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OPTIMAL PROFIT IN TWO-LEVEL TRADE CREDIT EOQ MODEL WITH DEFAULT RISK AND REMINDER COST UNDER FINITE TIME HORIZON HAVING TIME-DEPENDENT DEMAND AND DETERIORATION

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Abstract. Trade credit is a type of promotional activity that generally increases demand and revenue but also invites default risk due to dishonest customers. Due to default risk, revenue is lost, and to overcome this, an arrangement is made to remind the defaulters. A retailer dealing with a perishable item wants to exhaust the stock quickly after a certain deterioration level. Demand and deterioration of the products are normally dynamic. The business period of seasonal products is uncertain. Considering these facts, we formulate and analyze a two-level trade credit inventory model, where the wholesaler and retailer give credit periods to their corresponding downstream customers. After a certain level of deterioration, the retailer increases the credit period for the customers for early stock exhaustion, and to reduce default risk, a reminder cost is introduced. These activities increase the profit. The mathematical models under different circumstances are formulated for different time horizons. Some existing results are deduced. The models are numerically solved using a parametric study and the Generalised Reduced Gradient method through LINGO 19.0 software. Some lemmas and theorems are deduced to establish the analytical outcomes. Trade-offs between the number of the business cycle, trade credit, and reminder cost against optimum profit are separately demonstrated. The results with and without reminder cost are compared, and it is shown that the model with reminder cost fetches more profit. Profit under different uncertain environments are evaluated, and they differ marginally. Some beneficial impacts are discussed.

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1. Introduction

In the 21st century, promotional activities are essential to a successful business. Promotional activities attract new customers, retain current customers and increase the company's revenue. There are many promotional

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activity types, one of which is trade credit. Trade credit is an allowable delay time for the buyer's payment from the seller. In traditional economic order quantity, it is assumed that a retailer pays the total purchasing cost at the time of purchase/receipt. This acts as a financial constraint for retailers. In a trade credited system, if a wholesaler provides a trade credit to a retailer, the retailer does not pay at the time of purchase. He pays the amount at the end of the credit period, which helps the retailer acquire more business goods. In this system, if the retailer fails to clear all dues at the end of the specific credit period, the wholesaler charges an interest until it is cleared. In a two-level trade-credited system, like the wholesaler, the retailer also gives trade credit to the customers to settle the account, which helps the customers to purchase more. In this case, customers demand increases with the retailer's trade credit period [29, 39].

Nowadays, trade credit plays a crucial role in today's business, although it has some adverse effects. In a trade credited system, at the end of the specified credit period, some customers pay a portion of the total dues or do not pay at all. This is called default risk. This reduces the retailer's profit. Many researchers have considered a two-level trade credit system in their formulations. They assumed credit-dependent demand with a constant or time-dependent deterioration for a finite or infinite time horizon. Here some of these investigations are given.

Arcelus *et al.* [3] studied retailers' responses to special sales using two payment processes: price discount and trade credit. Abad and Jaggi [1] considered the seller-buyer channel, where the end demand is price sensitive, and the seller offers trade credit to the buyer. Teng *et al.* [33] considered time-dependent non-decreasing linear demand in which the supplier offers a credit period to the retailer to settle the account. Under order-dependent trade credit terms, the integrated inventory policies of single-supplier single-buyer were investigated by Shah and Pandey [30]. Impacts of green and preservation technology investments on a sustainable EPQ model during COVID-19 pandemic formulated by Barman *et al.* [4]. Inventory model involving reworking of faulty products under neutrosophic environment developed by Barman *et al.* [6]. Pervin *et al.* [24] introduce an EOQ model for non-instantaneous exponential deteriorating items with composite demand. Analysis of a dual-channel green supply chain game-theoretical model under carbon policy investigated by Barman *et al.* [5] All these studies are performed with the single-level trade credit under an infinite planning horizon.

Further, many authors have formulated inventory models under two-level trade credit. Teng *et al.* [33] considered two-level trade credit for a time-dependent increasing demand. For the same demand as Teng *et al.* [33, 34] formulated an inventory model where the retailer's upstream and downstream credit period is constant. Chung and Cárdenas-Barrón [13] simplified the solution procedure for deteriorating items under stock-dependent demand and two-level trade credit in the supply chain (SC) management, which is the generalized result of Min *et al.* [20]. Pramanik *et al.* [26] developed an *EOQ* model for price, credit period, and credit amount-dependent demand. Otrodi *et al.* [21] developed an inventory model for joint pricing and lot-sizing for a perishable item under two-level trade credit with multiple demand classes. In this model: the demand rate at each market depends on the selling price, credit period, and the price of other complementary and substitute products. For stochastic demand, Kaur *et al.* [15] developed an *EOQ* model under a two-level trade-credit policy. Das *et al.* [14] formulated location–allocation problem for green efficient two-stage vehicle-based logistics system: A type-2 neutrosophic multi-objective modeling approach. In this formulation, they have taken credit-dependent default risk at both the retailer and customer end.

Many researchers have considered deteriorated items in two-level trade credit inventory models. Chen and Teng [12] proposed an *EOQ* model for a retailer when their product deteriorates continuously and has a maximum lifetime. Their supplier offers a permissible delay in payments for a constant demand rate. Wang *et al.* [38] and Wu *et al.* [39] considered credit-dependent exponential demand and default risk. They proposed an *EOQ* model for a seller by incorporating the following relevant facts: (i) deteriorating products deteriorate continuously, having their maximum lifetime, and (ii) credit period increases demand and default risk. Tiwari *et al.* [35] established an inventory model for the impact of trade credit and inflation on retailers' ordering policies for non-instantaneous deteriorating items in a two-warehouse environment for constant demand. Their objective is to determine the retailer's optimal replenishment policies that maximize the present worth of total optimal profit per unit of time. Paul *et al.* [23] formulated an *EOQ* model in which customer's demand is nonlinear function of credit period while deterioration is a linearly time-dependent function. Rameswari and Uthayakumar

TABLE 1. Features of some trade credited inventory models.

References	Level of trade credit	Demand	Deterioration	Customer's credit	Default risk	Reminder cost	Time horizon
Liao [16]	Two-level	ct	ct	Fix	×	×	Infinite
Teng [31]	Two-level	ct	×	Fix	×	×	Infinite
Chang *et al.* [10]	Two-level	ct	ct	Fix	×	×	Infinite
Teng *et al.* [32]	Two-level	td	ct	Fix	×	×	Infinite
Teng *et al.* [33]	Two-level	td	×	Fix	✓	×	Infinite
Lou and Wang [17]	One-level	cd	×	Fix	✓	×	Infinite
Teng *et al.* [34]	Two-level	td	×	Fix	×	×	Infinite
Wang *et al.* [38]	Two-level	cd	td	Fix	✓	×	Infinite
Wu *et al.* [39]	Two-level	cd	td	Fix	✓	×	Infinite
Shah and Cárdenas-Barrón [29]	Two-level	cd	ct	Fix	✓	×	Infinite
Pakhira *et al.* [22]	Two-level	cd	×	Fix	×	×	Infinite
Zhang *et al.* [40]	Two-level	cd	×	Fix	✓	×	Infinite
Mahata *et al.* [18]	Two-level	cd & td	×	Fix	✓	×	Infinite
Tsao [37]	One-level	cd	×	Fix	✓	×	Infinite
Pramanik and Maiti [25]	Two-level	td	ct	Fix	×	×	Finite
Ali *et al.* [2]	×	td	ct	×	×	×	Infinite
This paper	Two-level	cd, r_cd & td	td	Changing	✓	✓	Finite & unct

Notes. ct: constant, td: time dependent, cd: credit dependent, r_cd: reminder level dependent, unct: uncertain.

[28] formulated two inventory models for deteriorating items under a two-level trade credit policy. In the first model, the demand rate is influenced by both the displayed stock and the selling price, while in others, the demand rate depends only on the selling price. In both models deterioration rate is constant. Mahata *et al.* [19] formulated an *EOQ* model for time-varying deteriorating items where the demand rate is an increasing function of the credit period. Tiwari *et al.* [36] developed an *EOQ* model for the retailer's credit and inventory decisions for the inferior quality and deteriorating items under two-level trade credit. This model explores the effects of deterioration and trade credit policy on inventory control of imperfect quality items. The above models are formulated under an infinite time horizon with two-level trade credited arrangement for wholesaler-retailer-customer SC.

Only a few investigations are available on the trade credited inventory model under a finite time horizon. Chang *et al.* [11] formulated a two-level inventory model in a finite time horizon. In that model, the demand rate is constant, and deterioration is a general function of time. Pramanik and Maiti [25] formulated an inventory model of a deteriorating item under a two-level partial trade credit policy incorporating inflation and the time value of money in a finite planning horizon.

Key features of the above investigations are presented in Table 1.

1.1. Research questions

This literature survey leads to some research questions.

(i) As discussed, demand/revenue increases with the credit period in the trade credited inventory system, but the default risk due to default customers also increases. Is there any measure to control the default risk? Till now, this aspect is yet to be considered in the trade credited inventory control system.

(ii) Perishable items, like vegetables, fruits, etc., deteriorate with time and the deterioration rate also increases fast towards the end. In the business of these items, after a specific time (reaching the deterioration to a certain level), it is economical to dispose of these units as early as possible; otherwise, the amount of deteriorated units will increase very fast. For some seasonal products, say potatoes, if the good potatoes come in contact with a rotten potato, the good ones are affected immediately. So, at what deterioration level is it advisable for the retailer to increase the customers' trade credit? Actually, at this point, the retailer is in a fix. If he does not increase the credit period, the deteriorated units will be more, which is a direct loss to him. If the retailer increases the credit period, the demand and the sale will increase, but the number of default customers will also increase. What will be the optimum increased credit period for maximum profit? Till now, this common practice has yet to be investigated.

(iii) The business period is finite for all seasonal products, like potatoes, onions, etc., with increasing demand over time. For example, in the market, the price of potatoes is lowest at harvest time. Because several people will have some potatoes from their own cultivation. After a few months, the potatoes of marginal farmers will be exhausted at different times, and then they go to the open market for purchase. So, as a result, demand increases. This continues up to the subsequent yield. In this situation, how many times will the retailer make the replenishment (time cycles) within a finite time horizon for maximum profit? Till now, none has investigated a finite time horizon inventory model with trended demand and time-dependent deterioration allowing a two-level trade credit system.

(iv) The business period of the seasonal products in a year mainly depends upon the season (harvesting time), which fluctuates due to the seasonal variation every year. Hence, these products business duration is finite (not infinite) but may be uncertain. In this case, how to represent this uncertainty of the business period? How to formulate and solve the two-level trade credited inventory model with dynamic demand and deterioration under an uncertain time horizon?

In this investigation, we formulate a two-level trade credit inventory model under a finite time horizon, where customers' demand and product deterioration rate increase with time. Here, demand and deterioration increase respectively throughout the time horizon and each time cycle. It is assumed that the retailer gets non-deteriorated units from the wholesaler's refrigerated godown at the beginning of each replenishment. In this formulation, the wholesaler provides a fixed credit period to the retailer on the purchasing of products. In each cycle, customers' two trade credit periods are offered. In the beginning, the retailer provides a credit period to the customers. At a specific level of deterioration, the retailer offers an increased credit period to increase the demand for early exhaust of the stock. Because of the credit period at the customer's end, there is some default risk for the retailer. If credit is more, the demand is more, but default risk also increases. To control the default risk, the retailer incurs some expenditures to make an arrangement (reminder level) to remind the defaulters. Some customers may not like this process of chasing; thus, the reminder level negatively affects the demand. The finite time horizon is divided into several time cycles. The model is formulated under different relations between the time period and credit periods provided by the wholesaler and retailer. The optimum customers' credit periods, reminder level, and the number of replenishment's are determined for maximum retailers profit by solving the mixed non-linear mathematical problem by iterative technique and the Generalised Reduced Gradient method through LINGO 19.0 software. Several analytical deductions are presented in the form of theorems and lemmas. The behavior of profit against the number of cycles, trade credits, and reminder costs are illustrated. Some managerial conclusions are drawn.

1.2. Novelties in this investigation are as follows

In this investigation, we have tried to answer the research questions raised earlier.

- In rural India, money lenders engage some people to chase/remind the defaulters. Mimicking this, for the first time, we introduce the reminder cost to remind the default customers to reduce the default risk, though it has a negative effect on the demand. Here, the optimum reminder cost is evaluated for maximum profit.
- Here, two credit periods within a time cycle are considered for the item with time-dependent increasing deterioration to avoid more loss due to deterioration. At a particular level of deterioration, the credit period is increased for early exhaust of the stock. Both initial and increased credit periods are determined for maximum profit. This real-life practice is introduced in the inventory control system of deteriorated units for the first time.
- We also assume that during the time horizon (business period), time cycles are finite in number and of equal length. As the demand increases with time, the replenished amount increases in successive cycles to satisfy the increased demands. Assuming that the retailer purchases the good units at each replenishment

(the wholesaler has the preservation facility), we find the optimal number (integer) of replenishment for maximum profit.
- For the uncertain time horizon, rough, fuzzy, and type-2 fuzzy representations are used. The horizons are expressed in deterministic forms using trust, possibility, necessity, credibility measures, and cv-based reduction methods.
- Trade-offs due to number of cycles, trade credit, and reminder level against maximum profit are presented.

