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WOLFE TYPE DUALITY ON QUASIDIFFERENTIABLE MATHEMATICAL PROGRAMS WITH VANISHING CONSTRAINTS

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Abstract. This article is devoted to the study of duality results for optimization problems with vanishing constraints in nonsmooth case. We formulate Wolfe type dual and establish weak, strong, converse, restricted converse and strict converse duality results for mathematical programs with vanishing constraints involving quasidifferentiable functions. Under the assumption of invex and strictly invex functions with respect to a convex compact set.

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1. Introduction

In 2008 Achtziger and Kanzow [1] introduced and formulated a difficult class of optimization problem known as Mathematical programs with vanishing constraints (MPVC) which is a generalization of Mathematical programming problems with equilibrium constraints. MPVC has various applications such as optimal topology design problems in mechanical constructions, see, for instance [1, 14, 29, 30]. In several applications, some constraints are often seen to "vanish" in certain feasible elements due to this property it's named as "vanishing constraints". Many authors have discussed several theoretical properties and numerical techniques for MPVCs for the smooth case see [1, 10, 14]. In 2018, Kazemi and Kanzi [18] derived the Karush–Kuhn–Tucker (KKT)-type necessary optimality conditions for nonsmooth MPVCs by using Clarke subdifferential. Kazemi *et al.* [19] studied Frechet normal cone and discussed some stationary conditions in the case of nonsmooth feasible region in MPVCs. For more details see [13, 21, 22].

The concept of duality is very important in optimization to search the optimal solution (see [4, 15–17, 28]). In 1961, Wolfe [36] introduced the concept of classical Wolfe duality. Mond and Weir [27] introduced a new type of dual based on the Wolfe type of dual and derived usual duality theorems for smooth functions. Mishra *et al.* [25] have discussed duality results for MPVCs for the smooth case. Later many authors explored both these duality for non-smooth case [23, 24]. Further, Ghobadzadeh *et al.* [12] studied Wolfe type duality for MPVCs in nonsmooth case by using Clarke subdifferentials.

Demyanov [6] was the first who studied the concept of quasidifferentiability. Many authors studied some basic optimality results regarding quasidifferentiable optimization, see [5, 11, 31, 32]. In 2005, Uderzo [35] established mean value inequalities using semicontinuous quasidifferentiable functions. Baier *et al.* [3] discussed the

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relation between the Rubinov subdifferential and Clarke, Dini, Michel-Penot and Mordukhovich subdifferentials. Recently, Singh and Laha [33] formulated vector variational-like inequalities of Stampacchia and Minty types in terms of quasidifferential calculus. In all the above mentioned works, the function involved in MPVCs is either continuously differentiable or locally lipschitz for the optimal solution. Our purpose is to obtain duality results for MPVCs using quasidifferentiable analysis.

The paper is organized as follows:: in Section 2, we recall some definitions and basic results for primal MPVC connecting to quasidifferentiable analysis which will be used in this paper. In Section 3, we formulate Wolfe type dual and derive weak, strong, converse, restricted converse and strict converse duality results under a suitable choice of quasidifferentiable invex and strictly invex functions. In Section 4, we give conclusion of results of this paper.

2. Preliminaries

In this section, we provide some definitions and results which will be used in this paper.

We consider a mathematical program with inequality, equality and vanishing constraints involving quasidifferentiable functions which was studied by Laha *et al.* [20]:

$$\begin{aligned} &\text{(QMPVC)} \quad \min \mathcal{J}(\aleph) \\ &\quad \text{subject to: } \mathcal{K}_j(\aleph) \le 0, && \forall j \in M = \{1,2,\dots,m\}, \\ &\qquad \mathcal{L}_k(\aleph) = 0, && \forall k \in N = \{1,2,\dots,n\}, \\ &\qquad \mathcal{N}_i(\aleph) \ge 0, && \forall i \in \pounds = \{1,2,\dots,l\}, \\ &\qquad \mathcal{M}_i(\aleph)\mathcal{N}_i(\aleph) \le 0, && \forall i \in \pounds = \{1,2,\dots,l\}, \end{aligned} \tag{1}$$

where, $\mathcal{J}, \mathcal{K}_j (j \in M), \mathcal{L}_k (k \in N), \mathcal{M}_i (i \in \pounds), \mathcal{N}_i (i \in \pounds)$ are quasidifferentiable functions on $\mathbb{R}^p$.

Let $\aleph \in X$ be any feasible solution of the QMPVC(1), we use the following index sets as follows:

$$\begin{aligned} &\pounds_{\mathcal{K}}(\aleph) := \{j \in \{1,2,\dots,m\} : \mathcal{K}_j(\aleph) = 0\}, \quad \pounds_{\mathcal{L}}(\aleph) := \{1,2,\dots,n\}, \\ &\pounds_{+}(\aleph) := \{i \in \{1,2,\dots,l\} : \mathcal{N}_i(\aleph) > 0\}, \\ &\pounds_{0}(\aleph) := \{i \in \{1,2,\dots,l\} : \mathcal{N}_i(\aleph) = 0\}. \end{aligned} \tag{2}$$

The index set $\pounds_{+}(\aleph)$ further divided as:

$$\begin{aligned} \pounds_{+0}(\aleph) &= \{i \in \{1,2,\dots,l\} : \mathcal{N}_i(\aleph) > 0, \mathcal{M}_i(\aleph) = 0\}, \\ \pounds_{+-}(\aleph) &= \{i \in \{1,2,\dots,l\} : \mathcal{N}_i(\aleph) > 0, \mathcal{M}_i(\aleph) < 0\}, \\ \pounds_{00}(\aleph) &:= \{i \in \{1,2,\dots,l\} : \mathcal{N}_i(\aleph) = 0, \mathcal{M}_i(\aleph) = 0\}, \\ \pounds_{0+}(\aleph) &= \{i \in \{1,2,\dots,l\} : \mathcal{N}_i(\aleph) = 0, \mathcal{M}_i(\aleph) = 0\}, \\ \pounds_{0-}(\aleph) &= \{i \in \{1,2,\dots,l\} : \mathcal{N}_i(\aleph) = 0, \mathcal{M}_i(\aleph) < 0\}. \end{aligned} \tag{3}$$

The following theorem is Theorem 3 in [20], which gives Fritz-John type necessary optimality conditions.

Theorem 2.1. *Suppose that the feasible point $\bar{\aleph}$ is a minimizer of the QMPVC such that $\mathcal{L}_k(k \in K)$ and $\mathcal{N}_i(i \in \pounds_{0+}(\bar{\aleph}) \cup \pounds_{00}(\bar{\aleph}))$ are locally Lipschitz in a neighborhood of $\bar{\aleph}$. Then, for any set of $w_0 \in \bar{\partial}\mathcal{J}(\bar{\aleph})$, $w_j \in \bar{\partial}\mathcal{K}_j(\bar{\aleph})(j \in J)$, $w_i \in \bar{\partial}\mathcal{M}_i(\bar{\aleph})(i \in \pounds)$, $v_i \in \bar{\partial}(-\mathcal{N}_i)(\bar{\aleph})(i \in \pounds_{0-}(\bar{\aleph}) \cup \pounds_{+}(\bar{\aleph}))$, there exists scalars $\beta_0 \ge 0$, $\bar{\ell}_j \ge 0 (j \in J)$, $\bar{\mu_k} \ge 0$, $\bar{\ell}_i^{\mathcal{N}} \ge 0$, $\bar{\ell}_i^{\mathcal{M}} \ge 0$, not all zero, such that*

$$0 \in \beta_0\big(\underline{\partial}\mathcal{J}(\bar{\aleph}) + w_0\big) + \sum_{j \in J} \ell_j\big(\underline{\partial}\mathcal{K}_j(\bar{\aleph}) + w_j\big) + \sum_{k=1}^{n} \mu_k \partial^{\circ}\mathcal{L}_k(\bar{\aleph})$$

$$-\sum_{i=1}^{l}\ell_i^{\mathcal{N}}\big(\underline{\partial}(-\mathcal{N}_i(\bar{\aleph}))+v_i\big)+\sum_{i=1}^{l}\ell_i^{\mathcal{M}}\big(\underline{\partial}\mathcal{M}_i(\bar{\aleph})+w_i\big)+\sum_{i\in\pounds_{0+}(\bar{\aleph})\cup\pounds_{00}(\bar{\aleph})}\ell_i^{\mathcal{N}}\partial^{\circ}\mathcal{N}_i(\bar{\aleph}) \tag{4}$$

$$\ell_j\mathcal{K}_j=0(j\in J),\quad \ell_i^{\mathcal{N}}\geq 0,\quad \forall i\in\pounds_{0-}(\bar{\aleph}),\quad \ell_i^{\mathcal{N}}=0,\quad \forall i\in\pounds_{+}(\bar{\aleph}),$$
$$\ell_i^{\mathcal{M}}=0\big(i\in\pounds_{+-}(\bar{\aleph})\cup\pounds_{0-}(\bar{\aleph})\cup\pounds_{0+}(\bar{\aleph})\big),\quad \ell_i^{\mathcal{M}}\geq 0,\quad \forall i\in\pounds_{+0}(\bar{\aleph})\cup\pounds_{00}(\bar{\aleph}). \tag{5}$$

The following KKT optimality condition for the QMPVC(1) were derived by Laha *et al.* [20] by assuming no nonzero abnormal multipliers exists:

Theorem 2.2. *Suppose that the feasible point $\bar{\aleph}$ is a minimizer of the QMPVC such that $\mathcal{L}_k(k\in K)$ and $\mathcal{N}_i(i\in\pounds_{0+}(\bar{\aleph})\cup\pounds_{00}(\bar{\aleph}))$ are locally Lipschitz in a neighborhood of $\bar{\aleph}$. Then, for any set of $w_0\in\bar{\partial}\mathcal{J}(\bar{\aleph})$, $w_j\in\bar{\partial}\mathcal{K}_j(\bar{\aleph})(j\in J)$, $w_i\in\bar{\partial}\mathcal{M}_i(\bar{\aleph})(i\in\pounds)$, $v_i\in\bar{\partial}(-\mathcal{N}_i)(\bar{\aleph})(i\in\pounds_{0-}(\bar{\aleph})\cup\pounds_{+}(\bar{\aleph}))$, there exists scalars $\beta_0\geq 0$, $\bar{\ell}_j\geq 0(j\in J)$, $\bar{\mu_k}\geq 0$, $\bar{\ell}_i^{\mathcal{N}}\geq 0$, $\bar{\ell}_i^{\mathcal{M}}\geq 0$, not all zero, such that (4) satisfied. Further, assume that there is no nonzero abnormal multiplier, that is,*

$$0\in\sum_{j\in J}\ell_j\big(\underline{\partial}\mathcal{K}_j(\bar{\aleph})+w_j\big)+\sum_{k=1}^{n}\mu_k\partial^{\circ}\mathcal{L}_k(\bar{\aleph})-\sum_{i\in\pounds_{0-}(\bar{\aleph})}\ell_i^{\mathcal{N}}\big(\underline{\partial}(-\mathcal{N}_i(\bar{\aleph}))+v_i\big)$$
$$+\sum_{i\in\pounds_{+0}(\bar{\aleph})\cup\pounds_{00}(\bar{\aleph})}^{l}\ell_i^{\mathcal{M}}\big(\underline{\partial}\mathcal{M}_i(\bar{\aleph})+w_i\big)+\sum_{i\in\pounds_{0+}(\bar{\aleph})\cup\pounds_{00}(\bar{\aleph})}\ell_i^{\mathcal{N}}\partial^{\circ}\mathcal{N}_i(\bar{\aleph}), \tag{6}$$
$$\ell_j\geq 0,\quad \ell_i^{\mathcal{N}}\geq 0,\quad \forall i\in\pounds_{0-}(\bar{\aleph}),\quad \ell_i^{\mathcal{M}}\geq 0,\quad \forall i\in\pounds_{0+}(\bar{\aleph})\cup\pounds_{00}(\bar{\aleph}),$$

which implies that

$$\ell_j=0,\quad \ell_i^{\mathcal{N}}=0,\quad \forall i\in\pounds_{0-}(\bar{\aleph}),\quad \ell_i^{\mathcal{M}}=0,\quad \forall i\in\pounds_{0+}(\bar{\aleph})\cup\pounds_{00}(\bar{\aleph}),\quad \mu_k=0(k\in K),$$
$$\ell_i^{\mathcal{N}}=0,\quad \forall i\in\pounds_{0+}(\bar{\aleph})\cup\pounds_{00}(\bar{\aleph}).$$

Then, one has $\beta_0>0$.

