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NON-DIFFERENTIABLE SECOND-ORDER SYMMETRIC MULTIOBJECTIVE FRACTIONAL VARIATIONAL PROGRAMMING WITH CONES CONSTRAINTS

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Abstract. In this article, we build a pair non-differentiable second-order symmetric multiobjective fractional variational programming models with cone constraints, where each objective function component has a support function for a compact convex set. The (C,ρ,θ)-convexity/(C,ρ,θ)-pseudo-convexity/(C,ρ,θ)-quasiconvexity functions are defined and also, constructed concrete numerical examples for existing such type of functions. The duality results are established by using these aforesaid assumptions.

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1. Introduction

Multiobjective programming, also known as vector optimization, has advanced greatly in various directions in the domains of game theory, Pareto optimality, variational inequalities and equilibria, as can be shown in Chinchuluun *et al.* [4] and Oveisiha and Zafarani [23]. Dorn [6] developed symmetric duality in nonlinear programming, in which the dual of the dual is primal. Mishra and Mukherjee [20] introduce a parametric technique to connect efficient solutions of the primal and dual problems under the assumption of concavity on the functions involved. A symmetric dual pair for a class of nondifferentiable multi-objective fractional variational problems, according to Mishra *et al.* [21], is as follows:

$$\text{Min } \frac{\int_c^d \left\{\Phi(t,\nu,\dot{\nu},\kappa,\dot{\kappa}) + s(\nu|A) - \kappa^T z\right\} \mathrm{d}t}{\int_c^d \left\{\Psi(t,\nu,\dot{\nu},\kappa,\dot{\kappa}) - s(\nu|B) + \kappa^T r\right\} \mathrm{d}t} \tag{MNFP}$$
$$= \left(\frac{\int_c^d \left\{\Phi_1(t,\nu,\dot{\nu},\kappa,\dot{\kappa}) + s(\nu|A_1) - \kappa^T z_1\right\} \mathrm{d}t}{\int_c^d \left\{\Psi_1(t,\nu,\dot{\nu},\kappa,\dot{\kappa}) - s(\nu|B_1) + \kappa^T r_1\right\} \mathrm{d}t}, \ldots, \frac{\int_c^d \left\{\Phi_k(t,\nu,\dot{\nu},\kappa,\dot{\kappa}) + s(\nu|A_k) - \kappa^T z_k\right\} \mathrm{d}t}{\int_c^d \left\{\Psi_k(t,\nu,\dot{\nu},\kappa,\dot{\kappa}) - s(\nu|B_k) + \kappa^T r_k\right\} \mathrm{d}t} \right),$$

subject to

$$\nu(c) = 0 = \nu(d), \quad \kappa(c) = 0 = \kappa(d),$$
$$\dot{\nu}(c) = 0 = \dot{\nu}(d), \quad \dot{\kappa}(c) = 0 = \dot{\kappa}(d),$$

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$$\sum_{i=1}^{k}\tau_i\Big\{[\Phi_{i\kappa}-D\Phi_{i\dot\kappa}+z_i]G_i^*(\nu,\kappa)-[\Psi_{i\kappa}-D\Psi_{i\dot\kappa}+r_i]F_i^*(\nu,\kappa)\Big\}\leq 0,$$

$$\int_c^d\kappa^T\sum_{i=1}^{k}\tau_i\Big\{[\Phi_{i\kappa}-D\Phi_{i\dot\kappa}+r_i]G_i^*(\nu,\kappa)-[\Psi_{i\kappa}-D\Psi_{i\dot\kappa}+r_i]F_i^*(\nu,\kappa)\Big\}\geq 0,$$

$$\tau\geq 0,\quad \tau^Te=1,\quad t\in I,$$

$$z_i\in X_i,\quad r_i\in Y_i,\quad i\in\tilde N.$$

$$\text{Max}\ \frac{\int_c^d\{\Phi(t,\mu,\dot\mu,\varsigma,\dot\varsigma)-s(\varsigma|X)+\mu^Tw\}\,\mathrm{d}t}{\int_c^d\{\Psi(t,\mu,\dot\mu,\varsigma,\dot\varsigma)+s(\varsigma|Y)-\mu^Ts\}\,\mathrm{d}t} \tag{MNFD}$$

$$=\frac{\int_c^d\{\Phi_1(t,\mu,\dot\mu,\varsigma,\dot\varsigma)-s(\varsigma|X_1)+\mu^Tw_1\}\,\mathrm{d}t}{\int_c^d\{\Psi_1(t,\mu,\dot\mu,\varsigma,\dot\varsigma)+s(\varsigma|Y_1)-\mu^Ts_1\}\,\mathrm{d}t},\ldots,\frac{\int_c^d\{\Phi_k(t,\mu,\dot\mu,\varsigma,\dot\varsigma)-s(\varsigma|X_k)+\mu^Tw_k\}\,\mathrm{d}t}{\int_c^d\{\Psi_k(t,\mu,\dot\mu,\varsigma,\dot\varsigma)+s(\varsigma|Y_k)-\mu^Ts_k\}\,\mathrm{d}t}$$

subject to

$$\mu(c)=0=\mu(d),\quad \varsigma(c)=0=\varsigma(d),$$

$$\dot\mu(c)=0=\dot\mu(d),\quad \dot\varsigma(c)=0=\dot\varsigma(d),$$

$$\sum_{i=1}^{k}\tau_i\Big\{[\Phi_{iu}-D\Phi_{i\dot\mu}+w_i]G_i^*(\mu,\varsigma)-[\Psi_{iu}-D\Psi_{i\dot\mu}-s_i]F_i^*(\mu,\varsigma)\Big\}\geq 0,$$

$$\int_c^d\mu^T\sum_{i=1}^{k}\tau_i\Big\{[\Phi_{iu}-D\Phi_{i\dot\mu}+r_i]G_i^*(\mu,\varsigma)-[\Psi_{iu}-D\Psi_{i\dot\mu}-s_i]F_i^*(\mu,\varsigma)\Big\}\leq 0,$$

$$\tau\geq 0,\quad \tau^Te=1,\quad t\in I,$$

$$w_i\in A_i,\quad s_i\in B_i,\quad i\in\tilde N,$$

where $\Phi_i: I\times R^n\times R^n\times R^m\times R^m\to R_+$ and $\Psi_i: I\times R^n\times R^n\times R^m\times R^m\to R_+\backslash\{0\}$ are continuously differentiable function and

$$F_i(\nu,\kappa)=\int_c^d\Big\{\Phi_i(t,\nu,\dot\nu,\kappa,\dot\kappa)+s(\nu|A_i)-\kappa^Tz_i\Big\}\,\mathrm{d}t,$$

$$G_i(\nu,\kappa)=\int_c^d\Big\{\Psi_i(t,\nu,\dot\nu,\kappa,\dot\kappa)-s(\nu|B_i)+\kappa^Tr_i\Big\}\,\mathrm{d}t,$$

$$F_i^*(\mu,\varsigma)=\int_c^d\Big\{\Phi_i(t,\mu,\dot\mu,\varsigma,\dot\varsigma)-s(\varsigma|X_i)+\mu^Tw_i\Big\}\,\mathrm{d}t,$$

and

$$G_i^*(\mu,\varsigma)=\int_c^d\Big\{\Psi_i(t,\mu,\dot\mu,\varsigma,\dot\varsigma)+s(\varsigma|Y_i)-\mu^Ts_i\Big\}\,\mathrm{d}t.$$

Kang *et al.* [18] developed a pair of symmetric dual problems for nondifferentiable multiobjective fractional variational problems with cone constraints over arbitrary cones in his paper. We develop symmetric duality relations for Mond-Weir-type issues under invexity and pseudoinvexity assumptions using weak efficiency. It is a continuation of Mishra *et al.* [21]. Gulati and Mehndiratta [13] first modify a converse duality theorem for a scalar variational problem of second-order duality. Consider its multiobjective analog as well, and find the required optimality criteria and duality relations. Jayswal *et al.* [16] constructs an example to test the presence of the newly introduced class of functions, which is more general than previously specified functions. Several duality results, such as weak, strong, and strict converse duality results, are created to relate the primal variational problem and its second-order dual. For variational problems, the concept of second-order (F,α,ρ,d)-convexity is as follows.

By neglecting the equality condition, have a look at the following variational problem (CP):

$$\text{Min} \int_c^d \Phi(t,\nu,\dot{\nu})\,\mathrm{d}t \tag{CP}$$

subject to

$$\nu(c) = 0 = \nu(d), \tag{1}$$

$$\Psi(t,\nu,\dot{\nu}) \leq 0, \quad t \in I. \tag{2}$$

Gulati and Mehndiratta [13] formulation of the second order dual problem for (CP) is given below

$$\text{Max} \int_c^d \left(\Phi(t,\mu,\dot{\mu}) - \frac{1}{2}\beta^T F \beta\right)\mathrm{d}t \tag{CD}$$

subject to

$$\mu(c) = 0 = \mu(d),$$

$$\Phi_\nu(t,\mu,\dot{\mu}) + \Psi_\nu(t,\mu,\dot{\mu})\kappa - D(\Phi_{\dot{\nu}}) + \Psi_{\dot{\nu}}(t,\mu,\dot{\mu}\kappa) + (F+H)\beta = 0, \quad t \in I,$$

$$\kappa^T \Psi(t,\mu,\dot{\mu}) - \frac{1}{2}\beta^T H \beta \geq 0, \quad t \in I,$$

$$\kappa \geq 0, \quad t \in I,$$

where

$$F(t,\mu,\dot{\mu},\ddot{\mu},\dddot{\mu},\ddddot{\mu}) = \Phi_{\nu\nu}(t,\mu,\dot{\mu}) - 2D\Phi_{\nu\dot{\nu}}(t,\mu,\dot{\mu}) + D^2\Phi_{\dot{\nu}\dot{\nu}}(t,\mu,\dot{\mu}) - D^3\Phi_{\dot{\nu}\ddot{\nu}}(t,\mu,\dot{\mu}), \quad t, \in I$$

and

$$H(t,\mu,\dot{\mu},\ddot{\mu},\dddot{\mu},\ddddot{\mu},\kappa,\dot{\kappa},\ddot{\kappa},\dddot{\kappa}) = (\Psi_\nu(t,\mu,\dot{\mu})\kappa)_\nu - 2D(\Psi_\nu(t,\mu,\dot{\mu})\kappa)_{\dot{\nu}} + D^2(\Psi_{\dot{\nu}}(t,\mu,\dot{\mu})\kappa)_{\dot{\nu}} - D^3(\Psi_{\dot{\nu}}(t,\mu,\dot{\mu})\kappa)_{\ddot{\nu}}, \; t \in I.$$

Jayswal and Jha [15] presented a set of second order fractional symmetric variational programming under cone constraints and established weak, strong, and converse duality theorems under second order F-convexity assumptions. A pair of multiobjective second-order symmetric dual variational problems were introduced by Sachdev *et al.* [26]. Under the premise of η-bonvexity/η-pseudobonvexity, we demonstrate weak, strong, and converse duality theorems for this pair. For additional information, see [7–12, 14, 17, 19, 24, 27]. In a vector optimization issue, Prasad *et al.* [25] developed a pair of nondifferentiable multiobjective symmetric fractional duality models with cone function, where each component of the objective function has support function of a compact convex set.

Primal Problem (NFVP)

$$\text{Min} \frac{\int_c^d \left(\Phi(t,\nu,\dot{\nu},\kappa,\dot{\kappa}) - \frac{1}{2}p^T X p + s(\nu|E) - \kappa^T z\right)\mathrm{d}t}{\int_c^d \left(\Psi(t,\nu,\dot{\nu},\kappa,\dot{\kappa}) - \frac{1}{2}p^T Y p - s(\nu|F) + \kappa^T r\right)\mathrm{d}t}$$

subject to

$$\nu(c) = 0 = \nu(d), \quad \dot{\nu}(c) = 0 = \dot{\nu}(d),$$

$$\kappa(c) = 0 = \kappa(d), \quad \dot{\kappa}(c) = 0 = \dot{\kappa}(d),$$

$$\{G(\nu,\kappa)[\Phi_\kappa - D\Phi_{\dot{\kappa}} + Xp - z] - F(\nu,\kappa)[\Psi_\kappa - D\Psi_{\dot{\kappa}} + Yp + r]\} \in C_2^*,$$

$$\int_c^d \kappa^T \{G(\nu,\kappa)[\Phi_\kappa - D\Phi_{\dot{\kappa}} + Xp - z] - F(\nu,\kappa)[\Psi_\kappa - D\Psi_{\dot{\kappa}} + Yp + r]\}\,\mathrm{d}t \geq 0,$$

$$\nu \in C, \quad t \in I,$$

$$z \in J, \quad r \in K.$$

Dual Problem (NFVD)

$$\text{Max } \frac{\int_c^d \big(\Phi(t,\mu,\dot{\mu},\varsigma,\dot{\varsigma}) - \frac{1}{2}q^T W q - s(\varsigma|J) + \mu^T w\big)\,\mathrm{d}t}{\int_c^d \big(\Psi(t,\mu,\dot{\mu},\varsigma,\dot{\varsigma}) - \frac{1}{2}q^T Z q + s(\varsigma|K) - \mu^T s\big)\,\mathrm{d}t}$$

subject to

$$\begin{aligned}
&\mu(c) = 0 = \mu(d), \quad \dot{\mu}(c) = 0 = \dot{\mu}(d),\\
&\varsigma(c) = 0 = \varsigma(d), \quad \dot{\varsigma}(c) = 0 = \dot{\varsigma}(d),\\
&-\{G^*(\mu,\varsigma)[\Phi_\nu - D\Phi_{\dot{\nu}} + Wq + w] - F^*(\mu,\varsigma)[\Psi_\nu - D\Psi_{\dot{\nu}} + Zq - s]\} \in C_2^*,\\
&\int_c^d \mu^T\{G^*(\mu,\varsigma)[\Phi_\nu - D\Phi_{\dot{\nu}} + Wq + w] - F^*(\mu,\varsigma)[\Psi_\nu - D\Psi_{\dot{\nu}} + Zq - s]\}\,\mathrm{d}t \geq 0,\\
&\varsigma \in C_2, \quad w \in E, \quad s \in F,
\end{aligned}$$

where

(i) $f : I \times C \times C \times C_2 \times C_2 \to R_+$, and $g : I \times C \times C \times C_2 \times C_2 \to R_+\backslash\{0\}$,

