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(G, V)-INVEXITY NONDIFFERENTIABLE GENERALIZED MINIMAX FRACTIONAL PROGRAMMING

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Abstract. In this paper, we study the nondifferentiable generalized minimax fractional programming problem under (G, V)-invexity. G-sufficient optimality conditions for the considered nondifferentiable generalized minimax fractional programming problem are established under the concept of (G, V)-invexity. Further, two types of dual models are formulated and G-duality theorems relating to the primal minimax fractional programming problem and dual problems are established. These results extend several known results to a wider class of programs.

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1. Introduction

Convexity plays a central role in minimax programming. However, due to the limitations of convexity, in order to solve more practical problems, Heason [1] introduced the concept of differentiable invexity function. Craven [2] named it as an invexity function. Mond *et al.* [3] introduced the concept of preconvex function, which is a special case of convexity.

In recent years, the derivation of optimality and duality theorems in minimax programming has also been a research hotspot. Liu [4] discussed the sufficient optimality conditions for minimax fractional programming under the assumption of (F, ρ)-invexity, and [5] discussed its duality. On the basis of [4], Yang *et al.* [6] gave the concept of generalized (F, ρ)-convexity and established sufficient optimality conditions and duality for weakly convex generalized minimax fractional programming problems under its assumptions. Other scholars have studied optimality conditions and dual results for minimax fractional programming problems using the generalized convexity assumption, see for example [7–20] and the references cited therein. Recently, Antczak [21] extended the invexity of scalar differentiable functions to G-invexity. In [22], the concept of vector-valued G-differentiable function is given, and the optimality conditions of differentiable multiobjective programming with equality and inequality constraints are established. According to the concept of vector G-invexity function, the G-duality result of nonlinear differentiable multiobjective programming problem is further proved [23]. Antczak [24] introduced non-smooth (G, V)-invexity functions and proved the sufficient optimality theorem for nonsmooth multi-objective programming problems by using the concept of nondifferentiable generalized invexity. Yuan *et al.*

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[25] proved the sufficient optimality conditions and G-duality of nondifferentiable (G,β)-invexity functions in generalized minimax programming, and further discussed similarly in minimax fractional programming [26].

Inspired by the work of [24–26], this paper considers a class of nondifferentiable generalized minimax fractional programming problems (FP) with (G,V)-invexity functions. According to the necessary optimality conditions of the given nondifferentiable optimization problem, an auxiliary minimax fractional programming problem $(G-P_k)$ is constructed. On this basis, the G-necessary optimality conditions are established. Furthermore, we construct two duality problems (DI) and (DII) related to the nondifferentiable generalized minimax fractional programming problem (FP). Under the assumption of (G,V)-invexity, the G-duality results are obtained.

2. Notations and preliminaries

This section gives the relevant definitions and theorems of the paper. The following convention for equalities and inequalities will be used throughout the paper. For any $x=(x_1,x_2,\dots,x_n)^T, y=(y_1,y_2,\dots,y_n)^T$, we define:

(i) $x=y$ if and only if $x_i=y_i$ for all $i=1,2,\dots,n$;
(ii) $x>y$ if and only if $x_i>y_i$ for all $i=1,2,\dots,n$;
(iii) $x\geqq y$ if and only if $x_i\geqq y_i$ for all $i=1,2,\dots,n$;
(iv) $x\geq y$ if and only if $x_i>y_i$ and $x\neq y$.

Let R^n be the n-dimensional Euclidean space and R^n_+ be its nonnegative orthant, X be a non-empty open subset of R^n, and $u\in x$. Let $R^+=\{x\in R^n|x>0\}$. Further, let $f=(f_1,\dots,f_s):X\to R^s$ where each f_i is a locally Lipschitz function on X and $I_{f_i}(x), i\in I=\{1,\dots,s\}$, be the range of f_i, that is, the image of X under f_i. $I(X,Y)$ is the range of $f_i,g_i,i=1,\dots,s$, that is, the image of X under f_i,g_i.

Definition 2.1 ([27]). Let $d\in R^n$, X be a non-empty set of R^n and $f:X\to R$. If

$$f^0(x,d):=\lim_{y\to x_{\mu\downarrow 0}}\sup\frac{1}{\mu}(f(y+\mu d)-f(y)),$$

exists and then $f^0(x;d)$ is called the Clarke derivative of at x in the direction d. If this limit super exists for all $d\in R^n$, then f is called the Clarke differentiable at x. The set

$$\partial f(x)=\left\{f^0(x,d)\le\langle\zeta.d\rangle,\forall d\in R^n\right\}$$

is called the Clarke subdifferential of f at x.

Note that if a given function f is locally Lipschitz, then the Clarke subdifferential $\partial f(x)$ exists.

Lemma 2.2 ([24]). *Let Ψ be a real-valued Lipschitz continuous function defined on X and denote the image of X under Ψ by $I_\Psi(x)$; let $\varphi:I_\Psi(x)\to R$ be a differentiable function such that $\varphi'(\gamma)$ is continuous on $I_\Psi(x)$ and $\varphi'(\gamma)\le 0$ for each $\gamma\in\Psi(x)$. Then the chain rule*

$$(\varphi\circ\Psi)^0(x,d)=\varphi'(\Psi(x))\Psi^0(x,d),$$

holds for each $d\in R^n$. Therefore

$$\partial(\varphi\circ\Psi)(x)=\varphi'(\Psi(x))\partial(\Psi)(x).$$

Definition 2.3 ([23]). If there exist $G_f=(G_{f_1},\dots,G_{f_s}):R\to R^n$ such that any its component $G_{f_i}:I_{f_i}(x)\to R, i=1,\dots,s$, is a strictly increasing differentiable real-valued function on its domain $I_{f_i}(x)$, a vector-valued function $\alpha_f=(\alpha_{f_1},\dots,\alpha_{f_s}):X\times X\to R^n$, where $\alpha_{f_i}:X\times X\to R\backslash\{0\}, i=1,\dots,s$, and a vector-valued function $\eta:X\times X\to R^n$ such that the inequalities

$$G_{f_i}(f_i(x))-G_{f_i}(f_i(u))\geqq\alpha_{f_i}(x,u)G'_{f_i}(f_i(u))\langle\xi_i,\eta(x,u)\rangle,\quad i\in I$$

hold for all $x \in X$ and each $\xi_i \in \partial f_i(u), i = 1, \dots, s$, then f is said to be a nondifferentiable (G, V)-invex function at on X with respect to η, G_f and α_f. If the direction of the above inequality is changed to the opposite direction, then we call f is a nondifferentiable (G, V)-incave function at on X with respect to η, G_f and α_f.

3. Optimality conditions

In this section, we consider the following nondifferentiable generalized minimax fractional programming problem (FP):

$$\min\left\{ F(x) := \sup_{y\in Y}\left\{ \phi(x,y) := \frac{f(x,y) + \langle x, Ax\rangle^{1/2}}{g(x,y) - \langle x, Bx\rangle^{1/2}} \right\}\right\}$$
$$\text{s.t. } h_j(x) \le 0, \quad j = 1, \dots, p,$$

where Y is a compact subset of R^m, $f(\cdot,\cdot), g(\cdot,\cdot) : R^n \times R^m \to R$ and $h_j(\cdot) : R^n \to R (j \in J = \{1, \dots, p\})$ are C^1-mappings. Let A and B be $n * n$ positive semidefinite matrices. The problem (FP) is a nondifferentiable optimization problem if either A or B is nonzero. If A and B are null matrices, problem (FP) is a differentiable generalized minimax fractional programming problem.