2. Notations and assumptions

For the development of proposed models, we use the following notations and assumptions.

2.1. Notations

The following notations are used in the development of the mathematical models which is given below

- h : holding cost per unit quantity per period (\$/unit);
- O : ordering cost (OC) per period (\$/order);
- M : credit period offered to the retailer by the wholesaler (in year);
- T : optimal cycle length of each period (in year);
- H : finite time horizon (total business time) (in year);
- s : selling price per unit quantity (\$/unit);
- c : purchasing price per unit quantity (\$/unit);
- a : base demand per cycle;
- a_1, a_3 : demand coefficients for credit period;
- a_2 : demand coefficient for reminder level;
- c_1 : reminder level coefficient for default risk;
- c_2, c_3 : credit period coefficients for default risk;
- e : lifetime of the product (in year);
- I_c : annual interest rate paid by the retailer (\$/unit/year);
- I_e : annual interest rate earned by the retailer (\$/unit/year);
- ρ : investment parameter;
- RC : reminder cost ($= \rho r_c^2$ in \$);
- wrt: with respect to.

Decision variables

- $N(N_1$ and $N_2)$: credit period offered to the customer by the retailer (in year);
- n : number of cycle (integer);
- r_c : reminder level.

2.2. Assumptions

(i) Time horizon is finite and divided into n equal time cycles.
(ii) The model is considered for a single time-dependent deteriorated item with time-dependent demand.
(iii) At the time of replenishment of the item in each cycle, the units are of good quality, *i.e.*, not deteriorated. It is assumed that the item is kept under preservation technology by the wholesaler. But when the item is kept in the retailer's storage, it gets deteriorated as the retailer's warehouse is not a cold storage.
(iv) In ith cycle, deterioration increases continuously with time. The expression of deterioration function in ith cycle is

$$\theta_i(t) = \frac{1}{1 + e + (i-1)T - t}, \quad (i-1)T \le t \le iT. \tag{2.1}$$

(v) For increased sales, both wholesaler and retailer give trade credits M and N to their downstream customers (retailer and common public), respectively.

(vi) To avoid more loss due to the deteriorated item, the wholesaler gives a fixed credit period of M to the retailer, and initially, the retailer provides credit period $N = N_1$ and after pre-assigned deterioration level θ_0, credit period changes to $N = N_2$ for the customers

$$N = \begin{cases} N_1, & \text{if} \quad \theta_0 \leq \theta_i(t) \\ N_2, & \text{if} \quad \theta_0 \geq \theta_i(t) \end{cases} \tag{2.2}$$

where $N_1 \leq N_2$.

(vii) The retailer has to clear all dues of the wholesaler within the credit period M, otherwise the wholesaler charges interest on the dues. On the other hand, the retailer also earns some interest on the revenue from the sold item during the period M.

(viii) Credit period given at the customer end invites risk on the payment like some customers may not pay *i.e.*, become defaulter [29]. This is termed default risk, and due to this, the retailer's profit goes down. To reduce this risk, it is assumed here that the retailer adopts some kind of strategy like sending people to remind the customers (reminder level), making phone calls, etc. to chase the defaulters for payment and by this, some default customers will pay back their dues.

(ix) If the retailer adopts the strategy as mentioned in the assumption (viii) to reduce the default risk, he incurs some extra cost due to it, which is termed as reminder cost and given as

$$RC = \rho r_c^2, \tag{2.3}$$

where $\rho > 0$ is an investment parameter and r_c is reminder level.

(x) Though this process (as discussed in assumption (viii)) reduced the number of default customers, all the customers may not like it. Thus, this chasing activity will have some adverse effects on demand. From the next cycle, some customers may go away from the retailer.

(xi) The customer demand is a function of credit period, reminder level, and time, given by

$$D(t) = \begin{cases} D_{1i}(t) = A_1 + Bt, & \text{if} \quad (i-1)T \leq t \leq (i-1)T + t_1 \\ D_{2i}(t) = A_2 + Bt, & \text{if} \quad (i-1)T + t_1 \leq t \leq iT \end{cases} \tag{2.4}$$

where $A_1 = a(1 + a_1 N_1 - a_2 r_c)$ and $A_2 = a(1 + a_3 N_2 - a_2 r_c)$. a_2, a_3 and B are positive constants, t_1 is the time where deterioration level id θ_0. Clearly, from the above expressions, demand is the increasing function of credit period and time while decreasing with reminder level.

(xii) If r_c is the reminder level, then the expression of default risk is

$$d = \begin{cases} d_1 = 1 - \exp(c_1 r_c - c_2 N_1), & \text{if} \quad \theta_0 \leq \theta_i(t) \\ d_2 = 1 - \exp(c_1 r_c - c_3 N_2), & \text{if} \quad \theta_0 \geq \theta_i(t) \end{cases} \tag{2.5}$$

where c_1, c_2, c_3 are positive constants and $c_1 r_c - c_2 N_1 \leq 0$, $c_1 r_c - c_3 N_2 \leq 0$. If either $N_1 = 0$ or $N_2 = 0$ then obviously $r_c = 0$ (as there is no credit period) and the corresponding default risk is zero. Clearly, from the expression, default risk is a increasing and decreasing function of credit period and reminder level respectively.

3. Mathematical formulations under crisp finite time horizon

Here, a wholesaler-retailer-customers supply chain is considered, where a deteriorating seasonal item is sold. Both item deterioration and customers' demand are time-dependent. The wholesaler offers trade credit, M, to increase the retailer's sales. The retailer gives customers two trade credits - initially N_1 and later N_2 ($N_2 > N_1$),

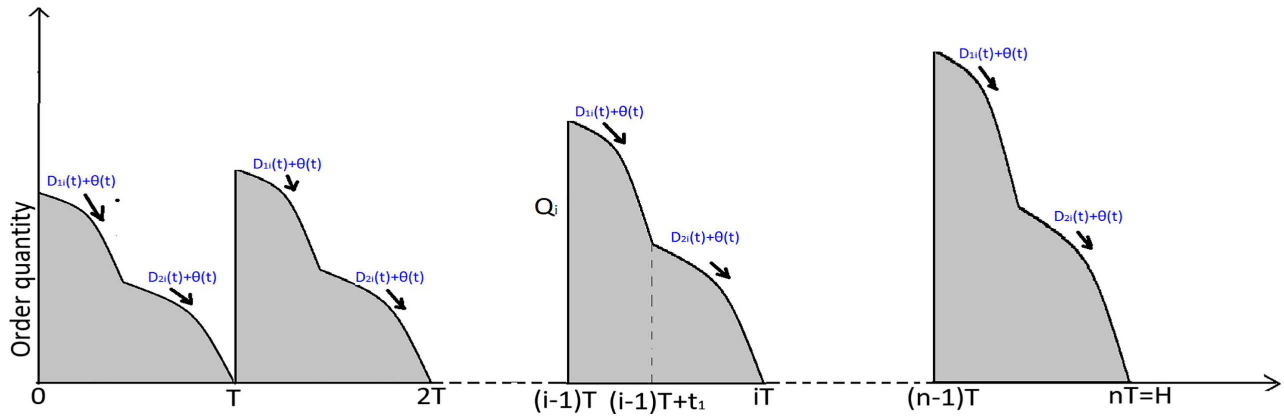


FIGURE 1. Pictorial representation of inventory level.

making changes at a particular level of deterioration. Due to the offer of trade credit to customers, there is a default risk, *i.e.*, some customers become defaulters. An arrangement is made to remind the defaulters, and some cost is incurred depending upon the reminder level, r_c. Of course, some customers may prefer something other than this system of reminding and leaving the system. So, customers' demand is a function of time, N_1 (or N_2), and r_c. As the product is seasonal, its time horizon depends on its one harvesting to another, which is normally uncertain. This uncertainty can be expressed through rough, fuzzy, or type-2 fuzzy variables. Let the retailer does the business in n equal cycles during the business period. Here, as the time cycle during each replenishment is fixed and the same in length, the replenished amount gradually increases with time.

The pictorial diagram of the proposed two-level *EOQ* model is given in Figure 1. In this representation, both demand and deterioration are included. As items demand and deterioration increase with time, retailers also increase the stock level or purchase more goods to sell in the next cycle. On the basis of the assumptions in Section 2.2, we derive the following expression.

At any time t, inventory level in ith cycle is

$$\begin{cases} \frac{dq_{1i}(t)}{dt} = -D_{1i}(t) - \theta(t)q_{1i}, & (i-1)T \le t \le (i-1)T + t_1 \\ \frac{dq_{2i}(t)}{dt} = -D_{2i}(t) - \theta(t)q_{2i}, & (i-1)T + t_1 \le t \le iT \end{cases} \tag{3.1}$$

with the condition $q_{1i}((i-1)T) = Q_i$, $q_{1i}((i-1)T + t_1) = q_{2i}((i-1)T + t_1)$ and $q_{2i}(iT) = 0$ for $i = 1 \dots n$. Solving equation 3.1 we have

$$\begin{cases} q_{1i}(t) &= (1+e+(i-1)T-t)\left\{K_{1i}\log\left(\frac{1+e+(i-1)T-t}{1+e-t_1}\right) - B\left\{((i-1)T+t_1)-t\right\} + K_{3i}\right\} \\ q_{2i}(t) &= (1+e+(i-1)T-t)\left\{K_{2i}\log\left(\frac{1+e+(i-1)T-t}{1+e-T}\right) - B(iT-t)\right\} \end{cases} \tag{3.2}$$

where K_{1i}, K_{2i} and K_{3i} are independent of time, and given in Appendix A. The retailer's initial order quantity in ith cycle is

$$Q_i = q_{1i}((i-1)T) = (1+e)\left\{K_{1i}\log\left(\frac{1+e}{1+e-t_1}\right) - Bt_1 + K_{3i}\right\}. \tag{3.3}$$

Therefore, in ith cycle, the retailer purchasing and selling prices are respectively

$$RPP_i = c(1+e)\left\{K_{1i}\log\left(\frac{1+e}{1+e-t_1}\right) - Bt_1 + K_{3i}\right\} \quad \text{and}$$

$$RSP_i = s\left[(1-d_1)\left\{A_1t_1 + \frac{B}{2}\left(2(i-1)T+t_1\right)t_1\right\} + (1-d_2)\left\{A_2(T-t_1) + \frac{B}{2}\left((2i-1)T+t_1\right)(T-t_1)\right\}\right]. \tag{3.4}$$

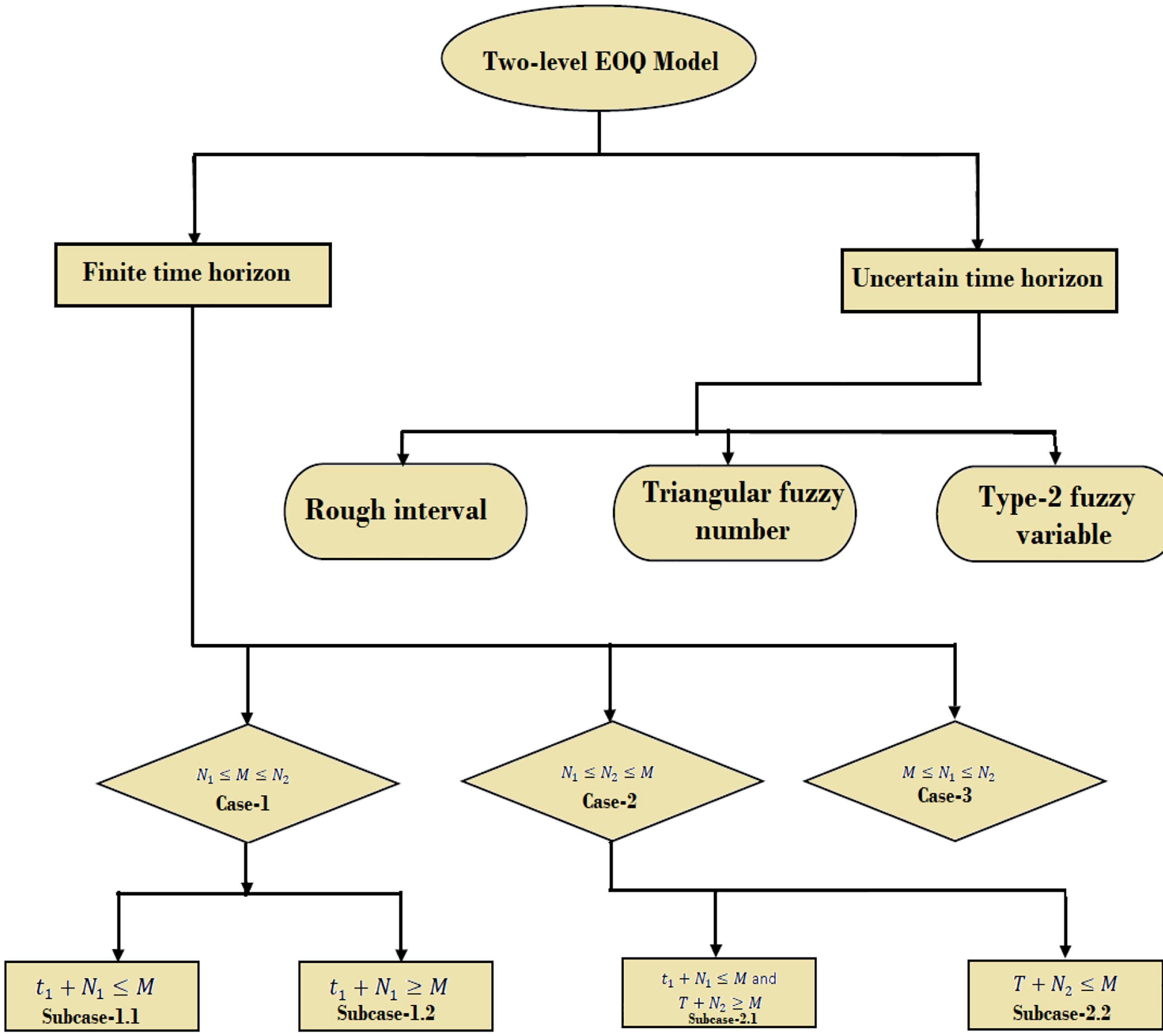


FIGURE 2. Pictorial representation for different case and sub-cases.

