14. $\mathrm{dem}(L(W_4)) = 4$.

Proof. Let $M = \{v_1u, v_2u, v_1v_3, v_2v_3\} \subseteq V(L(W_4))$. From Lemma 2.10, we have $\{(v_1v_2)(v_1u), (v_2u)(v_1u), (v_3u)(v_1u), (v_1v_3)(v_1u)\} \subseteq \mathrm{EM}(v_1u)$, $\{(v_1v_2)(v_2u), (v_2u)(v_3u), (v_2v_3)(v_2u)\} \subseteq \mathrm{EM}(v_2u)$, $\{(v_1v_2)(v_1v_3), (v_1v_3)(v_3u), (v_2v_3)(v_1v_3)\} \subseteq \mathrm{EM}(v_1v_3)$ and $\{(v_1v_2)(v_2v_3), (v_2v_3)(v_3u)\} \subseteq \mathrm{EM}(v_2v_3)$. Therefore, $\mathrm{EM}(v_1u) \cup \mathrm{EM}(v_2u) \cup \mathrm{EM}(v_1v_3) \cup \mathrm{EM}(v_2v_3) = E(L(W_4))$, and so $\mathrm{dem}(L(W_4)) \leq 4$.

To show that $\mathrm{dem}(L(W_4)) \geq 4$, let Q be a DEM set of $L(W_4)$ with $|Q| = 3$. If $|\{v_iu \mid 1 \leq i \leq 3\} \cap Q| = 1$, says v_1u, then it follows from Lemma 3.12 and Theorem 1.8 that the edge $(v_2u)(v_3u)$ cannot be monitored by the vertex in Q, which is a contradiction. If $|\{v_iu \mid 1 \leq i \leq 3\} \cap Q| = 2$, say v_1u and v_2u belong to this intersection, then at least one of the two edges $(v_3u)(v_1v_3)$ and $(v_3u)(v_2v_3)$ cannot be monitored by the vertex in Q from Lemma 3.13. If $|\{v_iu \mid 1 \leq i \leq 3\} \cap Q| = 3$, then the edge $(v_1v_2)(v_2v_3)$ cannot be monitored by the vertex in Q by Lemma 3.11. If $|\{v_iu \mid 1 \leq i \leq 3\} \cap Q| = 0$, then $Q = \{v_iv_{i+1} \mid 1 \leq i \leq 3\}$, and hence the edge $(v_1u)(v_2u)$ cannot be monitored by the vertex in Q by Lemma 3.12. Therefore, $\mathrm{dem}(L(W_4)) \geq 4$. ☐

Proposition 3.15. $\text{dem}(L(W_6)) = 6$.

Proof. Let $M = \{v_2u, v_3u, v_4u, v_5u, v_1v_2, v_1v_5\} \subseteq V(L(W_6))$. By Lemma 2.10, we know that $\{(v_iu)(v_ju) \mid 1 \leq i \leq 5, i \neq j\} \cup \{(v_ju)(v_{j-1+n}v_j), (v_ju)(v_jv_{j+1})\} \subseteq \text{EM}(v_ju)$ for any $2 \leq j \leq 5$, $\{(v_1v_2)(v_1v_5), (v_1v_2)(v_2v_3), (v_2v_3)(v_3v_4), (v_1v_2)(v_1u)\} \subseteq \text{EM}(v_1v_2)$ and $\{(v_3v_4)(v_4v_5), (v_4v_5)(v_1v_5), (v_1v_5)(v_1u)\} \subseteq \text{EM}(v_1v_5)$. Therefore, $\cup_{x\in M}\text{EM}(x) = E(L(W_6))$, and so $\text{dem}(L(W_6)) \leq 6$.