The following concept of Weak stationary point was studied by Laha *et al.* [20].

Definition 2.3. A feasible point $\bar{\aleph}$ is said to be weak stationary point of the QMPVC, iff there exists scalars $\bar{\ell}_j\geq 0(j\in J)$, $\bar{\mu_k}\geq 0$, $\bar{\ell}_i^{\mathcal{N}}\geq 0(i\in\pounds_{0-}(\bar{\aleph})\cup\pounds_{+}(\bar{\aleph}))$, $\bar{\ell}_i^{\mathcal{M}}\geq 0(i\in\pounds)$, $\bar{\ell}_i^{\mathcal{N}}(i\in\pounds_{0+}(\bar{\aleph})\cup\pounds_{00}(\bar{\aleph}))$, such that

$$0\in\beta_0\big\{\underline{\partial}\mathcal{J}(\bar{\aleph})+\bar{\partial}\mathcal{J}(\bar{\aleph})\big\}+\sum_{j\in J}\ell_j\big\{\underline{\partial}\mathcal{K}_j(\bar{\aleph})+\bar{\partial}\mathcal{K}_j(\bar{\aleph})\big\}+\sum_{k=1}^{n}\mu_k\big\{\underline{\partial}\mathcal{L}_k(\bar{\aleph})+\bar{\partial}\mathcal{L}_k(\bar{\aleph})\big\}$$
$$-\sum_{i=1}^{l}\ell_i^{\mathcal{N}}\big\{\underline{\partial}\mathcal{N}_i(\bar{\aleph})+\bar{\partial}\mathcal{N}_i(\bar{\aleph})\big\}+\sum_{i=1}^{l}\ell_i^{\mathcal{M}}\big\{\underline{\partial}\mathcal{M}_i(\bar{\aleph})+\bar{\partial}\mathcal{M}_i(\bar{\aleph})\big\}+\sum_{i\in\pounds_{0+}(\bar{\aleph})\cup\pounds_{00}(\bar{\aleph})}\ell_i^{\mathcal{N}}\big\{\underline{\partial}\mathcal{N}_i(\bar{\aleph})+\bar{\partial}\mathcal{N}_i(\bar{\aleph})\big\}$$
$$\ell_j\mathcal{K}_j=0(j\in J),\quad \ell_i^{\mathcal{N}}\geq 0,\quad \forall i\in\pounds_{0-}(\bar{\aleph}),\quad \ell_i^{\mathcal{N}}=0,\quad \forall i\in\pounds_{+}(\bar{\aleph}),$$
$$\ell_i^{\mathcal{M}}=0\big(i\in\pounds_{+-}(\bar{\aleph})\cup\pounds_{0-}(\bar{\aleph})\cup\pounds_{0+}(\bar{\aleph})\big),\quad \ell_i^{\mathcal{M}}\geq 0,\quad \forall i\in\pounds_{+0}(\bar{\aleph})\cup\pounds_{00}(\bar{\aleph}). \tag{7}$$

Definition 2.4. A function $\mathcal{L}:\mathbb{R}^p\to\mathbb{R}$ is known as *directionally differentiable* at $\bar{\aleph}\in\mathbb{R}^p$ in a direction $d\in\mathbb{R}^p$, iff the limit

$$\mathcal{L}'(\bar{\aleph}:d):=\lim_{\alpha\downarrow 0}\frac{\mathcal{L}(\bar{\aleph}+\alpha d)-\mathcal{L}(\bar{\aleph})}{\alpha}$$

exists and is finite.

In 1980, Demyanov [6] gave the definition of quasidifferentiable functions as follows:

Definition 2.5. A real-valued function $\mathcal{L} : \mathbb{R}^n \to \mathbb{R}$ is known as *quasidifferentiable* at $\bar{\aleph} \in \mathbb{R}^n$ iff $\mathcal{L}$ is directionally differentiable at $\bar{\aleph}$ and there exists a pair of convex compact sets $\underline{\partial}\mathcal{L}(\bar{\aleph}) \subset \mathbb{R}^n$ and $\bar{\partial}\mathcal{L}(\bar{\aleph}) \subset \mathbb{R}^n$ such that

$$\mathcal{L}'\left(\bar{\aleph}; d\right) := \max_{v \in \underline{\partial}\mathcal{L}(\bar{\aleph})} \langle v, d\rangle + \min_{w \in \bar{\partial}\mathcal{L}(\bar{\aleph})} \langle w, d\rangle,$$

where $\underline{\partial}\mathcal{L}(\bar{\aleph})$ and $\bar{\partial}\mathcal{L}(\bar{\aleph})$ are called *subdifferential* and *superdifferential* of $\mathcal{L}$ at $\bar{\aleph}$, respectively. Further, the pair of sets $D_{\mathcal{L}}(\bar{\aleph}) := [\underline{\partial}\mathcal{L}(\bar{\aleph}),\ \bar{\partial}\mathcal{L}(\bar{\aleph})]$ is called a *quasidifferential* of the function $\mathcal{L}$ at $\bar{\aleph}$.

Remark 2.6. The pair of sets, for the quasidifferential to a function $\mathcal{L}$ at a point $\bar{\aleph}$ is not unique, because if $D_{\mathcal{L}}(\bar{\aleph}) = [\underline{\partial}\mathcal{L}(\bar{\aleph}),\ \bar{\partial}\mathcal{L}(\bar{\aleph})]$ is a quasidifferentiable of $\mathcal{L}$ at $\bar{\aleph}$, then, for any nonempty convex compact set V, the pair of sets $[\underline{\partial}\mathcal{L}(\bar{\aleph}) + V,\ \bar{\partial}\mathcal{L}(\bar{\aleph}) - V]$ is also its quasidifferential.

Some important property of quasidifferentiable functions are listed below:

Proposition 2.7 ([8])**.** *Suppose two real-valued functions $\mathcal{L}_1$ and $\mathcal{L}_2$ on $\mathbb{R}^n$ be quasidifferentiable at a point $\bar{\aleph} \in \mathbb{R}^n$ with quasidifferentials $D_{\mathcal{L}_1}(\bar{\aleph}) = [\underline{\partial}\mathcal{L}_1(\bar{\aleph}),\ \bar{\partial}\mathcal{L}_1(\bar{\aleph})]$ and $D_{\mathcal{L}_2}(\bar{\aleph}) = [\underline{\partial}\mathcal{L}_2(\bar{\aleph}),\ \bar{\partial}\mathcal{L}_2(\bar{\aleph})]$, respectively, and let $\alpha \in \mathbb{R}$. Then, the following statements are true:*

(a) *The function $\mathcal{L} := \mathcal{L}_1 + \mathcal{L}_2$ is also quasidifferentiable at $\bar{\aleph}$ with quasidifferential*

$$D_{\mathcal{L}}(\bar{\aleph}) = \left[\underline{\partial}\mathcal{L}_1(\bar{\aleph}) + \underline{\partial}\mathcal{L}_2(\bar{\aleph}),\ \bar{\partial}\mathcal{L}_1(\bar{\aleph}) + \bar{\partial}\mathcal{L}_2(\bar{\aleph})\right].$$

(b) *The function $\mathcal{L} := \alpha\mathcal{L}_1$ is also quasidifferentiable at $\bar{\aleph}$ with quasidifferential*

$$D_{\mathcal{L}}(\bar{\aleph}) = \begin{cases} \left[\alpha\underline{\partial}\mathcal{L}_1(\bar{\aleph}),\ \alpha\bar{\partial}\mathcal{L}_1(\bar{\aleph})\right], & \alpha \geq 0 \\ \alpha\bar{\partial}\mathcal{L}_1(\bar{\aleph}),\ \alpha\underline{\partial}\mathcal{L}_1(\bar{\aleph}), & \alpha < 0. \end{cases}$$

(c) *The set $S_{\mathcal{L}}(\bar{\aleph}) := \underline{\partial}\mathcal{L}(\bar{\aleph}) + \bar{\partial}\mathcal{L}(\bar{\aleph})$ is a convex compact set. Also, for any $\alpha \in \mathbb{R}$, $S_{\alpha\mathcal{L}}(\bar{\aleph}) = \alpha S_{\mathcal{L}}(\bar{\aleph})$.*

Theorem 2.8 ([7])**.** *Suppose the functions $\mathcal{L}, \mathcal{L}_1, \mathcal{L}_2 : \mathbb{R}^n \to \mathbb{R}$ be quasidifferentiable at $\aleph \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}^n$. Then the sum and the scalar multiple of these functions are quasidifferentiable at $\aleph$ and also*

$$D(\mathcal{L}_1 + \mathcal{L}_2)(\aleph) = D(\mathcal{L}_1)(\aleph) + D(\mathcal{L}_2)(\aleph)$$
$$D(\alpha\mathcal{L})(\aleph) = \alpha D\mathcal{L}(\aleph).$$

Definition 2.9 ([26])**.** Let $K \neq \emptyset \subseteq \mathbb{R}^n$ and let $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be a vector-valued function. The set K is called an *invex set* wrt η iff for any $\aleph, \varrho \in K$ and for any $\ell \in [0, 1]$, one has $\aleph + \ell\eta(\varrho, \aleph) \in K$.