(ii) $p : I \to R^m$ and $q : I \to R^n$,

(iii) $X(t,\nu,\dot{\nu},\kappa,\dot{\kappa}) = \Phi_{\kappa\kappa}(t,\nu,\dot{\nu},\kappa,\dot{\kappa}) - 2D\Phi_{\kappa\dot{\kappa}}(t,\nu,\dot{\nu},\kappa,\dot{\kappa}) + D^2\Phi_{\dot{\kappa}\dot{\kappa}}(t,\nu,\dot{\nu},\kappa,\dot{\kappa}) - D^3\Phi_{\dot{\kappa}\ddot{\kappa}}(t,\nu,\dot{\nu},\kappa,\dot{\kappa}), t \in I$,

(iv) $Y(t,\nu,\dot{\nu},\kappa,\dot{\kappa}) = \Psi_{\kappa\kappa}(t,\nu,\dot{\nu},\kappa,\dot{\kappa}) - 2D\Psi_{\kappa\dot{\kappa}}(t,\nu,\dot{\nu},\kappa,\dot{\kappa}) + D^2\Psi_{\dot{\kappa}\dot{\kappa}}(t,\nu,\dot{\nu},\kappa,\dot{\kappa}) - D^3\Psi_{\dot{\kappa}\ddot{\kappa}}(t,\nu,\dot{\nu},\kappa,\dot{\kappa}), t \in I$,

(v) $W(t,\mu,\dot{\mu},\varsigma,\dot{\varsigma}) = \Phi_{\nu\nu}(t,\mu,\dot{\mu},\varsigma,\dot{\varsigma}) - 2D\Phi_{\nu\dot{\nu}}(t,\mu,\dot{\mu},\varsigma,\dot{\varsigma}) + D^2\Phi_{\dot{\nu}\dot{\nu}}(t,\mu,\dot{\mu},\varsigma,\dot{\varsigma}) - D^3\Phi_{\dot{\nu}\ddot{\nu}}(t,\mu,\dot{\mu},\varsigma,\dot{\varsigma}), t \in I$,

(vi) $Z(t,\mu,\dot{\mu},\varsigma,\dot{\varsigma}) = \Psi_{\nu\nu}(t,\mu,\dot{\mu},\varsigma,\dot{\varsigma}) - 2D\Psi_{\nu\dot{\nu}}(t,\mu,\dot{\mu},\varsigma,\dot{\varsigma}) + D^2\Psi_{\dot{\nu}\dot{\nu}}(t,\mu,\dot{\mu},\varsigma,\dot{\varsigma}) - D^3\Psi_{\dot{\nu}\ddot{\nu}}(t,\mu,\dot{\mu},\varsigma,\dot{\varsigma}), t \in I$,

(vii) $F(\nu,\kappa) = \int_c^d \big(\Phi(t,\nu,\dot{\nu},\kappa,\dot{\kappa}) - \frac{1}{2}p^T X p + s(\nu|E) - \kappa^T z\big)\,\mathrm{d}t$,

(viii) $G(\nu,\kappa) = \int_c^d \big(\Psi(t,\nu,\dot{\nu},\kappa,\dot{\kappa}) - \frac{1}{2}p^T Y p - s(\nu|F) + \kappa^T r\big)\,\mathrm{d}t$,

(ix) $F^*(\mu,\varsigma) = \int_c^d \big(\Phi(t,\mu,\dot{\mu},\varsigma,\dot{\varsigma}) - \frac{1}{2}p^T W p + s(\varsigma|J) - \mu^T w\big)\,\mathrm{d}t$,

(x) $G^*(\mu,\varsigma) = \int_c^d \big(\Psi(t,\mu,\dot{\mu},\varsigma,\dot{\varsigma}) - \frac{1}{2}p^T Z p - s(\varsigma|K) + \mu^T s\big)\,\mathrm{d}t$,

(xi) E and F are compact convex sets in R^n,

(xii) J and K are compact convex sets in R^m.

2. Preliminaries and definitions

Let $\tilde{N} = \{1,2,\ldots,k\}$ and for $r \in \tilde{N}$, the set $K_r = \tilde{N} - \{r\}$. The following convention for vector inequalities will be used: for $a, b \in R^n$,

$$\begin{aligned}
a \geqq b &\iff a_i \geqq b_i, \quad i = 1,2,\ldots,n;\\
a \geq b &\iff a_i \geq b_i \text{ and } a \neq b;\\
a > b &\iff a_i > b_i, \quad i = 1,2,\ldots,n.
\end{aligned}$$

The following variational problem is taken into consideration in this paper:

$$\text{Min } \left(\frac{\int_c^d \Theta_1(t,\nu,\dot{\nu})\,\mathrm{d}t}{\int_c^d \Omega_1(t,\nu,,\dot{\nu})\,\mathrm{d}t}, \frac{\int_c^d \Theta_2(t,\nu,\dot{\nu})\,\mathrm{d}t}{\int_c^d \Omega_2(t,\nu,,\dot{\nu})\,\mathrm{d}t}, \ldots, \frac{\int_c^d \Theta_r(t,\nu,\dot{\nu})\,\mathrm{d}t}{\int_c^d \Omega_r(t,\nu,,\dot{\nu})\,\mathrm{d}t}\right) \tag{FVP}$$

subject to $\nu(c) = \alpha, \quad \nu(d) = \beta$,

$$g(t,\nu,\dot{\nu}) = (g_1(t,\nu,\dot{\nu}), g_1(t,\nu,\dot{\nu}), \ldots, g_p(t,\nu,\dot{\nu})) \leq 0, \quad t \in I.$$

Taking $\int_c^d \Theta_i(t,\nu,\dot{\nu})\,\mathrm{d}t \geq 0$ and $\int_c^d \Omega_i(t,\nu,\dot{\nu})\,\mathrm{d}t > 0$, $\forall\ i \in \{1,2,\ldots,r\}$ and $\forall\ \nu \in X$. Consider $X_0 = \{\nu \in X|\ g(t,\nu,\dot{\nu}) \leqq 0, t \in I, \nu(c) = \alpha, \nu(d) = \beta\}$ be the set of all feasible solutions of (FVP), where $\Theta : I \times R^n \times R^n \to R$ and $\Omega : I \times R^n \times R^n \to R^m$ are continuously differentiable function and $\nu, t \in I = [c,d]$ is an n-dimensional piece wise smooth continuous function, whose derivative is represented by the symbol $\dot{\nu}$. For the sake of notation, $p,\ q,\ \kappa,\ \dot{\kappa},\ \varsigma,\ \dot{\varsigma},\ \mu,\ \dot{\mu},\ \nu$ and $\dot{\nu}$ written in the place of $p(t), q(t),\ \kappa(t),\ \dot{\kappa}(t),\ \varsigma(t),\ \dot{\varsigma}(t),\ \mu(t),\ \dot{\mu}(t),\ \nu(t)$ and $\dot{\nu}(t)$. The gradient vectors of Θ w.r.t. ν and $\dot{\nu}$ are referred to by Θ_ν and $\Theta_{\dot{\nu}}$, *i.e.*

$$\Theta_\nu = \left(\frac{\partial\Theta}{\partial\nu^1}, \frac{\partial\Theta}{\partial\nu^2}, \dots, \frac{\partial\Theta}{\partial\nu^n}\right)^T \text{ and } \Theta_{\dot{\nu}} = \left(\frac{\partial\Theta}{\partial\dot{\nu}^1}, \frac{\partial\Theta}{\partial\dot{\nu}^2}, \dots, \frac{\partial\Theta}{\partial\dot{\nu}^n}\right)^T.$$

Similarly, $\Theta_{\nu\nu}$ denote the $n \times n$ matrix w.r.t. ν. Let $M(t,\nu,\dot{\nu}) = \Theta_{\nu\nu} - 2D\Theta_{\nu\dot{\nu}} + D^2\Theta_{\dot{\nu}\dot{\nu}} - D^3\Theta_{\dot{\nu}\ddot{\nu}}\ , t \in I$.

The symbol for the space of a piecewise smooth function is $C(I,R)$, $\forall\ \nu \in C(I,R)$, we specify its norm by

$$\|\nu\| = \|\nu\|_\infty + \|D\nu\|_\infty,$$

where D represents the differential operator specified by

$$\mu = D\nu \iff \nu = \alpha + \int_0^t \mu(s)\,\mathrm{d}s,$$

where the boundary value α is given. So, $\frac{\mathrm{d}}{\mathrm{d}t} \equiv D$ (except at discontinuous).

Definition 1. Let $C \subset R^n$ is called cone if it satisfy conditions:

$$0 \le \xi \in R,\ \nu \in C \implies \xi\nu \in C.$$

Definition 2. In general polar cone C^* for cone C is defined as

$$C^* = \left\{z : \nu^T z \le 0\ \forall\, \nu \in C\right\}.$$

Definition 3. Let $C : I \times X \times X \times X \times X \times R^n \to R(X \subseteq R^n)$ be a function which satisfies $C_{t,\nu,\dot{\nu},\mu,\dot{\mu}}(0) = 0, \forall(\nu,\dot{\nu},\mu,\dot{\mu}) \in X \times X \times X \times X$. The function C is thus considered to be convex on R^n with respect to the third parameter iff for every given $(\nu,\dot{\nu},\mu,\dot{\mu}) \in X \times X \times X \times X$,

$$C_{t,\nu,\dot{\nu},\mu,\dot{\mu}}(\lambda\nu_1 + (1-\lambda)\nu_2) \le \lambda C_{t,\nu,\dot{\nu},\mu,\dot{\mu}}(\nu_1) + (1-\lambda)C_{t,\nu,\dot{\nu},\mu,\dot{\mu}}(\nu_2), \quad \forall\ \lambda \in (0,1),\ \forall\ \nu_1,\ \nu_2 \in R^n.$$

Definition 4. For a compact convex set $C \subset R^n$ support function is defined as

$$s(\nu|C) = \max\left\{\nu^T\kappa : \kappa \in C\right\}.$$

Any set $S \subset R^n$ normal cone is described by

$$N_S(\nu) = \left\{\kappa \in R^n : \kappa^T(z-\nu) \le 0, \forall\ z \in S\right\}.$$

Definition 5 ([19])**.** $\bar{\nu} \in X_0$ is said to be a efficient solution for (FVP), if there is no other $\nu \in X_0$ such that

$$\frac{\int_c^d \Theta_i(t,\nu,\dot{\nu})\,\mathrm{d}t}{\int_c^d \Omega_i(t,\nu,,\dot{\nu})\,\mathrm{d}t} \le \frac{\int_c^d \Theta_i(t,\bar{\nu},\dot{\bar{\nu}})\,\mathrm{d}t}{\int_c^d \Omega_i(t,\bar{\nu},\dot{\bar{\nu}})\,\mathrm{d}t},$$

$\forall\ i \in \{1,2,\dots,r\}$ with strict inequality for at least one i

Definition 6 ([19])**.** $\bar{\nu} \in X_0$ is said to be a weak efficient solution for (FVP), if there is no other $\nu \in X_0$ such that

$$\frac{\int_c^d \Theta_i(t,\nu,\dot{\nu})\,\mathrm{d}t}{\int_c^d \Omega_i(t,\nu,,\dot{\nu})\,\mathrm{d}t} < \frac{\int_c^d \Theta_i(t,\bar{\nu},\dot{\bar{\nu}})\,\mathrm{d}t}{\int_c^d \Omega_i(t,\bar{\nu},\dot{\bar{\nu}})\,\mathrm{d}t}, \qquad \forall\ i \in \{1,2,\dots,r\}.$$

Definition 7. The vector of functionals $\int_c^d \Theta = \left(\int_c^d \Theta_1, \ldots, \int_c^d \Theta_k\right)$ is defined as second order (C, ρ, θ)-convex at $\mu \in X$, $\exists$ functions $q \in C(I, R^n)$, $\rho \in R$ and $\theta : I \times R^n \times R^n \to R^n$ such that $\forall\ \nu \in X$ and $\forall\ i \in \tilde{N}$, we have

$$\int_c^d \left(\Theta_i(t, \nu, \dot{\nu}) - \Theta_i(t, \mu, \dot{\mu}) + \frac{1}{2} q^T M_i q - \rho\theta^2(t, \nu, \mu)\right) \mathrm{d}t \geq \int_c^d C_{(t,\nu,\dot{\nu},\mu,\dot{\mu})}(\Theta_{i\nu}(t, \mu, \dot{\mu}) - D\Theta_{i\dot{\nu}}(t, \mu, \dot{\mu}) + M_i q) \, \mathrm{d}t.$$