During the ith cycle, the holding cost is

$$HC_i = h\left\{\int_{(i-1)T}^{(i-1)T+t_1} q_{1i}(t)\,\mathrm{d}t + \int_{(i-1)T+t_1}^{iT} q_{2i}(t)\,\mathrm{d}t\right\} = h\Big(H_{1i} + H_{2i} + H_{3i} + H_{4i} + H_{5i}\Big) \tag{3.5}$$

where $H_{1i}, H_{2i}, H_{3i}, H_{4i}$ and H_{5i} are given in Appendix A.

On the basis of the numerical value of N_1, N_2 and M, we develop the mathematical model under some case and sub-cases as in Figure 2.

3.1. Case 1. If $N_1 \le M \le N_2$

Initially, The retailer provides N_1 to the customer up to t_1 (say), and after that, at $t = t_1$, N_2 up to the end of the cycle, but he clears all the wholesaler's dues at $t = M$. Here, we assume $N_1 \le M \le N_2$. Since $M \le N_2$, in this situation, the retailer definitely pays interest on the purchasing amount in each cycle under $N_1 \le M \le N_2$.

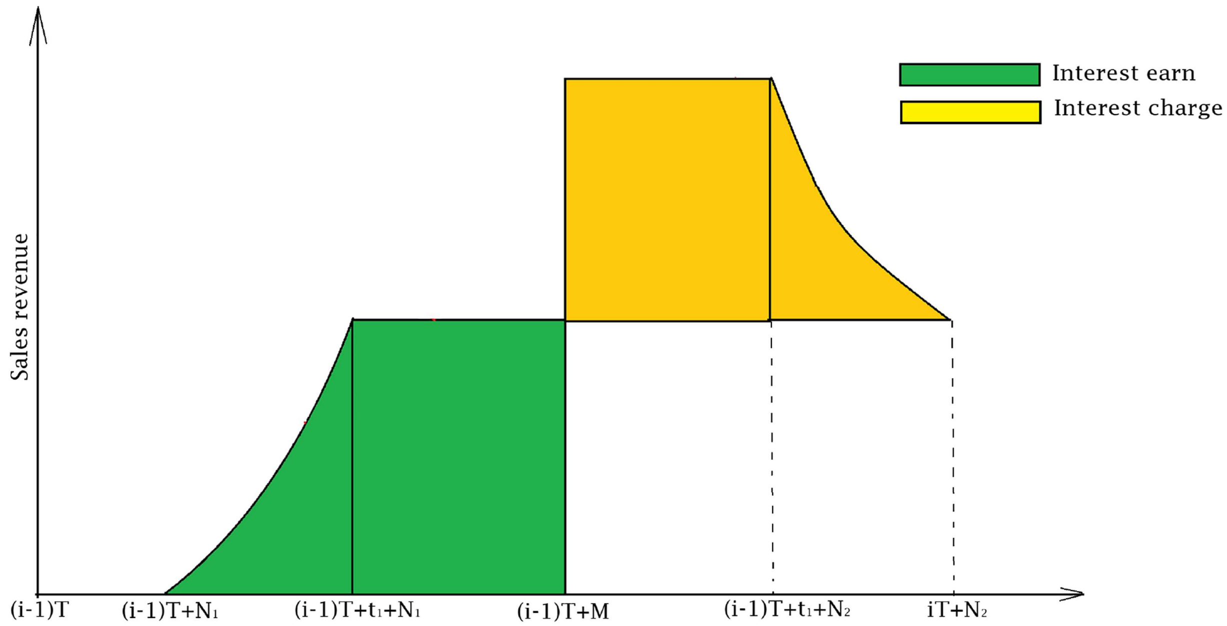


FIGURE 3. Pictorial representation for sub-case 1.1.

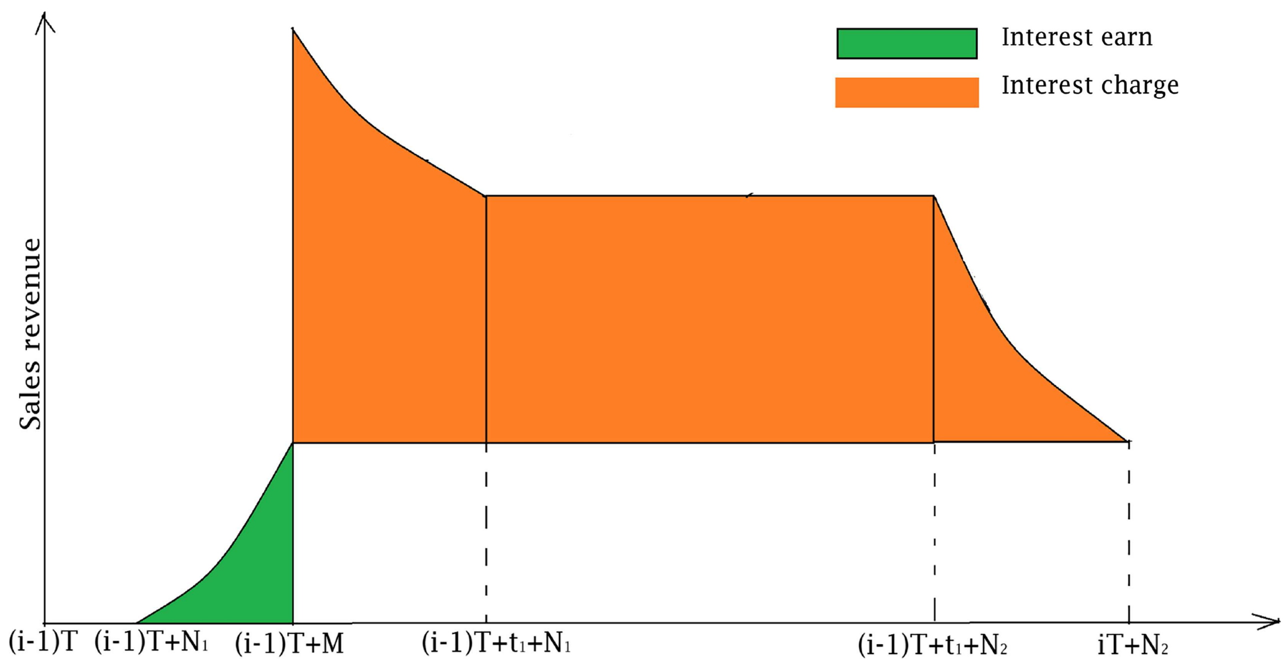


FIGURE 4. Pictorial representation for sub-case 1.2.

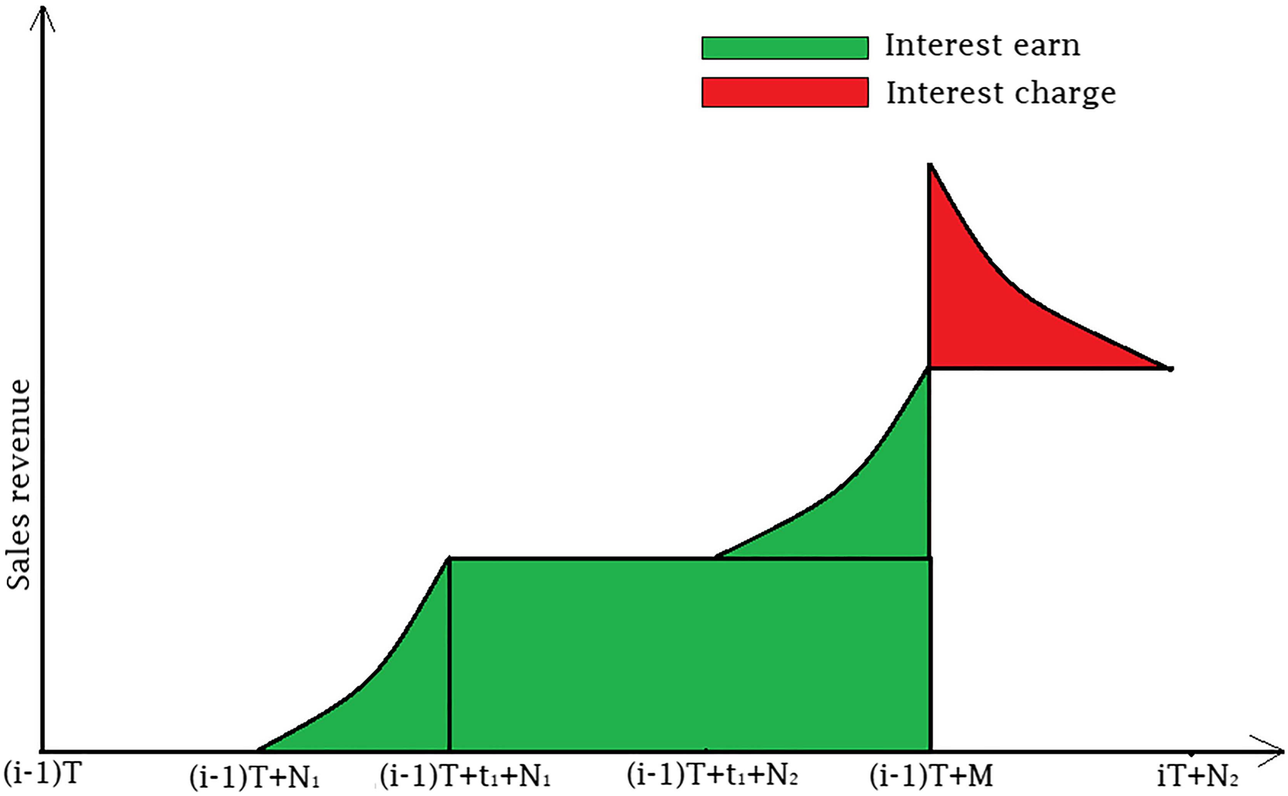


FIGURE 5. Pictorial representation for sub-case 2.1.

Based on N_1 and t_1 values, there are two potential sub-cases, *i.e.*, (i) $t_1 + N_1 \leq M$ and (ii) $t_1 + N_1 \geq M$. In both sub-cases, the retailer earns interest only on the sold amount before t_1 in each cycle.

3.1.1. Sub-case 1.1. If $t_1 + N_1 \leq M$

In this situation, the retailer receives all revenues for the sold amount within M against the demand $D_{1i}(t)$ in each cycle. So, he earns interest on it from $(i-1)T + N_1$ to $(i-1)T + M$. On the other hand, as $M \leq N_2$, the retailer starts receiving revenue for demand $D_{2i}(t)$ form $iT + t_1 + N_2$ and receives the last payment at $iT + N_2$ paying interest on the whole purchased amount from $(i-1)T + M$ to $iT + N_2$ (shown in Fig. 3).