Definition 2.10 ([2], Def. 2.3)**.** Let $\bar{\aleph} \in \mathbb{R}^n$, let $\mathcal{L} : \mathbb{R}^n \to \mathbb{R}$ be a real-valued function, let $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be a vector-valued function, and let $S_{\mathcal{L}}(\bar{\aleph})$ be a nonempty convex compact subset of $\mathbb{R}^n$. Then the function $\mathcal{L}$ is said to be *quasidifferentiable invex* at $\bar{\aleph}$ wrt $S_{\mathcal{L}}(\bar{\aleph})$ and η on $\mathbb{R}^n$ iff for all $\aleph \in \mathbb{R}^n$ and for all $\omega \in S_{\mathcal{L}}(\bar{\aleph})$, one has

$$\mathcal{L}(\aleph) \geq \mathcal{L}(\bar{\aleph}) + \langle \omega, \eta(\aleph, \bar{\aleph})\rangle. \tag{8}$$

If, for each $\aleph \in \mathbb{R}^n$, there exists a convex compact set $S_{\mathcal{L}}(\aleph) \subseteq \mathbb{R}^n$ such that the inequality (8) holds at each $\aleph$ wrt the same η, then $\mathcal{L}$ is said to be *quasidifferentiable invex* function on $\mathbb{R}^n$ wrt $S_{\mathcal{L}}(\aleph)$ and η.

3. Wolfe dual type-model

In this section, we formulate a Wolfe dual (WD) to the QMPVC(1) depending on a feasible point $\aleph \in X$, denoted by VC-WD($\aleph$), as follows:

$$\max \psi\big(\varrho, \ell, \mu, \ell^{\mathcal{N}}, \ell^{\mathcal{M}}\big) = \mathcal{J}(\varrho) + \sum_{j=1}^{m} \ell_j \mathcal{K}_j(\varrho) + \sum_{k=1}^{n} \mu_k \mathcal{L}_k(\varrho) - \sum_{i=1}^{l} \ell_i^{\mathcal{N}} \mathcal{N}_i(\varrho) + \sum_{i=1}^{l} \ell_i^{\mathcal{M}} \mathcal{M}_i(\varrho), \tag{9}$$

for any sets of $w_0 \in \bar{\partial}\mathcal{J}(\varrho)$, $w_j \in \bar{\partial}\mathcal{K}_j(\varrho)$, $w_k \in \bar{\partial}\mathcal{L}_k(\varrho)$, $w_i \in \bar{\partial}\mathcal{M}_i(\varrho)$, $v_i \in \bar{\partial}\mathcal{N}_i(\varrho)$.

Subject to:

$$\begin{aligned} 0 \in\, & \underline{\partial}\mathcal{J}(\varrho) + w_0 + \sum_{j\in J} \ell_j(\underline{\partial}\mathcal{K}_j(\varrho) + w_j) + \sum_{k=1}^{n} \mu_k(\underline{\partial}\mathcal{L}_k(\varrho) + w_k) - \sum_{i=1}^{l} \ell_i^{\mathcal{N}}(\underline{\partial}\mathcal{N}_i(\varrho) + v_i) \\ & + \sum_{i=1}^{l} \ell_i^{\mathcal{M}}(\underline{\partial}\mathcal{M}_i(\varrho) + w_i) \end{aligned} \tag{10}$$

$$\ell_j \geq 0 \quad \forall i \notin \pounds_{\mathcal{K}}(\aleph), \quad \ell_i^{\mathcal{N}} \geq 0, \quad \forall i \in \pounds_{+}(\aleph), \quad \ell_i^{\mathcal{M}} \leq 0, \quad \forall i \in \pounds_{0+}(\aleph), \quad \ell_i^{\mathcal{M}} \geq 0, \quad \forall i \in \pounds_{0-}(\aleph) \cup \pounds_{+-}(\aleph).$$

Let $S_w(\aleph) \subseteq \mathbb{R}^p \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^l$ given by

$$S_w(\aleph) = \Bigg\{ \big(\varrho, \ell, \mu, \ell^{\mathcal{N}}, \ell^{\mathcal{M}}\big) : 0 \in \underline{\partial}\mathcal{J}(\varrho) + w_0 + \sum_{j\in J} \ell_j(\underline{\partial}\mathcal{K}_j(\varrho) + w_j) + \sum_{k=1}^{n} \mu_k(\underline{\partial}\mathcal{L}_k(\varrho) + w_k)$$

$$- \sum_{i=1}^{l} \ell_i^{\mathcal{N}}(\underline{\partial}\mathcal{N}_i(\varrho) + v_i) + \sum_{i=1}^{l} \ell_i^{\mathcal{M}}(\underline{\partial}\mathcal{M}_i(\varrho) + w_i)$$

$$\ell_j \geq 0, \quad \forall i \notin \pounds_{\mathcal{K}}(\aleph), \quad \ell_i^{\mathcal{N}} \geq 0, \quad \forall i \in \pounds_{+}(\aleph), \quad \ell_i^{\mathcal{M}} \geq 0\,(i \in \pounds_{+-}(\aleph) \cup \pounds_{0-}(\aleph)), \quad \ell_i^{\mathcal{M}} \leq 0, \quad \forall i \in \pounds_{0+}(\aleph) \Bigg\},$$

denote the set of all feasible points of the VC-WD($\aleph$).

We define the following index sets:

$$\begin{aligned} \pounds_{\mathcal{K}}^{+}(\aleph) &= \{i \in \{1, 2, \ldots, m\} : \ell_j > 0\}, \\ \pounds_{\mathcal{L}}^{+} &= \{k \in \pounds_{\mathcal{L}} : \mu_k > 0\}, \quad \pounds_{\mathcal{L}}^{-} = \{k \in \pounds_{\mathcal{L}} : \mu_k < 0\}, \\ \pounds_{+}^{+} &= \big\{i \in \pounds_{+} : \ell_i^{\mathcal{N}} > 0\big\}, \quad \pounds_{+-}^{+} = \big\{i \in \pounds_{+} : \ell_i^{\mathcal{M}} > 0\big\}, \\ \pounds_{0}^{+} &= \big\{i \in \pounds_{0} : \ell_i^{\mathcal{N}} > 0\big\}, \quad \pounds_{0}^{-} = \big\{i \in \pounds_{0} : \ell_i^{\mathcal{N}} < 0\big\}, \\ \pounds_{0+}^{-} &= \big\{i \in \pounds_{0+} : \ell_i^{\mathcal{M}} < 0\big\}, \quad \pounds_{00}^{-} = \big\{i \in \pounds_{00} : \ell_i^{\mathcal{M}} < 0\big\}, \\ \pounds_{+0}^{-} &= \big\{i \in \pounds_{+0} : \ell_i^{\mathcal{M}} < 0\big\}, \quad \pounds_{00}^{+} = \big\{i \in \pounds_{00} : \ell_i^{\mathcal{M}} > 0\big\}, \\ \pounds_{+0}^{+} &= \big\{i \in \pounds_{+0} : \ell_i^{\mathcal{M}} > 0\big\}, \quad \pounds_{0-}^{+} = \big\{i \in \pounds_{0-} : \ell_i^{\mathcal{M}} > 0\big\}. \end{aligned} \tag{11}$$

Remark 3.1. If $\mathcal{J}, \mathcal{K}_j$, $\mathcal{L}_k$, $\mathcal{N}_i$, $\mathcal{M}_i$ are locally Lipschitz at ϱ and $S_{\mathcal{J}(\varrho)}, S_{\mathcal{K}_j}(\varrho), S_{\mathcal{L}_k}(\varrho), S_{\mathcal{N}_i}(\varrho), S_{\mathcal{M}_i}(\varrho)$ are equal to its Clark subdifferentials *i.e.* $\partial^{\circ}\mathcal{J}(\varrho), \partial^{\circ}\mathcal{K}_j(\varrho), \partial^{\circ}\mathcal{L}_k(\varrho), \partial^{\circ}\mathcal{N}_i(\varrho), \partial^{\circ}\mathcal{M}_i(\varrho)$ respectively, then this dual problem reduces to Wolfe type duality studied in [12]. Further, if all the functions are differentiable and convex then this duality is a generalization of the dual problem VC-WD($\aleph$) that is defined in [25].

The relationship between a QMPVC(1) feasible point and its associated Wolfe type dual feasible point is given by the following theorem.

Theorem 3.2 (Weak duality)**.** *Let $\aleph \in X$ and $(\varrho, \ell, \mu, \ell^{\mathcal{N}}, \ell^{\mathcal{M}}) \in S_w(\aleph)$ be feasible points for the QMPVC(1) and the VC-WD($\aleph$) respectively, suppose that $\mathcal{J}, \mathcal{K}_j (j \in \pounds^+_{\mathcal{K}}(\aleph))$, $\mathcal{L}_k (k \in \pounds^+_{\mathcal{L}})$, $-\mathcal{L}_k (k \in \pounds^-_{\mathcal{L}})$, $-\mathcal{N}_i (i \in \pounds^+_+(\aleph) \cup \pounds^+_0(\aleph))$, $\mathcal{N}_i (i \in \pounds^-_0(\aleph))$, $-\mathcal{M}_i (i \in \pounds^-_{0+}(\aleph) \cup \pounds^-_{00}(\aleph) \cup \pounds^-_{+0}(\aleph))$, $\mathcal{M}_i (i \in \pounds^+_{00}(\aleph) \cup \pounds^+_{0-}(\aleph) \cup \pounds^+_{+0}(\aleph) \cup \pounds^+_{+-}(\aleph))$ are quasidifferentiable invex at ϱ on $X \cup S_w(\aleph)$. Then*

$$\mathcal{J}(\aleph) \geq \mathcal{J}(\varrho) + \sum_{j=1}^{m} \ell_j \mathcal{K}_j(\varrho) + \sum_{k=1}^{n} \mu_k \mathcal{L}_k(\varrho) - \sum_{i=1}^{l} \ell_i^{\mathcal{N}} \mathcal{N}_i(\varrho) + \sum_{i=1}^{l} \ell_i^{\mathcal{M}} \mathcal{M}_i(\varrho).$$