Let $D := \{x \in R^n \mid h_j(x) \le 0, j \in J\}$ be the set of all feasible solutions of the problem (FP) and $(x, y) \in X \times Y, f(x, y) + \langle x, Ax\rangle^{1/2} \ge 0$ and $g(x, y) - \langle x, Bx\rangle^{1/2} > 0$. For each $x \in D$, we define the following sets:

$J(x) := \{j \in J \mid h_j(x) = 0\}$,

$Y(x) := \left\{ y \in Y \mid \frac{f(x,y)+\langle x,Ax\rangle^{1/2}}{g(x,y)-\langle x,Bx\rangle^{1/2}} = \sup_{z\in Y} \frac{f(x,z)+\langle x,Ax\rangle^{1/2}}{g(x,z)-\langle x,Bx\rangle^{1/2}} \right\}$,

$K(x) := \{(s, t, \bar{y}) \in N \times R^s_+ \times R^{ms} : 1 \le s \le n+1, t = (t_1, t_2, \dots, t_s)$ with $\sum_{i=1}^s t_i = 1$ and $\bar{y} = (\bar{y}_1, \bar{y}_2, \dots, \bar{y}_m)$ with $\bar{y}_i \in Y(x), i = 1, 2, \dots, s\}$.

Since f and g are continuously differentiable and Y is a compact subset of R^m, it follows that, for each $x^* \in D$, $Y(x^*) \ne \varnothing$, $y_i \in Y(x^*)$, there is a positive constant k_0, $k_0 = \phi(x^*, y_i)$. The following generalized Schwartz inequality will be needed in our considerations:

$$\langle x, Av\rangle \le \langle x, Ax\rangle^{1/2} \langle v, Av\rangle^{1/2}, \qquad \forall x, v \in R^n. \tag{3.1}$$

The equality holds if $Ax = \lambda Av$, for some $\lambda \ge 0$. Hence, if $\langle v, Av\rangle^{1/2} \le 1$, we have

$$\langle x, Av\rangle \le \langle x, Ax\rangle^{1/2}. \tag{3.2}$$

Now, we will also use the following auxiliary programming problem $(G - P_k)$:

$$\min\sup_{y\in Y} G \circ \left(f(x,y) + \langle x, Ax\rangle^{1/2}\right) - G \circ \left(k\left(g\left((x,y) - \langle x, Bx\rangle^{1/2}\right)\right)\right)$$
$$\text{s.t. } G_{h_j} h_j(x) \le G_{h_j}(0), \quad j = 1, \dots, p.$$

We denote by $D_{G-P} := \{x \in R^n \mid G_{h_j} h_j(x) \le G_{h_j}(0), j = 1, \dots, p\}$, $J'(\bar{x}) = \{j \in J \mid G_{h_j} h_j(\bar{x}) = G_{h_j}(0)\}$. Assume the function G_{h_j} is strictly increasing on $I_{h_j}(x)$ for each $j \in J$, then $D = D_{G-P}$ and $J(\bar{x}) = J'(\bar{x})$.

Lemma 3.1 ([25]). *Let x^* be an optimal solution for (FP) and $k_0 := F(x^*)$. If the functions G increases strictly in R, then*

(i) *x^* is an optimal solution to the problem $(G - P_{k_0})$ and $\phi(x^*, k_0) = 0$;*
(ii) *$G \circ (f(x^*, \bar{y}_i) + \langle x^*, Au\rangle^{1/2}) - G \circ (k_0(g(x^*, \bar{y}_i) + \langle x^*, Bv\rangle^{1/2})) = 0$, whenever $\bar{y} \in Y(x^*)$.*

Theorem 3.2 (Necessary optimality conditions [26])**.** *Let $\bar{x}$ be an optimal; solution for (FP) satisfying $\langle \bar{x}, A\bar{x}\rangle > 0$, $\langle \bar{x}, B\bar{x}\rangle > 0$ and $\partial h_j(\bar{x}), j \in J(\bar{x})$ are linearly independent. Then, there exist $(s, t^*, \bar{y}) \in K(\bar{x}), k_0 \in R^+, u, v \in R^n$ and $\mu^* \in R^p_+$, such that*

$$\sum_{i=1}^{s} t_i^* \{\partial f(\bar{x}, \bar{y}_i) + Au - k_0(\partial g(\bar{x}, \bar{y}_i) - Bv)\} + \sum_{j=1}^{p} \mu_j^* \partial h(\bar{x}) = 0, \tag{3.3}$$

$$f(\bar{x}, \bar{y}_i) + \langle \bar{x}, A\bar{x}\rangle^{1/2} - k_0\Big(g(\bar{x}, \bar{y}_i) - \langle \bar{x}, B\bar{x}\rangle^{1/2}\Big) = 0, \quad i = 1, \ldots, s, \tag{3.4}$$

$$\langle \mu^*, h(\bar{x})\rangle = 0, \tag{3.5}$$

$$t_i^* \geq 0, \quad i = 1, \ldots, s, \text{with} \sum_{i=1}^{s} t_i^* = 1, \tag{3.6}$$

$$\langle u, Au\rangle \leq 1, \langle v, Bv\rangle \leq 1, \tag{3.7}$$

$$\langle \bar{x}, Au\rangle = \langle \bar{x}, A\bar{x}\rangle^{1/2}, \langle \bar{x}, Bv\rangle = \langle \bar{x}, B\bar{x}\rangle^{1/2}. \tag{3.8}$$

According to Theorem 3.2, the G-necessary optimality conditions of the problem (FP) can be derived as follows, see Theorem 3.3.

Theorem 3.3 (G-necessary optimality conditions)**.** *Let problem (FP) is satisfying $< \bar{x}, A\bar{x} > 0$, $\langle \bar{x}, B\bar{x}\rangle > 0$. Let $\bar{x}$ be an optimal solution of (FP). Assume that G is both continuously differentiable and strictly increasing on $I(X, Y)$. If G_{h_j} is both continuously differentiable and strictly increasing on $I_{h_j}(X)$ with $G'_{h_j}(h_j(\bar{x})) > 0$ for each $j \in J$, then there exists $(s, t^*, \bar{y}) \in K(\bar{x}), k_0 \in R^+, u, v \in R^n$ and $\mu^* \in R^p_+$, such that*

$$\begin{aligned} 0 \in & \sum_{i=1}^{s} t_i^* G'(f(\bar{x}, \bar{y}_i) + \langle \bar{x}, Au\rangle)(\partial_x f(\bar{x}, \bar{y}_i) + Au) \\ & - \sum_{i=1}^{s} t_i^* k_0 G'(k_0(g((\bar{x}, \bar{y}_i) - \langle \bar{x}, Bv\rangle)))(\partial_x g(\bar{x}, \bar{y}_i) - Bv) + \sum_{j=1}^{p} \mu_j^* G'(h(\bar{x}))\partial_x h_j(\bar{x}). \end{aligned} \tag{3.9}$$

$$G \circ \Big(f(\bar{x}, \bar{y}_i) + \langle \bar{x}, Au\rangle^{1/2}\Big) - G \circ \Big(k_0\Big(g\Big((\bar{x}, \bar{y}_i) - \langle \bar{x}, Bv\rangle^{1/2}\Big\rangle\Big)\Big) = 0, \quad i = 1, \ldots, s, \tag{3.10}$$

$$\mu_j^*(G \circ h_j(\bar{x}) - G \circ h_j(0)) = 0, \mu_j^* \geq 0, j \in J, \tag{3.11}$$

$$t_i^* \geq 0, \quad i = 1, \ldots, s, \text{with} \sum_{i=1}^{s} t_i^* = 1, \tag{3.12}$$

$$\langle u, Au\rangle \leq 1, \langle v, Bv\rangle \leq 1, \tag{3.13}$$

$$\langle \bar{x}, Au\rangle = \langle \bar{x}, A\bar{x}\rangle^{1/2}, \langle \bar{x}, Bv\rangle = \langle \bar{x}, B\bar{x}\rangle^{1/2}. \tag{3.14}$$