Definition 8. The vector of functionals $\int_c^d \Theta = \left(\int_c^d \Theta_1, \ldots, \int_c^d \Theta_k\right)$ is defined as second order (C, ρ, θ)-pseudoconvex at $\mu \in X$, $\exists$ functions $q \in C(I, R^n)$, $\rho \in R$ and $\theta : I \times R^n \times R^n \to R^n$ such that $\forall\ \nu \in X$ and $\forall\ i \in \tilde{N}$, we have

$$\begin{aligned} &\int_c^d C_{(t,\nu,\dot{\nu},\mu,\dot{\mu})}(\Theta_{i\nu}(t, \mu, \dot{\mu}) - D\Theta_{i\dot{\nu}}(t, \mu, \dot{\mu}) + M_i q) \, \mathrm{d}t \geq 0 \\ &\Rightarrow \int_c^d \left(\Theta_i(t, \nu, \dot{\nu}) - \Theta_i(t, \mu, \dot{\mu}) + \tfrac{1}{2} q^T M_i q - \rho\theta^2(t, \nu, \mu)\right) \mathrm{d}t \geq 0. \end{aligned}$$

Definition 9. The vector of functionals $\int_c^d \Theta = \left(\int_c^d \Theta_1, \ldots, \int_c^d \Theta_k\right)$ is defined as second order (C, ρ, θ)-quasiconvex at $\mu \in X$, $\exists$ functions $q \in C(I, R^n)$, $\rho \in R$ and $\theta : I \times R^n \times R^n \to R^n$ such that for all $\nu \in X$ and $\forall\ i \in \tilde{N}$, we have

$$\begin{aligned} &\int_c^d \left\{\Theta_i(t, \nu, \dot{\nu}) - \Theta_i(t, \mu, \dot{\mu}) + \tfrac{1}{2} q^T M_i q - \rho\theta^2(t, \nu, \mu)\right\} \mathrm{d}t \geq 0 \\ &\Rightarrow \int_c^d C_{(t,\nu,\dot{\nu},\mu,\dot{\mu})}(\Theta_{i\nu}(t, \mu, \dot{\mu}) - D\Theta_{i\dot{\nu}}(t, \mu, \dot{\mu}) + M_i q) \, \mathrm{d}t \geq 0. \end{aligned}$$

We will try to constract an example which is second order (C, ρ, θ)-convex, but not (C, ρ)-convex at a given point.

Example 1. Assume $I = [0, 1]$. Suppose the functional $\Theta : I \times R \times R \to R$ be defined by

$$\Theta_1(t, \nu, \dot{\nu}) = \nu^3 + \nu^2 + 1 \text{ and } \Theta_2(t, \nu, \dot{\nu}) = 2\nu^2 - \nu + 3.$$

Suppose $C : I \times R \times R \times R \times R \times R^n \to R$, and $\theta : I \times R \times R \to R$ be given by

$$\begin{aligned} C_{t,\nu,\dot{\nu},\mu,\dot{\mu}}(c) &= c(\nu - \mu), \\ \theta(t, \nu, \mu) &= \nu - \mu, \ \ \nu = t \text{ and } q = t + 3. \end{aligned}$$

Then, the functional $\int_0^1 \Theta(t, \nu, \dot{\nu}) \, \mathrm{d}t$ is second order (C, ρ, θ)-convex at the point $\mu = 0$, for $\nu = t, q = t + 3$ and $\rho = 3$, but it is not (C, ρ)-convex at same point.

Solution.

$$\text{Let} \quad \psi_1 = \int_0^1 \left[\left\{\Theta_1(t, \nu, \dot{\nu}) - \Theta_1(t, \mu, \dot{\mu}) + \frac{1}{2} q^T M q - \rho\theta^2(t, \nu, \mu)\right\}\right] \mathrm{d}t.$$

Substituting the values of Θ_1, q, ρ and θ, we get after simplifying, we have

$$\psi_1 = \frac{193}{6},$$

and

$$\psi_2 = \int_0^1 C_{t,\nu,\dot{\nu},\mu,\dot{\mu}}(\Theta_{1\nu}(t, \mu, \dot{\mu}) - D\Theta_{1\dot{\nu}}(t, \mu, \dot{\mu}) + qM) \, \mathrm{d}t.$$

Putting the values of Θ_1, q and $C_{t,\nu,\dot{\nu},\mu,\dot{\mu}}(c)$, and after simplifying, we get

$$\psi_2 = \int_0^1 \left((t^2 + 3t)(6t + 2)\right) \mathrm{d}t,$$

after simplifying, we have

$$\psi_2 = \frac{67}{6}\cdot$$

$$\text{Next, Let} \quad \psi_3 = \int_0^1 \left[\left\{\Theta_2(t,\nu,\dot{\nu}) - \Theta_2(t,\mu,\dot{\mu}) + \frac{1}{2}q^T Mq - \rho\theta^2(t,\nu,\mu)\right\}\right] \mathrm{d}t.$$

Substituting the values of Θ_2, q, ρ and θ, we get

$$\psi_3 = \int_0^1 \left(-t - t^2 + 2(t+3)^2\right),$$

after simplifying, we have

$$\psi_3 = \frac{143}{6},$$

and

$$\text{Let} \quad \psi_4 = \int_0^1 C_{t,\nu,\dot{\nu},\mu,\dot{\mu}}\Big(\Theta_{2\nu}(t,\mu,\dot{\mu}) - D\Theta_{2\dot{\nu}}(t,\mu,\dot{\mu}) + qM\Big) \,\mathrm{d}t.$$

Substituting the values of Θ_2, q and $C_{t,\nu,\dot{\nu},\mu,\dot{\mu}}(c)$, and after simplifying, we have the following expression

$$\psi_4 = \int_0^1 \left(4t^2 + 11t\right) \mathrm{d}t$$

or

$$\psi_4 = \frac{41}{6}\cdot$$

Obviously,

$$\int_0^1 \left[\left\{\Theta_i(t,\nu,\dot{\nu}) - \Theta_i(t,\mu,\dot{\mu}) + \frac{1}{2}q^T Mq - \rho\theta^2(t,\nu,\mu)\right\}\right] \mathrm{d}t$$
$$\geq \int_0^1 C_{t,\nu,\dot{\nu},\mu,\dot{\mu}}(\Theta_{i\nu}(t,\mu,\dot{\mu}) - D\Theta_{i\dot{\nu}}(t,\mu,\dot{\mu}) + qM) \,\mathrm{d}t, \quad \text{for i} = 1,2.$$

Therefore, the functional $\int_0^1 \Theta(t,\nu,\dot{\nu}) \,\mathrm{d}t$ is second order (C,ρ,θ)-convex at $\mu = 0$.

Next, we will try to claim that $\int_0^1 \Theta_i(t,\nu,\dot{\nu}) \,\mathrm{d}t$ is not (C,ρ)-convex at the same point, as shown below. Currently, to check (C,ρ)-convexity, we take $q = 0$.

$$\text{Consider } \psi_5 = \int_0^1 (\Theta_1(t,\nu,\dot{\nu}) - \Theta_1(t,\mu,\dot{\mu}) - \rho\theta^2(t,\nu,\mu)) \,\mathrm{d}t.$$

Substituting values of Θ_1, ρ and θ, it yield

$$\psi_5 = \int_0^1 \left(t^3 - 2t^2\right) \mathrm{d}t.$$

After simplifying, we have

$$\psi_5 = -\frac{5}{12}\cdot$$

$$\text{Let} \quad \psi_6 = \int_0^1 (\Theta_2(t,\nu,\dot{\nu}) - \Theta_2(t,\mu,\dot{\mu}) - \rho\theta^2(t,\nu,\mu)) \,\mathrm{d}t.$$

Substituting the values of Θ_2, q and $C_{t,\nu,\dot{\nu},\mu,\dot{\mu}}(c)$, and simplifying, we get

$$\psi_6 = -\frac{5}{6}\cdot$$

$$\text{Consider} \quad \psi_7 = \int_0^1 C_{t,\nu,\dot{\nu},\mu,\dot{\mu}}(\Theta_{1\nu}(t,\mu,\dot{\mu}) - D\Theta_{1\dot{\nu}}(t,\mu,\dot{\mu}))\,\mathrm{d}t.$$

Substituting values of Θ_1, ρ and θ, we have

$$\psi_7 = 0.$$

And

$$\psi_8 = \int_0^1 C_{t,\nu,\dot{\nu},\mu,\dot{\mu}}(\Theta_{2\nu}(t,\mu,\dot{\mu}) - D\Theta_{2\dot{\nu}}(t,\mu,\dot{\mu}))\,\mathrm{d}t.$$

Putting the values of Θ_2, q and $C_{t,\nu,\dot{\nu},\mu,\dot{\mu}}(c)$, we find that

$$\psi_8 = -\frac{1}{2}\cdot$$

After simplifying the following expression, reduced in the following form:

$$\int_0^1 (\Theta_i(t,\nu,\dot{\nu}) - \Theta_i(t,\mu,\dot{\mu}) - \rho\theta^2(t,\nu,\mu))\,\mathrm{d}t$$
$$\ngeq \int_0^1 C_{t,\nu,\dot{\nu},\mu,\dot{\mu}}(\Theta_{i\nu}(t,\mu,\dot{\mu}) - D\Theta_{i\dot{\nu}}(t,\mu,\dot{\mu}))\,\mathrm{d}t,\ \ i = 1,2.$$

Hence, $\int_0^1 \Theta_i(t,\nu,\dot{\nu})\,\mathrm{d}t$ is not (C,ρ)-convex.

In contrast, to the class of (C,ρ)-convex functions, the class of second order (C,ρ,θ)-convex functions is more inclusive, as demonstrated by the aforementioned example.

Example 2. Let $I = [0,1]$ and $X = [0,10^{-2}] \times [0,10^{-2}] \subset R^2$. Consider the functional $\varphi : I \times X \times X \to R$ which is defined by

$$\varphi(t,\nu,\dot{\nu}) = \sin^2 \nu_1 + \nu_2.$$

Let the functional $C : I \times X \times X \times X \times X \times R^n \to R$ be given by

$$C_{t,\nu,\dot{\nu},\mu,\dot{\mu}}(a) = |a|(\nu_1 + \nu_2 - \mu),\ \ \theta(t,\nu,\mu) = \nu\mu.$$

For $\rho = 1, q = \left(\frac{1}{2}, 1\right)$ and $\mu = (0,0)$, we have

Solution

Let $\phi_1 = \int_a^b \{\varphi(t,\nu,\dot{\nu}) - \varphi(t,\mu,\dot{\mu}) + \frac{1}{2}q^T M_i q\}\,\mathrm{d}t - \int_a^b C_{t,\nu,\dot{\nu},\mu,\dot{\mu}}(\varphi_\nu(t,\mu,\dot{\mu}) - D\varphi_{\dot{\nu}}(t,\mu,\dot{\mu}) + M_i q)\,\mathrm{d}t + \int_a^b \rho\theta^2(t,\nu,\mu)\,\mathrm{d}t$.

Substitute the value of $\varphi, C_{t,\nu,\dot{\nu},\mu,\dot{\mu}}(a), \rho$ and θ, the following expression reduces as

$$\phi_1 = \int_0^1 \left(\sin^2 \nu_1 + \nu_2 + \frac{1}{4}\cos 2\nu_1\right)\mathrm{d}t \ - \int_0^1 C_{t,\nu,\dot{\nu},\mu,\dot{\mu}}(cos2\nu_1)\ \mathrm{d}t$$

or

$$\phi_1 = \int_0^1 \left(\sin^2 \nu_1 + \nu_2 + \frac{1}{4}\cos 2\nu_1\right)\mathrm{d}t \ - \int_0^1 |\cos 2\nu_1|\ (\nu_1 + \nu_2)\,\mathrm{d}t.$$

This follows that

$$\phi_1 \geq 0, \forall\ \nu \in X \text{ (as seen by Fig. 1).}$$

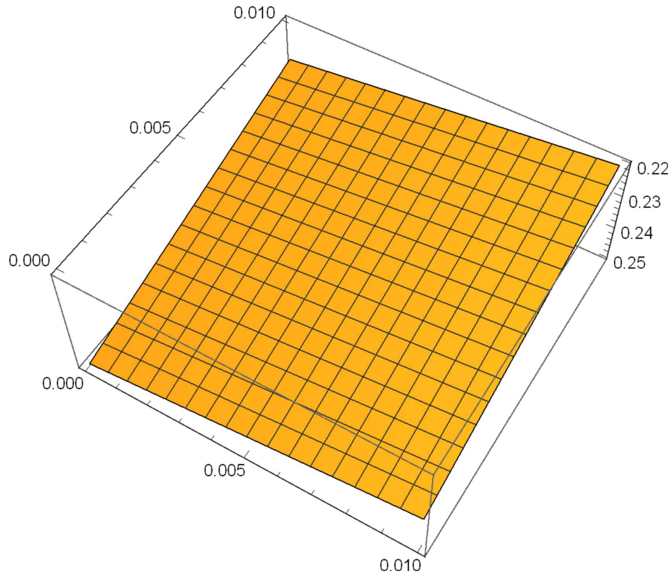

FIGURE 1. $\left(\sin^2 \nu_1 + \nu_2 + \frac{1}{4}\cos 2\nu_1 - |\cos 2\nu_1|\ (\nu_1+\nu_2)\right)$.