Proof. Since $\mathcal{K}_j (j \in \pounds^+_{\mathcal{K}}(\aleph))$, $\mathcal{L}_k (k \in \pounds^+_{\mathcal{L}})$, $-\mathcal{L}_k (k \in \pounds^-_{\mathcal{L}})$, $-\mathcal{N}_i (i \in \pounds^+_+(\aleph) \cup \pounds^+_0(\aleph))$, $\mathcal{N}_i (i \in \pounds^-_0(\aleph))$, $-\mathcal{M}_i (i \in \pounds^-_{0+}(\aleph) \cup \pounds^-_{00}(\aleph) \cup \pounds^-_{+0}(\aleph))$, $\mathcal{M}_i (i \in \pounds^+_{00}(\aleph) \cup \pounds^+_{0-}(\aleph) \cup \pounds^+_{+0}(\aleph) \cup \pounds^+_{+-}(\aleph))$ are quasidifferentiable invex at ϱ on $X \cup S_w(\aleph)$, the feasibility of $\aleph$ for the QMPVC(1), the feasibility of $(\varrho, \ell, \mu, \ell^{\mathcal{N}}, \ell^{\mathcal{M}})$ for the VC-WD($\aleph$) and the definitions of the index sets, one has

$$\begin{aligned}
&\mathcal{K}_j(\varrho) + \langle w_j^*, \eta(\aleph, \varrho)\rangle \leq \mathcal{K}_j(\aleph) \leq 0, && \forall w_j^* \in S_{\mathcal{K}_j}(\varrho), \quad \forall j \in \pounds^+_{\mathcal{K}}(\aleph), \ell_j > 0,\\
&\mathcal{L}_k(\varrho) + \langle w_k^*, \eta(\aleph, \varrho)\rangle \leq \mathcal{L}_k(\aleph) = 0, && \forall w_k^* \in S_{\mathcal{L}_k}(\varrho), \quad \forall k \in \pounds^+_{\mathcal{L}}, \mu_k > 0,\\
&\mathcal{L}_k(\varrho) + \langle w_k^*, \eta(\aleph, \varrho)\rangle \geq \mathcal{L}_k(\aleph) = 0, && \forall w_k^* \in S_{\mathcal{L}_k}(\varrho), \quad \forall k \in \pounds^-_{\mathcal{L}}, \mu_k < 0,\\
&-\mathcal{N}_i(\varrho) - \langle v_i^*, \eta(\aleph, \varrho)\rangle \leq -\mathcal{N}_i(\aleph) < 0, && \forall v_i^* \in S_{\mathcal{N}_i}(\varrho), \quad \forall i \in \pounds^+_+(\aleph), \ell_i^{\mathcal{N}} > 0,\\
&-\mathcal{N}_i(\varrho) - \langle v_i^*, \eta(\aleph, \varrho)\rangle \leq -\mathcal{N}_i(\aleph) = 0, && \forall v_i^* \in S_{\mathcal{N}_i}(\varrho), \quad \forall i \in \pounds^+_0(\aleph), \ell_i^{\mathcal{N}} > 0,\\
&-\mathcal{N}_i(\varrho) - \langle v_i^*, \eta(\aleph, \varrho)\rangle \geq -\mathcal{N}_i(\aleph) = 0, && \forall v_i^* \in S_{\mathcal{N}_i}(\varrho), \quad \forall i \in \pounds^-_0(\aleph), \ell_i^{\mathcal{N}} < 0,\\
&\mathcal{M}_i(\varrho) + \langle w_i^*, \eta(\aleph, \varrho)\rangle \geq \mathcal{M}_i(\aleph) > 0, && \forall w_i^* \in S_{\mathcal{M}_i}(\varrho), \quad \forall i \in \pounds^-_{0+}(\aleph), \ell_i^{\mathcal{M}} < 0,\\
&\mathcal{M}_i(\varrho) + \langle w_i^*, \eta(\aleph, \varrho)\rangle \geq \mathcal{M}_i(\aleph) = 0, && \forall w_i^* \in S_{\mathcal{M}_i}(\varrho), \quad \forall i \in \pounds^-_{00}(\aleph) \cup \pounds^-_{+0}(\aleph), \ell_i^{\mathcal{M}} < 0,\\
&\mathcal{M}_i(\varrho) + \langle w_i^*, \eta(\aleph, \varrho)\rangle \leq \mathcal{M}_i(\aleph) = 0, && \forall w_i^* \in S_{\mathcal{M}_i}(\varrho), \quad \forall i \in \pounds^+_{00}(\aleph) \cup \pounds^+_{+0}(\aleph), \ell_i^{\mathcal{M}} > 0,\\
&\mathcal{M}_i(\varrho) + \langle w_i^*, \eta(\aleph, \varrho)\rangle \leq \mathcal{M}_i(\aleph) < 0, && \forall w_i^* \in S_{\mathcal{M}_i}(\varrho), \quad \forall i \in \pounds^+_{0-}(\aleph) \cup I^+_{+-}(\aleph), \ell_i^{\mathcal{M}} > 0,
\end{aligned}$$

which implies that

$$\begin{aligned}
&\sum_{j=1}^{m} \ell_j \mathcal{K}_j(\varrho) + \sum_{k=1}^{n} \mu_k \mathcal{L}_k(\varrho) - \sum_{i=1}^{l} \ell_i^{\mathcal{N}} \mathcal{N}_i(\varrho) + \sum_{i=1}^{l} \ell_i^{\mathcal{M}} \mathcal{M}_i(\varrho)\\
&\qquad + \left\langle \sum_{j=1}^{m} \ell_j w_j^* + \sum_{k=1}^{n} \mu_k w_k^* - \sum_{i=1}^{l} \ell_i^{\mathcal{N}} v_i^* + \sum_{i=1}^{l} \ell_i^{\mathcal{M}} w_i^*, \eta(\aleph, \varrho) \right\rangle \leq 0. \qquad (12)
\end{aligned}$$

Also $\mathcal{J}$ is quasidifferentiable invex at ϱ on $X \cup S_w(\aleph)$, one has

$$\mathcal{J}(\varrho) + \langle w_0^*, \eta(\aleph, \varrho)\rangle \leq \mathcal{J}(\aleph), \qquad \forall w_0^* \in S_{\mathcal{J}}(\varrho). \qquad (13)$$

Adding (12) and (13), we get

$$\begin{aligned}
&\mathcal{J}(\varrho) + \sum_{j=1}^{m} \ell_j \mathcal{K}_j(\varrho) + \sum_{k=1}^{n} \mu_k \mathcal{L}_k(\varrho) - \sum_{i=1}^{l} \ell_i^{\mathcal{N}} \mathcal{N}_i(\varrho) + \sum_{i=1}^{l} \ell_i^{\mathcal{M}} \mathcal{M}_i(\varrho)\\
&\quad + \left\langle \sum_{j=1}^{m} \ell_j w_j^* + \sum_{k=1}^{n} \mu_k w_k^* - \sum_{i=1}^{l} \ell_i^{\mathcal{N}} v_i^* + \sum_{i=1}^{l} \ell_i^{\mathcal{M}} w_i^*, \eta(\aleph, \varrho) \right\rangle + \langle w_0^*, \eta(\aleph, \varrho)\rangle \leq \mathcal{J}(\aleph) \qquad (14)
\end{aligned}$$

from inequality (10), there exists $\bar{\xi}_0 \in \underline{\partial}\mathcal{J}(\varrho) + w_0$, $\bar{\xi}_j^{\mathcal{K}} \in \underline{\partial}\mathcal{K}_j(\varrho) + w_j$, $\bar{\xi}_k^{\mathcal{L}} \in \underline{\partial}\mathcal{L}_k(\varrho) + w_k$, $\bar{\xi}_i^{\mathcal{M}} \in \underline{\partial}\mathcal{M}_i(\varrho) + w_i$ and $\bar{\xi}_i^{\mathcal{N}} \in \underline{\partial}\mathcal{N}_i(\varrho) + v_i$ such that

$$\bar{\xi}_0 + \sum_{j \in \pounds_{\mathcal{K}}} \ell_j \bar{\xi}_j^{\mathcal{K}} + \sum_{k=1}^{n} \mu_k \bar{\xi}_k^{\mathcal{L}} + \sum_{i=1}^{l} \ell_i^{\mathcal{M}} \bar{\xi}_i^{\mathcal{M}} - \sum_{i=1}^{l} \ell_i^{\mathcal{N}} \bar{\xi}_i^{\mathcal{N}} = 0. \qquad (15)$$

Using inequality (15) in (14), we get

$$\mathcal{J}(\aleph) \geq \mathcal{J}(\varrho) + \sum_{j=1}^{m} \ell_j \mathcal{K}_j(\varrho) + \sum_{k=1}^{n} \mu_k \mathcal{L}_k(\varrho) - \sum_{i=1}^{l} \ell_i^{\mathcal{N}} \mathcal{N}_i(\varrho) + \sum_{i=1}^{l} \ell_i^{\mathcal{M}} \mathcal{M}_i(\varrho).$$

□

Remark 3.3. Theorem 3.2 improves Theorem 3.3 [12] which is for locally Lipschitz Clark subdifferentials functions involving η-invex functions. Moreover, if all the functions are differentiable and convex then this results reduces to the weak duality results given by Mishra *et al.* [25].

The condition under which the associated Wolfe dual may be solved and the global maximum is attained is given by the ensuing result.

Theorem 3.4 (Strong duality). *Let $\aleph^* \in X$ be a local minimum of the QMPVC(1) such that the NNAMCQ holds at $\aleph^*$. Then their exist some vectors $\bar{\ell} \in \mathbb{R}^m$, $\bar{\mu} \in \mathbb{R}^n$, $\bar{\ell}^{\mathcal{N}}$, $\bar{\ell}^{\mathcal{M}} \in \mathbb{R}^l$ such that $(\aleph^*, \bar{\ell}, \bar{\mu}, \bar{\ell}^{\mathcal{N}}, \bar{\ell}^{\mathcal{M}})$ is a feasible point of the VC-WD$(\aleph^*)$ and*

$$\sum_{j=1}^{m} \bar{\ell}_j \mathcal{K}_j(\aleph^*) + \sum_{k=1}^{n} \bar{\mu}_k \mathcal{L}_k(\aleph^*) - \sum_{i=1}^{l} \bar{\ell}_i^{\mathcal{N}} \mathcal{N}_i(\aleph^*) + \sum_{i=1}^{l} \bar{\ell}_i^{\mathcal{M}} \mathcal{M}_i(\aleph^*) = 0. \tag{16}$$

Furthermore, if all the function $\mathcal{J}, \mathcal{K}_j(j \in \pounds_{\mathcal{K}}^+(\aleph^*))$, $\mathcal{L}_k(k \in \pounds_{\mathcal{L}}^+)$, $-\mathcal{L}_k(k \in \pounds_{\mathcal{L}}^-)$, $-\mathcal{N}_i(i \in \pounds_+^+(\aleph^*) \cup \pounds_0^+(\aleph^*))$, $\mathcal{N}_i(i \in \pounds_0^-(\aleph^*))$, $-\mathcal{M}_i(i \in \pounds_{0+}^-(\aleph^*) \cup \pounds_{00}^-(\aleph^*) \cup \pounds_{+0}^-(\aleph^*))$, $\mathcal{M}_i(i \in \pounds_{00}^+(\aleph^*) \cup \pounds_{0-}^+(\aleph^*) \cup \pounds_{+0}^+(\aleph^*) \cup \pounds_{+-}^+(\aleph^*))$ are quasidifferentiable invex at ϱ on $X \cup S_w(\aleph^*)$. Then $(\aleph^*, \bar{\ell}, \bar{\mu}, \bar{\ell}^{\mathcal{N}}, \bar{\ell}^{\mathcal{M}})$ is a global maximum of the VC-WD$(\aleph^*)$, that is,

$$\psi\big(\aleph^*, \bar{\ell}, \bar{\mu}, \bar{\ell}^{\mathcal{N}}, \bar{\ell}^{\mathcal{M}}\big) \geq \psi\big(\varrho, \ell, \mu, \ell^{\mathcal{N}}, \ell^{\mathcal{M}}\big), \qquad \forall\big(\varrho, \ell, \mu, \ell^{\mathcal{N}}, \ell^{\mathcal{M}}\big) \in S_w(\aleph^*),$$

and

$$\mathcal{J}(\aleph^*) = \psi\big(\aleph^*, \bar{\ell}, \bar{\mu}, \bar{\ell}^{\mathcal{N}}, \bar{\ell}^{\mathcal{M}}\big).$$

Proof. Since $\aleph^*$ is a local minimum of QMPVC(1) and NNAMCQ is satisfied at $\aleph^*$ and by Theorem 2.1 it follows that, for any sets of $w_0 \in \bar{\partial}\mathcal{J}(\aleph^*)$, $w_j \in \bar{\partial}\mathcal{K}_j(\aleph^*)$, $w_k \in \bar{\partial}\mathcal{L}_k(\aleph^*)$, $w_i \in \bar{\partial}\mathcal{M}_i(\aleph^*)$, $v_i \in \bar{\partial}\mathcal{N}_i(\aleph^*)$, there exists scalars $\beta_0 \geq 0$, $\bar{\ell}_j \geq 0 (j \in J)$, $\bar{\mu_k} \geq 0$, $\bar{\ell}_i^{\mathcal{N}} \geq 0$, $\bar{\ell}_i^{\mathcal{M}} \geq 0$ such that (4) and (5) is satisfied.