Proof. Since $\bar{x}$ is an optimal solution to problem (FP), by Lemma 3.1, $\bar{x}$ is an optimal solution to problem $(G - Pk_0)$ and E_q. We can deduce from (3.3) to (3.6) and constraints [25] are satisfied for problem $(G - Pk_0)$, then there exist positive integers and vectors $\bar{y}_i \in Y(x^*)$ together with scalars $t_i^*, i = 1, \ldots, s$, and $\mu_j^*(j \in J)$ such that (3.11), (3.12) and

$$\begin{aligned} 0 \in & \sum_{i=1}^{s} t_i^* \{\partial_x(G \circ (f(\bar{x}, \bar{y}_i) + \langle \bar{x}, Au\rangle)) - \partial_x(G \circ (k_0(g(\bar{x}, \bar{y}_i) - \langle \bar{x}, Bv\rangle))))\} \\ & + \sum_{j=1}^{p} \mu_j^* \partial_x\big(G_{h_j} \circ h_j(\bar{x})\big) \end{aligned} \tag{3.15}$$

hold, by Lemma 2.2 and the continuity of G and G_{h_j}, we have

$$\partial_x(G \circ (f(\bar{x}, \bar{y}_i) + \langle \bar{x}, Au\rangle)) - \partial_x(G \circ (k_0(g(\bar{x}, \bar{y}_i) - \langle \bar{x}, Bv\rangle)))$$

$$= G^{'}(f(\bar{x}, \bar{y}_i) + \langle \bar{x}, Au\rangle)(\partial_x f(\bar{x}, \bar{y}_i) + Au) - k_0 G^{'}(k_0(g((\bar{x}, \bar{y}_i) - \langle \bar{x}, Bv\rangle)))$$
$$(\partial_x g(\bar{x}, \bar{y}_i) - Bv), \quad i = 1, \dots, s,$$
$$\partial_x \big(G_{h_j} \circ h_j\big)(\bar{x}) = G^{'}(h(\bar{x}))\partial_x h_j(\bar{x}), j \in J.$$

According to the above equation and (3.15), we get (3.9), the proof is complete. □

Theorem 3.4 (G-sufficient optimality conditions)**.** *Suppose that there $x^* \in D$ is a feasible solution for (FP) and exist $k_0 \in R^+, (s, t^*, \bar{y}) \in K(x^*), u, v \in R^n$ and $\mu \in R^p_+$ satisfies (3.9)–(3.14). Let G is both continuously differentiable and strictly increasing on $I(X, Y)$, G_{h_j} is both continuously differentiable and strictly increasing on $I_{h_j}(X)$ for each $j \in J$. Let $f(\cdot, \bar{y}_i) + \langle \cdot, Au\rangle$ and $k(g(\cdot, \bar{y}_i) - \langle \cdot, Bv\rangle)$ is (G, V)-invexity and (G, V)-incave function at $\bar{x} \in D$, respectively; and $h_j(\cdot), j = 1, \dots, p$ is (G, V)-invexity at $\bar{x}$ on D, then $\bar{x}$ is an optimal solution to (FP).*

Proof. Suppose that $\bar{x}$ is not an optimal solution to (FP). Hence, there exists $x^* \in D$ such that

$$\sup_{y\in Y} \frac{f(x^*, y) + \langle x^*, Ax^*\rangle^{1/2}}{g(x^*, y) - \langle x^*, Bx^*\rangle^{1/2}} < \sup_{y\in Y} \frac{f(\bar{x}, y) + \langle \bar{x}, A\bar{x}\rangle^{1/2}}{g(\bar{x}, y) - \langle \bar{x}, B\bar{x}\rangle^{1/2}}.$$

We observe that

$$\sup_{y\in Y} \frac{f(\bar{x}, y) + \langle \bar{x}, A\bar{x}\rangle^{1/2}}{g(\bar{x}, y) - \langle \bar{x}, B\bar{x}\rangle^{1/2}} = \frac{f(\bar{x}, \bar{y}_i) + \langle \bar{x}, A\bar{x}\rangle^{1/2}}{g(\bar{x}, \bar{y}_i) - \langle \bar{x}, B\bar{x}\rangle^{1/2}} = k_0,$$

for $\bar{y}_i \in Y(x^*), i = 1, \dots, s$ and

$$\frac{f(x^*, \bar{y}_i) + \langle x^*, Ax^*\rangle^{1/2}}{g(x^*, \bar{y}_i) - \langle x^*, Bx^*\rangle^{1/2}} \le \sup_{y\in Y} \frac{f(x^*, y) + \langle x^*, Ax^*\rangle^{1/2}}{g(x^*, y) - \langle x^*, Bx^*\rangle^{1/2}}.$$

Thus, we have

$$\frac{f(x^*, \bar{y}_i) + \langle x^*, Ax^*\rangle^{1/2}}{g(x^*, \bar{y}_i) - \langle x^*, Bx^*\rangle^{1/2}} < k_0, \qquad i = 1, \dots, s,$$

hold,

$$f(x^*, \bar{y}_i) + \langle x^*, Ax^*\rangle^{1/2} - k_0\Big(g(x^*, \bar{y}_i) - \langle x^*, Bx^*\rangle^{1/2}\Big) < 0, \qquad i = 1, \dots, s,$$

According to the monotonicity of G, we have.

$$G \circ \Big(f(x^*, \bar{y}_i) + \langle x^*, Ax^*\rangle^{1/2}\Big) - G \circ \Big(k_0\Big(g(x^*, \bar{y}_i) - \langle x^*, Bx^*\rangle^{1/2}\Big)\Big) < 0, \qquad i = 1, \dots, s, \tag{3.16}$$

According to (3.1), (3.4), (3.6)–(3.8) and (3.16), we have

$$\begin{aligned} &G\circ(f(x^*, \bar{y}_i) + \langle x^*, Au\rangle) - G \circ (k_0(g(x^*, \bar{y}_i) - \langle x^*, Bv\rangle)) \\ &\le G \circ \Big(f(x^*, \bar{y}_i) + \langle x^*, Ax^*\rangle^{1/2}\Big) - G \circ \Big(k_0\Big(g(x^*, \bar{y}_i) - \langle x^*, Bx^*\rangle^{1/2}\Big)\Big) < 0 \\ &= G \circ \Big(f(\bar{x}, \bar{y}_i) + \langle \bar{x}, A\bar{x}\rangle^{1/2}\Big) - G \circ \Big(f(\bar{x}, \bar{y}_i) + \langle \bar{x}, B\bar{x}\rangle^{1/2}\Big) \\ &= G \circ (f(\bar{x}, \bar{y}_i) + \langle \bar{x}, Au\rangle) - G \circ (f(\bar{x}, \bar{y}_i) + \langle \bar{x}, Bv\rangle). \end{aligned} \tag{3.17}$$

So, because $\alpha(x^*, \bar{x}) > 0$

$$\left\{\frac{G \circ (f(x^*, \bar{y}_i) + \langle x^*, Au\rangle) - G \circ (f(\bar{x}, \bar{y}_i) + \langle \bar{x}, Au\rangle)}{\alpha(x^*, \bar{x})}\right\} + \left\{\frac{-(G \circ (k_0(g(x^*, \bar{y}_i) - \langle x^*, Bv\rangle)) - G \circ (k_0(g(\bar{x}, \bar{y}_i) - \langle \bar{x}, Bv\rangle)))}{\alpha(x^*, \bar{x})}\right\} < 0,$$

by (3.11) and $x \in D$, it follows that

$$\left\{\frac{G\circ(f(x^*,\bar{y_i})+\langle x^*,Au\rangle)-G\circ(f(\bar{x},\bar{y_i})+\langle\bar{x},Au\rangle)}{\alpha(x^*,\bar{x})}\right\} + \left\{\frac{-(G\circ(k_0(g(x^*,\bar{y_i})-\langle x^*,Bv\rangle))-G\circ(k_0(g(\bar{x},\bar{y_i})-\langle\bar{x},Bv\rangle)))}{\alpha(x^*,\bar{x})}\right\} + \frac{G\circ h_j(x^*)-G\circ h_j(\bar{x})}{\beta(x^*,\bar{x})}<0.$$