Hence $\phi_1 \geq 0$. This means, the functional $\int_0^1 \varphi(t,\nu,\dot{\nu})\,\mathrm{d}t$ is second-order (C,ρ,θ)-convex at $\mu=(0,0)$. Further, note that $\int_0^1 \varphi(t,\nu,\dot{\nu})\,\mathrm{d}t$ is not (C,ρ)-convex at $\mu=(0,0)$, since, for $\rho=1$ and $q=\left(\frac{1}{2},1\right)$, we obtain.

Consider $\phi_2 = \int_0^1 (\varphi(t,\nu,\dot{\nu}) - \varphi(t,\mu,\dot{\mu}))\,\mathrm{d}t - \int_0^1 C_{t,\nu,\dot{\nu},\mu,\dot{\mu}}(\varphi_\nu(t,\mu,\dot{\mu}) - D\varphi_{\dot{\nu}}(t,\mu,\dot{\mu}))\,\mathrm{d}t + \int_0^1 \rho\theta^2(t,\nu,\mu)\,\mathrm{d}t.$

Substituting the values of φ_1,ρ,θ and $C_{t,\nu,\dot{\nu},\mu,\dot{\mu}}(a)$, we get

$$\phi_2 = \int_0^1 \left(\sin^2 \nu_1\right)\mathrm{d}t \ - \int_0^1 C_{(t,\nu,\dot{\nu},\mu,\dot{\mu})}\ (\cos 2\nu_1)\,\mathrm{d}t.$$

It follows that

$$\phi_2 = \int_0^1 \left(\sin^2 \nu_1 + \nu_2\right)\mathrm{d}t \ - \int_0^1 |\cos 2\nu_1|\ (\nu_1+\nu_2)\,\mathrm{d}t$$

or

$$\phi_2 \ngeq 0, \forall\ \nu \in X \text{ (as it can be seen from Fig. 2).}$$

Hence, $\int_0^1 \varphi(t,\nu,\dot{\nu})\,\mathrm{d}t$ is not (C,ρ)-convex.

In contrast to the class of (C,ρ)-convex functions, the class of second order (C,ρ,θ)-convex functions is more inclusive, as demonstrated by the aforementioned example.

3. Second order nondifferentiable multiobjective symmetric duality model

In the past several years, there has been a lot of development in the field of duality in variational programming problems. A constrained variational issue was introduced by Mond *et al.* [22] as a mathematical programming problem, along with its Wolfe type dual, which was used to validate different duality results under convexity. By adopting a continuous form of invexity, Smart and Mond [28] extended the symmetric duality results to variational problems. Duality theorems for variational programming problems with reduced convexity assumptions were further studied by Bector *et al.* [3]. Bector *et al.* [2] found duality relations for

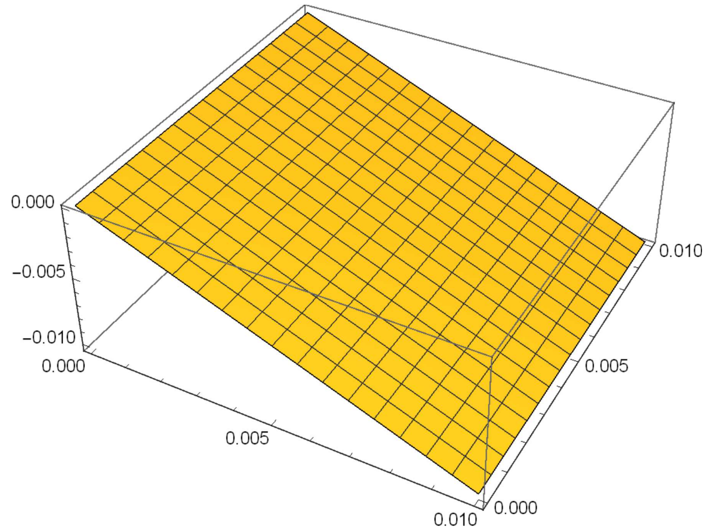

FIGURE 2. $\left(\sin^2 \nu_1 + \nu_2 - |\cos 2\nu_1|(\nu_1 + \nu_2)\right)$.

Wolfe and Mond-Weir type duals for multiobjective variational problems under convexity by extending the results seen in Bector *et al.* [3], Dantzig [5] to the multiobjective case. Now, we consider the problem of finding functions $\nu : [c, d] \to R^n$ and $\kappa : [c, d] \to R^m$, where $(\dot{\nu}, \dot{\kappa})$ is piecewise smooth on $[c, d]$, to solve the following pair of symmetric dual multiobjective nondifferentiable fractional variational problems over arbitrary cones introduced as follows:

Primal Problem (NFVP)

$$\text{Minimize} \left(\frac{\int_c^d \left(\Phi_1(t, \nu, \dot{\nu}, \kappa, \dot{\kappa}) - \frac{1}{2} p_1^T A_1 p_1 + s(\nu|E_1) - \kappa^T z_1\right) \mathrm{d}t}{\int_c^d \left(\Psi_1(t, \nu, \dot{\nu}, \kappa, \dot{\kappa}) - \frac{1}{2} p_1^T B_1 p_1 - s(\nu|F_1) + \kappa^T r_1\right) \mathrm{d}t}, \ldots, \right.$$
$$\left. \frac{\int_c^d \left(\Phi_k(t, \nu, \dot{\nu}, \kappa, \dot{\kappa}) - \frac{1}{2} p_k^T A_k p_k + s(\nu|E_k) - \kappa^T z_k\right) \mathrm{d}t}{\int_c^d \left(\Psi_k(t, \nu, \dot{\nu}, \kappa, \dot{\kappa}) - \frac{1}{2} p_k^T B_k p_k - s(\nu|F_k) + \kappa^T r_k\right) \mathrm{d}t} \right),$$

subject to

$$\nu(c) = 0 = \nu(d), \quad \dot{\nu}(c) = 0 = \dot{\nu}(d),$$
$$\kappa(c) = 0 = \kappa(d), \quad \dot{\kappa}(c) = 0 = \dot{\kappa}(d),$$
$$-\sum_{i=1}^{k} \lambda_i \{G_i(\nu, \kappa)[\Phi_{i\kappa} - D\Phi_{i\dot{\kappa}} + A_i p_i - z_i] - F_i(\nu, \kappa)[\Psi_{i\kappa} - D\Psi_{i\dot{\kappa}} + B_i p_i + r_i]\} \in C_2^*,$$
$$\int_c^d \kappa^T \sum_{i=1}^{k} \lambda_i \{G_i(\nu, \kappa)[\Phi_{i\kappa} - D\Phi_{i\dot{\kappa}} + A_i p_i - z_i] - F_i(\nu, \kappa)[\Psi_{i\kappa} - D\Psi_{i\dot{\kappa}} + B_i p_i + r_i]\} \,\mathrm{d}t \geq 0,$$
$$\lambda > 0, \quad \lambda^T e_k = 1, \quad \nu \in C_1, \quad t \in I,$$
$$z_i \in J_i, \quad r_i \in K_i, \quad i \in \tilde{N}.$$

Dual Problem (NFVD)

$$\text{Maximize}\ \left(\frac{\int_c^d\big(\Phi_1(t,\mu,\dot\mu,\varsigma,\dot\varsigma)-\frac{1}{2}q_1^TY_1q_1-s(\varsigma|J_1)+\mu^Tw_1\big)\,\mathrm{d}t}{\int_c^d\big(\Psi_1(t,\mu,\dot\mu,\varsigma,\dot\varsigma)-\frac{1}{2}q_1^TZ_1q_1+s(\varsigma|K_1)-\mu^Ts_1\big)\,\mathrm{d}t},\dots,\right.$$
$$\left.\frac{\int_c^d\big(\Phi_k(t,\mu,\dot\mu,\varsigma,\dot\varsigma)-\frac{1}{2}q_k^TY_kq_k-s(\varsigma|J_k)+\mu^Tw_k\big)\,\mathrm{d}t}{\int_c^d\big(\Psi_k(t,\mu,\dot\mu,\varsigma,\dot\varsigma)-\frac{1}{2}q_k^TZ_kq_k+s(\varsigma|K_k)-\mu^Ts_k\big)\,\mathrm{d}t}\right),$$

subject to

$$\mu(c)=0=\mu(d),\quad \varsigma(c)=0=\varsigma(d),$$
$$\dot\mu(c)=0=\dot\mu(d),\quad \dot\varsigma(c)=0=\dot\varsigma(d),$$
$$\sum_{i=1}^k\lambda_i\{G_i^*(\mu,\varsigma)[\Phi_{i\nu}-D\Phi_{i\dot\nu}+Y_iq_i+w_i]-F_i^*(\mu,\varsigma)[\Psi_{i\nu}-D\Psi_{i\dot\nu}+Z_iq_i-s_i]\}\in C_2^*,$$
$$\int_c^d\mu^T\sum_{i=1}^k\lambda_i\{G_i^*(\mu,\varsigma)[\Phi_{i\nu}-D\Phi_{i\dot\nu}+Y_iq_i+w_i]-F_i^*(\mu,\varsigma)[\Psi_{i\nu}-D\Psi_{i\dot\nu}+Z_iq_i-s_i]\}\,\mathrm{d}t\geq 0,$$
$$\lambda>0,\ \lambda^Te_k=1,\varsigma\in C_2,$$
$$w_i\in E_i,\ s_i\in F_i,\ i\in\tilde N,$$

where

(i) $\Phi_i: I\times C_1\times C_1\times C_2\times C_2\to R_+$, and $\Psi_i: I\times C_1\times C_1\times C_2\times C_2\to R_+\backslash\{0\}$,
(ii) $A_i(t,\nu,\dot\nu,\kappa,\dot\kappa)=\Phi_{i\kappa\kappa}(t,\nu,\dot\nu,\kappa,\dot\kappa)-2D\Phi_{i\kappa\dot\kappa}(t,\nu,\dot\nu,\kappa,\dot\kappa)+D^2\Phi_{i\dot\kappa\dot\kappa}(t,\nu,\dot\nu,\kappa,\dot\kappa)-D^3\Phi_{i\dot\kappa\ddot\kappa}(t,\nu,\dot\nu,\kappa,\dot\kappa), t\in I$,
(iii) $B_i(t,\nu,\dot\nu,\kappa,\dot\kappa)=\Psi_{i\kappa\kappa}(t,\nu,\dot\nu,\kappa,\dot\kappa)-2D\Psi_{i\kappa\dot\kappa}(t,\nu,\dot\nu,\kappa,\dot\kappa)+D^2\Psi_{i\dot\kappa\dot\kappa}(t,\nu,\dot\nu,\kappa,\dot\kappa)-D^3\Psi_{i\dot\kappa\ddot\kappa}(t,\nu,\dot\nu,\kappa,\dot\kappa), t\in I$,
(iv) $Y_i(t,\mu,\dot\mu,\varsigma,\dot\varsigma)=\Phi_{i\nu\nu}(t,\mu,\dot\mu,\varsigma,\dot\varsigma)-2D\Phi_{i\nu\dot\nu}(t,\mu,\dot\mu,\varsigma,\dot\varsigma)+D^2\Phi_{i\dot\nu\dot\nu}(t,\mu,\dot\mu,\varsigma,\dot\varsigma)-D^3\Phi_{i\dot\nu\ddot\nu}(t,\mu,\dot\mu,\varsigma,\dot\varsigma), t\in I$,
(v) $Z_i(t,\mu,\dot\mu,\varsigma,\dot\varsigma)=\Psi_{i\nu\nu}(t,\mu,\dot\mu,\varsigma,\dot\varsigma)-2D\Psi_{i\nu\dot\nu}(t,\mu,\dot\mu,\varsigma,\dot\varsigma)+D^2\Psi_{i\dot\nu\dot\nu}(t,\mu,\dot\mu,\varsigma,\dot\varsigma)-D^3\Psi_{i\dot\nu\ddot\nu}(t,\mu,\dot\mu,\varsigma,\dot\varsigma), t\in I$,
(vi) $p_i: I\to R^m$, $q_i: I\to R^n$,
(vii) $F_i(\nu,\kappa)=\int_c^d\big(\Phi_i(t,\nu,\dot\nu,\kappa,\dot\kappa)-\frac{1}{2}p_i^TA_ip_i+s(\nu|E_i)-\kappa^Tz_i\big)\,\mathrm{d}t$,
(viii) $G_i(\nu,\kappa)=\int_c^d\big(\Psi_i(t,\nu,\dot\nu,\kappa,\dot\kappa)-\frac{1}{2}p_i^TB_ip_i-s(\nu|F_i)+\kappa^Tr_i\big)\,\mathrm{d}t$,
(ix) $F_i^*(\mu,\varsigma)=\int_c^d\big(\Phi_i(t,\mu,\dot\mu,\varsigma,\dot\varsigma)-\frac{1}{2}p_i^TY_ip_i+s(\varsigma|J_i)-\mu^Tw_i\big)\,\mathrm{d}t$,
(x) $G_i^*(\mu,\varsigma)=\int_c^d\big(\Psi_i(t,\mu,\dot\mu,\varsigma,\dot\varsigma)-\frac{1}{2}p_i^TZ_ip_i-s(\varsigma|K_i)+\mu^Ts_i\big)\,\mathrm{d}t$,
(xi) E_i and F_i are compact convex sets in R^n,
(xii) J_i and K_i are compact convex sets in R^m.