Hence $(\aleph^*, \bar{\ell}, \bar{\mu}, \bar{\ell}^{\mathcal{N}}, \bar{\ell}^{\mathcal{M}})$ is a feasible point of the VC-WD$(\aleph^*)$.

Also, by condition (5) and definition of index sets (2) and (3) the equality (16) follows, as proceeding in the Theorem 3.2, it follows that

$$\mathcal{J}(\aleph^*) \geq \mathcal{J}(\varrho) + \sum_{j=1}^{m} \ell_j \mathcal{K}_j(\varrho) + \sum_{k=1}^{n} \mu_k \mathcal{L}_k(\varrho) - \sum_{i=1}^{l} \ell_i^{\mathcal{N}} \mathcal{N}_i(\varrho) + \sum_{i=1}^{l} \ell_i^{\mathcal{M}} \mathcal{M}_i(\varrho), \qquad \forall\big(\varrho, \ell, \mu, \ell^{\mathcal{N}}, \ell^{\mathcal{M}}\big) \in S_w(\aleph^*),$$

and hence by condition (16) one has,

$$\psi\big(\aleph^*, \bar{\ell}, \bar{\mu}, \bar{\ell}^{\mathcal{N}}, \bar{\ell}^{\mathcal{M}}\big) \geq \psi\big(\varrho, \ell, \mu, \ell^{\mathcal{N}}, \ell^{\mathcal{M}}\big) \qquad \forall\big(\varrho, \ell, \mu, \ell^{\mathcal{N}}, \ell^{\mathcal{M}}\big) \in S_w(\aleph^*),$$

that is, $(\aleph^*, \bar{\ell}, \bar{\mu}, \bar{\ell}^{\mathcal{N}}, \bar{\ell}^{\mathcal{M}})$ is a global maximum of the VC-WD$(\aleph^*)$ and the local minimum of the QMPVC(1) and the global maximum of VC-WD$(\aleph^*)$ are equal. □

Remark 3.5. If we consider all the functions are differentiable and convex and replace NNAMCQ to VC-ACQ then the Theorem 3.4 reduces to Theorem 4 in [25].

The condition under which a feasible point of the related Wolfe dual produces a global minimum of the primal QMPVC is shown in the following finding.

Theorem 3.6 (Converse duality)**.** *Let $\aleph \in X$ be any feasible solution of the QMPVC and let $(\varrho^*, \bar{\ell}, \bar{\mu}, \bar{\ell}^{\mathcal{N}}, \bar{\ell}^{\mathcal{M}})$ be a feasible point of the VC-WD($\aleph$) such that*

$$\begin{aligned}
&\bar{\ell}_j \mathcal{K}_j(\varrho^*) \geq 0, && \forall j = 1, 2, \ldots, m\\
&\bar{\mu}_k \mathcal{L}_k(\varrho^*) = 0, && \forall k = 1, 2, \ldots, n\\
&-\bar{\ell}_i^{\mathcal{N}} \mathcal{N}_i(\varrho^*) \geq 0, && \forall i = 1, 2, \ldots, l\\
&\bar{\ell}_i^{\mathcal{M}} \mathcal{M}_i(\varrho^*) \geq 0, && \forall i = 1, 2, \ldots, l.
\end{aligned}$$

Suppose that $\mathcal{J}, \mathcal{K}_j(j \in \pounds_{\mathcal{K}}^{+}(\aleph))$, $\mathcal{L}_k(k \in \pounds_{\mathcal{L}}^{+})$, $-\mathcal{L}_k(k \in \pounds_{\mathcal{L}}^{-})$, $-\mathcal{N}_i(i \in \pounds_{+}^{+}(\aleph) \cup \pounds_{0}^{+}(\aleph))$, $\mathcal{N}_i(i \in \pounds_{0}^{-}(\aleph))$, $-\mathcal{M}_i(i \in \pounds_{0+}^{-}(\aleph) \cup \pounds_{00}^{-}(\aleph) \cup \pounds_{+0}^{-}(\aleph))$, $\mathcal{M}_i(i \in \pounds_{00}^{+}(\aleph) \cup \pounds_{0-}^{+}(\aleph) \cup \pounds_{+0}^{+}(\aleph) \cup \pounds_{+-}^{+}(\aleph))$ are quasidifferentiable invex at ϱ^* on $X \cup S_w(\aleph)$. Then ϱ^* is a global maximum of the QMPVC(1).

Proof. Suppose to the contrary that ϱ^* is not a global minimum of the QMPVC(1), that is, there exists $\widetilde{\aleph} \in X$ such that

$$\mathcal{J}\left(\widetilde{\aleph}\right) < \mathcal{J}(\varrho^*).$$

By the feasibility of $\widetilde{\aleph}$ for the QMPVC(1), the feasibility of $(\varrho^*, \bar{\ell}, \bar{\mu}, \bar{\ell}^{\mathcal{N}}, \bar{\ell}^{\mathcal{M}})$ for the VC-WD($\aleph$), by the definition of index sets (2), (3) and (11) and by the assumption in the theorem, we have

$$\begin{aligned}
&\mathcal{K}_j\left(\widetilde{\aleph}\right) \leq \mathcal{K}_j(\varrho^*), && \forall j \in \pounds_{\mathcal{K}}^{+}\left(\widetilde{\aleph}\right),\\
&\mathcal{L}_k\left(\widetilde{\aleph}\right) = \mathcal{L}_k(\varrho^*), && \forall k \in \pounds_{\mathcal{L}}^{+}\left(\widetilde{\aleph}\right) \cup \pounds_{\mathcal{L}}^{-}\left(\widetilde{\aleph}\right)\\
&-\mathcal{N}_i\left(\widetilde{\aleph}\right) \leq -\mathcal{N}_i(\varrho^*), && \forall i \in \pounds_{+}^{+}\left(\widetilde{\aleph}\right) \cup \pounds_{0}^{+}\left(\widetilde{\aleph}\right),\\
&-\mathcal{N}_i\left(\widetilde{\aleph}\right) \geq -\mathcal{N}_i(\varrho^*), && \forall i \in \pounds_{0}^{-}\left(\widetilde{\aleph}\right),\\
&\mathcal{M}_i\left(\widetilde{\aleph}\right) \geq \mathcal{M}_i(\varrho^*), && \forall i \in \pounds_{0+}^{-}\left(\widetilde{\aleph}\right) \cup \pounds_{00}^{-}\left(\widetilde{\aleph}\right) \cup \pounds_{+0}^{-}\left(\widetilde{\aleph}\right),\\
&\mathcal{M}_i\left(\widetilde{\aleph}\right) \leq \mathcal{M}_i(\varrho^*), && \forall i \in \pounds_{00}^{+}\left(\widetilde{\aleph}\right) \cup \pounds_{0-}^{+}\left(\widetilde{\aleph}\right) \cup \pounds_{+0}^{+}\left(\widetilde{\aleph}\right) \cup \pounds_{+-}^{+}\left(\widetilde{\aleph}\right).
\end{aligned}$$

Since $\mathcal{K}_j(j \in \pounds_{\mathcal{K}}^{+}(\widetilde{\aleph}))$, $\mathcal{L}_k(k \in \pounds_{\mathcal{L}}^{+})$, $-\mathcal{L}_k(k \in \pounds_{\mathcal{L}}^{-})$, $-\mathcal{N}_i(i \in \pounds_{+}^{+}(\widetilde{\aleph}) \cup \pounds_{0}^{+}(\widetilde{\aleph}))$, $\mathcal{N}_i(i \in \pounds_{0}^{-}(\widetilde{\aleph}))$, $-\mathcal{M}_i(i \in \pounds_{0+}^{-}(\widetilde{\aleph}) \cup \pounds_{00}^{-}(\widetilde{\aleph}) \cup \pounds_{+0}^{-}(\widetilde{\aleph}))$, $\mathcal{M}_i(i \in \pounds_{00}^{+}(\widetilde{\aleph}) \cup \pounds_{0-}^{+}(\widetilde{\aleph}) \cup \pounds_{+0}^{+}(\widetilde{\aleph}) \cup \pounds_{+-}^{+}(\widetilde{\aleph}))$ are quasidifferentiable invex at ϱ^* on $X \cup S_w(\widetilde{\aleph})$ and the definition of index sets it follows that

$$\begin{aligned}
&\left\langle w_j^*, \eta(\aleph, \varrho)\right\rangle \leq 0, && \forall w_j^* \in S_{\mathcal{K}_j}(\varrho^*), \quad \forall j \in \pounds_{\mathcal{K}}^{+}\left(\widetilde{\aleph}\right), \bar{\ell}_j > 0,\\
&\langle w_k^*, \eta(\aleph, \varrho)\rangle \leq 0, && \forall w_k^* \in S_{\mathcal{L}_k}(\varrho^*), \quad \forall k \in \pounds_{\mathcal{L}}^{+}\left(\widetilde{\aleph}\right), \bar{\mu}_k > 0,\\
&\langle w_k^*, \eta(\aleph, \varrho)\rangle \geq 0, && \forall w_k^* \in S_{\mathcal{L}_k}(\varrho^*), \quad \forall k \in \pounds_{\mathcal{L}}^{-}\left(\widetilde{\aleph}\right), \bar{\mu}_k < 0,\\
&-\langle v_i^*, \eta(\aleph, \varrho)\rangle \leq 0, && \forall v_i^* \in S_{\mathcal{N}_i}(\varrho^*), \quad \forall i \in \pounds_{+}^{+}\left(\widetilde{\aleph}\right) \cup \pounds_{0}^{+}\left(\widetilde{\aleph}\right), \bar{\ell}_i^{\mathcal{N}} > 0,\\
&-\langle v_i^*, \eta(\aleph, \varrho)\rangle \geq 0, && \forall v_i^* \in S_{\mathcal{N}_i}(\varrho^*), \quad \forall i \in \pounds_{0}^{-}\left(\widetilde{\aleph}\right), \bar{\ell}_i^{\mathcal{N}} < 0,\\
&\langle w_i^*, \eta(\aleph, \varrho)\rangle \geq 0, && \forall w_i^* \in S_{\mathcal{M}_i}(\varrho^*), \quad \forall i \in \pounds_{0+}^{-}\left(\widetilde{\aleph}\right) \cup \pounds_{00}^{-}\left(\widetilde{\aleph}\right) \cup \pounds_{+0}^{-}\left(\widetilde{\aleph}\right), \bar{\ell}_i^{\mathcal{M}} < 0,\\
&\langle w_i^*, \eta(\aleph, \varrho)\rangle \leq 0, && \forall w_i^* \in S_{\mathcal{M}_i}(\varrho^*), \quad \forall i \in \pounds_{00}^{+}\left(\widetilde{\aleph}\right) \cup \pounds_{+0}^{+}\left(\widetilde{\aleph}\right) \cup \pounds_{+-}^{+}\left(\widetilde{\aleph}\right), \bar{\ell}_i^{\mathcal{M}} > 0,
\end{aligned}$$