By the generalized invexity assumptions of $f(\cdot,y_i)+\langle\cdot,Au\rangle$, $k(g(\cdot,y_i)-\langle\cdot,Bv\rangle)$ and $h_j(\cdot)$, we have

$$G\circ(f(x^*,\bar{y_i})+\langle x^*,Au\rangle)-G\circ(f(\bar{x},\bar{y_i})+\langle\bar{x},Au\rangle)\geqq\alpha(x^*,\bar{x}) G^{'}(f(\bar{x},\bar{y_i})+\langle\bar{x},Au\rangle)\left\langle\xi_i^f,\eta(x^*,\bar{x})\right\rangle, \quad \forall\xi_i^f\in\partial_x f(\bar{x},\bar{y_i})+Au, i=1,\dots,s, \tag{3.18}$$

$$G\circ(k_0(g(x^*,\bar{y_i}))-\langle x^*,Bv\rangle)-G\circ(k_0(g(\bar{x},\bar{y_i})-\langle\bar{x},Bv\rangle)) \leqq k_0\alpha(x^*,\bar{x}) G^{'}(k_0(g(\bar{x},\bar{y_i})-\langle\bar{x},Bv\rangle))\langle\xi_i^g,\eta(x^*,\bar{x})\rangle, \quad \forall\xi_i^g\in\partial_x g(\bar{x},\bar{y_i})-Bv, i=1,\dots,s, \tag{3.19}$$

$$G\circ h_j(x^*)-G\circ h_j(\bar{x})\geqq\beta(x^*,\bar{x})G^{'}(h(\bar{x}))\langle\xi_j,\eta(x^*,\bar{x})\rangle, \quad \forall\xi_j\in\partial_x h_j(\bar{x}), j=1,\dots,p, \tag{3.20}$$

Since $t_i^*\geq 0, i=1,\dots,s$ and $\sum_{i=1}^s t_i^*=1$, then

$$\sum_{i=1}^s t_i^*\left\{\frac{G\circ(f(x^*,\bar{y_i})+\langle x^*,Au\rangle)-G\circ(f(\bar{x},\bar{y_i})+\langle\bar{x},Au\rangle)}{\alpha(x^*,\bar{x})}\right\} + \sum_{i=1}^s t_i^*\left\{\frac{-(G\circ(k_0(g(x^*,\bar{y_i})-\langle x^*,Bv\rangle))-G\circ(k_0(g(\bar{x},\bar{y_i})-\langle\bar{x},Bv\rangle)))}{\alpha(x^*,\bar{x})}\right\} + \sum_{j=1}^p \mu_j^*\left(\frac{G\circ h_j(x^*)-G\circ h_j(\bar{x})}{\beta(x^*,\bar{x})}\right)<0. \tag{3.21}$$

By (3.17)–(3.21), we deduce that

$$\sum_{i=1}^s t_i^*\left\{G^{'}(f(\bar{x},\bar{y_i})+\langle\bar{x},Au\rangle)\left\langle\xi_i^f,\eta(x^*,\bar{x})\right\rangle-k_0G^{'}(k_0(g(\bar{x},\bar{y_i})-\langle\bar{x},Bv\rangle))\langle\xi_i^g,\eta(x^*,\bar{x})\rangle\right\} + \sum_{j=1}^p\mu_j^*G^{'}(h(\bar{x}))\langle\xi_j,\eta(x^*,\bar{x})\rangle<0,$$

or

$$\sum_{i=1}^s t_i^*\left\{G^{'}(f(\bar{x},\bar{y_i})+\langle\bar{x},Au\rangle)\xi_i^f-k_0G^{'}(k_0(g(\bar{x},\bar{y_i})-\langle\bar{x},Bv\rangle))\xi_i^g\right\}+\sum_{j=1}^p G^{'}(h(\bar{x}))\xi_j<0.$$

This implies that

$$0\notin\sum_{i=1}^s t_i^*G^{'}(f(\bar{x},\bar{y_i})+\langle\bar{x},Au\rangle)(\partial_x f(\bar{x},\bar{y_i})+Au) - \sum_{i=1}^s t_i^*k_0G^{'}(k_0g((\bar{x},\bar{y_i})-\langle\bar{x},Bv\rangle))(\partial_x g(\bar{x},\bar{y_i})-Bv)+\sum_{j=1}^p\mu_j^*G^{'}(h(\bar{x}))\partial_x h_j(\bar{x}).$$

This contradicts equation (3.9), so the assumption is wrong and the conclusion holds. the proof is complete. □

4. Duality model I

In this section, we consider the maximal minimum programming problem (FP) and establish the connection between the original problem (FP) and the duality problem (DI). The duality problem that we consider is as follows:

$$\max_{(s,t,\bar{y})\in K(z)} \sup_{(z,\mu,k,u,v)\in H_1(s,t,\bar{y})} k$$

where $H_1(s,t,\bar{y})$ denotes the set of all triplets $(z,\mu,k,u,v)\in R^n\times R^p_+\times R_+\times R^n\times R^n$ satisfying

$$\begin{aligned} 0 \in &\sum_{i=1}^{s} t_i G^{'}(f(z,\bar{y}_i)+\langle z,Au\rangle)(\partial_z f(z,\bar{y}_i)+Au) \\ &-\sum_{i=1}^{s} t_i k_0 G^{'}(k_0 g((z,\bar{y}_i)-\langle z,Bv\rangle))(\partial_z g(z,\bar{y}_i)-Bv)+\sum_{j=1}^{p}\mu_j G^{'}(h(z))\partial_z h_j(z) \end{aligned} \tag{4.1}$$

$$f(z,\bar{y}_i)+\langle z,Au\rangle - k_0 g((z,\bar{y}_i)-\langle z,Bv\rangle)\geq 0, \tag{4.2}$$

$$\langle \mu, h(z)\rangle \geq 0, \tag{4.3}$$

$$(s,t,\bar{y})\in K(z), \tag{4.4}$$

$$\langle u,Au\rangle\leq 1, \langle v,Bv\rangle \leq 1, \tag{4.5}$$

If $H_1(s,t,\bar{y})$ is empty for some triplet $(s,t,\bar{y})\in K(z)$, then $\sup_{(z,\mu,k,u,v)\in H_1(s,t,\bar{y})} k=-\infty$.