The retailer's total earned interest in ith cycle is

$$TIE_{1i} = p(1-d_1)I_e\left[\frac{A_1}{2}\Big(2(i-1)T + t_1\Big)t_1 + \frac{B}{3}\left\{\Big((i-1)T + t_1\Big)^3 - \Big((i-1)T\Big)^3\right\} + \left\{A_1 t_1 + \frac{B}{2}\Big(2(i-1)T + t_1\Big)t_1\right\}(M - t_1 - N_1)\right]. \tag{3.6}$$

Up to $(i-1)T + t_1$, the retailer does not pay any interest on the sold items, but he does not have any revenue within $(i-1)T + M$ on the sold amount after $(i-1)T + t_1$; therefore he pays interest on the sold sold amounts after t_1. Total interest charged to the retailer in ith cycle is

$$TIC_{1i} = cI_c\Big[(1 + e - t_1)\Big\{\Big(A_2 + B\Big(1 + e + (i-1)T\Big)\Big)\log\left(\frac{1 + e - t_1}{1 + e - T}\right) - B(T - t_1)\Big\}(t_1 + N_2 - M) + \Big(H_4 + H_5\Big)\Big]. \tag{3.7}$$

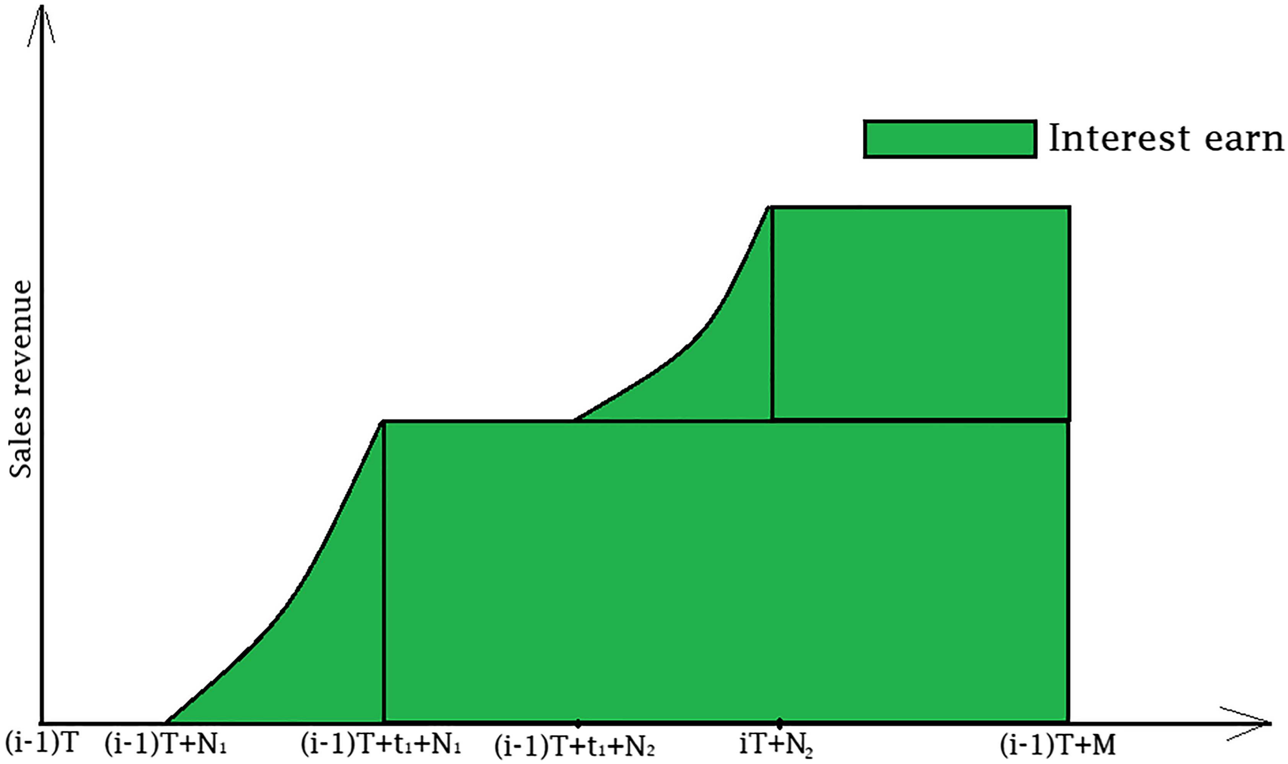


FIGURE 6. Pictorial representation for sub-case 2.2.

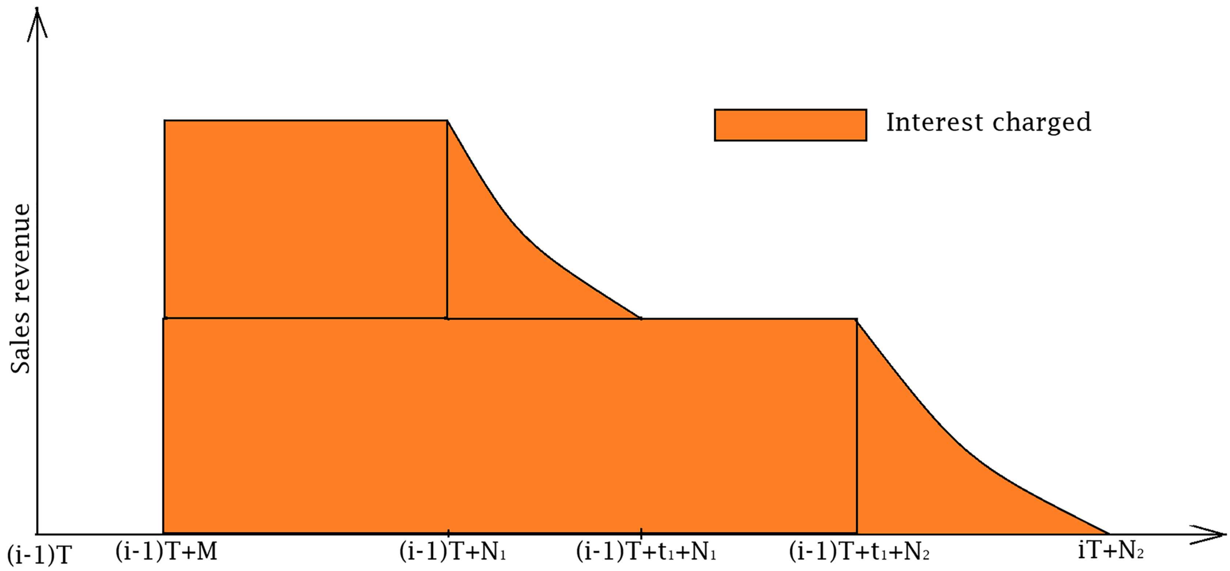


FIGURE 7. Pictorial representation for case 3.

Therefore, the retailer's total profit is

$$RTP_1(N_1, N_2, r_c, n) = \sum_{i=1}^{n} \Big(RSP_i + TEI_{1i} - RPP_i - TIC_{1i} - HC_i - OC - RC\Big) \tag{3.8}$$

under the conditions: $c_1r_c - c_2N_1 \leq 0$, $c_1r_c - c_3N_2 \leq 0$, $t_1 + N_1 \leq M$, $nT = H$, $N_1 \leq M \leq N_2$ and $N_1, N_2, T, r_c \geq 0$.

3.1.2. Sub-case 1.2. If $t_1 + N_1 \geq M$

Here before $(i-1)T + t_1$, the retailer receives the revenue from $(i-1)T + N_1$ to $(i-1)T + t_1 + N_1$. So he earns interest only on sold units from $(i-1)T$ to $(i-1)T + M - N_1$ (*cf.* in Fig. 4). Total earned interest in ith cycle is

$$TEI_{2i} = s(1-d_1)I_e \left[\frac{A_1}{2} + \frac{B(M-N_1)}{3}\right](M-N_1)^2. \tag{3.9}$$

The retailer also gets some revenue after $(i-1)T + M$ on sold amount before $(i-1)T + t_1$ *i.e.*, sold amount from $(i-1)T + M - N_1$ to $(i-1)T + t_1$. Therefore he pays interest on it from $(i-1)T + M$ to $(i-1)T + t_1 + N_1$ and on whole sold amount after $(i-1)T + t_1$ (*cf.* in Fig. 4). Total interest charge for the retailer in ith cycle is

$$TIC_{2i} = cI_c \left[\frac{TIC_{11i}}{cI_c} + \big(I_{21i} + I_{22i} + I_{23i}\big)\right] \tag{3.10}$$

where the expressions of I_{21i}, I_{22i} and I_{23i} are given in Appendix A. Therefore, the retailer's total profit is

$$RTP_2(N_1, N_2, r_c, n) = \sum_{i=1}^{n} \Big(RSP_i + TEI_{2i} - RPP_i - TIC_{2i} - HC_i - OC - RC\Big) \tag{3.11}$$

under the conditions: $c_1r_c - c_2N_1 \leq 0$, $c_1r_c - c_3N_2 \leq 0$, $t_1 + N_1 \geq M$, $nT = H$, $N_1 \leq M \leq N_2$ and $N_1, N_2, T, r_c \geq 0$.

3.2. Case 2. If $N_1 \leq N_2 \leq M$

In this case, the retailer gets the first payment at N_1 on the sold amount before t_1 and at N_2 on sold amount after t_1. As both N_1 and N_2 are less than M, so he earns interest on the sold amount before t_1 and as well as after t_1. On the basis of N_2, t_1 and T values, there are two potential sub-cases (i) $t_1 + N_2 \leq M$ with $T + N_2 \geq M$ (ii) $T + N_2 \leq M$. In both sub-cases, the retailer may pay the interest charged on the sold amount after t_1.

3.2.1. Sub-case 2.1. If $t_1 + N_2 \leq M$ and $T + N_2 \geq M$

Against the sold units before $(i-1)T + t_1$, the retailer gets first payment at $(i-1)T + N_1$ and the last payment at $(i-1)T + t_1 + N_1$ which is less than $(i-1)T + M$ as $N_1 \leq N_2$. So he earns interest on the sold units before $(i-1)T + t_1$. But, as $t_1 + N_2 \leq M$ and $T + N_2 \geq M$, the retailer earns interest from $(i-1)T + t_1 + N_2$ to $(i-1)T + M$ on the sold amount from $(i-1)T + t_1$ to $(i-1)T + M - N_2$ (as shown in Fig. 5). In this scenario, the retailer's total earned interest in ith cycle is

$$\begin{aligned} TEI_{3i} = sI_e \Bigg[\frac{TIE_{1i}}{sI_e} + (1-d_2)\Bigg[\frac{(A_2 - BN_2)}{2}\Big\{\Big(2(i-1)T + t_1 + N_1 + M\Big)(M - t_1 - N_1) - A_2N_2(M - t_1 - N_1)\Big\} \\ + \frac{B}{3}\Big\{\Big((i-1)T + M\Big)^3 - \Big((i-1)T + t_1 + N_1\Big)^3\Big\}\Bigg]\Bigg]. \end{aligned} \tag{3.12}$$

The retailer pays interest charge only from $(i-1)T+M$ to $iT+N_2$ on the sold amount $(i-1)T+M-N_2$ to iT (*cf.* Fig. 5). Total interest charge paid by the retailer in the ith cycle is

$$\begin{aligned} TIC_{3i} = cI_c \Bigg[\frac{1}{4}\Bigg[(1+e-T-N_2)^2 \left\{ 1 - 2\log\left(\frac{1+e-T-N_2}{1+e-T}\right)\right\} - (1+e-M)^2 \left\{1-2\log\left(\frac{1+e-M}{1+e-T}\right)\right\}\Bigg]\\ + B\Bigg[\frac{1}{3}\Big\{(iT+N_2)^3 - ((i-1)T+M)^3\Big\} - \frac{1}{2}\Big(1+e+(2i-1)T-2N_2\Big)\Big((2i-1)T+M+N_2\Big)(T+N_2-M)\\ +\Big(1+e+(i-1)T-N_2\Big)(iT-N_2)(T+N_2-M)\Bigg]\Bigg]. \end{aligned} \tag{3.13}$$

Therefore the retailer's total profit is

$$RTP_3(N_1,N_2,r_c,n) = \sum_{i=1}^{n}\Big(RSP_i + TEI_{3i} - RPP_i - TIC_{3i} - HC_i - OC - RC\Big) \tag{3.14}$$

under the conditions: $c_1r_c - c_2N_1 \le 0$, $c_1r_c - c_3N_2 \le 0$, $t_1 + N_2 \le M$, $T+N_2 \ge M$, $nT = H$, $N_1 \le N_2 \le M$ and $N_1, N_2, T, r_c \ge 0$.

3.2.2. Sub-case 2.2. If $T + N_2 \le M$

In this scenario, the retailer gets the first payment at $(i-1)T+N_1$ and the last payment at $iT+N_2$, which is less than $(i-1)T+M$. So he has all the revenue within $(i-1)T+M$. Therefore, he earns interest on the whole sold amount, and the total interest charge is zero (*cf.* in Fig. 6).

The retailer's total earn interest in ith cycle is

$$\begin{aligned} TEI_{4i} = sI_e\Bigg(\frac{TIE_{1i}}{sI_e} + (1-d_2)\Bigg[\frac{A_2}{2}\Big((2i-1)T+t_1\Big)(T-t_1) + \frac{B}{3}\Big\{(iT)^3 - \Big((i-1)T+t_1\Big)^3\Big\}\\ +\Big\{A_2(T-t_1) + \frac{B}{2}\Big((2i-1)T+t_1\Big)(T-t_1)\Big\}(M-T-N_2)\Bigg]\Bigg). \end{aligned} \tag{3.15}$$

Retailer's total interest charge in this scenario is zero *i.e.*, $TIC_{4i} = 0$. Therefore the retailer's total profit amount is

$$RTP_4(N_1,N_2,r_c,n) = \sum_{i=1}^{n}\Big(RSP_i + TEI_{4i} - RPP_i - TIC_{4i} - HC_i - OC - RC\Big) \tag{3.16}$$

under the conditions: $c_1r_c - c_2N_1 \le 0$, $c_1r_c - c_3N_2 \le 0$, $T+N_2 \le M$, $nT = H$, $N_1 \le N_2 \le M$ and $N_1, N_2, T, r_c \ge 0$.