To show that $\text{dem}(L(W_6)) \geq 6$, let Q be a DEM set of $L(W_6)$ with $|Q| = 5$. If $|\{v_iu \mid 1 \leq i \leq 5\} \cap Q| = 5$, then it follows from Lemma 3.11 that the edges of C'_n cannot be monitored by the vertex in Q. If $|\{v_iu \mid 1 \leq i \leq 5\} \cap Q| = 4$ and $|\{v_iv_{i+1} \mid 1 \leq i \leq 5\} \cap Q| = 1$, then let $\{v_iv_{i+1} \mid 1 \leq i \leq 5\} \cap Q = \{v_1v_2\}$, without loss of generality, and hence the edge $(v_3v_4)(v_4v_5)$ cannot be monitored by the vertex in Q. If $|\{v_iu \mid 1 \leq i \leq 5\} \cap Q| < 4$, then it follows from Lemma 3.12 and Theorem 1.8 that there exists an edge xy which cannot be monitored by the vertex in Q, where $x, y \in \{v_iu \mid 1 \leq i \leq 5\} \setminus Q = \emptyset$, a contradiction. Therefore, $\text{dem}(L(W_6)) \geq 6$. ☐

Proposition 3.16. $\text{dem}(L(W_{n+1})) = n + 2$ *for* $n = 4$ *or* $6 \leq n \leq 8$.

Proof. Let the vertex set $M = \{v_iu \mid 1 \leq i \leq n\} \cup \{v_1v_2, v_{1+t}v_{2+t}\}$, where $t = \lceil n/2 \rceil$. Let $E_i = \{(v_iu)(v_iu) \mid 1 \leq i \leq j, i \neq j\} \cup \{(v_iu)(v_iv_{i+1}), (v_iu)(v_iv_{i-1+n})\}$, which is the set of edges adjacent with v_iu for each i, $1 \leq i \leq n$. By Lemma 2.10, we have $E_i \subseteq \text{EM}(v_iu)$. It can be seen from the construction of $L(W_{n+1})$ that the graph induced by the vertices in $\{v_iv_{i+1} \mid 1 \leq i \leq n\}$ is an n-cycle C'_n. Let $R_1 = \{(v_iv_{i+1})(v_{i+1}v_{i+2}) \mid 1 \leq i \leq k, k+t+1 \leq i \leq n\}$ and $R_2 = \{(v_iv_{1+i})(v_{1+i}v_{2+i}) \mid t \leq i \leq t+k\}$, where $k = \lceil n/4 \rceil$ and the subscripts of v_i are in the sense of modulo n. By Definition 1.2 and Lemma 1.9, we have $R_1 \subseteq \text{EM}(v_1v_2)$ and $R_2 \subseteq \text{EM}(v_{t+1}v_{t+2})$, where $t = \lceil n/2 \rceil$. Therefore, $(\cup_{i=1}^{n} E_i) \cup R_1 \cup R_2 = E(L(W_{n+1}))$, and so $\text{dem}(L(W_{n+1})) \leq n + 2$.

To show that $\text{dem}(L(W_{n+1})) \geq n+2$, let Q be a DEM set of $L(W_{n+1})$ with $|Q| = n+1$. Take $U_1 = \{v_iu \mid 1 \leq i \leq n\}$ and $U_2 = \{v_iv_{i+1} \mid 1 \leq i \leq n\}$. If $|Q \cap U_1| < n - 1$, then there exists an edge xy such that $x, y \in U_1$ but $\{x, y\} \cap Q = \emptyset$, by Lemma 3.12 and Theorem 1.8, and hence the edge xy cannot be monitored by Q, which is a contradiction.