We assume that the numerators and denominators of the primal and dual problems, as stated above, are nonnegative and positive in order to properly characterise the problem, respectively. First, we introduce Γ and Υ, described follows, the objective function's optimum value matches the value of the objective function in the previously specified situation. This turns our problem into a parametric problem. Let's take

$$\Gamma_i=\frac{\int_c^d\big(\Phi_i(t,\nu,\dot\nu,\kappa,\dot\kappa)-\frac{1}{2}p_i^TA_ip_i+s(\nu|E_i)-\kappa^Tz_i\big)\,\mathrm{d}t}{\int_c^d\big(\Psi_i(t,\nu,\dot\nu,\kappa,\dot\kappa)-\frac{1}{2}p_i^TB_ip_i-s(\nu|F_i)+\kappa^Tr_i\big)\,\mathrm{d}t}$$

and

$$\Upsilon_i=\frac{\int_c^d\big(\Phi_i(t,\mu,\dot\mu,\varsigma,\dot\varsigma)-\frac{1}{2}q_i^TY_iq_i-s(\varsigma|J_i)+\mu^Tw_i\big)\,\mathrm{d}t}{\int_c^d\big(\Psi_i(t,\mu,\dot\mu,\varsigma,\dot\varsigma)-\frac{1}{2}q_i^TZ_iq_i+s(\varsigma|K_i)-\mu^Ts_i\big)\,\mathrm{d}t},\ i\in\tilde N.$$

Equivalently, the aforesaid problems may be expressed as follows.

Primal Problem (NFVP′)

$$
\begin{aligned}
&\text{Min } \Gamma = (\Gamma_1, \Gamma_2, \ldots, \Gamma_k)\\
&\text{subject to}\\
&\nu(c) = 0 = \nu(d), \quad \kappa(c) = 0 = \kappa(d),\\
&\dot{\nu}(c) = 0 = \dot{\nu}(d), \quad \dot{\kappa}(c) = 0 = \dot{\kappa}(d),\\
&\int_c^d \left(\Phi_i(t, \nu, \dot{\nu}, \kappa, \dot{\kappa}) - \frac{1}{2} p_i^T A_i p_i + s(\nu|E_i) - \kappa^T z_i \right) \mathrm{d}t\\
&\qquad - \Gamma_i \int_c^d \left(\Psi_i(t, \nu, \dot{\nu}, \kappa, \dot{\kappa}) - \frac{1}{2} p_i^T B_i p_i - s(\nu|F_i) + \kappa^T r_i \right) \mathrm{d}t = 0, \qquad (3)\\
&- \sum_{i=1}^k \lambda_i \Big\{ \big[\Phi_{i\kappa} - D\Phi_{i\kappa} + A_i p_i - z_i \big] - \Gamma_i \big[\Psi_{i\kappa} - D\Psi_{i\kappa} + B_i p_i + r_i \big] \Big\} \in C_2^*, \qquad (4)\\
&\kappa^T \sum_{i=1}^k \lambda_i \Big\{ \big[\Phi_{i\kappa} - D\Phi_{i\kappa} + A_i p_i - z_i \big] - \Gamma_i \big[\Psi_{i\kappa} - D\Psi_{i\kappa} + B_i p_i + r_i \big] \Big\} \geq 0, \qquad (5)\\
&\lambda > 0, \quad \lambda^T e_k = 1, \quad \nu \in C_1, \quad t \in I, \qquad (6)\\
&z_i \in J_i, \quad r_i \in K_i, \quad i \in \tilde{N}. \qquad (7)
\end{aligned}
$$

Dual Problem (NFVD′)

$$
\begin{aligned}
&\text{Max } \Upsilon = (\Upsilon_1, \Upsilon_2, \ldots, \Upsilon_k)\\
&\text{subject to}\\
&\mu(c) = 0 = \mu(d), \quad \varsigma(c) = 0 = \varsigma(d),\\
&\dot{\mu}(c) = 0 = \dot{\mu}(d), \quad \dot{\varsigma}(c) = 0 = \dot{\varsigma}(d),\\
&\int_c^d \left(\Phi_i(t, \mu, \dot{\mu}, \varsigma, \dot{\varsigma}) - \frac{1}{2} q_i^T Y_i q_i - s(\varsigma|J_i) - \mu^T w_i \right) \mathrm{d}t\\
&\qquad - \Upsilon_i \int_c^d \left(\Psi_i(t, \mu, \dot{\mu}, \varsigma, \dot{\varsigma}) - \frac{1}{2} q_i^T Z_i q_i + s(\varsigma|K_i) - \mu^T s_i \right) \mathrm{d}t = 0, \qquad (8)\\
&\sum_{i=1}^k \lambda_i \Big\{ \big[\Phi_{i\nu} - D\Phi_{i\dot{\nu}} + Y_i q_i + w_i \big] - \Upsilon_i \big[\Psi_{i\nu} - D\Psi_{i\dot{\nu}} + Z_i q_i - s_i \big] \Big\} \in C_1^*, \qquad (9)\\
&\mu^T \sum_{i=1}^k \lambda_i \Big\{ \big[\Phi_{i\nu} - D\Phi_{i\dot{\nu}} + Y_i q_i + w_i \big] - \Upsilon_i \big[\Psi_{i\nu} - D\Psi_{i\dot{\nu}} + Z_i q_i - s_i \big] \Big\} \leq 0, \qquad (10)\\
&\lambda > 0, \quad \lambda^T e_k = 1, \quad \varsigma \in C_2, \quad t \in I, \qquad (11)\\
&w_i \in E_i, \quad s_i \in F_i, \quad i \in \tilde{N}. \qquad (12)
\end{aligned}
$$

Let V^* and W^* be the sets of feasible solutions of primal (NFVP′) and dual (NFVD′), respectively.

Theorem 1 (Weak Duality)**.** *Let* $(\nu, \kappa, \Gamma, p, z, r) \in V^*$ *and* $(\mu, \varsigma, \Upsilon, q, w, s) \in W^*$*. Suppose that*

(i) $\int_c^d \big(\Phi_i(t, ., ., \varsigma, \dot{\varsigma}) + (.)^T w_i - \Upsilon_i (\Psi_i(t, ., ., \varsigma, \dot{\varsigma}) - (.)^T s_i) \big) \, \mathrm{d}t$ *be second order* (C, ρ, θ)*-convex at* μ *for fixed* ς*,*

(ii) $-\int_c^d \big(\Phi_i(t,\nu,\dot{\nu},.,.) - (.)^T z_i - \Gamma_i(\Psi_i(t,\nu,\dot{\nu},.,.) - (.)^T r_i)\big)\,\mathrm{d}t$ *be second order* $(\bar{C},\bar{\rho},\bar{\theta})$*-convex at* κ *for fixed* ν,

(iii) $C_{t,\nu,\dot{\nu},\mu,\dot{\mu}}(c) + c^T\mu \geq 0, \forall\ c \in C_1^*,\ t \in I,\ \bar{C}_{t,\varsigma,\dot{\varsigma},\kappa,\dot{\kappa}}(d) + d^T\kappa \geq 0, \forall\ d \in C_2^*,\ t \in I,$

(iv) $\sum_{i=1}^k \lambda_i \int_c^d \big(\Psi_i(t,\nu,\dot{\nu},\varsigma,\dot{\varsigma}) + \varsigma^T r_i - \nu^T s_i\big)\,\mathrm{d}t \geq 0,$

(v) $\sum_{i=1}^k \lambda_i \int_c^d \big\{\rho\theta^2(t,\nu,\mu) + \bar{\rho}\bar{\theta}^2(t,\varsigma,\kappa)\big\}\,\mathrm{d}t \geq 0.$

Then, $\Gamma \geq \Upsilon$.

Proof. Using the constraint (9) and hypothesis (iii), we have

$$C_{t,\nu,\dot{\nu},\mu,\dot{\mu}}\left(\sum_{i=1}^k \lambda_i\{[\Phi_{i\nu} - D\Phi_{i\dot{\nu}} + Y_i q_i + w_i] - \Upsilon_i[\Psi_{i\nu} - D\Psi_{i\dot{\nu}} + Z_i q_i - s_i]\}\right)$$
$$+ \mu^T \sum_{i=1}^k \lambda_i\{[\Phi_{i\nu} - D\Phi_{i\dot{\nu}} + Y_i q_i + w_i] - \Upsilon_i[\Psi_{i\nu} - D\Psi_{i\dot{\nu}} + Z_i q_i - s_i]\} \geq 0,$$

which by the virtue of (10) reduces in the following form:

$$\int_c^d C_{t,\nu,\dot{\nu},\mu,\dot{\mu}}\left(\sum_{i=1}^k \lambda_i\{[\Phi_{i\nu} - D\Phi_{i\dot{\nu}} + Y_i q_i + w_i] - \Upsilon_i[\Psi_{i\nu} - D\Psi_{i\dot{\nu}} + Z_i q_i - s_i]\}\right) \geq 0. \tag{13}$$

As $\int_c^d (\Phi_i(t,.,.,\varsigma,\dot{\varsigma}) + (.)^T w_i - \Upsilon_i(\Psi_i(t,.,.,\varsigma,\dot{\varsigma}) - (.)^T s_i))\,\mathrm{d}t$ is second order (C,ρ,θ)-convex at μ for fixed ς, we have

$$\begin{aligned}\int_c^d \bigg\{ &\bigg(\Phi_i(t,\nu,\dot{\nu},\varsigma,\dot{\varsigma}) + \nu^T w_i - \Phi_i(t,\mu,\dot{\mu},\varsigma,\dot{\varsigma}) + \frac{1}{2} q_i^T Y_i q_i - \mu^T w_i - \rho\theta^2(t,\nu,\mu)\bigg) \\ &- \Upsilon_i\bigg(\Psi_i(t,\nu,\dot{\nu},\varsigma,\dot{\varsigma}) - \nu^T s_i - \Psi_i(t,\mu,\dot{\mu},\varsigma,\dot{\varsigma}) + \mu^T s_i + \frac{1}{2} q_i^T Z_i q_i - \rho\theta^2(t,\nu,\mu)\bigg)\bigg\}\,\mathrm{d}t \\ &\geq \int_c^d C_{t,\nu,\dot{\nu},\mu,\dot{\mu}}(\{(\Phi_{i\nu} - D\Phi_{i\dot{\nu}} + Y_i q_i + w_i) - \Upsilon_i(\Psi_{i\nu} - D\Psi_{i\dot{\nu}} + Z_i q_i - s_i)\})\,\mathrm{d}t.\end{aligned}$$

Since, $\lambda^T e_k = 1$, we get

$$\begin{aligned}\sum_{i=1}^k \lambda_i \int_c^d \bigg\{ &\bigg(\Phi_i(t,\nu,\dot{\nu},\varsigma,\dot{\varsigma}) + \nu^T w_i - \Phi_i(t,\mu,\dot{\mu},\varsigma,\dot{\varsigma}) + \frac{1}{2} q_i^T Y_i q_i - \mu^T w_i - \rho\theta^2(t,\nu,\mu)\bigg) \\ &- \Upsilon_i\bigg(\Psi_i(t,\nu,\dot{\nu},\varsigma,\dot{\varsigma}) - \nu^T s_i - \Psi_i(t,\mu,\dot{\mu},\varsigma,\dot{\varsigma}) + \mu^T s_i + \frac{1}{2} q_i^T Z_i q_i - \rho\theta^2(t,\nu,\mu)\bigg)\bigg\}\,\mathrm{d}t \\ &\geq \int_c^d C_{t,\nu,\dot{\nu},\mu,\dot{\mu}}\left(\sum_{i=1}^k \lambda_i\{(\Phi_{i\nu} - D\Phi_{i\dot{\nu}} + Y_i q_i + w_i) - \Upsilon_i(\Psi_{i\nu} - D\Psi_{i\dot{\nu}} + Z_i q_i - s_i)\}\right)\mathrm{d}t,\end{aligned}$$

which due to (13) reduce to

$$\begin{aligned}\sum_{i=1}^k \lambda_i \int_c^d \bigg\{ &\bigg(\Phi_i(t,\nu,\dot{\nu},\varsigma,\dot{\varsigma}) + \nu^T w_i - \Phi_i(t,\mu,\dot{\mu},\varsigma,\dot{\varsigma}) + \frac{1}{2} q_i^T Y_i q_i - \mu^T w_i\bigg) \\ &- \Upsilon_i\bigg(\Psi_i(t,\nu,\dot{\nu},\varsigma,\dot{\varsigma}) - \nu^T s_i - \Psi_i(t,\mu,\dot{\mu},\varsigma,\dot{\varsigma}) + \mu^T s_i + \frac{1}{2} q_i^T Z_i q_i\bigg)\bigg\}\,\mathrm{d}t \geq 2\sum_{i=1}^k \lambda_i \int_c^d \rho\theta^2(t,\nu,\mu)\,\mathrm{d}t.\end{aligned}$$