which implies that

$$\left\langle \sum_{j=1}^{m} \bar{\ell}_j w_j^* + \sum_{k=1}^{n} \bar{\mu}_k w_k^* - \sum_{i=1}^{l} \bar{\ell}_i^{\mathcal{N}} v_i^* + \sum_{i=1}^{l} \bar{\ell}_i^{\mathcal{M}} w_i^*, \eta(\aleph, \varrho) \right\rangle \leq 0. \tag{17}$$

from inequality (10), there exists $\bar{\xi}_0 \in \underline{\partial}\mathcal{J}(\varrho^*) + w_0$, $\bar{\xi}_j^{\mathcal{K}} \in \underline{\partial}\mathcal{K}_j(\varrho^*) + w_j$, $\bar{\xi}_k^{\mathcal{L}} \in \underline{\partial}\mathcal{L}_k(\varrho^*) + w_k$, $\bar{\xi}_i^{\mathcal{M}} \in \underline{\partial}\mathcal{M}_i(\varrho^*) + w_i$ and $\bar{\xi}_i^{\mathcal{N}} \in \underline{\partial}\mathcal{N}_i(\varrho^*) + v_i$ such that

$$\bar{\xi}_0 + \sum_{j \in I_{\mathcal{K}}} \bar{\ell}_j \bar{\xi}_j^{\mathcal{K}} + \sum_{k=1}^{n} \bar{\mu}_k \bar{\xi}_k^{\mathcal{L}} + \sum_{i=1}^{l} \bar{\ell}_i^{\mathcal{M}} \bar{\xi}_i^{\mathcal{M}} - \sum_{i=1}^{l} \bar{\ell}_i^{\mathcal{N}} \bar{\xi}_i^{\mathcal{N}} = 0. \tag{18}$$

Using inequality (18) in inequality (17), we get

$$\left\langle \bar{\xi}_0, \eta(\aleph, \varrho) \right\rangle \geq 0. \tag{19}$$

Also $\mathcal{J}$ is quasidifferentiable invex at ϱ^* on $X \cup S_w(\aleph)$ it follows that

$$\mathcal{J}\left(\widetilde{\aleph}\right) - \mathcal{J}(\varrho^*) \geq \langle \bar{\xi}_0, \eta(\aleph, \varrho) \rangle$$

from inequality (19), we get

$$\mathcal{J}\left(\widetilde{\aleph}\right) \geq \mathcal{J}(\varrho^*)$$

contradict to our hypothesis and hence the result. □

The results below provide a sufficient condition for the primal QMPVC feasible point to be a global minimum when employing the Wolfe dual.

Theorem 3.7 (Restricted converse duality). *Let $\aleph^* \in X$ be a feasible point of the QMPVC(1) and let $(\varrho^*, \bar{\ell}, \bar{\mu}, \bar{\ell}^{\mathcal{N}}, \bar{\ell}^{\mathcal{M}})$ be a feasible point of the VC-WD($\aleph$) such that $\mathcal{J}(\aleph^*) = \psi(\varrho^*, \bar{\ell}, \bar{\mu}, \bar{\ell}^{\mathcal{N}}, \bar{\ell}^{\mathcal{M}})$. Suppose that $\mathcal{J}, \mathcal{K}_j (j \in \pounds_{\mathcal{K}}^{+}(\aleph^*))$, $\mathcal{L}_k (k \in \pounds_{\mathcal{L}}^{+})$, $-\mathcal{L}_k (k \in \pounds_{\mathcal{L}}^{-})$, $-\mathcal{N}_i (i \in \pounds_{+}^{+}(\aleph^*) \cup \pounds_{0}^{+}(\aleph^*))$, $\mathcal{N}_i (i \in \pounds_{0}^{-}(\aleph^*))$, $-\mathcal{M}_i (i \in \pounds_{0+}^{-}(\aleph^*) \cup \pounds_{00}^{-}(\aleph^*) \cup \pounds_{+0}^{-}(\aleph^*))$, $\mathcal{M}_i (i \in \pounds_{00}^{+}(\aleph^*) \cup \pounds_{0-}^{+}(\aleph^*) \cup \pounds_{+0}^{+}(\aleph^*) \cup \pounds_{+-}^{+}(\aleph^*))$ are quasidifferentiable invex at ϱ^* on $X \cup S_w(\aleph^*)$. Then, $\aleph^*$ is a global minimum of the QMPVC(1).*

Proof. Suppose to the contrary that $\aleph^* \in X$ is not a global minimum of the QMPVC(1), then there exists $\widetilde{\aleph} \in X$ such that

$$\mathcal{J}\left(\widetilde{\aleph}\right) < \mathcal{J}(\aleph^*).$$

By the assumption in the theorem, it follows that

$$\mathcal{J}\left(\widetilde{\aleph}\right) < \psi(\varrho^*, \bar{\ell}, \bar{\mu}, \bar{\ell}^{\mathcal{N}}, \bar{\ell}^{\mathcal{M}}),$$

a contradiction to the weak duality Theorem 3.2 and hence the result. □

Remark 3.8. It should be noted that Theorems 3.6 and 3.7 are generalizations of Theorems 5 and 6 given in [25] to nonsmooth quasidifferentiable mathematical programs with vanishing constraints.

The following findings provide the need for uniqueness for a local minimum of the primal QMPVC and a global maximum of the associated Wolfe type dual.

Theorem 3.9 (Strict converse duality). *Let $\aleph^* \in X$ be a local minimum of the QMPVC(1) such that the NNAMCQ holds at $\aleph^*$ and the strong duality between the QMPVC(1) and the VC-WD($\aleph^*$) as in Theorem 3.4 holds. Also, let $(\varrho^*, \widetilde{\ell}, \widetilde{\mu}, \widetilde{\ell}^{\mathcal{N}}, \widetilde{\ell}^{\mathcal{M}})$ be a global maximum of the VC-WD($\aleph^*$). Suppose that $\mathcal{J}$ is quasidifferentiable strictly invex and $\mathcal{K}_j(j \in \pounds^+_{\mathcal{K}}(\aleph^*))$, $\mathcal{L}_k(k \in \pounds^+_{\mathcal{L}})$, $-\mathcal{L}_k(k \in \pounds^-_{\mathcal{L}})$, $-\mathcal{N}_i(i \in \pounds^+_+(\aleph^*) \cup \pounds^+_0(\aleph^*))$, $\mathcal{N}_i(i \in \pounds^-_0(\aleph^*))$, $-\mathcal{M}_i(i \in \pounds^-_{0+}(\aleph^*) \cup \pounds^-_{00}(\aleph^*) \cup \pounds^-_{+0}(\aleph^*))$, $\mathcal{M}_i(i \in \pounds^+_{00}(\aleph^*) \cup \pounds^+_{0-}(\aleph^*) \cup \pounds^+_{+0}(\aleph^*) \cup \pounds^+_{+-}(\aleph^*))$ are quasidifferentiable invex at ϱ^* on $X \cup S_w(\aleph^*)$ respectively. Then,*

$$\aleph^* = \varrho^*.$$

Proof. Suppose to the contrary that $\aleph^* \neq \varrho^*$. Then by the strong duality Theorem, there exists $\bar{\ell}_j \in \mathbb{R}^m$, $\bar{\mu}_k \in \mathbb{R}^n$, $\bar{\ell}_i^{\mathcal{N}}$, $\bar{\ell}_i^{\mathcal{M}} \in \mathbb{R}^l$ such that $(\aleph^*, \bar{\ell}_j, \bar{\mu}_k, \bar{\ell}_i^{\mathcal{N}}, \bar{\ell}_i^{\mathcal{M}})$ is a global maximum of the VC-WD($\aleph^*$) and hence

$$\mathcal{J}(\aleph^*) = \psi\big(\aleph^*, \bar{\ell}, \bar{\mu}, \bar{\ell}^{\mathcal{N}}, \bar{\ell}^{\mathcal{M}}\big) = \psi\Big(\varrho^*, \widetilde{\ell}, \widetilde{\mu}, \widetilde{\ell}^{\mathcal{N}}, \widetilde{\ell}^{\mathcal{M}}\Big). \tag{20}$$

Since $\mathcal{J}$ is quasidifferentiable strictly invex at ϱ^* on $X \cup S_w(\aleph^*)$, one has

$$\mathcal{J}(\aleph^*) - \mathcal{J}(\varrho^*) > \langle w_0^*, \eta(\aleph, \varrho)\rangle, \qquad \forall w_0^* \in S_{\mathcal{J}}(\varrho^*). \tag{21}$$

Also, $\mathcal{K}_j(j \in \pounds^+_{\mathcal{K}}(\aleph^*))$, $\mathcal{L}_k(k \in \pounds^+_{\mathcal{L}})$, $-\mathcal{L}_k(k \in \pounds^-_{\mathcal{L}})$, $-\mathcal{N}_i(i \in \pounds^+_+(\aleph^*) \cup \pounds^+_0(\aleph^*))$, $\mathcal{N}_i(i \in \pounds^-_0(\aleph^*))$, $-\mathcal{M}_i(i \in \pounds^-_{0+}(\aleph^*) \cup \pounds^-_{00}(\aleph^*) \cup \pounds^-_{+0}(\aleph^*))$, $\mathcal{M}_i(i \in \pounds^+_{00}(\aleph^*) \cup \pounds^+_{0-}(\aleph^*) \cup \pounds^+_{+0}(\aleph^*) \cup \pounds^+_{+-}(\aleph^*))$ are quasidifferentiable invex at ϱ^* on $X \cup S_w(\aleph^*)$, the feasibility of $\aleph^*$ for the QMPVC(1), the feasibility of $(\varrho^*, \widetilde{\ell}, \widetilde{\mu}, \widetilde{\ell}^{\mathcal{N}}, \widetilde{\ell}^{\mathcal{M}})$ for the VC-WD($\aleph^*$) and by the definition of index sets (13) one has,