3.3. Case 3. If $M \le N_1 \le N_2$

In this case, the retailer starts getting revenue from the time $(i-1)T+N_1$ which is more than $(i-1)T+M$, *i.e.*, he does not have any revenue within the period $(i-1)T+M$. So the total earned interest is zero ($TEI_{5i} = 0$) and he pays interest charge on the whole purchasing amount (*cf.* Fig. 7).

Total charged interest in ith cycle is

$$TIC_{5i} = cI_c\Big(H_{1i} + H_{2i} + H_{3i} + H_{4i} + H_{5i} + RPP_i(N_2 - M)\Big). \tag{3.17}$$

Therefore the retailer's total profit amount is

$$RTP_5(N_1,N_2,r_c,n) = \sum_{i=1}^{n}\Big(RSP_i + TEI_{5i} - RPP_i - TIC_{5i} - HC_i - OC - RC\Big) \tag{3.18}$$

under the conditions: $c_1r_c - c_2N_1 \le 0$, $c_1r_c - c_3N_2 \le 0$, $M \le N_1 \le N_2$, $nT = H$, and $N_1, N_2, T, r_c \ge 0$.

4. Deduction of expression of an earlier investigation (*cf.* [9]) from the present model

To tally with the Bhunia and Maiti [9] model, we make the following additional assumptions in the present model.

(i) There is no credit period for the retailer and customers *i.e.*, $M = N = 0$ where $N = (N_1, N_2)$.
(ii) There is no reminder cost *i.e.*, $r_c = 0$.
(iii) Deterioration rate is constant *i.e.*, $\theta(t) = \theta$ (say).

With these assumptions, from the model under sub case 2.1, profit expression 3.14 reduces to the cost function which is minimized. In this expression, we have

- $RSP_i = TEI_{3i} = 0$
- $RPP_i = \left[\frac{c}{\theta^2}\left\{\left\{\theta(a+BiT)-B\right\}\exp(\theta T) - \left\{\theta\left(a+B(i-1)T\right)-B\right\}\right\}\right.$
- $TC_{3i} = 0$
- $HC_i = \frac{h}{\theta^2}\left\{\frac{\theta(a+BiT)-B}{\theta}\left\{\exp(\theta iT)-1\right\} - (a\theta - B)T - \frac{B(2i-1)T^2}{2}\right\}$
- $OC_i = O$
- $RC = 0$

Summing these expressions over i, we get the expression in (18) of Bhunia and Maiti [9] investigation. Hence, with the input data as in [9], the results of Bhunia and Maiti [9] can be obtained.

5. Analytical results

Theorems and lemmas for some particular values of (N_1, N_2, r_c)

5.1. For sub-case 1.1

Theorem 1. *For given positive values of N_2 and r_c, if $K_1 \le 0$, we have*

(i) RTP_1 is a strictly pseudo-concave function in N_1, and hence a unique optimum solution $N^*_{1_1}$ exists.
(ii) If $\delta^1_{N_1} \le 0$, then RTP_1 is maximized at $N^*_{1_1} = 0$.
(iii) If $\delta^1_{N_1} \ge 0$, then there exist a unique $N^*_{1_1}(> 0)$ such that at which RTP_1 is maximized.

where K_1 is the second order partial derivative of RTP_1 wrt N_1 and proof is given in Appendix B.

Theorem 2. *For given positive values of N_1 and r_c, if $K_2 \le 0$, then we have*

(i) RTP_1 is a strictly pseudo-concave function in N_2, and hence a unique optimum solution $N^*_{2_1}$ exists.
(ii) If $\delta^1_{N_2} \le 0$, then RTP_1 is maximized at $N^*_{2_1} = 0$.
(iii) If $\delta^1_{N_2} \ge 0$, then there exist a unique $N^*_{2_1}(> 0)$ such that at which RTP_1 is maximized.

where K_2 is the second order partial derivative of RTP_1 wrt N_2 and proof is similar to Theorem 1.

Theorem 3. *For given values of $(N_1, N_2) > 0$, if $K_3 \le 0$, then we have*

(i) RTP_1 is a strictly pseudo-concave function in r_c, and hence a unique optimum solution $r^*_{c_1}$ exists.
(ii) If $\delta^1_{r_c} \le 0$, then RTP_1 is maximized at $r^*_{c_1} = 0$.
(iii) If $\delta^1_{r_c} \ge 0$, then there exist a unique $r^*_{c_1}(> 0)$ such that at which RTP_1 is maximized.

where K_3 is the second order partial derivative of RTP_1 wrt r_c and proof is similar to Theorem 1.

Lemma 4. *The profit function RTP_1 is concave in N_1, N_2 and r_c if the Hessian matrix*

$$H(RTP_1; N_1, N_2, r_c) = \begin{bmatrix} \frac{\partial^2 RTP_1}{\partial N_1^2} & \frac{\partial^2 RTP_1}{\partial N_1 \partial N_2} & \frac{\partial^2 RTP_1}{\partial N_1 \partial r_c} \\ \frac{\partial^2 RTP_1}{\partial N_1 \partial N_2} & \frac{\partial^2 RTP_1}{\partial N_2^2} & \frac{\partial^2 RTP_1}{\partial N_2 \partial r_c} \\ \frac{\partial^2 RTP_1}{\partial N_1 \partial r_c} & \frac{\partial^2 RTP_1}{\partial N_2 \partial r_c} & \frac{\partial^2 RTP_1}{\partial r_c^2} \end{bmatrix} = \left[\frac{\partial^2 RTP_1}{\partial j_1 \partial j_2} \right] \tag{5.1}$$

where $(j_1, j_2) = \big((N_1, N_2, r_c), (N_2, r_c, N_1)\big)$,is negative definite while convex if the Hessian matrix is positive definite.

Since, expression of the Hessian matrix is highly non-linear, we prove it numerically which is given in Appendix C.

5.2. For sub-case 1.2, 2.1, 2.2 and case 3

Theorem 5. *Following the Theorems 1, 2 and 3 for sub-case 1.1, For given positive values of J_1 and J_2, if $K_{J_3} \leq 0$, then we have*

(i) RTP_{J_4} is a strictly pseudo-concave function in J_3, and hence a unique optimum solution ${J_3}^*_{1_2}$ exists.
(ii) If $\delta^{J_4}_{J_3} \leq 0$, then RTP_{J_4} is maximized at ${J_3}^*_{1_2} = 0$.
(iii) If $\delta^{J_4}_{J_3} \geq 0$, then there exist a unique ${J_3}^*_{1_2} (> 0)$ such that at which RTP_{J_4} is maximized. Where

$(J_1, J_2, J_3) \equiv \big((N_1, N_2, r_c), (N_2, r_c, N_1), (r_c, N_1, N_2)\big)$, $J_4 = 2, 3, 4, 5$ and $K_{J_3} = \frac{\partial^2 RTP_{J_4}}{\partial J_3^2}$. Proof is similar to the Theorems 1, 2 and 3.

Lemma 6. *The profit function RTP_{J_4} is concave in J_1, J_2 and J_3 if the Hessian matrix $H(RTP_{J_4}; J_1, J_2, J_3) = \left[\frac{\partial^2 RTP_{J_4}}{\partial J_{j_1} \partial J_{j_2}}\right]$, where $(J_1, J_2, J_3) \equiv \big((N_1, N_2, r_c)\ , (N_2, r_c, N_1),\ (r_c, N_1, N_2)\big)$ $J_4 = 2, 3, 4, 5$ $j_1 = 1, 2, 3$ and $j_2 = 1, 2, 3$ is negative definite and convex if Hessian matrix is positive definite.*

6. Mathematical formulations under uncertain time horizon

In this section, we consider time horizon is finite but uncertain. If $\widetilde{H}$ is uncertain, then the objective function is

$$\begin{aligned} &\text{Maximize} = RTP_j \\ &\text{Subject to} \\ &c_1 r_c - c_2 N_1 \leq 0 \\ &c_1 r_c - c_3 N_2 \leq 0, \end{aligned} \tag{6.1}$$

$$\text{and } \ nT \leq \widetilde{H} \tag{6.2}$$

where, $j = 1 \cdots 5$ and $\widetilde{H}$ is represented by rough, fuzzy or type-2 fuzzy variable. We use trust measure (Tr), possibility measure and and CV-based reduction method (*cf.* Bera *et al.* [7,8], Qin *et al.* [27]) respectively to get the deterministic expressions of equation 6.2.

6.1. Time horizon is rough variable

Let $\widetilde{H} = \big([h_1, h_2], [h_3, h_4]\big)$ is rough variable with $h_3 \leq h_1 < h_2 \leq h_4$, then for a predefined α, $0 < \alpha \leq 1$, $Tr\{\widetilde{H} \geq nT\} \geq \alpha$ is equivalent to

(i) $(1 - 2\alpha)h_4 + 2\alpha h_3 \geq nT$, when $\alpha \leq \frac{h_4 - h_2}{2(h_4 - h_3)}$,
(ii) $2(1 - \alpha)h_4 + (2\alpha - 1)h_3 \geq nT$, when $\alpha \geq \frac{2h_4 - h_1 - h3}{2(h_4 - h_3)}$,
(iii) $\frac{h_4(h_2 - h_1) + h_2(h_4 - h_3) - 2\alpha(h_2 - h_1)(h_4 - h_3)}{(h_2 - h_1) + (h_4 - h_3)} \geq nT$, otherwise.

6.2. Time horizon is a fuzzy number

If $\widetilde{H} = \big(h_1, h_2, h_3\big)$ be a triangular fuzzy number (TFN) with $0 < h_1$ and nT is a crisp number, $Pos(\widetilde{H} \geq nT) \geq \alpha$, $Nes(\widetilde{H} \geq nT) \geq \alpha$, and credibility ($Cr$) measures of $\widetilde{H}$ are equivalent to $nT \leq (1-\alpha)h_3 + \alpha h_2$ in Pos. sense, $nT \leq (1-\alpha)h_2 + \alpha h_1$ in Nes. sense
and $Cr = \frac{1}{2}(pos(\widetilde{H} \geq nT) + nes(\widetilde{H} \geq nT))$ respectively, where $\alpha \in (0, 1)$.

6.3. Time horizon is type-2 fuzzy variable

Let the type-2 triangular fuzzy variable $\widetilde{H} = \big(h_1, h_2, h_3; \theta_l, \theta_r\big)$. By CV reduction method H, then $\tilde{C}r\{H \geq nT\} \geq \alpha$ is equivalent to

(i) $\frac{(1-2\alpha+(1-4\alpha)\theta_l)h_3+2\alpha h_2}{1+(1-4\alpha)\theta_l} \geq nT$ if $\alpha \in (0, 0.25]$,
(ii) $\frac{(1-2\alpha)h_3+(2\alpha+(4\alpha-1)\theta_r)h_2}{1+(4\alpha-1)\theta_r} \geq nT$ if $\alpha \in (0.25, 0.50]$,
(iii) $\frac{(2\alpha-1)h_1+(2(1-\alpha)+(3-4\alpha)\theta_r)h_2}{1+(3-4\alpha)\theta_r} \geq nT$ if $\alpha \in (0.50, 0.75]$,
(iv) $\frac{(2\alpha-1+(4\alpha-3)\theta_l)h_1+2(1-\alpha)h_2}{1+(4\alpha-3)\theta_l} \geq nT$ if $\alpha \in (0.75, 1]$.

6.4. Total profit with uncertain time horizon

Now our maximization problem is reduced to

$$\begin{aligned} &\text{Maximize} \quad = RTP_j \\ &\text{Subject to} \\ &\qquad c_1 r_c - c_2 N_1 \leq 0 \\ &\qquad c_1 r_c - c_3 N_2 \leq 0 \\ &\qquad m \geq \alpha \end{aligned} \tag{6.3}$$

where, $j = 1 \cdots 5$ and $m = Tr\{\widetilde{H} \geq nT\}, pos(\widetilde{H} \geq nT)$ and $\tilde{C}r\{H \geq nT\}$ for rough, fuzzy and type-2 fuzzy time horizon respectively.

7. Particular cases

Here we shall consider some particular cases under the sub-case 1.1 (Sect. 3.1.1), which shown in Figure 8.

7.1. The retailer does not give any credit period to the customers

If the retailer does not give any credit period to the customers, then there is no risk in getting the payment from the customers, so both default risk and reminder cost are zero, i.r., $d = 0, r_c = 0$. In this case, formulation is similar to the case-2 and total profit is obtained from equations 3.14 and 3.16 with $N_1 = 0$, $N_2 = 0$ and $r_c = 0$. In this situation, demand is only time-dependent.

7.2. Initially no credit period is allowed, but later changes credit period provided by the retailer

If the retailer does not give any credit period initially, but when the deterioration rate is equal to or more than a particular level, then to exhaust the stock early, the retailer provides a delayed payment opportunity to the customers. In this case, a mathematical formulation is similar to case 1.1 and case 2 (sub-case 2.1 and 2.2) with the value of $N_1 = 0, d_1 = 0$.