If $|U_1 \cap Q| = n-1$ and $|U_2 \cap Q| = 2$, then there is a vertex $x \in U_1 \setminus Q$. Without loss of generality, let $x = v_1u$. From Lemma 3.13, the two edges $(v_1u)(v_1v_2)$ and $(v_1u)(v_nv_1)$ cannot be monitored by the vertex in $U_1 \cap Q$ and $(v_1u)(v_1v_2)$ and $(v_1u)(v_nv_1)$ can only be monitored by the vertices v_1v_2 and v_nv_1 in C'_n, respectively. Then, we claim that $U_2 \cap Q = \{v_1v_2, v_nv_1\}$. Otherwise one of $(v_1u)(v_1v_2)$ and $(v_1u)(v_nv_1)$ cannot be monitored by the vertex in Q. From Definition 1.2 and Lemma 1.9, the edge $(v_tv_{t+1})(v_{t+1}v_{t+2})$ cannot be monitored by the vertex in U_1, which is a contradiction. Note that $t = \lceil n/2 \rceil$.

If $|U_1 \cap Q| = n$ and $|U_2 \cap Q| = 1$, then, without loss of generality, let $U_2 \cap Q = \{v_1v_2\}$. By Lemma 3.11, $\{(v_iv_{i+1})(v_{i+1}v_{i+2}) \mid 1 \leq i \leq n\} \cap (\cup_{i=1}^{n}\text{EM}(v_iu)) = \emptyset$. By Definition 1.2 and Lemma 1.9, the edge $(v_tv_{t+1})(v_{t+1}v_{t+2})$ cannot be monitored by the vertex in Q. Therefore, $\text{dem}(L(W_{n+1})) \geq n + 2$. ☐

From Definition 1.2 and Lemma 1.9, we have the following lemma immediately.

Lemma 3.17. *For each vertex* v_iv_{i+1}, $1 \leq i \leq n$ *and* $n \geq 9$, *we have* $\text{EM}(v_iv_{i+1}) \cap E(C'_n) = \{(v_iv_{1+i})(v_{i+1}v_{2+i}), (v_{i+1}v_{2+i})(v_{2+i}v_{3+i}), (v_{i-1+n}v_i)(v_{i-2+n}v_{i-1}), (v_{i-2+n}v_{i-1+n})(v_{i-3+n}v_{i-2+n})\}$, *where the subscripts of* v_i *are in the sense of modulo* n.

Theorem 3.18. *For the wheel graph* W_{n+1} *with* $n \geq 3$,

$$\text{dem}(L(W_{n+1})) = \begin{cases} n+1 & \text{if } n = 3 \text{ or } 5, \\ n+2 & \text{if } n = 4 \text{ or } 6 \leq n \leq 8, \\ n + \lceil \frac{n}{4} \rceil - 1 & \text{if } n \geq 9, n \equiv 1 \pmod 4, \\ n + \lceil \frac{n}{4} \rceil & \text{if } n \geq 9, n \equiv 0, 2, 3 \pmod 4). \end{cases}$$

Proof. By Propositions 3.14, 3.15 and 3.16, the result follows for $3 \leq n \leq 8$. Now, we consider $n \geq 9$.

Suppose $n \equiv 0, 2, 3 \pmod 4$. Choose the vertex set $R = \{v_iu \mid 1 \leq i \leq n\} \cup \{v_jv_{j+1} \mid 1 \leq j \leq n, j \equiv 1 \pmod 4\}$, and let $Q_i = \{(v_iu)(v_ju) \mid 1 \leq i, j \leq n, i \neq j\} \cup \{(v_iu)(v_iv_{i+1}), (v_iu)(v_iv_{i-1+n})\}$ and

$Q'_j = \{(v_jv_{1+j})(v_{j+1}v_{2+j}), (v_{j+1}v_{2+j})(v_{2+j}v_{3+j}), (v_{j-1+n}v_j)(v_{j-2+n}v_{j-1+n}), (v_{j-2+n}v_{j-1+n})(v_{j-3+n}\ v_{j-2+n})\}$ for $1 \leq i, j \leq n$, where the subscripts of v_i are in the sense of modulo n. By Lemmas 2.10 and 3.17, we have $Q_i \subseteq \mathrm{EM}(v_iu)$ for $1 \leq i \leq n$ and $Q'_j \subseteq \mathrm{EM}(v_jv_{1+j})$ for $1 \leq j \leq n$, $j \equiv 1 \pmod 4$. Since $(\cup_{i=1}^{n} Q_i) \cup (\cup_{1\leq j\leq n, j\equiv 1 (\bmod 4)} Q'_j) = E(L(W_{n+1}))$, it follows that $\mathrm{dem}(L(W_{n+1})) \leq |R| = n + \lceil n/4 \rceil$.