Theorem 4.1 (G-weak duality)**.** *Let x and $(z,\mu,u,v,s,t,\bar{y})$ be (FP)-feasible solution and (DI)-feasible solution, respectively; let G be both continuously differentiable and strictly increasing on $I(X,Y)$, G_{h_j} is both continuously differentiable and strictly increasing on $I_{h_j}(X)$ for each $j\in J$. Let $f(\cdot,\bar{y}_i)+\langle\cdot,Au\rangle$ and $k(g(\cdot,\bar{y}_i)-\langle\cdot,Bv\rangle)$ is (G,V)-invexity and (G,V)-incave function at $z\in D$, respectively; and $h_j(\cdot), j=1,\dots,p$ is (G,V)-invexity function at $z\in D$, then*

$$\sup_{y\in Y}\frac{f(x,y)+\langle x,Ax\rangle^{1/2}}{g(x,y)-\langle x,Bx\rangle^{1/2}}\geq k.$$

Proof. Suppose to the contrary that

$$\sup_{y\in Y}\frac{f(x,y)+\langle x,Ax\rangle^{1/2}}{g(x,y)-\langle x,Bx\rangle^{1/2}}< k.$$

We have

$$f(x,\bar{y}_i)+\langle x,Ax\rangle^{1/2}-k\Big(g(x,\bar{y}_i)-\langle x,Bx\rangle^{1/2}\Big)<0, \bar{y}_i\in Y, \qquad i=1,\dots,s,$$

according to the monotonicity of G

$$G\circ(f(x,\bar{y}_i)+\langle x,Ax\rangle^{1/2})-G\circ\Big(k\Big(g(x,\bar{y}_i)-\langle x,Bx\rangle^{1/2}\Big)\Big)<0, \bar{y}_i\in Y, \qquad i=1,\dots,s.$$

Since $t_i\geq 0, i=1,\dots,s$ and $\sum_{i=1}^{s} t_i=1$, then

$$t_i\Big\{G\circ\Big(f(x,\bar{y}_i)+\langle x,Ax\rangle^{1/2}\Big)-G\circ\Big(k\Big(g(x,\bar{y}_i)-\langle x,Bx\rangle^{1/2}\Big)\Big)\Big\}\leq 0. \tag{4.6}$$

Then there is at least one strict inequality because $t=t_1,\dots,t_s\neq 0$.

From (3.1), (4.1), (4.4) and (4.6), we have

$$\begin{aligned}&\sum_{i=1}^{s} t_i\{f(x,\bar{y}_i)+\langle x,Au\rangle-k(g(x,\bar{y}_i)-\langle x,Bv\rangle)\}\\&\quad\leq \sum_{i=1}^{s} t_i\Big\{f(x,\bar{y}_i)+\langle x,Ax\rangle^{1/2}-k\big(g(x,\bar{y}_i)-\langle x,Bx\rangle^{1/2}\Big\}\\&\quad<0\leq \sum_{i=1}^{s} t_i\{f(z,\bar{y}_i)+\langle z,Au\rangle-k(g(z,\bar{y}_i)-\langle z,Bv\rangle)\}.\end{aligned}$$

Hence

$$\begin{aligned}&\sum_{i=1}^{s} t_i\{f(x,\bar{y}_i)+\langle x,Au\rangle-f(z,\bar{y}_i)+\langle z,Au\rangle\}\\&\quad+\sum_{i=1}^{s} t_i k\{-g(x,\bar{y}_i)-\langle x,Bv\rangle+(g(z,\bar{y}_i)-\langle z,Bv\rangle)\}<0.\end{aligned}$$

Because x is a feasible solution to (FP) and (4.3), we have

$$\begin{aligned}&\sum_{i=1}^{s} t_i^*\left\{\frac{G\circ(f(x,\bar{y}_i)+\langle x,Au\rangle)-G\circ(f(z,\bar{y}_i)+\langle z,Au\rangle)}{\alpha(x,z)}\right\}\\&\quad+\sum_{i=1}^{s} t_i^*\left\{\frac{-(G\circ(k_0(g(x,\bar{y}_i)-\langle x,Bv\rangle))-G\circ(k_0(g(z,\bar{y}_i)-\langle z,Bv\rangle)))}{\alpha(x,z)}\right\}\\&\quad+\sum_{j=1}^{p}\mu_j^*\left(\frac{G\circ h_j(x)-G\circ h_j(z)}{\beta(x,z)}\right)<0\end{aligned}\tag{4.7}$$

similar to the proof of Theorem 3.4, based on the convexity assumptions of $f(\cdot,\bar{y}_i)+\langle\cdot,Au\rangle$, $k(g(\cdot,\bar{y}_i)-\langle\cdot,Bv\rangle)$ and h_j, we get

$$\begin{aligned}0\notin&\sum_{i=1}^{s} t_i^* G^{'}(f(z,\bar{y}_i)+\langle z,Au\rangle)(\partial_z f(z,\bar{y}_i)+Au)\\&-\sum_{i=1}^{s} t_i^* k_0 G^{'}(k_0 g((z,\bar{y}_i)-\langle z,Bv\rangle))(\partial_z g(z,\bar{y}_i)-Bv)+\sum_{j=1}^{p}\mu_j^* G^{'}(h(z))\partial_z h_j(\bar{z}).\end{aligned}$$

This contradicts equation (4.1), so $\sup_{y\in Y}\phi(x,y)\geq k$. the proof is complete. □

Theorem 4.2 (G-strong duality)**.** *Let problem (FP) satisfy constrains [25], and x be an optimal solution of problem (FP). Let G be both continuously differentiable and strictly increasing on $I(X,Y)$, G_{h_j} is both continuously differentiable and strictly increasing on $I_{h_j}(X)$ with $G^{'}(h_j(\bar{x}))>0$ for each $j\in J$. If the hypothesis of Theorem 4.1 holds for all (DI) feasible points $(\bar{z},\bar{\mu},\bar{v},\bar{q},\bar{\lambda},\bar{y}^*)$, then there exist $(\bar{s},\bar{t},\bar{y}^*)\in K(z)$ and $(\bar{x},\bar{\mu},\bar{k},\bar{u},\bar{v})\in H_1(\bar{s},\bar{t},\bar{y}^*)$ such that $(\bar{x},\bar{\mu},\bar{k},\bar{u},\bar{v},\bar{s},\bar{t},\bar{y}^*)$ is a (DI) optimal solution, and the two problems (FP) and (DI) have the same optimal values.*

Proof. Since x is an optimal solution in the nondifferentiable minimax fractional programming problem (FP), there exists

$$\bar{k}=\frac{f(\bar{x},y_i^*)+\langle\bar{x},A\bar{x}\rangle^{1/2}}{g(\bar{x},y_i^*)-\langle\bar{x},B\bar{x}\rangle^{1/2}},\qquad i=1,\ldots,s,$$

satisfying the requirements specified in the theorem, by weak duality (Thm. 4.1), for any feasible point $(\bar{x},\bar{\mu},\bar{k},\bar{u},\bar{v},\bar{s},\bar{t},\bar{y}^*)$ of (DI). we have

$$\sup_{y\in Y}\frac{f(\bar{x},\bar{y}_i^*)+\langle\bar{x},A\bar{x}\rangle^{1/2}}{g(\bar{x},\bar{y}_i^*)-\langle\bar{x},B\bar{x}\rangle^{1/2}}\geq\bar{k}.$$

This implies that $(\bar{x},\bar{\mu},\bar{k},\bar{u},\bar{v},\bar{s},\bar{t},\bar{y}^*)$ is optimal in (DI). □

Theorem 4.3 (G-strict converse duality)**.** *Let $\bar{x}$ and $(z,\bar{\mu},\bar{k},\bar{u},\bar{v},\bar{s},\bar{t},\bar{y}^*)$ be (FP)-feasible solution and (DI)-feasible solution, respectively; let G be both continuously differentiable and strictly increasing on $I(X,Y)$, G_{h_j} is both continuously differentiable and strictly increasing on $I_{h_j}(X)$ for each $j\in J$. If $f(\cdot,\bar{y}^*)+\langle\cdot,A\bar{u}\rangle$ and $\bar{k}(g(\cdot,\bar{y}^*)-\langle\cdot,B\bar{v}\rangle), i=1,\ldots,s$ is strictly (G,V)-invexity and strictly (G,V)-incave function at $z\in D$, respectively; and $h_j(\cdot), j\in J$ is (G,V)-invexity function at $z\in D$, then $\bar{x}=z$; that is, z is a (FP)-optimal solution and*

$$\sup_{y\in Y}\frac{f(z,\bar{y}^*)+\langle z,Az\rangle^{1/2}}{g(z,\bar{y}^*)-\langle z,Bz\rangle^{1/2}}=\bar{k}.$$