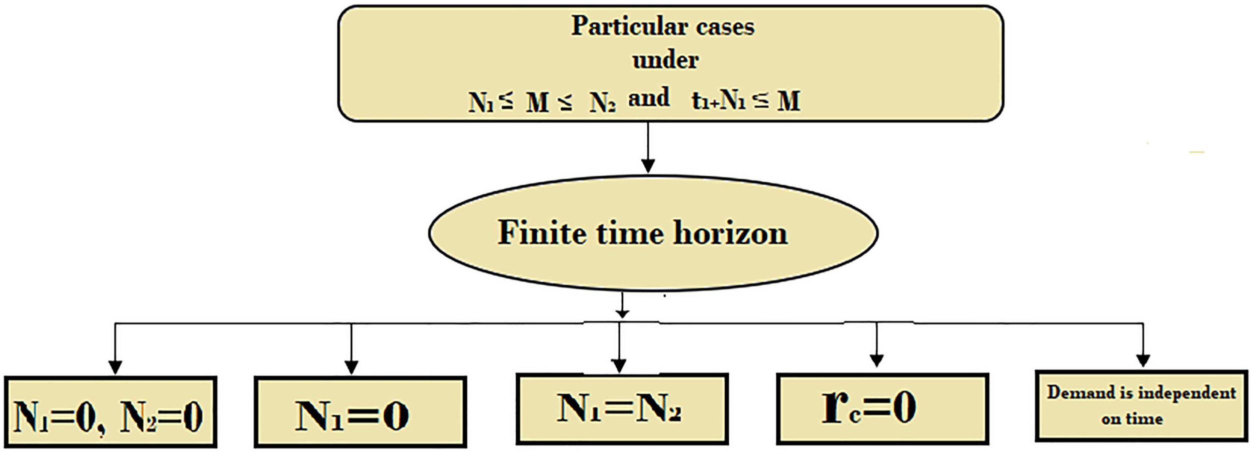


FIGURE 8. Pictorial representation for particular cases.

7.3. Retailer's credit period is same throughout the cycle

Here, the retailer gives the customers only one uniform delay payment opportunity in the whole cycle. In this case, a mathematical formulation is similar to case 2 (sub-case 2.1 and 2.2), case 3 and model formulation in Section 4 with constant deterioration.

7.4. Reminder cost is zero

Here, the retailer does not make any arrangements to reduce default risk. In this case, mathematical expression can be obtained from all cases-1 to 3, with $r_c = 0$.

7.5. Demand is independent of time

Here we consider the parameter $B = 0$, *i.e.*, demand is independent of time and is only a function of credit period and reminder level. Following cases 1 to 3, the mathematical expression can be deduced in this scenario.

There are so many specific cases that are possible for different cases, and their optimal solution procedure is similar to the above following cases and particular cases.

8. NUMERICAL EXPERIMENTS

We perform the numerical experiments for all cases and sub-cases under finite and uncertain time horizons and maximize the retailer's profit using the parametric study and generalized reduced gradient (GRG) method through LINGO 19.0 software for some parametric values given below.

8.1. Input data

The input parametric values to solve the problems under finite and uncertain time horizons are given in Table 2. The uncertain data is collected with the helps of experts and experienced people.

8.2. Experimental results: for finite time horizon

8.2.1. Optimal cycle number

Here, considering the Experiment 1 for the sub-case 1.1, we find the optimal number of cycles for the maximum profit is determined by parametric study, which is presented in Table 3.

Table 2. Parametric values for problems under all cases/sub-cases.

TH	Expt.	Sec.	C/S	a	a_1	a_2	a_3	b	c	c_1	c_2	c_3	e	M	H	ρ	s	O	h	I_e	I_c	Q_0
	1	3.1.1	1.1	500	0.550	0.03	0.08	5	1.5	0.100	0.220	0.18	4.150	0.30	7	140	2.8	45	0.200	6.0%	9 %	20.01%
	2	3.1.2	1.2	500	3.950	0.03	0.08	5	1.5	0.100	0.220	0.18	4.150	0.25	7	140	2.8	45	0.200	6.0%	10 %	20.015%
Ft	3	3.2.1	2.1	750	0.435	0.032	0.79	6	1.8	0.200	0.229	0.18	4.133	0.25	7	145	3.1	45	0.199	6.5%	10 %	20.01%
	4	3.2.2	2.2	500	0.500	0.030	0.08	5	1.5	0.100	0.220	0.93	4.100	0.30	7	140	2.8	45	0.200	6.0%	9 %	20.01%
	5	3.3	3	750	0.105	0.022	4.39	6	1.8	0.143	0.229	0.43	4.133	0.20	7	145	3.1	45	0.199	5.5%	10 %	20.01%
	6	6.1	1.1	650	0.550	0.03	0.80	5	1.5	0.100	0.235	0.18	4.15	0.30	$\widetilde{H}_1$	140	2.8	50	0.200	6.0%	9 %	20.01%
Unct	7	6.2	1.1	700	0.370	0.02	0.80	5	1.5	0.020	0.140	0.18	4.15	0.30	$\widetilde{H}_2$	140	2.8	50	0.200	6.0%	9 %	20.01%
	8	6.3	1.1	700	0.370	0.02	0.80	5	1.5	0.020	0.140	0.18	4.15	0.30	$\widetilde{H}_3$	140	2.8	50	0.200	6.0%	9 %	20.01%

Notes. $\widetilde{H}_1 = ([7,8],[6,9])$, $\widetilde{H}_2 = (7.0, 7.5, 8.0)$, $\widetilde{H}_3 = (7.5, 8.0, 8.5; 0.34, 0.43)$.
TH: Time horizon, Expt: Experiments, Sec.: Section, C/S: cases/sub-cases, Ft: Finite, Unct: uncertain.
*All prices are in $ and time unit in years.

Table 3. Optimal profit of the retailer for different numbers of cycle.

n	28	29	30	31	32	**33**	34	35	36	37	38
N_1	0.1242	0.1237	0.1232	0.1228	0.1224	**0.1220**	0.1216	0.1213	0.1209	0.1206	0.1203
N_2	0.3681	0.3700	0.3718	0.3734	0.3749	**0.3764**	0.3777	0.3790	0.3809	0.3814	0.3824
r_c	0.1414	0.1365	0.1318	0.1275	0.1234	**0.1196**	0.1160	0.1126	0.1094	0.1064	0.1035
HC	110.51	106.19	102.13	98.31	94.708	91.29	88.05	84.98	82.07	79.28	76.63
OC	1400	1450	1400	1550	1600	1650	1700	1750	1800	1850	1900
OP	4605.356	4607.878	4609.691	4610.864	4611.458	**4611.525**	4611.112	4610.262	4609.011	4607.391	4605.432

Notes. HC: Holding cost, OC: Ordering cost, OP: Optimal profit.

Table 4. Optimal solutions of the problems under finite time horizon.

Case	Expt.	Sub-case	N_1^*	N_2^*	t_1	T	n	r_c^*	Optimum profit
1	1	1.1	0.1220	0.3764	0.1525	0.2121	33	0.1196	4611.525
	2	1.2	0.0963	0.3720	0.1537	0.2121	33	0.1313	4055.828
2	3	2.1	0.1097	0.1145	0.1355	0.2800	25	0.1256	6991.112
	4	2.2	0.0525	0.0879	0.1025	0.2121	33	0.1155	5263.130
3	5		0.2275	0.2896	0.1355	0.2800	25	0.3644	7191.144

Notes. Expt.: Experiments.

8.2.2. Optimum profit of the retailer

Against different experiments, we find the optimum values of the decision variables under different cases/sub-cases for retailer's maximum profit, which are given in Table 4.

8.3. Experimental results: for uncertain time horizon

Here, for the input data of experiments 7–9 in Table 2, we derive the optimum results for sub-case 1.1 only. First, we consider the time horizon as the rough, triangular fuzzy number type-2 fuzzy parameter. Experimental results for different levels of uncertainty are given in Table 5. Similarly, the the results of other cases/sub-cases can be obtained.

8.4. Comparison of numerical result for crisp and uncertain model

For comparison, we consider the sub-case 1.1, with the following parameters: $a = 500$, $a_1 = 0.55$, $a_2 = 0.03$, $a_3 = 0.008$, $b = 5$, $c = 1.5$, $c_1 = 0.10$, $c_2 = 0.220$, $c_3 = 0.180$, $e = 4.150$, $M = 0.300$, $\rho = 140$, $s = 2.8$, $O = 45$, $h = 0.2$, $I_e = 6.0\%$, $I_c = 9.0\%$, $Q_o = 20.01\%$, $\alpha = 0.003$, time horizon for crisp model $H = 7$, and for uncertain

TABLE 5. Optimal solutions for different uncertainty level of rough interval.

Time horizon	Experiment	α	N_1^*	N_2^*	r_c^*	T^*	Optimal profit
		0.10	0.1050	0.3780	0.1900	0.2545	8089.961
		0.14	0.0981	0.3798	0.1835	0.2473	7806.453
Rough	7	0.60	0.0741	0.3859	0.1617	0.2227	6848.960
		0.80	0.0654	0.3881	0.1536	0.2136	6494.090
		0.93	0.0578	0.3927	0.1358	0.1945	5748.272
		0.20	0.1250	0.3500	0.0243	0.2394	8037.087
Fuzzy	8	0.50	0.1186	0.3523	0.0237	0.2348	7853.217
		0.80	0.1121	0.3542	0.0230	0.2303	7669.157
		0.17	0.1443	0.3450	0.0262	0.2529	8583.494
		0.25	0.1401	0.3462	0.0258	0.2500	8465.373
		0.29	0.1374	0.3470	0.0255	0.2480	8386.379
		0.45	0.1307	0.3489	0.0249	0.2443	8196.345
Type-2 fuzzy	9	0.57	0.1275	0.3498	0.0245	0.2411	8105.908
		0.65	0.1246	0.3506	0.0242	0.2390	8021.633
		0.73	0.1201	0.3519	0.0238	0.2359	7895.304
		0.81	0.1154	0.3532	0.0234	0.2326	7762.027
		0.89	0.1118	0.3543	0.0230	0.2301	7659.833

TABLE 6. Comparison of numerical result for crisp and uncertain time horizon.

Time horizon		N_1^*	N_2^*	r_c^*	T^*	Optimal profit
Crisp		0.1220	0.3764	0.1196	0.2545	4611.525
Rough		0.1190	0.3838	0.1287	0.2485	4615.614
	Possibility	0.1363	0.3795	0.1135	0.248	4719.167
Fuzzy	Credibility	–	–	–	–	4,610.549
	Necessity	0.1114	0.3801	0.1272	0.2547	4501.932
Type-2 fuzzy		0.1203	0.3868	0.1202	0.2455	4618.150

TABLE 7. Profit of particular problems under finite time horizon.

Sections ↓	Particular Values	Experiment 1 Sub-case 1.1	Experiment 2 Sub-case 1.2	Experiment 3 Sub-case 2.1	Experiment 4 Sub-case 2.2	Experiment 5 Case 3
7.1	$(N_1, N_2, r_c) = 0$	–	–	–	5104.748	–
7.2	$N_1 = 0$	4537.948	–	6830.738	5201.312	–
7.3	$N_1 = N_2$	–	–	6981.738	5260.575	7163.402
7.4	$r_c = 0$	4545.854	3977.039	6796.541	5198.107	7084.193
7.5	$b = 0$	4401.538	3911.656	6758.543	5013.193	7042.143
3.2.2	N_1^*, N_2^* (*cf.* Tab. 4)	–	–	–	5263.130	–

models are $\widetilde{H}_1 = ([6,8],[5,9])$, (For rough) $\widetilde{H}_2 = (6,7,8)$, (for fuzzy) and $\widetilde{H}_3 = (6,7,8;0.34,0.43)$ (for type-2 fuzzy). The optimal results for different models are given in Table 6.

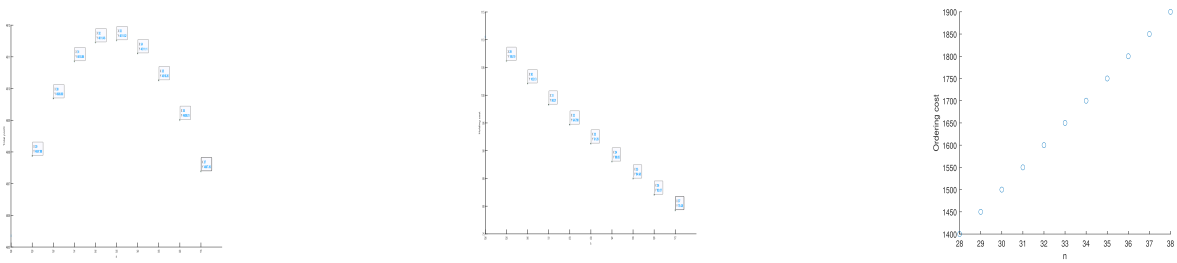
8.5. Experimental results: for particular problems

The experimental results for the particular cases as discussed in Section 7 under finite time horizon are given in Table 7.

TABLE 8. Effects of (N_1, N_2, r_c) on profit components for sub-case 1.1.