This can be rewritten as

$$\sum_{i=1}^{k}\lambda_i\int_c^d\Big\{\Big(\Phi_i(t,\nu,\dot\nu,\varsigma,\dot\varsigma)+\nu^Tw_i-\Phi_i(t,\mu,\dot\mu,\varsigma,\dot\varsigma)+\frac{1}{2}q_i^TY_iq_i-\mu^Tw_i\Big)$$
$$+\Upsilon_i\Big(\Psi_i(t,\mu,\dot\mu,\varsigma,\dot\varsigma)+\varsigma^Tr_i-\mu^Ts_i-\frac{1}{2}q_i^TZ_iq_i-\Upsilon_i(\Psi_i(t,\nu,\dot\nu,\varsigma,\dot\varsigma)+\varsigma^Tr_i-\nu^Ts_i)\Big)\Big\}\,\mathrm{d}t$$
$$\geq 2\sum_{i=1}^{k}\lambda_i\int_c^d\rho\theta^2(t,\nu,\mu)\,\mathrm{d}t.$$

Using (8) together with $\varsigma^Tr_i\leq s(\varsigma|K_i)$ in the above inequality, we get

$$\sum_{i=1}^{k}\lambda_i\int_c^d\big(\Phi_i(t,\nu,\dot\nu,\varsigma,\dot\varsigma)+\nu^Tw_i-s(\varsigma|J_i)-\Upsilon_i\Psi_i(t,\nu,\dot\nu,\varsigma,\dot\varsigma)+\varsigma^Tr_i-\nu^Ts_i\big)\,\mathrm{d}t$$
$$\geq 2\sum_{i=1}^{k}\lambda_i\int_c^d\rho\theta^2(t,\nu,\mu)\,\mathrm{d}t. \tag{14}$$

Likewise, as $-\int_c^d\big(\Phi_i(t,\nu,\dot\nu,.,.)-(.)^Tz_i-\Gamma_i(\Psi_i(t,\nu,\dot\nu,.,.)-(.)^Tr_i)\big)\,\mathrm{d}t$ is second order (C,ρ,θ)-convex at κ for fixed ν, we get

$$\sum_{i=1}^{k}\lambda_i\int_c^d\big(-\Phi_i(t,\nu,\dot\nu,\varsigma,\dot\varsigma)+\varsigma^Tz_i-s(\nu|E_i)+\Gamma_i\Psi_i(t,\nu,\dot\nu,\varsigma,\dot\varsigma)-\nu^Ts_i+\varsigma^Tr_i\big)\,\mathrm{d}t$$
$$\geq 2\sum_{i=1}^{k}\lambda_i\int_c^d\bar\rho\bar\theta^2(t,\varsigma,\kappa)\,\mathrm{d}t. \tag{15}$$

On adding (14) and (15) and applying (iv), we have

$$\sum_{i=1}^{k}\lambda_i\int_c^d\big((\varsigma^Tz_i-s(\varsigma|J_i)+\nu^Tw_i-s(\nu|E_i))+(\Gamma_i-\Upsilon_i)(\Psi_i(t,\nu,\dot\nu,\varsigma,\dot\varsigma)+\varsigma^Tr_i-\nu^Ts_i)\big)\,\mathrm{d}t\geq 0.$$

Since $\varsigma^Tz_i\leq s(\varsigma|J_i),\nu^Tw_i\leq s(\nu|E_i)$ the above inequality yields

$$\sum_{i=1}^{k}\lambda_i\int_c^d(\Gamma_i-\Upsilon_i)\big(\Psi_i(t,\nu,\dot\nu,\varsigma,\dot\varsigma)+\varsigma^Tr_i-\nu^Ts_i\big)\,\mathrm{d}t\geq 0. \tag{16}$$

Suppose, if possible, that $\Gamma_i<\Upsilon_i$ for all i, then from $\lambda^Te=1$ and (v) gives

$$\sum_{i=1}^{k}\lambda_i\int_c^d(\Gamma_i-\Upsilon_i)\big(\Psi_i(t,\nu,\dot\nu,\varsigma,\dot\varsigma)+\varsigma^Tr_i-\nu^Ts_i\big)\,\mathrm{d}t<0, \tag{17}$$

which is in contradict to inequality (16). Hence, proof is completes. □

Example 3. Let $I=[0,1]$. Consider the functional $\Phi:I\times R\times R\to R$ be defined by

$$\Phi_1(t,\nu,\dot\nu)=\nu,\ \ \Psi_1(t,\nu,\dot\nu)=1.$$

Suppose, $C:I\times R\times R\times R\times R\times R^n\to R$, $\theta_1:I\times R\times R\to R$, and $\theta_2:I\times R\times R\to R$, be given by

$$C_{t,\nu,\dot\nu,\mu,\dot\mu}(c)=c(\nu-\mu),$$

$$\theta_1(t,\nu,\mu)=\sqrt{\nu},\quad \theta_2(t,\varsigma,\mu)=\sqrt{\varsigma},$$
$$\nu=t^2(t-1)^2,\quad \varsigma=t^2(t-1)^2.$$

Next, $E_1=J_1=K_1=\{0\}$, $F_1=[0,1], q_1=p_1=t+2,\ \ \rho_1=-29$ and $\bar{\rho}_1=30$.

Proof. **(A1).** Since we claim that $\int_0^1\big(\Phi_1(t,.,.,\varsigma,\dot{\varsigma})+(.)^Tw_1-\Upsilon_1(\Psi_1(t,.,.,\varsigma,\dot{\varsigma})-(.)^Ts_1)\big)\,\mathrm{d}t$ is second order (C,ρ,θ)-convex at $\mu=0$ for fixed ς. We have

$$\begin{aligned}\int_0^1\Bigg\{&\bigg(\Phi_1(t,\nu,\dot{\nu},\varsigma,\dot{\varsigma})+\nu^Tw_1-\Phi_1(t,\mu,\dot{\mu},\varsigma,\dot{\varsigma})+\frac{1}{2}q_1^TY_1q_1-\mu^Tw_1-\rho\theta^2(t,\nu,\mu)\bigg)\\&-\Upsilon_1\bigg(\Psi_1(t,\nu,\dot{\nu},\varsigma,\dot{\varsigma})-\nu^Ts_1-\Psi_1(t,\mu,\dot{\mu},\varsigma,\dot{\varsigma})+\mu^Ts_1+\frac{1}{2}q_1^TZ_1q_1-\rho\theta^2(t,\nu,\mu)\bigg)\Bigg\}\,\mathrm{d}t\\&\geq\int_0^1C_{t,\nu,\dot{\nu},\mu,\dot{\mu}}(\{(\Phi_{1\nu}-D\Phi_{1\dot{\nu}}+Y_1q_1+w_1)-\Upsilon_1(\Psi_{1\nu}-D\Psi_{1\dot{\nu}}+Z_1q_1-s_1)\})\,\mathrm{d}t,\end{aligned}\tag{18}$$

where $\Upsilon_1=\dfrac{\int_0^1\big(\Phi_1(t,\mu,\dot{\mu},\varsigma,\dot{\varsigma})-\frac{1}{2}q_1^TY_1q_1-s(\varsigma|J_1)+\mu^Tw_1\big)\,\mathrm{d}t}{\int_0^1\big(\Psi_1(t,\mu,\dot{\mu},\varsigma,\dot{\varsigma})-\frac{1}{2}q_1^TZ_1q_1+s(\varsigma|K_1)-\mu^Ts_1\big)\,\mathrm{d}t}.$

Substituting the values and simplifying, we obtain

$$\Upsilon_1=\frac{\int_0^1 0\,\mathrm{d}t}{\int_0^1\,\mathrm{d}t}$$

or

$$\Upsilon_1=0.$$

Let

$$\begin{aligned}\Lambda_1=\int_0^1\Bigg\{&\bigg(\Phi_1(t,\nu,\dot{\nu},\varsigma,\dot{\varsigma})+\nu^Tw_1-\Phi_1(t,\mu,\dot{\mu},\varsigma,\dot{\varsigma})+\frac{1}{2}q_1^TY_1q_1-\mu^Tw_1-\rho\theta^2(t,\nu,\mu)\bigg)\\&-\Upsilon_1\bigg(\Psi_1(t,\nu,\dot{\nu},\varsigma,\dot{\varsigma})-\nu^Ts_1-\Psi_1(t,\mu,\dot{\mu},\varsigma,\dot{\varsigma})+\mu^Ts_1+\frac{1}{2}q_1^TZ_1q_1-\rho\theta^2(t,\nu,\mu)\bigg)\Bigg\}\,\mathrm{d}t.\end{aligned}$$

Substituting the values and simplifying, it follows that

$$\Lambda_1=\int_0^1(30t^2(t-1)^2)\,\mathrm{d}t$$

or

$$\Lambda_1=1.$$

Let

$$\Lambda_2=\int_0^1C_{t,\nu,\dot{\nu},\mu,\dot{\mu}}(\{(\Phi_{1\nu}-D\Phi_{1\dot{\nu}}+Y_1q_1+w_1)-\Upsilon_1(\Psi_{1\nu}-D\Psi_{1\dot{\nu}}+Z_1q_1-s_1)\})\,\mathrm{d}t.$$

Putting values and after simplifying, we have

$$\Lambda_2=\int_0^1t^2(t-1)^2\,\mathrm{d}t$$

or

$$\Lambda_2=\frac{1}{30}.$$

Thus, inequality (18) satisfied.

Hence, $\int_0^1 \big(\Phi_1(t,.,.,\varsigma,\dot{\varsigma}) + (.)^T w_1 - \Upsilon_1(\Psi_1(t,.,.,\varsigma,\dot{\varsigma}) - (.)^T s_1)\big)\,\mathrm{d}t$ is second order (C,ρ,θ)-convex at $\mu = 0$ for fixed ς.

(A2). Next, we have to show that $-\int_0^1 \big(\Phi_1(t,\nu,\dot{\nu},.,.) - (.)^T z_1 - \Gamma_1(\Psi_1(t,\nu,\dot{\nu},.,.) - (.)^T r_1)\big)\,\mathrm{d}t$ is second order $(\bar{C},\bar{\rho},\bar{\theta})$-convex at $\kappa = 0$ for fixed ν, *i.e.*

$$\begin{aligned} &-\int_0^1 \Bigg\{ \bigg(\Phi_1(t,\nu,\dot{\nu},\varsigma,\dot{\varsigma}) - \varsigma^T z_1 - \Phi_1(t,\nu,\dot{\nu},\kappa,\dot{\kappa}) + \frac{1}{2} q_1^T A_1 q_1 - \kappa^T z_1 - \bar{\rho}\bar{\theta}^2(t,\nu,\mu) \bigg) \\ &\quad - \Gamma_1 \bigg(\Psi_1(t,\nu,\dot{\nu},\varsigma,\dot{\varsigma}) - \varsigma^T r_1 - \Psi_1(t,\nu,\dot{\nu},\kappa,\dot{\kappa}) - \kappa^T r_1 + \frac{1}{2} q_1^T B_1 q_1 - \bar{\rho}\bar{\theta}^2(t,\nu,\mu) \bigg) \Bigg\}\,\mathrm{d}t \\ &\quad \geq -\int_0^1 C_{t,\varsigma,\dot{\varsigma},\kappa,\dot{\kappa}}(\{(\Phi_{1\kappa} - D\Phi_{1\dot{\kappa}} + A_1 q_1 + z_1) - \Gamma_1(\Psi_{1\kappa} - D\Psi_{1\dot{\kappa}} + B_1 q_1 - r_1)\})\,\mathrm{d}t, \end{aligned} \tag{19}$$

$$\text{where} \quad \Gamma_1 = \frac{\int_0^1 \big(\Phi_1(t,\nu,\dot{\nu},\kappa,\dot{\kappa}) - \frac{1}{2} p_1^T A_1 p_1 + s(\nu|E_1) - \kappa^T z_1\big)\,\mathrm{d}t}{\int_0^1 \big(\Psi_1(t,\nu,\dot{\nu},\kappa,\dot{\kappa}) - \frac{1}{2} p_1^T B_1 p_1 - s(\nu|F_1) + \kappa^T r_1\big)\,\mathrm{d}t}.$$

Putting values and put $\nu = t^2(t^2-1)^2$, we have

$$\Gamma_1 = \frac{\int_0^1 \big(t^4 - 2t^3 + t^2\big)\,\mathrm{d}t}{\int_0^1 (1 - t^4 + 2t^3 - t^2)\,\mathrm{d}t},$$

or

$$\Gamma_1 = \frac{1}{29}.$$

Now let

$$\begin{aligned} \Lambda_3 &= -\int_0^1 \Bigg\{ \bigg(\Phi_1(t,\nu,\dot{\nu},\varsigma,\dot{\varsigma}) - \varsigma^T z_1 - \Phi_1(t,\nu,\dot{\nu},\kappa,\dot{\kappa}) + \frac{1}{2} q_1^T A_1 q_1 - \kappa^T z_1 - \bar{\rho}\bar{\theta}^2(t,\nu,\mu) \bigg) \\ &\quad - \Gamma_1 \bigg(\Psi_1(t,\nu,\dot{\nu},\varsigma,\dot{\varsigma}) - \varsigma^T r_1 - \Psi_1(t,\nu,\dot{\nu},\kappa,\dot{\kappa}) - \kappa^T r_1 + \frac{1}{2} q_1^T B_1 q_1 - \bar{\rho}\bar{\theta}^2(t,\nu,\mu) \bigg) \Bigg\}\,\mathrm{d}t, \\ \Lambda_3 &= \int_0^1 \bigg(30t^2(t-1)^2 - \frac{30}{29} t^2(t-1)^2 \bigg)\,\mathrm{d}t \end{aligned}$$

or

$$\Lambda_3 = \frac{28}{29},$$

and

$$\Lambda_4 = -\int_0^1 C_{t,\varsigma,\dot{\varsigma},\kappa,\dot{\kappa}}(\{(\Phi_{1\kappa} - D\Phi_{1\dot{\kappa}} + A_1 q_1 + z_1) - \Gamma_1(\Psi_{1\kappa} - D\Psi_{1\dot{\kappa}} + B_1 q_1 - r_1)\})\,\mathrm{d}t.$$

Substituting the value of Φ_1, Ψ_1, Γ_1 and $C_{t,\nu,\dot{\nu},\mu,\dot{\mu}}$, in above expression, we have

$$\Lambda_4 = 0.$$

Therefore, inequality (19) holds.