Proof. (Counter-evidence) suppose $x\neq z$. Similar to the proof of Theorem 4.1, the exist $\xi_i^f\in\partial_z f(z,\bar{y}_i^*)+\langle z,A\bar{u}\rangle$,$\xi_i^g\in\partial_z f(z,\bar{y}_i^*)+\langle z,B\bar{v}\rangle, i=1,\ldots,s$ and $\xi_j\in\partial_z h_j(z), j\in J$ such that

$$\begin{aligned}0&=\sum_{i=1}^{s}t_iG'(f(z,\bar{y}_i^*)+\langle z,A\bar{u}\rangle)\xi_i^f-\bar{k}G'\left(\bar{k}g((z,\bar{y}_i^*)-\langle z,B\bar{v}\rangle)\right)\xi_i^g+\sum_{j=1}^{p}\mu_jG'(h(z))\xi_j\\&<\sum_{i=1}^{s}\bar{t}_i\left\{\frac{G\circ(f(\bar{x},\bar{y}_i^*)+\langle\bar{x},A\bar{u}\rangle)-G\circ(f(z,\bar{y}_i^*)+\langle z,A\bar{u}\rangle)}{\alpha(\bar{x},z)}\right\}\\&\quad+\sum_{i=1}^{s}\bar{t}_i\left\{\frac{-\left(G\circ\left(\bar{k}(g(\bar{x},\bar{y}_i^*)-\langle\bar{x},B\bar{v}\rangle)\right)-G\circ\left(\bar{k}(g(z,\bar{y}_i^*)-\langle z,B\bar{v}\rangle)\right)\right)}{\alpha(\bar{x},z)}\right\}\\&\quad+\sum_{j=1}^{p}\bar{\mu}_j\frac{G\circ h_j(\bar{x})-G\circ h_j(z)}{\beta(\bar{x},z)}.\end{aligned}\tag{4.8}$$

By the feasibility of $\bar{x}$ and (4.2), we have

$$\sum_{j=1}^{p}\bar{\mu}_j\frac{G\circ h_j(\bar{x})-G\circ h_j(z)}{\beta(\bar{x},z)}\leq 0.\tag{4.9}$$

From (4.8) and (4.9) we get

$$\begin{aligned}&\sum_{i=1}^{s}\bar{t}_i\left\{\frac{G\circ(f(\bar{x},\bar{y}_i^*)+\langle\bar{x},A\bar{u}\rangle)-G\circ(f(z,\bar{y}_i^*)+\langle z,A\bar{u}\rangle)}{\alpha(\bar{x},z)}\right\}\\&\quad+\sum_{i=1}^{s}\bar{t}_i\left\{\frac{\bar{k}\{-\left(G\circ\left(\bar{k}(g(\bar{x},\bar{y}_i^*)-\langle\bar{x},B\bar{v}\rangle)\right)-G\circ\left(\bar{k}(g(z,\bar{y}_i^*)-\langle z,B\bar{v}\rangle)\right)\right)\}}{\alpha(\bar{x},z)}\right\}>0.\end{aligned}$$

Since (3.2) and (4.1), the above inequality implies that

$$\begin{aligned}&\sum_{i=1}^{s}\bar{t}_i\left\{G\circ\left(f(\bar{x},\bar{y}_i^*)+\langle\bar{x},A\bar{x}\rangle^{1/2}\right)-\bar{k}G\circ\left(g(\bar{x},\bar{y}_i^*)-\langle\bar{x},B\bar{x}\rangle^{1/2}\right)\right\}\\&\quad>\sum_{i=1}^{s}\bar{t}_i\left\{G\circ\left(f(z,\bar{y}_i^*)+\langle z,Az\rangle^{1/2}\right)-\bar{k}G\circ\left(g(z,\bar{y}_i^*)-\langle z,Bz\rangle^{1/2}\right)\right\}\geq 0.\end{aligned}$$

From the above inequality, we conclude that there exists a certain $i_0 = 1, \ldots, s$ such that

$$G \circ \left(f\left(\bar{x}, \bar{y}_{i_0}^*\right) + \langle \bar{x}, A\bar{x}\rangle^{1/2}\right) - \bar{k}G \circ \left(g\left(\bar{x}, \bar{y}_{i_0}^*\right) - \langle \bar{x}, B\bar{x}\rangle^{1/2}\right) > 0,$$

according to the monotonicity of G, we have

$$f\left(\bar{x}, \bar{y}_{i_0}^*\right) + \langle \bar{x}, A\bar{x}\rangle^{1/2} - \bar{k}\left(g\left(\bar{x}, \bar{y}_{i_0}^*\right) - \langle \bar{x}, B\bar{x}\rangle^{1/2}\right) > 0.$$

It follows that

$$\sup_{y\in Y} \frac{f(\bar{x}, \bar{y}^*) + \langle \bar{x}, A\bar{x}\rangle^{1/2}}{g(\bar{x}, \bar{y}^*) - \langle \bar{x}, B\bar{x}\rangle^{1/2}} \geq \frac{f\left(\bar{x}, \bar{y}_{i_0}^*\right) + \langle \bar{x}, A\bar{x}\rangle^{1/2}}{g\left(\bar{x}, \bar{y}_{i_0}^*\right) - \langle \bar{x}, B\bar{x}\rangle^{1/2}} > \bar{k}, \tag{4.10}$$

according to Theorem 4.1, we have

$$\sup_{y\in Y} \frac{f(\bar{x}, y) + \langle \bar{x}, A\bar{x}\rangle^{1/2}}{g(\bar{x}, y) - \langle \bar{x}, B\bar{x}\rangle^{1/2}} = \bar{k}.$$

Contradicts equation (4.10), therefore, $\bar{x} = z$, the proof is complete. □

5. Duality model II

In this section, the second type duality model (DII) is given by adjusting theorem 4.1.

Theorem 5.1 (Necessary conditions [28])**.** *Let $\bar{x}$ be an optimal solution for (FP) and let $\partial h_j(\bar{x}), j \in J(\bar{x})$ be linearly independent. Then, there exist $(\bar{s}, \bar{t}, \bar{y}) \in K(\bar{x})$ and $\mu^* \in R_+^p$, such that*

$$\nabla\left(\frac{\sum_{i=1}^{\bar{s}} \bar{t}_i(f(\bar{x}, \bar{y}_i) + \langle \bar{x}, Au\rangle) + \sum_{j=1}^{p} \mu_j^* h(\bar{x})}{\sum_{i=1}^{\bar{s}} \bar{t}_i(g(\bar{x}, \bar{y}_i) - \langle \bar{x}, Bv\rangle)}\right) = 0, \tag{5.1}$$

$$\sum_{j=1}^{p} \mu_j^* h(\bar{x}) = 0, \tag{5.2}$$

$$\mu^* \in R_+^p, \bar{t}_i \geq 0, \sum_{i=1}^{\bar{s}} \bar{t}_i = 1, \bar{y}_i \in Y(\bar{x}), \qquad i = 1, \ldots, \bar{s}, \tag{5.3}$$

$$\begin{aligned} &\langle u, Au\rangle \leq 1, \langle \bar{x}, A\bar{x}\rangle^{1/2} = \langle \bar{x}, Au\rangle, \\ &\langle v, Bv\rangle \leq 1, \langle \bar{x}, B\bar{x}\rangle^{1/2} = \langle \bar{x}, Bv\rangle. \end{aligned} \tag{5.4}$$