	DV	d_1	d_2	D_1	D_2	RSP	TEI	y_1	y_2	RPP	TIC	HC	RC	OC	y_3	y_4	Profit
	0.06	0.0017	0.0542	17003.34	21409.16	10964.33	1615.49	12579.82	–	6120.15	46.05	89.99	66.08	1650	7972.27	–	4607.55
	0.09	0.0082	0.0543	17275.59	21409.16	11030.33	1615.86	12646.19	0.5276	6183.36	46.05	90.63	66.08	1650	8036.13	0.8010	4610.06
	0.12	**0.0148**	**0.0542**	**11095.15**	**21409.16**	**11095.15**	**1615.73**	**12710.88**	**0.5153**	**6246.58**	**46.05**	**91.27**	**66.08**	**1650**	**8099.99**	**0.7947**	**4610.89**
N_1	0.15	0.0212	0.0542	17820.09	21409.16	11158.78	1615.13	12773.91	0.4959	6309.80	46.05	91.91	66.08	1650	8163.85	0.7884	4610.06
	0.18	0.0278	0.0542	18092.34	21409.16	11221.25	1614.06	12835.31	0.4807	6373.02	46.05	92.55	66.08	1650	8227.71	0.7822	4607.60
	0.21	0.0341	0.0542	18364.59	21409.16	11282.56	1612.53	12895.09	0.4657	6436.24	46.05	93.19	66.08	1650	8291.56	0.7760	4603.53
	0.24	0.0404	0.0542	18636.84	21409.16	11342.73	1610.54	12953.27	0.4512	6499.45	46.05	93.82	66.08	1650	8355.43	0.7703	4597.85
	0.28	0.0148	0.0388	17547.84	20221.16	10961.30	1615.73	12577.03	–	6136.48	28.41	88.64	66.08	1650	7969.61	–	4607.42
	0.31	0.0148	0.0440	17547.84	20617.16	11006.49	1615.73	12622.22	0.3593	6173.18	34.10	89.51	66.08	1650	8012.87	0.5428	4609.35
	0.34	0.0148	0.0491	17547.84	21013.16	11051.11	1615.73	12666.84	0.3535	6209.88	39.98	90.39	66.08	1650	8056.33	0.5424	4610.51
N_2	**0.37**	**0.0148**	**0.0543**	**17547.84**	**21409.16**	**11095.15**	**1615.73**	**12710.88**	**0.3477**	**6246.58**	**46.05**	**91.27**	**66.08**	**1650**	**8099.99**	**0.5419**	**4610.89**
	0.40	0.0148	0.0593	17547.84	21805.16	11138.61	1615.73	12754.34	0.3419	6283.28	52.32	92.14	66.08	1650	8143.83	0.5412	4610.51
	0.43	0.0148	0.0644	17547.84	22201.16	11181.51	1615.73	12797.24	0.3364	6319.99	58.78	93.02	66.08	1650	8187.87	0.5408	4609.37
	0.47	0.0148	0.0711	17547.84	22729.16	11227.82	1615.73	12833.55	0.2837	6368.92	67.69	94.19	66.08	1650	8226.88	0.4764	4606.67
	0.05	0.0207	0.0599	17577.54	21438.86	11045.86	1608.67	12654.53	–	6256.23	46.12	91.40	16.41	1650	8060.16	–	4594.37
	0.07	0.0187	0.0580	17567.64	21428.96	11062.27	1611.02	12673.29	0.1482	6253.01	46.10	91.36	29.27	1650	8069.74	0.1189	4603.55
	0.09	0.0167	0.0561	17557.74	21419.06	11078.70	1613.38	12692.08	0.1482	6249.80	46.08	91.31	45.83	1650	8083.03	0.1647	4609.05
r_c	**0.11**	**0.0148**	**0.0543**	**17547.84**	**21409.16**	**11095.15**	**1615.73**	**12710.88**	**0.1481**	**6246.58**	**46.05**	**91.27**	**66.08**	**1650**	**8099.99**	**0.2098**	**4610.89**
	0.13	0.0128	0.0524	17537.94	21399.26	11111.62	1618.09	12729.71	0.1481	6243.37	46.03	91.22	90.03	1650	8120.65	0.2551	4609.06
	0.15	0.0108	0.0504	17528.04	21389.36	11128.12	1620.45	12748.57	0.1481	6240.15	46.01	91.18	117.68	1650	8145.02	0.3001	4603.55
	0.17	0.0088	0.0486	17518.14	21379.46	11144.63	1622.82	12767.45	0.1481	6236.93	45.99	91.13	149.02	1650	8173.08	0.3445	4594.37

Notes. DV: decision variable, y_1 : sum of the retailer total sailing price (RSP) and earn interest (TEI), y_2 : % change in y_1, y_3 : Total expenditure of the retailer, y_4 : % change in y_3.



(A) (B) (C)

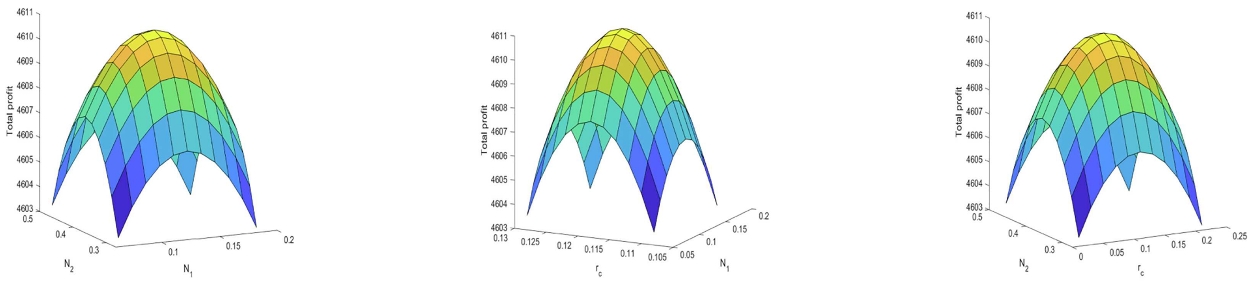
FIGURE 9. Nature of profit, holding cost and ordering cost with number of cycle n. (A) Nature of profit with n. (B) Nature of holding cost with n. (C) Nature of ordering cost with n.

9. DISSECTION OF EXPERIMENTAL RESULTS

(i) **Optimum profit is concave wrt number of cycle:** the number of cycles (n) for the optimum profit of sub-case 1.1 is determined and given in Table 3. By parametric studies method, it is shown that with the increase of n, the holding cost and ordering cost decrease and increase, respectively, as per expectation, and total profit increases up to $n = 33$ and then decreases. This system profit is concave wrt n. (*cf.* Figs. 9A–9C).

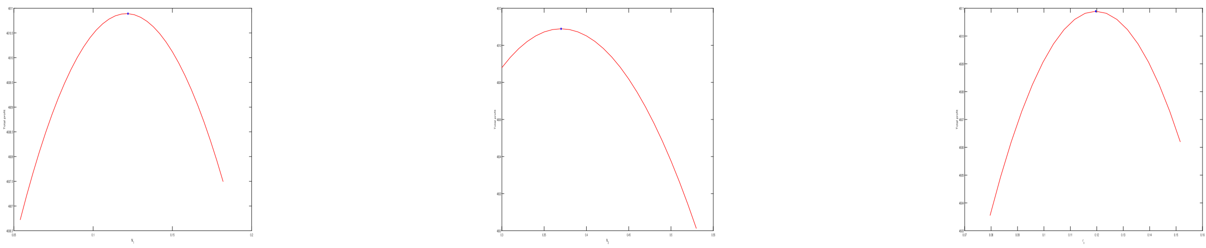
(ii) **3-Dimensional representation of Profit:** in this system, profit is a function of N_1, N_2, and r_c and concave wrt these three variables (*cf.* Appendix C). As four-dimensional graphs are not possible to draw, we present three 3-Dimensional (3 D) graphs to represent the concavity of profit function RTP_1 wrt two decision variables keeping the other variable constant at its optimum value. These are shown in Figures 10A–10C, *i.e.*, wrt N_1 and N_2 (for fixed r_c),N_1 and r_c (for fixed N_2) and N_2 and r_c (for fixed N_1) in Figures 10A–10C respectively. Similarly, we can plot a graph for other sub-cases.

(iii) **Impact of credit period on profit:** as both credit periods positively affect the demand (*i.e.*, the increases with the credit period), the retailer's total revenue increases with the credit period, and so makes the profit. On the other hand, default risk also increases with credit periods (N_1, N_2). So profit decreases after a certain value of N_1 and N_2. Profit is concave in N_1 and N_2 (*cf.* Tab. 8 and Figs. 11A, 11B).



(A) (B) (C)

FIGURE 10. 3 Dimensional representation of profit function. (A) Concavity of profit function in N_2 and N_1, (B) Concavity of profit function in r_c and N_1, (C) Concavity of profit function in N_2 and r_c.



(A) (B) (C)

FIGURE 11. Concavity of profit function wrt N_1, N_2 and r_c. (A) Concavity of profit function in N_1. (B) Concavity of profit function in N_2. (C) Concavity of profit function in r_c.

(iv) **Impact of reminder level on profit:** because of the credit period, the retailer faces some risk in payment from the customers' end, *i.e.*, does not receive revenue from the default customers. So, the retailer adopts a strategy to spend some amount (reminder cost) to get back money from the default customers. Therefore, profit initially increases with the reminder level (r_c), but after a certain reminder level, reminder cost dominates, and profit goes down. So, profit is concave with r_c (*cf.* Tab. 8 and Fig. 11c). Thus, spending money to chase the default customers is beneficial for the retailer, and the retailer's profit increases to a certain level.

It is clear from Tables 4 and 7 that if the retailer uses some strategy to reduce default risk, profit increases, *i.e.*, profit is more with reminder level ($r_c \neq 0$) than without reminder level ($r_c = 0$). This is true up to a certain level of r_c. Because, after the optimal value of r_c, reminder cost dominates. For example, in Experiment 2 (say), the profit with r_c is \$ 4055.828, and without r_c is \$3977.039, which is less than \$ 4055.828.

(v) **Benefits of introducing N_2:** here, the deterioration increases with time, and demand increases with the credit period. As the decay rate increases at an increasing rate, it is beneficial for the retailer to exhaust the stock early by increasing the demand, *i.e.*, offering more credit periods after a certain level of deterioration. Because deterioration invites more deterioration, the retailer's loss will be more if he takes more time to clear the stock. It is demonstrated from Table 7, (experiment 4, sub-case-2.2). When the credit period is the same throughout the cycle, the retailer profit amount is \$ 5222.219, but if we change the credit period from N_1 to N_2, then the profit amount is $ 5263.130 (*cf.* Table 4, experiment 4, sub-case-2.2). So, profit is more if the credit period increases during sales.

(vi) **Profit with time-independent demand:** if customers' demand is independent wrt time, then the retailers' total profit is less wrt time-dependent demand (*cf.* Tabs. 4 and 7 (b = 0)). This is as per expectation.

(vii) **Comparison of numerical result for crisp and uncertain model:** in this investigation, we have considered the seasonal products, whose business period is normally from harvesting time to next year's harvesting time. This time slightly changes, mainly due to weather and uncertainty. The optimal profit of the model under different time horizons changes slightly, *i.e.*, it is almost the same. (*cf.* Sect. 8.4 and Tab. 6).

10. Conclusions

10.1. General conclusions

Here we have formulated a two-level trade credit inventory model with time-dependent demand and deterioration under different (finite and uncertain) time horizons. The formulation considers different cases/sub-cases wrt retailer's and customers' credit periods and the length of time cycle. In this inventory control system, the retailer makes a change in customers' credit period within a time cycle. Retailer's profit sequentially increases for the cases when he offers no credit period, uniform credit period throughout the whole business cycle and initially a credit period, later increased credit period at a certain level of deterioration (*cf.* Tabs. 4 and 7).

For the first time, a strategy for the retailer is suggested to counter the default risk by introducing a reminder cost to remind the default customers. Wrt reminder cost, it helps the retailer initially to increases the profit but more investment on the reminder level is detrimental to the profit. For the first time, two-level changing trade credited EOQ models with time dependent demand and deterioration under a finite time horizon (crisp and uncertain) have been solved and illustrated numerically. Several necessary theorems and lemmas are derived and presented.

10.2. Managerial decisions

This investigation is helpful for the traders/retailers doing the business of seasonal products like apple, orange, tomato, potato etc. As, nowadays, trade credits are often given to the downstream customers for increased sale, how much credit could be allowed by a trader of the above businesses for maximum profit? For time dependent deteriorating items, is it necessary to increase the credit period within a time cycle for more profit? If so, how much? In developing countries, problem of default customers is faced in almost all businesses specially if credit is allowed. How to tackle this situation? For seasonal products' business, the business period always fluctuates. How to tackle this fluctuation? How many should be the time cycle of business within in a season for maximum profit? It is demonstrated that offers of trade credits (N_1 and N_2) and arrangement to reduce defectiveness (reminder cost) do not have unitary positive effects on the profit, these also invites some negative effects. Up to what levels of values, are these parameters allowed for maximum profit? All these questions are answered in this investigation, through a study with some assumed data.