To show $\mathrm{dem}(L(W_{n+1})) \geq n + \lceil n/4 \rceil$, let M be a DEM set of $L(W_{n+1})$ with $|M| = n + \lceil n/4 \rceil - 1$. Let $U_1 = \{v_iu \mid 1 \leq i \leq n\}$ and $U_2 = \{v_iv_{1+i} \mid 1 \leq i \leq n\}$, where $v_{n+1} = v_1$. If $|U_1 \cap M| < n - 1$, then there exists an edge xy such that $x, y \in U_1$ but $\{x, y\} \cap M = \emptyset$. By Lemma 3.12 and Theorem 1.8, the edge e cannot be monitored by the vertex in M, which is a contradiction. If $|U_1 \cap M| = n-1$ and $|U_2 \cap M| = \lceil n/4 \rceil$, without loss of generality, then there exists a vertex $v_1u \in U_1$ but $v_1u \notin M$. From Lemma 3.13, the two edges $(v_1u)(v_1v_2)$ and $(v_1u)(v_nv_1)$ cannot be monitored by $U_1 \cap M$. For the vertex in U_2, since $(v_1u)(v_1v_2)$ and $(v_1u)(v_nv_1)$ can only be monitored by v_1v_2 and v_nv_1, it follows that $v_1v_2, v_nv_1 \in M$, and hence $|(U_2 \setminus \{v_1v_2, v_nv_1\}) \cap M| = \lceil n/4 \rceil - 2$. From Lemma 3.17, $(\mathrm{EM}(v_1v_2) \cup \mathrm{EM}(v_nv_1)) \cap E(C'_n) = \{(v_iv_{i+1})(v_{i+1}v_{i+2}) \mid i = 1, 2, n-2, n-1, n\}$, where the subscripts of v_i are in the sense of modulo n. From Lemma 3.17, we know that $(U_2 \setminus \{v_1v_2, v_nv_1\}) \cap M$ can monitor $4(\lceil n/4 \rceil - 2)$ edges of C'_n, and so $|(\cup_{x\in M} \mathrm{EM}(x)) \cap E(C'_n)| = 4(\lceil n/4 \rceil - 2) + 5 < |E(C'_n)|$ for $n \geq 9, n \equiv 0, 2, 3 \pmod 4$, a contradiction.

If $|U_1 \cap M| = n$ and $|U_2 \cap M| = \lceil n/4 \rceil - 1$, then it follows from Lemma 3.17 that $U_2 \cap M$ can monitor $4(\lceil n/4 \rceil - 1)$ edges of C'_n, and hence $|(\cup_{x\in M} \mathrm{EM}(x)) \cap E(C'_n)| = 4(\lceil n/4 \rceil - 1) < |E(C'_n)|$, a contradiction. Thus, we have $\mathrm{dem}(L(W_{n+1})) \geq n + \lceil \frac{n}{4} \rceil$ for $n \geq 9, n \equiv 0, 2, 3 \pmod 4$.