Theorem 5.2 (G-Necessary conditions)**.** *Let problem (FP) satisfy satisfying $\langle \bar{x}, A\bar{x}\rangle$ >0, $\langle \bar{x}, B\bar{x}\rangle > 0$. Let $\bar{x}$ be an optimal solution of (FP). Assume that G is both continuously differentiable and strictly increasing on $I(X, Y)$. If G_{h_j} is both continuously differentiable and strictly increasing on $I_{h_j}(X)$ with $G'_{h_j}(h_j(\bar{x})) > 0$ for each $j \in J$, then there exists $(\bar{s}, \bar{t}, \bar{y}) \in K(\bar{x})$ and $\mu^* \in R_+^p$, such that*

$$\nabla\left(\frac{\sum_{i=1}^{\bar{s}} \bar{t}_i(G(f(\bar{x}, \bar{y}_i) + \langle \bar{x}, Au\rangle)) + \sum_{j=1}^{p} \mu_j^* G(h(\bar{x}))}{\sum_{i=1}^{\bar{s}} \bar{t}_i(G(g(\bar{x}, \bar{y}_i) - \langle \bar{x}, Bv\rangle))}\right) = 0, \tag{5.5}$$

$$\sum_{j=1}^{p} \mu_j^* G(h(\bar{x})) = 0, \mu_j^* \geq 0, \quad j \in J, \tag{5.6}$$

$$\bar{t}_i \geq 0, \quad i = 1, \ldots, \bar{s}, \text{with} \sum_{i=1}^{\bar{s}} = 1, \bar{y}_i \in Y(\bar{x}), \tag{5.7}$$

$$\langle u, Au\rangle \leq 1, \langle \bar{x}, A\bar{x}\rangle^{1/2} = \langle \bar{x}, Au\rangle,$$
$$\langle v, Bv\rangle \leq 1, \langle \bar{x}, B\bar{x}\rangle^{1/2} = \langle \bar{x}, Bv\rangle, \tag{5.8}$$

The duality problem that we consider is as follows (DII):

$$\max_{(s,t,\bar{y})\in K(z)} \sup_{(z,\mu,u,v)\in H_2(s,t,\bar{y})} \frac{\sum_{i=1}^{\bar{s}} \bar{t}_i(f(\bar{x},\bar{y}_i) + \langle \bar{x}, Au\rangle) + \sum_{j=1}^{p} \mu_j^* h(\bar{x})}{\sum_{i=1}^{\bar{s}} \bar{t}_i(g(\bar{x},\bar{y}_i) - \langle \bar{x}, Bv\rangle)}$$

where $H_2(\bar{s},\bar{t},\bar{y})$ *denotes the set of* $(z,\mu,u,v) \in R^n \times R_+^p \times R^n \times R^n$ *to satisfy*

$$\nabla\left(\frac{\sum_{i=1}^{\bar{s}} \bar{t}_i(G(f(z,\bar{y}_i) + \langle z, Au\rangle)) + \sum_{j=1}^{p} \mu_j^* G(h(z))}{\sum_{i=1}^{\bar{s}} \bar{t}_i(G(g(z,\bar{y}_i) - \langle z, Bv\rangle))}\right) = 0, \tag{5.9}$$

$$\sum_{j=1}^{p} \mu_j^* G(h(z)) = 0, \mu_j^* \geq 0, \quad j \in J, \tag{5.10}$$

$$\langle u, Au\rangle \leq 1, \langle \bar{x}, A\bar{x}\rangle^{1/2} = \langle \bar{x}, Au\rangle,$$
$$\langle v, Bv\rangle \leq 1, \langle \bar{x}, B\bar{x}\rangle^{1/2} = \langle \bar{x}, Bv\rangle, \tag{5.11}$$

we denote

$$\omega(\cdot) = \left(\sum_{i=1}^{s} t_i(f(\cdot,\bar{y}_i) + \langle \cdot, Au\rangle) + \sum_{j=1}^{p} \mu_j^* h(\cdot)\right)\left(\sum_{i=1}^{\bar{s}} \bar{t}_i(g(z,\bar{y}_i)\right.$$
$$\left. - \langle\ z, Bv\rangle))\right) - \left(\sum_{i=1}^{s} t_i(g(\cdot,\bar{y}_i) - \langle \cdot, Bv\rangle)\right)\left(\sum_{i=1}^{\bar{s}} \bar{t}_i(f(z,\bar{y}_i) + \langle z, Au\rangle) + \sum_{j=1}^{p} \mu_j^* h(z)\right)$$

and assume that

$$\sum_{i=1}^{s} t_i(f(z,\bar{y}_i) + \langle z, Au\rangle) + \sum_{j=1}^{p} \mu_j^* h(z) \geq 0$$

and

$$\sum_{i=1}^{s} t_i(g(z,\bar{y}_i) - \langle z, Bv\rangle) > 0$$

for all $(s,t,\tilde{y}) \in K(z)$, $(z,\mu,u,v) \in H_2(s,t,\tilde{y})$.

Theorem 5.3 (G-weak duality)**.** *Let* x *and* $(z,\mu,u,v,s,t,\bar{y})$ *be (FP)-feasible solution and (DII)-feasible solution, respectively; let* G *be both continuously differentiable and strictly increasing on* $I(X,Y)$, G_{h_j} *is both continuously differentiable and strictly increasing on* $I_{h_j}(X)$ *for each* $j \in J$, *Let* $f(\cdot,\bar{y}_i)+\langle \cdot, Au\rangle$ *and* $k(g(\cdot,\bar{y}_i)-\langle \cdot, Bv\rangle)$ *is* (G,V)*-invexity and* (G,V)*-incave function at* $z \in D$*, respectively; and* $h_j(\cdot), j = 1,\dots,p$ *is* (G,V)*-invexity function at* $z \in D$*,then*

$$\sup_{y\in Y} \frac{f(x,y) + \langle x, Ax\rangle^{1/2}}{g(x,y) - \langle x, Bx\rangle^{1/2}} \geq \frac{\sum_{i=1}^{\bar{s}} \bar{t}_i(f(z,\bar{y}_i) + \langle z, Au\rangle) + \sum_{j=1}^{p} \mu_j^* h(z)}{\sum_{i=1}^{\bar{s}} \bar{t}_i(g(z,\bar{y}_i) - \langle z, Bv\rangle)}.$$

Proof. Suppose to the contrary that

$$\sup_{y\in Y} \frac{f(x,y) + \langle x, Ax\rangle^{1/2}}{g(x,y) - \langle x, Bx\rangle^{1/2}} < \frac{\sum_{i=1}^{\bar{s}} \bar{t}_i(f(z,\bar{y}_i) + \langle z, Au\rangle) + \sum_{j=1}^{p} \mu_j^* h(z)}{\sum_{i=1}^{\bar{s}} \bar{t}_i(g(z,\bar{y}_i) - \langle z, Bv\rangle)} \tag{5.12}$$

we get

$$\begin{aligned}&\left(f(x,y)+\langle x,Ax\rangle^{1/2}\right)\left(\sum_{i=1}^{\bar{s}}\bar{t}_i(g(z,\bar{y}_i)-\langle z,Bv\rangle)\right)\\&\quad-\left(g(x,y)-\langle x,Bx\rangle^{1/2}\right)\left(\sum_{i=1}^{\bar{s}}\bar{t}_i(f(z,\bar{y}_i)+\langle z,Au\rangle)+\sum_{j=1}^{p}\mu_j^*h(z)\right)<0.\end{aligned}\tag{5.13}$$