$$\begin{aligned}
&\mathcal{K}_j(\varrho^*) + \big\langle w_j^*, \eta(\aleph, \varrho)\big\rangle \le \mathcal{K}_j(\aleph^*) \le 0, && \forall w_j^* \in S_{\mathcal{K}_j}(\varrho^*),\ \ \forall j \in \pounds^+_{\mathcal{K}}(\aleph^*), \widetilde{\ell}_j > 0,\\
&\mathcal{L}_k(\varrho^*) + \langle w_k^*, \eta(\aleph, \varrho)\rangle \le \mathcal{L}_k(\aleph^*) = 0, && \forall w_k^* \in S_{\mathcal{L}_k}(\varrho^*),\ \ \forall k \in \pounds^+_{\mathcal{L}}, \widetilde{\mu}_k > 0,\\
&\mathcal{L}_k(\varrho^*) + \langle w_k^*, \eta(\aleph, \varrho)\rangle \ge \mathcal{L}_k(\aleph^*) = 0, && \forall w_k^* \in S_{\mathcal{L}_k}(\varrho^*),\ \ \forall k \in \pounds^-_{\mathcal{L}}, \widetilde{\mu}_k < 0,\\
&-\mathcal{N}_i(\varrho^*) - \langle v_i^*, \eta(\aleph, \varrho)\rangle \le -\mathcal{N}_i(\aleph^*) < 0, && \forall v_i^* \in S_{\mathcal{N}_i}(\varrho^*),\ \ \forall i \in \pounds^+_+(\aleph^*), \widetilde{\ell}_i^{\mathcal{N}} > 0,\\
&-\mathcal{N}_i(\varrho^*) - \langle v_i^*, \eta(\aleph, \varrho)\rangle \le -\mathcal{N}_i(\aleph^*) = 0, && \forall v_i^* \in S_{\mathcal{N}_i}(\varrho^*),\ \ \forall i \in \pounds^+_0(\aleph^*), \widetilde{\ell}_i^{\mathcal{N}} > 0,\\
&-\mathcal{N}_i(\varrho^*) - \langle v_i^*, \eta(\aleph, \varrho)\rangle \ge -\mathcal{N}_i(\aleph^*) = 0, && \forall v_i^* \in S_{\mathcal{N}_i}(\varrho^*),\ \ \forall i \in \pounds^-_0(\aleph^*), \widetilde{\ell}_i^{\mathcal{N}} < 0,\\
&\mathcal{M}_i(\varrho^*) + \langle w_i^*, \eta(\aleph, \varrho)\rangle \ge \mathcal{M}_i(\aleph^*) > 0, && \forall w_i^* \in S_{\mathcal{M}_i}(\varrho^*),\ \ \forall i \in \pounds^-_{0+}(\aleph^*), \widetilde{\ell}_i^{\mathcal{M}} < 0,\\
&\mathcal{M}_i(\varrho^*) + \langle w_i^*, \eta(\aleph, \varrho)\rangle \ge \mathcal{M}_i(\aleph^*) = 0, && \forall w_i^* \in S_{\mathcal{M}_i}(\varrho^*),\ \ \forall i \in \pounds^-_{00}(\aleph^*) \cup \pounds^-_{+0}(\aleph^*), \widetilde{\ell}_i^{\mathcal{M}} < 0,\\
&\mathcal{M}_i(\varrho^*) + \langle w_i^*, \eta(\aleph, \varrho)\rangle \le \mathcal{M}_i(\aleph^*) = 0, && \forall w_i^* \in S_{\mathcal{M}_i}(\varrho^*),\ \ \forall i \in \pounds^+_{00}(\aleph^*) \cup \pounds^+_{+0}(\aleph^*), \widetilde{\ell}_i^{\mathcal{M}} > 0,\\
&\mathcal{M}_i(\varrho^*) + \langle w_i^*, \eta(\aleph, \varrho)\rangle \le \mathcal{M}_i(\aleph^*) < 0, && \forall w_i^* \in S_{\mathcal{M}_i}(\varrho^*),\ \ \forall i \in \pounds^+_{0-}(\aleph^*) \cup \pounds^+_{+-}(\aleph^*), \widetilde{\ell}_i^{\mathcal{M}} > 0,
\end{aligned}$$

which imples that

$$\begin{aligned}
&\sum_{j=1}^{m} \widetilde{\ell}_j \mathcal{K}_j(\varrho) + \sum_{k=1}^{n} \widetilde{\mu}_k \mathcal{L}_k(\varrho) - \sum_{i=1}^{l} \widetilde{\ell}_i^{\mathcal{N}} \mathcal{N}_i(\varrho) + \sum_{i=1}^{l} \widetilde{\ell}_i^{\mathcal{M}} \mathcal{M}_i(\varrho)\\
&\qquad + \left\langle \sum_{j=1}^{m} \widetilde{\ell}_j w_j^* + \sum_{k=1}^{n} \widetilde{\mu}_k w_k^* - \sum_{i=1}^{l} \widetilde{\ell}_i^{\mathcal{N}} v_i^* + \sum_{i=1}^{l} \widetilde{\ell}_i^{\mathcal{M}} w_i^*, \eta(\aleph, \varrho) \right\rangle \le 0.
\end{aligned} \tag{22}$$

Adding the inequalities (21) and (22), and using the duality constraint (10) of the VC-WD($\aleph^*$), we have

$$\psi\Big(\varrho^*, \widetilde{\ell}, \widetilde{\mu}, \widetilde{\ell}^{\mathcal{N}}, \widetilde{\ell}^{\mathcal{M}}\Big) < \mathcal{J}(\aleph^*),$$

a contradiction to the equality (20) and hence the required result. ☐

We have given following examples which shows that our results is also true for non Lipschitz functions. Here all of these are nonsmooth quasidifferentiable functions and satisfies all conditions. So it gives better results than previous work.

Example 1. Consider the following MPVC in $\mathbb{R}^2$:

$$\begin{aligned} \text{MPVC(1)} \quad & \min\ ||\aleph_1| + \aleph_2| \\ & \text{subject to: } 1 + ||\aleph_1| + \aleph_2| \geq 0, \\ & ||\aleph_1| - \aleph_2| \cdot (1 + ||\aleph_1| + \aleph_2|) \leq 0, \end{aligned} \tag{23}$$

where $\mathcal{J}(\aleph_1, \aleph_2) = ||\aleph_1| + \aleph_2|$, $\mathcal{M}_1(\aleph_1, \aleph_2) = ||\aleph_1| - \aleph_2|$, $\mathcal{N}_1(\aleph_1, \aleph_2) = 1 + ||\aleph_1| + \aleph_2|$.

Let $P := \{(\aleph_1, \aleph_2) :\ 1 + ||\aleph_1| + \aleph_2| \geq 0, ||\aleph_1| - \aleph_2|.(1 + ||\aleph_1| + \aleph_2|) \leq 0\}$ be feasible region of MPVC(1).

Here $\bar{\aleph} = (0, 0)$ is a feasible solution of MPVC(1). Since the directional derivatives are $\mathcal{J}'(\bar{\aleph}; d) = ||d_1| + d_2|$, $\mathcal{M}_1'(\bar{\aleph}; d) = ||d_1| - d_2|$ and $\mathcal{N}_1'(\bar{\aleph}; d) = ||d_1| + d_2|$ for any $d := (d_1, d_2) \in \mathbb{R}^2$.

The quasidifferentials of $\mathcal{J}, \mathcal{M}_1, \mathcal{N}_1$ at $\bar{\aleph}$ are given by $D_{\mathcal{J}}(\bar{\aleph}) = [\underline{\partial}\mathcal{J}(\bar{\aleph}), \bar{\partial}\mathcal{J}(\bar{\aleph})]$, $D_{\mathcal{M}_1}(\bar{\aleph}) = [\underline{\partial}\mathcal{M}_1(\bar{\aleph}), \bar{\partial}\mathcal{M}_1(\bar{\aleph})]$ and $D_{\mathcal{N}_1}(\bar{\aleph}) = [\underline{\partial}\mathcal{N}_1(\bar{\aleph}), \bar{\partial}\mathcal{N}_1(\bar{\aleph})]$, respectively, where

$$\begin{aligned} \underline{\partial}\mathcal{J}(\bar{\aleph}) &= co\{(0,0), (-2,-2), (2,2)\}, \quad \bar{\partial}\mathcal{J}(\bar{\aleph}) = co\{(1,-1), (-1,-1)\}, \\ \underline{\partial}\mathcal{M}_1(\bar{\aleph}) &= co\{(0,0), (0,-2), (-2,-2), (2,-2)\}, \quad \bar{\partial}\mathcal{M}_1(\bar{\aleph}) = co\{(-1,1), (1,1)\}, \\ \underline{\partial}\mathcal{N}_1(\bar{\aleph}) &= co\{(0,0), (2,-2), (2,2)\}, \quad \bar{\partial}\mathcal{N}_1(\bar{\aleph}) = co\{(-1,1), (-1,-1)\}, \end{aligned}$$

so, by Definition 2.5, $\mathcal{J}, \mathcal{M}_1, \mathcal{N}_1$ are quasidifferentiable functions at $\bar{\aleph}$.

We define $\ell : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ as follows:

$$\ell(\aleph, \varrho) := \begin{bmatrix} \frac{||\aleph_1|+\aleph_2|}{4} \\ \frac{||\aleph_1|+\aleph_2|}{4} \end{bmatrix}.$$

Hence, by Definition 2.5, we can easily verified that $\mathcal{J}, \mathcal{M}_1, \mathcal{N}_1$ are quasidifferentiable invex functions at $\bar{\aleph} = (0, 0)$. For any feasible $\aleph \in X$, the VC-WD($\aleph$) to the MPVC(23) is given by

$$\max\big(\psi, \ell^{\mathcal{N}}, \ell^{\mathcal{M}}\big) = ||\varrho_1| + \varrho_2| - \ell^{\mathcal{N}}(1 + ||\varrho_1| + \varrho_2|) + \ell^{\mathcal{M}}(||\varrho_1| - \varrho_2|)$$

for any sets of $w_0 \in \bar{\partial}\mathcal{J}(\varrho)$, $w_1 \in \bar{\partial}\mathcal{M}_1(\varrho)$, $v_1 \in \bar{\partial}\mathcal{N}_1(\varrho)$.