Suppose that $n \equiv 1 (\bmod 4)$. Let $T_i = \{(v_iu)(v_ju) \mid 1 \leq i \leq n, i \neq j\} \cup \{(v_ju)(v_{j-1+n}v_j), (v_ju)(v_jv_{j+1})\}$ be the edge set associated with v_iu and $T'_j = \{(v_jv_{1+j})(v_{j+1}v_{2+j}), (v_{j+1}v_{2+j})(v_{2+j}v_{3+j}), (v_{j-1+n}v_j)(v_{j-2+n}v_{j-1+n}), (v_jv_{j+1})(v_jv_{j-1+n})\}$. By Lemmas 2.10 and 3.17, we have $(v_1u)(v_1v_2) \in \mathrm{EM}(v_1u)$, $(v_nu)(v_1v_n) \in \mathrm{EM}(v_nu)$, $T_i \subseteq \mathrm{EM}(v_iu)$ for $2 \leq i \leq n$ and $T'_j \subseteq \mathrm{EM}(v_jv_{1+j})$ for $1 \leq j \leq n$, $j \equiv 1 \pmod 4$. Since $(\cup_{i=2}^{n} T_i) \cup (\cup_{1\leq j\leq n, j\equiv 1 (\bmod 4)} T'_i) \cup \{(v_1u)(v_1v_2), (v_nu)(v_1v_n)\} = E(L(W_{n+1}))$, it follows that $\mathrm{dem}(L(W_{n+1})) \leq n + \lceil n/4 \rceil - 1$.

To show that $\mathrm{dem}(L(W_{n+1})) \geq n + \lceil n/4 \rceil - 1$, let S be a DEM set of $L(W_{n+1})$ with $|S| = n + \lceil n/4 \rceil - 2$. Take $U_1 = \{v_iu \mid 1 \leq i \leq n\}$ and $U_2 = \{v_iv_{i+1} \mid 1 \leq i \leq n\}$. If $|U_1 \cap S| < n - 1$, then there exists an edge xy such that $x, y \in U_1 \setminus S$. From Lemma 3.12 and Theorem 1.8, the edge xy cannot be monitored by the vertex in S, a contradiction. If $|U_1 \cap S| = n - 1$ and $|U_2 \cap S| = \lceil n/4 \rceil - 1$, then, without loss of generality, let $v_1u \in U_1 \setminus M$. By Lemma 3.13, we have $(v_1u)(v_1v_2) \in \mathrm{EM}(v_1v_2)$, $(v_1u)(v_nv_1) \in \mathrm{EM}(v_nv_1)$ and $(v_1u)(v_1v_2), (v_1u)(v_nv_1) \notin \cup_{x\in U_1\cap S} \mathrm{EM}(x)$, and hence $v_1v_2, v_nv_1 \in S$, and so $|(U_2 \setminus \{v_1v_2, v_nv_1\}) \cap S| = \lceil n/4 \rceil - 3$. From Lemma 3.17, $(U_2 \setminus \{v_1v_2, v_nv_1\}) \cap M$ can monitor $4(\lceil n/4 \rceil - 3)$ edges of $E(C'_n)$ and $(\mathrm{EM}(v_1v_2) \cup \mathrm{EM}(v_nv_1)) \cap E(C'_n) = \{(v_iv_{i+1})(v_{i+1}v_{i+2}) \mid i = 1, 2, n-2, n-1, n\}$, where the subscripts of v_i are in the sense of modulo n. Therefore, $|\cup_{x\in M} \mathrm{EM}(x)) \cap E(C'_n)| = 4(\lceil n/4 \rceil - 3) + 5 < |E(C'_n)|$, a contradiction. If $|U_1 \cap S| = n$ and $|U_2 \cap S| = \lceil n/4 \rceil - 2$, then $U_2 \cap S$ can monitor at most $4(\lceil n/4 \rceil - 2)$ edges of C'_n by Lemma 3.17, where $4(\lceil n/4 \rceil - 2) < |E(C'_n)|$, a contradiction. Therefore, $\mathrm{dem}(L(W_{n+1})) \geq n + \lceil \frac{n}{4} \rceil - 1$ for $n \geq 9, n \equiv 1 (\bmod 4)$. □

4. Application

It would be beneficial to discuss the significance of studying lower and upper bounds for the line graphs of these families when considering distance-edge-monitoring as a real-world foundation. In the section, we obtain the exact value of the DEM number of a new network by a line graph operation of the small networks.