Since $t_i \geq 0, i = 1, \ldots, s$ and $\sum_{i=1}^{s} t_i = 1$, then

$$\begin{aligned}&\sum_{i=1}^{s}t_i\left\{\left(f(x,y)+\langle x,Ax\rangle^{1/2}\right)\left(\sum_{i=1}^{\bar{s}}\bar{t}_i(g(z,\bar{y}_i)-\langle z,Bv\rangle)\right)\right\}\\&\quad-\sum_{i=1}^{s}t_i\left\{\left(g(x,y)-\langle x,Bx\rangle^{1/2}\right)\left(\sum_{i=1}^{\bar{s}}\bar{t}_i(f(z,\bar{y}_i)+\langle z,Au\rangle)+\sum_{j=1}^{p}\mu_j^*h(z)\right)\right\}<0.\end{aligned}\tag{5.14}$$

Then there is at least one strict inequality because $t = t_1, \ldots, t_s \neq 0$. From (3.1), (3.3) and (5.11), we have

$$\begin{aligned}\omega(x)\leq&\left(\sum_{i=1}^{s}t_i(f(x,\bar{y}_i)+\langle x,Au\rangle)+\sum_{j=1}^{p}\mu_j^*h(x)\right)\left(\sum_{i=1}^{\bar{s}}\bar{t}_i\left(g(z,\bar{y}_i)-\langle z,Bv\rangle^{1/2}\right)\right)\\&-\left(\sum_{i=1}^{s}t_i\left(g(x,\bar{y}_i)-\langle x,Bv\rangle^{1/2}\right)\right)\left(\sum_{i=1}^{\bar{s}}\bar{t}_i(f(z,\bar{y}_i)+\langle z,Au\rangle)+\sum_{j=1}^{p}\mu_j^*h(z)\right)\\<&\sum_{j=1}^{p}\mu_j^*h(x)\sum_{i=1}^{\bar{s}}\bar{t}_i\left(g(z,\bar{y}_i)-\langle z,Bv\rangle^{1/2}\right)\\<&\,0=\omega(z).\end{aligned}$$

Hence,

$$\begin{aligned}&\sum_{i=1}^{\bar{s}}\bar{t}_i(g(z,\bar{y}_i)-\langle z,Bv\rangle)\sum_{i=1}^{\bar{s}}\bar{t}_i((f(x,\bar{y}_i)+\langle x,Au\rangle)-(f(z,\bar{y}_i)+\langle z,Au\rangle))\\&\quad+\left(\sum_{j=1}^{p}\mu_j^*h(x)-\sum_{j=1}^{p}\mu_j^*h(z)\right)+\left\{\sum_{i=1}^{\bar{s}}\bar{t}_i(\left(f(z,\bar{y}_i)+\langle z,Au\rangle+\sum_{j=1}^{p}\mu_j^*h(z)\right)\right\}\\&\sum_{i=1}^{\bar{s}}\bar{t}_i(-(g(x,\bar{y}_i)-\langle x,Bv\rangle)+(g(z,\bar{y}_i)-\langle z,Bv\rangle))<0.\end{aligned}$$

Because x is a feasible solution to (FP), and according to the monotonicity of G, we have

$$\begin{aligned}&\sum_{i=1}^{\bar{s}}t_i^*\left\{\frac{[g(z,\bar{y}_i)-\langle z,Bv\rangle]\{G\circ(f(x,\bar{y}_i)+\langle x,Au\rangle)-G\circ(f(z,\bar{y}_i)+\langle z,Au\rangle)\}}{\alpha(x,z)}\right\}\\&\quad+\sum_{i=1}^{\bar{s}}t_i^*\left\{\frac{\sum_{i=1}^{\bar{s}}\bar{t}_i(f(z,\bar{y}_i)+\langle z,Au\rangle)+\sum_{j=1}^{p}\mu_j^*h(z)}{\alpha(x,z)}\right\}\\&\quad+\sum_{i=1}^{\bar{s}}t_i^*\left\{\frac{-\{G\circ(g(x,\bar{y}_i)-\langle x,Bv\rangle)-G\circ(g(z,\bar{y}_i)-\langle z,Bv\rangle)\}}{\alpha(x,z)}\right\}\\&\quad+\sum_{j=1}^{p}\mu_j^*\frac{G\circ h_j(x)-G\circ h_j(z)}{\beta(x,z)}\end{aligned}\tag{5.15}$$

similar to the proof of Theorem 3.4, based on the convexity assumptions of $f(\cdot, \bar{y}_i) + \langle \cdot, Au\rangle, g(\cdot, \bar{y}_i) - \langle \cdot, Bv\rangle$ and h_j, we get

$$0 \notin \nabla\left(\frac{\sum_{i=1}^{\bar{s}} \bar{t}_i(G(f(z, \bar{y}_i) + \langle z, Au\rangle)) + \sum_{j=1}^{p} \mu_j^* G(h(z))}{\sum_{i=1}^{\bar{s}} \bar{t}_i(G(g(z, \bar{y}_i) - \langle z, Bv\rangle))}\right).$$

This contradicts equation (5.9), so the proof is complete. □

Theorem 5.4 (G-strong duality). *Let problem (FP) satisfy constrains [25], and x be an optimal solution of problem (FP). Let G be both continuously differentiable and strictly increasing on $I(X, Y)$. If the hypothesis of Theorem 5.3 holds for all (DII) feasible points $(\bar{z}, \bar{\mu}, \bar{v}, \bar{s}, \bar{t}, \bar{y}^*)$, then there exist $(\bar{s}, \bar{t}, \bar{y}^*) \in K(z)$ and $(\bar{x}, \bar{\mu}, \bar{u}, \bar{v}) \in H_2(\bar{s}, \bar{t}, \bar{y}^*)$ such that $(\bar{x}, \bar{\mu}, \bar{k}, \bar{u}, \bar{v}, \bar{s}, \bar{t}, \bar{y}^*)$ is a (DI) optimal solution, and the two problems (FP) and (DII) have the same optimal values.*

Theorem 5.5 (G-strict converse duality). *Let $\bar{x}$ and $(z, \bar{\mu}, \bar{u}, \bar{v}, \bar{s}, \bar{t}, \bar{y}^*)$ be (FP)-feasible solution and (DII)-feasible solution, respectively; let G be both continuously differentiable and strictly increasing on $I(X, Y)$, G_{h_j} is both continuously differentiable and strictly increasing on $I_{h_j}(X)$ for each $j \in J$. If $f(\cdot, y_i^*) + \langle \cdot, A\bar{u}\rangle$ and $k(g(\cdot, y_i^*) - \langle \cdot, B\bar{v}\rangle), i = 1, \ldots, s$ is strictly (G, V)-invexity and strictly (G, V)-incave function at $z \in D$, respectively; and $h_j(\cdot), j \in J$ is (G, V)-invexity function at $z \in D$, then $\bar{x} = z$; that is, z is a (FP)-optimal solution and*

$$\sup_{y \in Y} \frac{f(z, \bar{y}^*) + \langle z, Az\rangle^{1/2}}{g(z, \bar{y}^*) - \langle z, Bz\rangle^{1/2}} = \frac{\sum_{i=1}^{\bar{s}} \bar{t}_i(G(f(z, \bar{y}_i) + \langle z, Au\rangle)) + \sum_{j=1}^{p} \mu_j^* G(h(z))}{\sum_{i=1}^{\bar{s}} \bar{t}_i(G(g(z, \bar{y}_i) - \langle z, Bv\rangle))}.$$

6. Conclusions

This paper discusses the application of nondifferentiable (G, V)-invexity functions in nondifferentiable generalized minimax fractional programming (FP). Firstly, we construct an important auxiliary minimax fractional programming problem, which is necessary to discuss the nondifferentiable generalized minimax fractional programming problem (FP). Based on this, we establish G-necessary optimality conditions. Further, we construct two duality models (DI and DII) of the nondifferentiable generalized minimax fractional programming problem (FP), and obtain the G-weak duality, G-strong duality and G-strict inverse duality theorems between the primal problem (FP) and the duality problems (DI and DII) under the assumption of (G, V)-invexity.

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