Subject to:

$$\begin{aligned} 0 \in\ & co\{(0,0), (-2,-2), (2,2)\} + w_0 - \ell^{\mathcal{N}}(co\{(0,0), (2,2), (2,-2)\} + v_1) \\ & + \ell^{\mathcal{M}}(co\{(0,0), (0,-2), (2,-2), (-2,-2)\} + w_1). \end{aligned} \tag{24}$$

and

$$\begin{aligned} &\ell^{\mathcal{N}} \geq 0, \quad \text{if } 1 \in \pounds_+(\aleph), && \ell^{\mathcal{N}} \in \mathbb{R}, \quad \text{if } 1 \in \pounds_0(\aleph), \\ &\ell^{\mathcal{M}} \leq 0, \quad \text{if } 1 \in \pounds_{0+}(\aleph), && \ell^{\mathcal{M}} \in \mathbb{R}, \quad \text{if } 1 \in \pounds_{00}(\aleph) \cup \pounds_{+0}(\aleph), \\ &\ell^{\mathcal{M}} \geq 0, \quad \text{if } 1 \in \pounds_{0-}(\aleph) \cup \pounds_{+-}(\aleph). \end{aligned} \tag{25}$$

To show that any feasible point $\aleph^* \in X$ is a global minimum of the MPVC(23), we have to show that $\mathcal{J}(\aleph^*) = \psi(\varrho^*, \ell_1^{\mathcal{N}}, \ell_1^{\mathcal{M}})$ for some $(\varrho^*, \ell_1^{\mathcal{N}}, \ell_1^{\mathcal{M}}) \in S_W$ such that the hypothesis of Theorem 3.7 holds at ϱ^* on $X \cup S_W(\aleph)$. Now, we choose $w_0 = (1, -1)$, $v_1 = (-1, 1)$, $w_1 = (-1, 1)$ and suppose $\ell_1^{\mathcal{N}} = \alpha \geq 0$ and $\ell_1^{\mathcal{M}} = 0$, we get

$$\begin{aligned} 0 \in co\{&(1+\alpha, -1-\alpha), (1-\alpha, -1-3\alpha), (1-\alpha, -1+\alpha), (-1+\alpha, -3-\alpha), \\ &(-1-\alpha, -3-3\alpha), (-1-\alpha, -3+\alpha), (3+\alpha, 1-\alpha), (3-\alpha, 1-3\alpha), (3-\alpha, 1+\alpha)\} \end{aligned}$$

for $\alpha = 1$, $(0,0)$ lies in this set. So condition (24) is satisfied.

Also,

$$\mathcal{J}(\aleph_1, \aleph_2) = \psi\big(\varrho_1, \varrho_2, \ell_1^{\mathcal{N}}, \ell_1^{\mathcal{M}}\big) = \psi\big(0, 0, \ell_1^{\mathcal{N}}, \ell_1^{\mathcal{M}}\big)$$
$$||\aleph_1| + \aleph_2| = ||\varrho_1| + \varrho_2| - 1 \cdot (1 + ||\varrho_1| + \varrho_2|).$$

From this we observe that

$$||\aleph_1| + \aleph_2| = 0$$

this is holds for $(0,0), (1,-1), (2,-2), \ldots$, but only $\aleph = (0,0)$ is lies in the feasible region of MPVC(23).

Also, it may be verified the Theorem 3.7, holds at ϱ^* on $X \cup S_W(\aleph)$.

Hence, by Theorem 3.7, $\aleph^* = (0,0)$ is a global minimum of the given QMPVC(23).

Example 2. Consider the following MPVC in $\mathbb{R}^2$:

$$\begin{aligned} \text{MPVC(2)} \quad & \min \ \ln(|\aleph_1 + |\aleph_2|| + 1) \\ & \text{subject to: } ||\aleph_1| + \aleph_2| \geq 0, \\ & \qquad ||\aleph_2| - \aleph_1| \cdot (||\aleph_1| + \aleph_2|) \leq 0, \end{aligned} \tag{26}$$

where $\mathcal{J}(\aleph_1, \aleph_2) = \ln(|\aleph_1 + |\aleph_2|| + 1)$, $\mathcal{M}_1(\aleph_1, \aleph_2) = ||\aleph_2| - \aleph_1|$, $\mathcal{N}_1(\aleph_1, \aleph_2) = ||\aleph_1| + \aleph_2|$.

Let $P := \{(\aleph_1, \aleph_2) : \ ||\aleph_1| + \aleph_2| \geq 0, ||\aleph_2| - \aleph_1|.(||\aleph_1| + \aleph_2|) \leq 0\}$ be feasible region of MPVC(2).

Here $\bar{\aleph} = (0,0)$ is a feasible solution of MPVC(2). Since the directional derivatives are $\mathcal{J}'(\bar{\aleph}; d) = |d_1 + |d_2||$, $\mathcal{M}_1'(\bar{\aleph}; d) = ||d_2| - d_1|$ and $\mathcal{N}_1'(\bar{\aleph}; d) = ||d_1| + d_2|$ for any $d := (d_1, d_2) \in \mathbb{R}^2$.

The quasidifferentials of $\mathcal{J}, \mathcal{M}_1, \mathcal{N}_1$ at $\bar{\aleph}$ are given by $D_{\mathcal{J}}(\bar{\aleph}) = [\underline{\partial}\mathcal{J}(\bar{\aleph}), \bar{\partial}\mathcal{J}(\bar{\aleph})]$, $D_{\mathcal{M}_1}(\bar{\aleph}) = [\underline{\partial}\mathcal{M}_1(\bar{\aleph}), \bar{\partial}\mathcal{M}_1(\bar{\aleph})]$ and $D_{\mathcal{N}_1}(\bar{\aleph}) = [\underline{\partial}\mathcal{N}_1(\bar{\aleph}), \bar{\partial}\mathcal{N}_1(\bar{\aleph})]$, respectively, where

$$\begin{aligned} \underline{\partial}\mathcal{J}\big(\bar{\aleph}\big) &= co\{(0,0), (-2,2), (2,2)\}, \quad \bar{\partial}\mathcal{J}\big(\bar{\aleph}\big) = co\{(-1,-1), (1,-1)\}, \\ \underline{\partial}\mathcal{M}_1\big(\bar{\aleph}\big) &= co\{(0,0), (-2,-2), (2,-2)\}, \quad \bar{\partial}\mathcal{M}_1\big(\bar{\aleph}\big) = co\{(1,-1), (-1,-1)\}, \\ \underline{\partial}\mathcal{N}_1\big(\bar{\aleph}\big) &= co\{(0,0), (-2,-2), (2,2)\}, \quad \bar{\partial}\mathcal{N}_1\big(\bar{\aleph}\big) = co\{(1,-1), (-1,-1)\}, \end{aligned}$$

so, by Definition 2.5, $\mathcal{J}, \mathcal{M}_1, \mathcal{N}_1$ are quasidifferentiable functions at $\bar{\aleph}$.

We define $\ell : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ as follows:

$$\ell(\aleph, \varrho) := \begin{bmatrix} \frac{|\aleph_1 + |\aleph_2||}{3} \\ 0 \end{bmatrix}.$$

Hence, by Definition 2.5, we can easily verified that $\mathcal{J}, \mathcal{M}_1, \mathcal{N}_1$ are quasidifferentiable invex functions at $\bar{\aleph} = (0,0)$. For any feasible $\aleph \in X$, the VC-WD($\aleph$) to the MPVC(26) is given by

$$\max\big(\psi, \ell^{\mathcal{N}}, \ell^{\mathcal{M}}\big) = \ln(|\varrho_1 + |\varrho_2|| + 1) - \ell^{\mathcal{N}}(||\varrho_1| + \varrho_2|) + \ell^{\mathcal{M}}(||\varrho_2| - \varrho_1|)$$

for any sets of $w_0 \in \bar{\partial}\mathcal{J}(\varrho)$, $w_1 \in \bar{\partial}\mathcal{M}_1(\varrho)$, $v_1 \in \bar{\partial}\mathcal{N}_1(\varrho)$.

Subject to:

$$\begin{aligned} 0 \in co\{(0,0), (-2,2), (2,2)\} + w_0 - \ell^{\mathcal{N}}(co\{(0,0), (2,2), (-2,-2)\} + v_1) \\ + \ell^{\mathcal{M}}(co\{(0,0), (-2,-2), (2,-2)\} + w_1) \end{aligned} \tag{27}$$

and

$$\begin{aligned} &\ell^{\mathcal{N}} \geq 0, \quad \text{if } 1 \in \pounds_+(\aleph), && \ell^{\mathcal{N}} \in \mathbb{R}, \quad \text{if } 1 \in \pounds_0(\aleph), \\ &\ell^{\mathcal{M}} \leq 0, \quad \text{if } 1 \in \pounds_{0+}(\aleph), && \ell^{\mathcal{M}} \in \mathbb{R}, \quad \text{if } 1 \in \pounds_{00}(\aleph) \cup \pounds_{+0}(\aleph), \end{aligned}$$

$$\ell^{\mathcal{M}} \geq 0, \quad \text{if } 1 \in \pounds_{0-}(\aleph) \cup \pounds_{+-}(\aleph). \tag{28}$$

To show that any feasible point $\aleph^* \in X$ is a global minimum of the MPVC(26), we have to show that $\mathcal{J}(\aleph^*) = \psi(\varrho^*, \ell_1^{\mathcal{N}}, \ell_1^{\mathcal{M}})$ for some $(\varrho^*, \ell_1^{\mathcal{N}}, \ell_1^{\mathcal{M}}) \in S_W$ such that the hypothesis of Theorem 3.7 holds at ϱ^* on $X \cup S_W(\aleph)$. Now, we choose $w_0 = (1, -1)$, $v_1 = (1, -1)$, $w_1 = (-1, 1)$ and suppose $\ell_1^{\mathcal{N}} = \alpha \geq 0$ and $\ell_1^{\mathcal{M}} = 0$, we get

$$0 \in co\{(1-\alpha, -1+\alpha), (1-3\alpha, -1-\alpha), (1+\alpha, -1-\alpha), (1+\alpha, -1+3\alpha), (-1-\alpha, 1+\alpha), (-1-3\alpha, 1-\alpha), (-1+\alpha, 1+3\alpha), (3-\alpha, 1+\alpha), (3-3\alpha, 1-\alpha), (3+\alpha, 1+3\alpha)\}$$

for $\alpha = 1$, $(0, 0)$ lies in this set. So condition (27) is satisfied.

Also,

$$\mathcal{J}(\aleph_1, \aleph_2) = \psi(\varrho_1, \varrho_2, \ell_1^{\mathcal{N}}, \ell_1^{\mathcal{M}}) = \psi(0, 0, \ell_1^{\mathcal{N}}, \ell_1^{\mathcal{M}})$$
$$\ln(|\aleph_1 + |\aleph_2|| + 1) = \ln(|\varrho_1 + |\varrho_2|| + 1) - 1 \cdot (||\varrho_1| + \varrho_2|).$$

This is possible only for $\aleph = (0, 0)$ which is lies in the feasible region of MPVC(26).

Also, it may be verified the Theorem 3.7, holds at ϱ^* on $X \cup S_W(\aleph)$.

Hence, by Theorem 3.7, $\aleph^* = (0, 0)$ is a global minimum of the given QMPVC(26).

4. Conclusion

In this paper, we have considered a quasidifferentiable optimization problems with vanishing constraints. We have formulated Wolfe type duality results between the primal problem and its related dual and established weak, strong, converse, restricted converse, and strict converse for qusidifferentiable optimization problems with vanishing constraints. Under the suitable choice of invex and strictly invex functions with respect to a convex compact set based on the results obtained by Mishra *et al.* [25]. Further, Ghobadzadeh *et al.* [12] established this results using Clarke subdifferentials. We improve these results for quasidifferential functions because in case of quasidifferentials the nature of optimal points gives more accurately in comparison to the other subdifferentials. Also, quasidifferentiable functions play an important role in various fields of nonsmooth modeling in mechanics, engineering, economics [9, 34].

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Conflict of Interest

The authors declare that they have no competing interests.

Author contribution statement

All authors contributed equally to this work.

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