The DURT is one of the simplest models that can be constructed by a simple iterative algorithm [20]. Let us denote the DURT obtained after t iteration as U_t with N_t nodes and E_t edges, where $t = 0, 1, 2, \ldots, T-1$, and T is the total number of iterations, then the DURT generation process can be illustrated as follows [25]; see Figure 5.

Step 0. Initialization. Set $t = 0$, U_0 contains an edge that connects two nodes, and thus $N_0 = 2$ and $E_0 = 1$.
Step 1. Generation of U_{t+1} from U_t. For each node in U_t, a new node is linked to it. Thus we have $N_{t+1} = 2N_t$ and $E_{t+1} = E_t + N_t$.
Step 2. If $t < T - 1$, then we set $t = t + 1$ and go to Step 1. Otherwise, the algorithm is terminated.

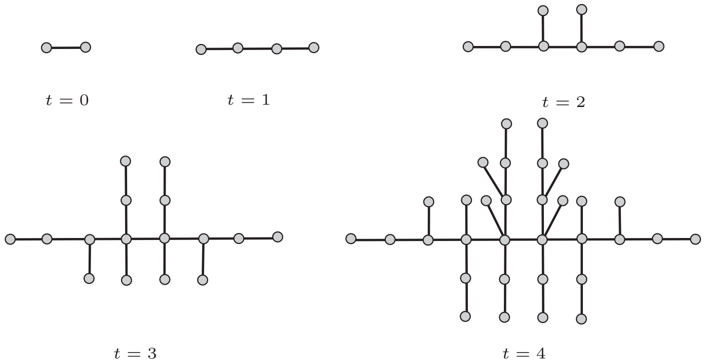

FIGURE 5. The first five iterations of the growth process of DURT.

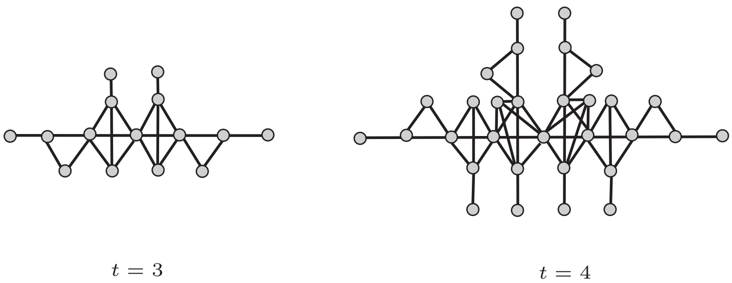

FIGURE 6. The third and fourth iterations of the growth process of $L(U_t)$.

For each iteration of the U_t, we can derive a new network by a line graph operation, denoted by $L(U_t)$; see Figure 6, and the following theorem is given.

Lemma 4.1 ([18]). *For the $L(U_t)$, where $t \geq 1$, the number of the complete subgraphs $K_2, K_3, \ldots, K_{t-1}, K_t, K_{t+1}$ equals to $2^{t-1}, 2^{t-2}, \ldots, 2^2, 2^1, 2^1$, respectively.*

Theorem 4.2. *For an integer $t \geq 1$, we have* $\text{dem}(L(U_t)) = 2^{t-1}$.