10.3. Limitations and future extensions

In this investigation, model development is restricted by the consideration of EOQ model without shortages. It can be extended for other inventory models (EPQ, News boy problem, etc.) with shortages. Depending on the level of deterioration, only one change in customers' credit period has been considered in this investigation. In future, this credit period may be taken as function of time, t *i.e.*, offered credit period increases as deterioration increases. Here, linear form of demand is taken for investigation. Other forms (stock dependent, exponential form, etc.) of demand can also be considered for model development and analysis.

Appendix A. Detail expression the following symbols used in mathematical formulations

$$\begin{cases} K_{1i} = A_1 + B\Big(1 + e + (i-1)T\Big) \\ K_{2i} = A_2 + B\Big(1 + e + (i-1)T\Big) \\ K_{3i} = K_{2i}\log\left(\frac{1+e-t_1}{1+e-T}\right) - B(T - t_1). \end{cases} \tag{A.1}$$

$$\begin{cases} H_{1i} & = \Big(A_1 + B(1 + e + (i-1)T)\Big)\left\{\frac{(1+e)^2}{2}\log\left(\frac{1+e}{1+e-t_1}\right) - \frac{1}{4}\Big(2(1+e) - t_1\Big)t_1\right\} \\ H_{2i} & = B\left[\frac{1}{2}\Big(1 + e + 2(i-1)T + t_1\Big)\Big(2(i-1)T + t_1\Big) - \Big(1 + e + (i-1)T\Big)\Big((i-1)T + t_1\Big)t_1\right. \\ & \quad \left. -\frac{1}{3}\left\{\Big((i-1)T + t_1\Big)^3 - \Big((i-1)T\Big)^3\right\}\right] \\ H_{3i} & = \frac{1}{2}\left[\left\{A_2 + B\Big(1 + e + (i-1)T\Big)\right\}\log\left(\frac{1+e-t_1}{1+e-T}\right) - B(T - t_1)\right]\Big(2(1+e) - t_1\Big)t_1 \\ H_{4i} & = \left\{A_2 + B\Big(1 + e + (i-1)T\Big)\right\}\left[\frac{(1+e-t_1)^2}{2}\log\left(\frac{1+e-t_1}{1+e-T}\right) - \frac{1}{4}\Big(2(1+e) - (T + t_1)\Big)(T - t_1)\right] \\ H_{5i} & = B\left[\frac{1}{2}\Big(1 + e + (2i-1)T\Big)\Big((2i-1)T + t_1\Big)(T - t_1) - i\Big(1 + e + (i-1)T\Big)(T - t_1)T\right. \\ & \quad \left. -\frac{1}{3}\left\{\Big(iT\Big)^3 - \Big((i-1)T + t_1\Big)^3\right\}\right]. \end{cases} \tag{A.2}$$

$$\begin{cases} I_{21i} & = K_{1i}\left[\frac{(1+e+N_1-M)}{2}\log\left(\frac{1+e+N_1-M}{1+e-t_1}\right) + \frac{1}{4}\Big((1+e-t_1)^2 - (1+e+N_1-M)^2\Big)\right] \\ I_{22i} & = B\left[\frac{1}{2}\Big(1 + e + 2(i-1)T + 2N_1 + t_1\Big)\Big(2(i-1)T + t_1 + N_1 + M\Big)(t_1 + N_1 - M) - \frac{1}{3}\left\{\Big((i-1)T + t_1 + N_1\Big)^3\right.\right. \\ & \quad \left.\left. -\Big((i-1)T + M\Big)^3\right\} - \Big(1 + e + (i-1)T + N_1\Big)\Big(N_1 + (i-1)T + t_1\Big)(t_1 + N_1 - M)\right] \\ I_{23i} & = \frac{K_{3i}}{2}\Big(2(1+e) - (t_1 - N_1 + M)\Big)(t_1 + N_1 - M). \end{cases} \tag{A.3}$$

Appendix B. Proof of the Theorem 1

Differentiating RTP_1 partially wrt N_1, we have

$$\begin{aligned} \frac{\partial RTP_1}{\partial N_1} = pe^{(c_1r_c - c_2N_1)}\Bigg[&\Big[-c_2\Big\{a(1 + a_1N_1 - a_2r_c) + \frac{B}{2}\Big(2(i-1)T + t_1\Big)\Big\}t_1 + aa_1t_1\Big] \\ & - c_2I_e\left[\frac{a}{2}(1 + a_1N_1 - a_2r_c)\Big(2(i-1)T + t_1\Big)t_1 + \frac{B}{3}\left\{\Big((i-1)T + t_1\Big)^3 - \Big((i-1)T\Big)^3\right\} + \Big\{a(1 + a_1N_1 - a_2r_c)t_1\right. \\ & \left. + \frac{B}{2}\Big(2(i-1)T + t_1\Big)t_1\Big\}(M - t_1 - N_1)\right] + I_e\left[\frac{aa_1}{2}\Big(2(i-1)T + t_1\Big)t_1 - \Big\{a(1 + a_1N_1 - a_2r_c)t_1\right. \\ & \left. + \frac{B}{2}\Big(2(i-1)T + t_1\Big)t_1\Big\} + aa_1(M - t_1 - N_1)t_1\right]\Bigg] - caa_1(1+e)\log\left(\frac{1+e}{1+e-t_1}\right) \\ & - haa_1\left\{\frac{(1+e)^2}{2}\log\left(\frac{1+e}{1+e-t_1}\right) - \frac{1}{4}\Big(2(1+e) - t_1\Big)t_1\right\} = \delta^1_{N_1} \quad (say). \end{aligned} \tag{B.1}$$

Again differentiating $\frac{\partial RTP_1}{\partial N_1}$ partially wrt N_1, we get

$$\begin{aligned}\frac{\partial^2 RTP_1}{\partial N_1^2} &= -pc_2e^{(c_1r_c-c_2N_1)}\Bigg[\Big[-c_2\Big\{a(1+a_1N_1-a_2r_c)+\frac{B}{2}\Big(2(i-1)T+t_1\Big)\Big\}t_1+aa_1t_1\Big]\\ &\quad -c_2I_e\Big[\frac{a}{2}(1+a_1N_1-a_2r_c)\Big(2(i-1)T+t_1\Big)t_1+\frac{B}{3}\Big\{\Big((i-1)T+t_1\Big)^3-\Big((i-1)T\Big)^3\Big\}\\ &\quad +\Big\{a(1+a_1N_1-a_2r_c)t_1+\frac{B}{2}\Big(2(i-1)T+t_1\Big)t_1\Big\}(M-t_1-N_1)\Big]+I_e\Big[\frac{aa_1}{2}\Big(2(i-1)T+t_1\Big)t_1\\ &\quad -\Big\{a(1+a_1N_1-a_2r_c)t_1+\frac{B}{2}\Big(2(i-1)T+t_1\Big)t_1\Big\}+aa_1(M-t_1-N_1)t_1\Big]\Bigg]\\ &\quad +pe^{(c_1r_c-c_2N_1)}\Big[-aa_1c_2t_1-c_2I_e\Big\{\frac{aa_1}{2}\Big(2(i-1)T+t_1\Big)t_1-\Big\{a(1+a_1N_1-a_2r_c)t_1+\frac{B}{2}\Big(2(i-1)T+t_1\Big)t_1\Big\}\\ &\quad +aa_1t_1(M-t_1-N_1)\Big\}-2aa_1t_1I_e\Big]=K_1 \quad (say).\end{aligned} \tag{B.2}$$

Proof. If $K_1 \le 0$, then proof of part (i) is obvious and $N_{1_1}^*$ can be obtained by solving $\delta_{N_1}^1 = 0$. For remaining part we have

$$\begin{aligned}\lim_{N_1\to\infty}\delta_{N_1}^1 &= pe^{(c_1r_c-c_2N_1)}\Bigg[\Big[-c_2\Big\{a(1+a_1N_1-a_2r_c)+\frac{B}{2}\Big(2(i-1)T+t_1\Big)\Big\}t_1+aa_1\Big]\\ &\quad -c_2I_e\Big[\frac{a}{2}(1+a_1N_1-a_2r_c)\Big(2(i-1)T+t_1\Big)t_1+\frac{B}{3}\Big\{\Big((i-1)T+t_1\Big)^3-\Big((i-1)T\Big)^3\Big\}\\ &\quad +\Big\{a(1+a_1N_1-a_2r_c)t_1+\frac{B}{2}\Big(2(i-1)T+t_1\Big)t_1\Big\}(M-t_1-N_1)\Big]\\ &\quad +I_e\Big[\frac{aa_1}{2}\Big(2(i-1)T+t_1\Big)t_1-\Big\{a(1+a_1N_1-a_2r_c)t_1+\frac{B}{2}\Big(2(i-1)T+t_1\Big)t_1\Big\}+aa_1(M-t_1-N_1)t_1\Big]\Bigg]\\ &\quad -caa_1(1+e)\log\left(\frac{1+e}{1+e-t_1}\right)-haa_1\left\{\frac{(1+e)^2}{2}\log\left(\frac{1+e}{1+e-t_1}\right)-\frac{1}{4}\Big(2(1+e)-t_1\Big)t_1\right\}=-\infty.\end{aligned} \tag{B.3}$$

If $\delta_{N_1}^1 \le 0$, then $\frac{\partial RTP_1}{\partial N_1} \le 0$ for all $N_1 \ge 0$ so RTP_1 decreasing function in N_1. Hence Hence the retailer optimal credit period $N_{1_1}^* = 0$, which complete the proof of part (ii).

Now, if $\delta_{N_1}^1 \ge 0$, and $\lim_{N_1\to\infty}\delta_{N_1}^1 \le 0$. By applying the Mean-value theorem and part (a), there exist a unique $N_{1_1}^*$ such that $\delta_{N_1}^1$=0, at $N_1 = N_{1_1}^*$. Consequently, RTP_1 is maximized at unique point $N_{1_1}^* > 0$. This complete the proof of Theorem 1. □

Appendix C. Numerical proof of Lemma 4

C.1. Numerical proof

In this section, we shall discuss the concavity of the profit function wrt all decision variables individually and for all three decision variables N_1, N_2, and r_c. Here we shall prove only for the 1st sub-case and similarly we can proof for remaining. Consider numerical data of Experiment 1, Hessian matrix for sub-case 1.1 is

$$H=\begin{bmatrix}\frac{\partial^2 RTP_1}{\partial N_1^2} & \frac{\partial^2 RTP_1}{\partial N_1\partial N_2} & \frac{\partial^2 RTP_1}{\partial N_1\partial r_c}\\ \frac{\partial^2 RTP_1}{\partial N_1\partial N_2} & \frac{\partial^2 RTP_1}{\partial N_2^2} & \frac{\partial^2 RTP_1}{\partial N_2\partial r_c}\\ \frac{\partial^2 RTP_1}{\partial N_1\partial r_c} & \frac{\partial^2 RTP_1}{\partial N_2\partial r_c} & \frac{\partial^2 RTP_1}{\partial r_c^2}\end{bmatrix}=\begin{bmatrix}-1841.754 & 0 & -940.751\\ 0 & -954.657 & 119.068\\ -940.751 & 119.0680 & -9211.499\end{bmatrix}. \tag{C.1}$$

If the Hessian matrix is negative definite then profit function is concave while positive definite for convex.

C.1.1. Concavity wrt single decision variable

It is clear from the above Hessian matrix all diagonal elements are negative, *i.e.*, 2nd order partial derivative of the profit function RTP_1 wrt each decision variable ($\frac{\partial^2 RTP_1}{\partial N_1^2} = -1841.754$, $\frac{\partial^2 RTP_1}{\partial N_2^2} = -954.657$, $\frac{\partial^2 RTP_1}{\partial r_c^2} = -9211.499$) is less than zero. Therefore profit function is concave in each decision variable individually.

C.1.2. Concavity wrt two decision variable

It is clear from the above Hessian matrix all 2nd order partial derivative of the profit function RTP_1 wrt each decision variable ($\frac{\partial^2 RTP_1}{\partial N_1^2} = -1841.754$, $\frac{\partial^2 RTP_1}{\partial N_2^2} = -954.657$, $\frac{\partial^2 RTP_1}{\partial r_c^2} = -9211.499$) is negative. If the determinant of 2×2 Hessian matrix concerning two decision variables keeping 3rd variable fix is positive, then the profit function is concave in two decision variables. Here the determinant of 2×2 matrix are

$H(N_1, N_2) = 1.8 \times 10^6$, ($r_c$ is fix), $H(N_1, r_c) = 1.6 \times 10^7$, ($N_2$ is fix) and $H(N_2, r_c) = 8.8 \times 10^6$, ($N_1$ is fix) which are positive. Hence profit function RTP_1 is concave in two decision variables.

C.1.3. Concavity in all decision variable

It is clear from Section C.1.1 all principal minors are negative, and all second-order principle minors are also positive, given in Section C.1.2. If the determinant of a Hessian matrix is negative, then the profit function is concave. Since the determinant of Hessian matrix is $-1.5 \times 10^{10} < 0$, therefore matrix is negative definite and hence RTP_1 is concave in N_1, N_2, and r_c.

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