Hence, $-\int_0^1 \big(\Phi_1(t,\nu,\dot{\nu},.,.) - (.)^T z_1 - \Gamma_1\big(\Psi_1(t,\nu,\dot{\nu},.,.) - (.)^T r_1\big)\big)\,\mathrm{d}t$ is second order $(\bar{C},\bar{\rho},\bar{\theta})$-convex at $\kappa = 0$ for fixed ν.

(A3). $C_{t,\nu,\dot{\nu},\mu,\dot{\mu}}(c) + c^T\mu = c\nu - c\mu + c\mu = c\nu \geq 0, \forall\ c \in C_1^*,\ t \in I$, $\bar{C}_{t,\varsigma,\dot{\varsigma},\kappa,\dot{\kappa}}(d) + d^T\kappa = d\varsigma - d\kappa + d\kappa = d\varsigma \geq 0, \forall\ d \in C_2^*,\ t \in I$.

(A4). $\lambda_1 \int_0^1 (\Psi_1(t,\nu,\dot{\nu},\varsigma,\dot{\varsigma}) + \varsigma^T r_1 - \nu^T s_1)\,\mathrm{d}t = \lambda_1 \int_0^1 (1 - \nu s_1)\,\mathrm{d}t \geq 0, \forall\ \nu,\ s_1 \in [0,1]$.

(A5). $\lambda_1 \int_0^1 \{\rho\theta^2(t,\nu,\mu) + \bar{\rho}\bar{\theta}^2(t,\varsigma,\kappa)\}\,\mathrm{d}t = \lambda_1 \int_0^1 \{-29t^2(t-1)^2 + 30t^2(t-1)^2\}\,\mathrm{d}t = \lambda_1 \int_0^1 t^2(t-1)^2\,\mathrm{d}t \geq 0.$ Hence, all the assumptions of weak duality are satisfied.

Validation. To validate weak duality theorem, it is enough to claim that $\Gamma \geq \Upsilon$ *i.e.* $\Gamma_1 \geq \Upsilon_1$,

$$\Gamma_1 = \frac{1}{29} \text{ and } \Upsilon_1 = 0.$$

$$\text{Therefore,} \quad \Gamma \geq \Upsilon.$$

Hence, this validates the weak duality theorem. □

Remark 1. Since every convex is pseudoconvex and quasiconvex so that the following results can be done on the lines of Theorem 1.

Theorem 2 (Weak Duality)**.** *Let* $(\nu,\kappa,\Gamma,p,z,r) \in V^*$ *and* $(\mu,\varsigma,\Upsilon,q,w,s) \in W^*$. *Suppose that*

(i) $\int_c^d \big(\Phi_i(t,.,.,\varsigma,\dot{\varsigma}) + (.)^T w_i - \Upsilon_i(\Psi_i(t,.,.,\varsigma,\dot{\varsigma}) - (.)^T s_i)\big)\,\mathrm{d}t$ *be second order* (C,ρ,θ)*-pseudoconvex at* μ *for fixed* ς,

(ii) $-\int_c^d \big(\Phi_i(t,\nu,\dot{\nu},.,.) - (.)^T z_i - \Gamma_i(\Psi_i(t,\nu,\dot{\nu},.,.) - (.)^T r_i)\big)\,\mathrm{d}t$ *be second order* $(\bar{C},\bar{\rho},\bar{\theta})$*-pseudoconvex at* κ *for fixed* ν,

(iii) $C_{t,\nu,\dot{\nu},\mu,\dot{\mu}}(c) + c^T\mu \geq 0, \forall c \in C_1^*,\ t \in I,\ \bar{C}_{t,\varsigma,\dot{\varsigma},\kappa,\dot{\kappa}}(d) + d^T\kappa \geq 0, \forall d \in C_2^*,\ t \in I,$

(iv) $\sum_{i=1}^k \lambda_i \int_c^d \big(\Psi_i(t,\nu,\dot{\nu},\varsigma,\dot{\varsigma}) + \varsigma^T r_i - \nu^T s_i\big)\,\mathrm{d}t \geq 0,$

(v) $\sum_{i=1}^k \lambda_i \int_c^d \{\rho\theta^2(t,\nu,\mu) + \bar{\rho}\bar{\theta}^2(t,\varsigma,\kappa)\}\,\mathrm{d}t \geq 0.$

Then, $\Gamma \geq \Upsilon$.

Proof. The proof is in accordance with Theorem 1. □

Theorem 3 (Weak Duality)**.** *Let* $(\nu,\kappa,\Gamma,p,z,r) \in V^*$ *and* $(\mu,\varsigma,\Upsilon,q,w,s) \in W^*$. *Let*

(i) $\int_c^d \big(\Phi_i(t,.,.,\varsigma,\dot{\varsigma}) + (.)^T w_i - \Upsilon_i(\Psi_i(t,.,.,\varsigma,\dot{\varsigma}) - (.)^T s_i)\big)\,\mathrm{d}t$ *be second order* (C,ρ,θ)*-quasiconvex at* μ *for fixed* ς,

(ii) $-\int_c^d \big(\Phi_i(t,\nu,\dot{\nu},.,.) - (.)^T z_i - \Gamma_i(\Psi_i(t,\nu,\dot{\nu},.,.) - (.)^T r_i)\big)\,\mathrm{d}t$ *be second order* $(\bar{C},\bar{\rho},\bar{\theta})$*-quasiconvex at* κ *for fixed* ν,

(iii) $C_{t,\nu,\dot{\nu},\mu,\dot{\mu}}(c) + c^T\mu \geq 0, \forall c \in C_1^*,\ t \in I,\ \bar{C}_{t,\varsigma,\dot{\varsigma},\kappa,\dot{\kappa}}(d) + d^T\kappa \geq 0, \forall d \in C_2^*,\ t \in I,$

(iv) $\sum_{i=1}^k \lambda_i \int_c^d \big(\Psi_i(t,\nu,\dot{\nu},\varsigma,\dot{\varsigma}) + \varsigma^T r_i - \nu^T s_i\big)\,\mathrm{d}t \geq 0,$

(v) $\sum_{i=1}^k \lambda_i \int_c^d \{\rho\theta^2(t,\nu,\mu) + \bar{\rho}\bar{\theta}^2(t,\varsigma,\kappa)\}\,\mathrm{d}t \geq 0.$

Then, $\Gamma \geq \Upsilon$.

Proof. The proof is in line with the Theorem 1. □

Theorem 4 (Strong Duality)**.** *Consider that all feasible solutions to* (NFVP′) *and* (NFVD′) *satisfy the Weak duality theorem's preconditions. Fix* $\lambda = \bar{\lambda}$. *Let*

(i) $(\bar{\nu},\bar{\kappa},\bar{\Gamma},\bar{\lambda},\bar{z}_1,\bar{z}_2,\ldots,\bar{z}_k,\bar{r}_1,\bar{r}_2,\ldots,\bar{r}_k,\bar{p})$ *be an efficient solution of* (NFVP′),

(ii) *the matrices* $A_i - \bar{\Gamma}_i B_i,\ i \in \tilde{N}$ *be non-singular,*

(iii) *the set* $\{(\Phi_{i\kappa} - \bar{z}_i) - \bar{\Gamma}_i(\Psi_{i\kappa} + \bar{r}_i) - D(\Phi_{i\dot{\kappa}} - \bar{\Gamma}_i\Psi_{i\dot{\kappa}}) + (A_i - \bar{\Gamma}_i B_i)\bar{p}_i,\ i \in \tilde{N},\ t \in I\}$ *be linearly independent, and*

(iv) *the matrix* $\sum_{i=1}^k \bar{\lambda}_i[(A_i\bar{p}_i)_\kappa - \bar{\Gamma}_i(B_i\bar{p}_i)_\kappa - D(A_i\bar{p}_i)_{\dot{\kappa}} + \bar{\Gamma}_i D(B_i\bar{p}_i)_{\dot{\kappa}} + D^2(A_i\bar{p}_i)_{\ddot{\kappa}} - \Gamma_i D^2(B_i\bar{p}_i)_{\ddot{\kappa}} - D^3(A_i\bar{p}_i)_{\dddot{\kappa}} + \bar{\Gamma}_i D^3(B_i\bar{p}_i)_{\dddot{\kappa}} + D^4(A_i\bar{p}_i)_{\ddddot{\kappa}} - \bar{\Gamma}_i D^4(B_i\bar{p}_i)_{\ddddot{\kappa}}],\ t \in I,$ *be positive or negative definite.*

Then, $(\bar{\nu},\bar{\kappa},\bar{\Gamma},\bar{\lambda},\bar{z}_1,\bar{z}_2,\ldots,\bar{z}_k,\bar{r}_1,\bar{r}_2,\ldots,\bar{r}_k,\bar{p}=0)$ *is an efficient solution of* $(\mathrm{NFVD}')_{\bar{\lambda}}$.

Proof. Since $(\bar{\nu},\bar{\kappa},\bar{\Gamma},\bar{\lambda},\bar{z}_1,\bar{z}_2,\ldots,\bar{z}_k,\bar{r}_1,\bar{r}_2,\ldots,\bar{r}_k,\bar{p})$ is an efficient solution of (NFVP$'$), there exist $\alpha\in R^k,\ \beta\in R^k,\ \gamma\in C_2,\ \xi\in R_+,\ \omega\in R$, and $\eta\in R^k$ satisfying the following Fritz John necessary conditions [1] at the point $(\bar{\nu},\bar{\kappa},\bar{\Gamma},\bar{\lambda},\bar{z}_1,\bar{z}_2,\ldots,\bar{z}_k,\bar{r}_1,\bar{r}_2,\ldots,\bar{r}_k,\bar{p})$ such that

$$\begin{aligned}&\Bigg[\sum_{i}^{k}\beta_i\Big\{(\Phi_{i\nu}+\bar{w}_i)-\bar{\Gamma}_i(\Psi_{i\nu}-\bar{s}_i)-D(\Phi_{i\dot{\nu}}-\bar{\Gamma}_i\Psi_{i\dot{\nu}})-\frac{1}{2}\big(\bar{p}_i^TA_i\bar{p}_i\big)_{\nu}+\frac{\bar{\Gamma}_i}{2}\big(\bar{p}_i^TB_i\bar{p}_i\big)_{\nu}+\frac{1}{2}D\big(\bar{p}_i^TA_i\bar{p}_i\big)_{\dot{\nu}}\\&-\frac{\bar{\Gamma}_i}{2}D\big(\bar{p}_i^TB_i\bar{p}_i\big)_{\dot{\nu}}-\frac{1}{2}D^2\big(\bar{p}_i^TA_i\bar{p}_i\big)_{\ddot{\nu}}+\frac{\bar{\Gamma}_i}{2}D^2\big(\bar{p}_i^TB_i\bar{p}_i\big)_{\ddot{\nu}}+\frac{1}{2}D^3\big(\bar{p}_i^TA_i\bar{p}_i\big)_{\dddot{\nu}}-\frac{\bar{\Gamma}_i}{2}D^3\big(\bar{p}_i^TB_i\bar{p}_i\big)_{\dddot{\nu}}\\&-\frac{1}{2}D^4\big(\bar{p}_i^TA_i\bar{p}_i\big)_{\ddddot{\nu}}+\frac{\bar{\Gamma}_i}{2}D^4(\bar{p}_i^TB_i\bar{p}_i)_{\ddddot{\nu}}\Big\}+(\gamma-\xi\bar{\kappa})^T\sum_{i=1}^{k}\bar{\lambda}_i\big\{\Phi_{iyx}-\bar{\Gamma}_i\Psi_{iyx}\\&-D\big(\Phi_{i\kappa\dot{\nu}}-\bar{\Gamma}_i\Psi_{i\kappa\dot{\nu}}\big)-D\big(\Phi_{i\dot{\kappa}\nu}-\bar{\Gamma}_i\Psi_{i\dot{\kappa}\nu}\big)+D^2\big(\Phi_{i\dot{\kappa}\dot{\nu}}-\bar{\Gamma}_i\Psi_{i\dot{\kappa}\dot{\nu}}\big)\\&-D^3(\Phi_{i\dot{\kappa}\ddot{\nu}}-\bar{\Gamma}_i\Psi_{i\dot{\kappa}\ddot{\nu}})+(A_i\bar{p}_i)_{\nu}-\bar{\Gamma}_i(B_i\bar{p}_i)_{\nu}-D((A_i\bar{p}_i)_{\dot{\nu}}-\bar{\Gamma}_i(B_i\bar{p}_i)_{\dot{\nu}})+D^2\big((A_i\bar{p}_i)_{\ddot{\nu}}-\bar{\Gamma}_i(B_i\bar{p}_i)_{\ddot{\nu}}\big)\\&-D^3\left((A_i\bar{p}_i)_{\dddot{\nu}}-\bar{\Gamma}_i(B_i\bar{p}_i)_{\dddot{\nu}}\right)+D^4\big((A_i\bar{p}_i)_{\ddddot{\nu}}-\bar{\Gamma}_i(B_i\bar{p}_i)_{\ddddot{\nu}}\big)\big\}\Bigg](\nu-\bar{\nu})^T\geq 0,\quad t\in I,\forall\ \nu\in C_1,\end{aligned}\tag{20}$$