Proof. Let the graph $G = L(U_t)$. To show the upper bound, let M be the set of vertices with the smallest degree selected in each complete subgraph of order $r \geq 3$ for the graph G. Since each complete subgraph has only one vertex with the smallest degree, it follows from Lemma 4.1 that $|M| = 2^{t-2} + \cdots + 2^2 + 2^1 + 2^1 = 2^{t-1}$. For every bridge e of G, by Lemma 2.2, e can be monitored by any vertex in M. For any complete subgraph H with order $r \geq 3$ of G, let $U = M \cap V(H)$. From the construction of G, we have $|U| = 1$ (says $U = \{v\}$) and let $v' \in V(H)$ such that $d_G(v') = d_G(v) + 1$. Note that the vertex v is not a cut vertex and the vertices in $V(H) \setminus v$ are all cut vertices. Let U_3 be the set of vertices with the smallest degree selected in each complete subgraph in G with order 3. Since there exists a vertex $y \in U_3$ such that $V(P_{x,y}) \cap V(H) = \{x\}$ for any vertex $x \in V(H) \setminus \{v, v'\}$, it follows that $(\cup_{x \in U_3} \text{EM}(x)) \cap E(H) = \cup_{x \in V(H) \setminus \{v,v'\}} \text{EM}(x)$, where $P_{x,y}$ is the shortest path from x to y. By Theorem 1.8, $E(H) = \cup_{x \in V(H) \setminus \{v,v'\}} \text{EM}(x) \cup \text{EM}(v)$, and hence $E(H) \subseteq \cup_{x \in U_3} \text{EM}(x) \cup \text{EM}(v)$, and so $E(G)$ can be monitored by the vertex in M. Therefore, $\text{dem}(G) \leq |M| = 2^{t-1}$.

To show the lower bound, let Q be a DEM set with $|Q| = 2^{t-1} - 1$. From the construction of G, let u be the leaf hanging on the vertex v' for any complete subgraph H with order $r \geq 3$ of G, where $v' \in V(H) \setminus v$ and v is the vertex with the smallest degree in H. Then, we claim that the edge vv' can only be monitored by the vertices

TABLE 2. $\text{dem}(G_{a,b})$ and $\text{dem}(L(G_{a,b}))$.

G	$\text{dem}(G_{a,b})$	$\text{dem}(L(G_{a,b}))$
$G_{2,b}$	a	4
F_a	a	$2a$
H	a	$2a+t$ $(t \geq 1)$

in $\{v, v', u\}$. Clearly, $\text{EM}(u) \cap E(H) = \text{EM}(v') \cap E(H)$ and $vv' \in \text{EM}(v) \cap \text{EM}(v')$, and hence the edge vv' can be monitored by the vertices in $\{v, v', u\}$. For any vertex $w \in V(G) \backslash \{v, v', u\}$, we have $d_G(w, v) = d_G(w, v')$, and hence $vv' \notin \text{EM}(w)$. Let H_1, H_2 be two different complete subgraphs with order $r \geq 3$ and S_1, S_2 be the vertex sets shaped like $\{v, v', u\}$ of H_1 and H_2, respectively. Since $S_1 \cap S_2 = \emptyset$, it follows that $|\{v, v', u\} \cap Q| \geq 1$ for each complete subgraph H with order $r \geq 3$, and hence $|Q| \geq 2^{t-2} + \cdots + 2^2 + 2^1 + 2^1 = 2^{t-1}$. Therefore, we have $\text{dem}(G) = 2^{t-1}$. □

For the small network, because distances are extremely short and the network is structurally robust, it is necessary to take about half of all vertices to build a DEM set. It would be an interesting future work to verify whether this holds true for other small-world networks.

5. Concluding remark

In this paper, we study a classical problem: for each $a, b > 1$, is there a graph $G_{a,b}$ such that $\text{dem}(G_{a,b}) = a$ and $\text{dem}(L(G_{a,b})) = b$? The ideal situation is to solve it completely for all a and b. For any $a > 1$, we present $G_{a,b}$ such that $\text{dem}(G_{a,b}) = a$ and $\text{dem}(L(G_{a,b})) \in \{4, 2a+t\}$ for $t \geq 0$; see Table 2. Thus we have accomplished this for almost all cases, except for a relatively small range. To be precise, it remains to find graphs for $\text{dem}(G_{a,b}) = a$ but $\text{dem}(L(G_{a,b})) \in \{x \mid a+1 \leq x \leq 2a-1, x \neq 4\}$.

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