$$\begin{aligned}&\sum_{i=1}^{k}\Big\{(\beta_i-\xi\bar{\lambda}_i)\big\{(\Phi_{i\kappa}-D\Phi_{i\dot{\kappa}}-\bar{z}_i)-\bar{\Gamma}_i(\Psi_{i\kappa}-D\Psi_{i\dot{\kappa}}+\bar{r}_i)\big\}+\beta_i\Big\{-\frac{1}{2}\big(\bar{p}_i^TA_ip_i\big)_{\kappa}+\frac{\bar{\Gamma}_i}{2}\big(\bar{p}_i^TB_ip_i\big)_{\kappa}\\&+\frac{1}{2}D\big(\bar{p}_i^TA_i\bar{p}_i\big)_{\dot{\kappa}}-\frac{\bar{\Gamma}_i}{2}D\big(\bar{p}_i^TB_i\bar{p}_i\big)_{\dot{\kappa}}-\frac{1}{2}D^2\big(\bar{p}_i^TA_i\bar{p}_i\big)_{\ddot{\kappa}}+\frac{\bar{\Gamma}_i}{2}D^2\big(\bar{p}_i^TB_i\bar{p}_i\big)_{\ddot{\kappa}}+\frac{1}{2}D^3\big(\bar{p}_i^TA_i\bar{p}_i\big)_{\dddot{\kappa}}\\&-\frac{\bar{\Gamma}_i}{2}D^3\big(\bar{p}_i^TB_i\bar{p}_i\big)_{\dddot{\kappa}}-\frac{1}{2}D^4\big(\bar{p}_i^TA_i\bar{p}_i\big)_{\ddddot{\kappa}}+\frac{\bar{\Gamma}_i}{2}D^4\big(\bar{p}_i^TB_i\bar{p}_i\big)_{\ddddot{\kappa}}\Big\}+(\gamma-\xi\bar{\kappa})^T\bar{\lambda}_i(A_i-\bar{\Gamma}_iB_i+(A_i\bar{p}_i)_{\kappa}\\&-\bar{\Gamma}_i(B_i\bar{p}_i)_{\kappa}-D(A_i\bar{p}_i)_{\dot{\kappa}}+\bar{\Gamma}_iD(B_i\bar{p}_i)_{\dot{\kappa}}+D^2(A_i\bar{p}_i)_{\ddot{\kappa}}-\bar{\Gamma}_iD^2(B_i\bar{p}_i)_{\ddot{\kappa}}-D^3(A_i\bar{p}_i)_{\dddot{\kappa}}+\bar{\Gamma}_iD^3(B_i\bar{p}_i)_{\dddot{\kappa}}\\&+D^4(A_i\bar{p}_i)_{\ddddot{\kappa}}-\bar{\Gamma}_iD^4(B_i\bar{p}_i)_{\ddddot{\kappa}})-\xi(-A_i\bar{p}_i+\bar{\Gamma}_iB_i\bar{p}_i)\big\}=0,\quad t\in I,\end{aligned}\tag{21}$$

$$(\gamma-\xi\bar{\kappa})^T\big\{(\Phi_{i\kappa}-\bar{z}_i)-\bar{\Gamma}_i(\Psi_{i\kappa}+\bar{r}_i)-D(\Phi_{i\dot{\kappa}}-\bar{\Gamma}_i\Psi_{i\dot{\kappa}})+A_i\bar{p}_i-\bar{\Gamma}_iB_i\bar{p}_i\big\}-\eta+\omega e_k=0,\tag{22}$$

$$\alpha_i-\beta_i\left(\Psi_i-\frac{\bar{\Gamma}_i}{2}\bar{p}_i^TB_i\bar{p}_i-s(\bar{\nu}|F)+\kappa^T\bar{r}_i\right)+(\gamma-\xi\bar{\kappa})(-\Psi_{i\kappa}+D\Psi_{i\dot{\kappa}}-B_i\bar{p}_i-\bar{r}_i)=0,\quad t\in I,\tag{23}$$

$$-\beta_i^T(A_i\bar{p}_i-\bar{\Gamma}_iB_i\bar{p}_i)+\bar{\lambda}_i(\gamma-\xi\bar{\kappa})^T(A_i-\bar{\Gamma}_iB_i)=0,\quad t\in I,\tag{24}$$

$$\gamma\sum_{i=1}^{k}\bar{\lambda}_i\big[(\Phi_{i\kappa}-\bar{z}_i)-\bar{\Gamma}_i(\Psi_{i\kappa}+\bar{r}_i)-D(\Phi_{i\dot{\kappa}}-\bar{\Gamma}_i\Psi_{i\dot{\kappa}})+A_i\bar{p}_i-\bar{\Gamma}_iB_i\bar{p}_i\big]=0,\tag{25}$$

$$\xi\bar{\kappa}^T\sum_{i=1}^{k}\bar{\lambda}_i\big((\Phi_{i\kappa}-\bar{z}_i)-\bar{\Gamma}_i(\Psi_{i\kappa}+\bar{r}_i)-D\big(\Phi_{i\dot{\kappa}}-\bar{\Gamma}_i\Psi_{i\dot{\kappa}}\big)+A_i\bar{p}_i-\bar{\Gamma}_iB_i\bar{p}_i\big)=0,\tag{26}$$

$$\bar{\lambda}^T\eta=0,\tag{27}$$

$$\omega\big(\bar{\lambda}^Te_k-1\big)=0,\tag{28}$$

$$s(\bar{\nu}|E_i)=\bar{\nu}^T\bar{w}_i,\ \bar{w}_i\in E_i,\tag{29}$$

$$s(\bar{\nu}|F_i)=\bar{\nu}^T\bar{s}_i,\ \bar{s}_i\in F_i,\tag{30}$$

$$\beta_i\kappa^T+\bar{\lambda}_i[\gamma-\xi\bar{\kappa}]\bar{\lambda}_i\in N_{J_i}(\bar{z}_i),\tag{31}$$

$$\bar{\Gamma}_i\big[\beta_i\kappa^T+\bar{\lambda}_i[\gamma-\xi\bar{\kappa}]\big]\in N_{K_i}(\bar{r}_i),\tag{32}$$

$$(\alpha,\beta,\gamma,\xi,\omega,\eta)\neq 0,\ (\alpha,\beta,\gamma,\xi,\omega,\eta)\geq 0,\ t\in I.\tag{33}$$

Using assumption (ii), in equation (24), it yields

$$\lambda_i(\gamma-\xi\bar{\kappa})=\beta_i\bar{p}_i.\tag{34}$$

Converting (21) into a suitable form, we get

$$\sum_{i}^{k}(\beta_i-\xi\bar{\lambda}_i)((\Phi_{i\kappa}-\bar{z}_i)-\bar{\Gamma}_i(\Psi_{i\kappa}+\bar{r}_i)-D(\Phi_{i\dot{\kappa}}-\bar{\Gamma}_i\Psi_{i\dot{\kappa}})+(A_i-\bar{\Gamma}_iB_i)(\gamma-\xi\bar{\kappa}-\xi\bar{p}_i))$$
$$+((A_i\bar{p}_i)_\kappa-\bar{\Gamma}_i(B_i\bar{p}_i)_\kappa-D(A_i\bar{p}_i)_{\dot{\kappa}}+\bar{\Gamma}_iD(B_i\bar{p}_i)_{\dot{\kappa}}+D^2(A_i\bar{p}_i)_{\ddot{\kappa}}-\bar{\Gamma}_iD^2(B_i\bar{p}_i)_{\ddot{\kappa}}$$
$$-D^3(A_i\bar{p}_i)_{\dddot{\kappa}}+\bar{\Gamma}_iD^3(B_i\bar{p}_i)_{\dddot{\kappa}}+D^4(A_i\bar{p}_i)_{\ddddot{\kappa}}-\bar{\Gamma}_iD^4(B_i\bar{p}_i)_{\ddddot{\kappa}})\left((\gamma-\xi\bar{\kappa})\lambda_i-\frac{1}{2}\beta\bar{p}_i\right)=0.$$

From equation (34), it follows that

$$\sum_{i=1}^{k}(\beta_i-\xi)\lambda_i((\Phi_{i\kappa}-\bar{z}_i)-\bar{\Gamma}_i(\Psi_{i\kappa}+\bar{r}_i)-D(\Phi_{i\dot{\kappa}}-\bar{\Gamma}_i\Psi_{i\dot{\kappa}})+(A_i-\bar{\Gamma}_iB_i)\bar{p}_i)$$
$$+\frac{1}{2}\lambda_i((A_i\bar{p}_i)_\kappa-\bar{\Gamma}_i(B_i\bar{p}_i)_\kappa-D(A_i\bar{p}_i)_{\dot{\kappa}}+\bar{\Gamma}_iD(B_i\bar{p}_i)_{\dot{\kappa}}+D^2(A_i\bar{p}_i)_{\ddot{\kappa}}$$
$$-\bar{\Gamma}_iD^2(B_i\bar{p}_i)_{\ddot{\kappa}}-D^3(A_i\bar{p}_i)_{\dddot{\kappa}}+\bar{\Gamma}_iD^3(B_i\bar{p}_i)_{\dddot{\kappa}}+D^4(A_i\bar{p}_i)_{\ddddot{\kappa}}-\bar{\Gamma}_iD^4(B_i\bar{p}_i)_{\ddddot{\kappa}})(\gamma-\xi\bar{\kappa})^T=0. \quad (35)$$

Premultiplying (35) by $(\gamma-\xi\bar{\kappa})$ and using (25) and (26), the above relation follows that

$$\frac{1}{2}(\gamma-\xi\bar{\kappa})((A_i\bar{p}_i)_\kappa-\bar{\Gamma}_i(B_i\bar{p}_i)_\kappa-D(A_i\bar{p}_i)_{\dot{\kappa}}+\bar{\Gamma}_iD(B_i\bar{p}_i)_{\dot{\kappa}}+D^2(A_i\bar{p}_i)_{\ddot{\kappa}}-\bar{\Gamma}_iD^2(B_i\bar{p}_i)_{\ddot{\kappa}}$$
$$-D^3(A_i\bar{p}_i)_{\dddot{\kappa}}+\bar{\Gamma}_iD^3(B_i\bar{p}_i)_{\dddot{\kappa}}+D^4(A_i\bar{p}_i)_{\ddddot{\kappa}}-\bar{\Gamma}_iD^4(B_i\bar{p}_i)_{\ddddot{\kappa}})=0,$$

as a result of supposition (iv) gives

$$\gamma=\xi\bar{\kappa}. \quad (36)$$

Using (36) in (35), we have

$$(\beta_i-\xi\lambda_i)((\Phi_\kappa-\bar{z}_i)-\bar{\Gamma}(\Psi_\kappa+\bar{r}_i)-D\big(\Phi_{\dot{\kappa}}-\bar{\Gamma}\Psi_{\dot{\kappa}}\big)+\big(A_i-\bar{\Gamma}B_i\big)\bar{p})=0. \quad (37)$$

Using hypothesis (iii), it follows that

$$\beta_i=\xi\bar{\lambda}_i. \quad (38)$$

Since $\bar{\lambda}>0$, and $\eta\geq 0$, (27) implies that $\eta=0$.

Now, if we replace $\xi=0$ in (38), we get, $\beta=0$, this results in $\gamma=0$ on utilising (36). Additionally, we using (23) to obtain $\alpha=0$ and use (36) and $\eta=0$ in (22), we obtain $\omega=0$. Ultimately, we come to $(\alpha,\beta,\gamma,\xi,\omega,\eta)=0,\ t\in I$ contradicting (33). Therefore, we take $\bar{\lambda}_i>0$, $\xi>0,\ t\in I$ and hence $\beta>0$. Given that $\xi>0, t\in I$, and (36) give,

$$\bar{\kappa}=\frac{\gamma}{\xi}\in C_2,\quad t\in I.$$

We derive the following by using the relations (36) and (38) in (20), we find that

$$\sum_{i=1}^{k}\beta_i\big((\Phi_{i\nu}+\bar{w}_i)-\bar{\Gamma}_i(\Psi_{i\nu}-\bar{s}_i)-D(\Phi_{i\dot{\nu}}-\bar{\Gamma}_i\Psi_{i\dot{\nu}})\big)(\nu-\bar{\nu})\geq 0,\quad t\in I,\forall\ \nu\in C_1. \quad (39)$$

As both β_i and $\bar{\lambda}_i$ are greater than zero, $i\in\tilde{N}$ the relation (38) indicates that $\xi>0$. Now, utilising (38) and $\xi>0$, relation (39) reduces to

$$\sum_{i=1}^{k}\lambda_i\Big((\Phi_{i\nu}+\bar{w}_i)-\bar{\Gamma}_i(\Psi_{i\nu}-\bar{s}_i)-D(\Phi_{i\dot{\nu}}-\bar{\Gamma}_i\Psi_{i\dot{\nu}})(\nu-\bar{\nu})\Big)\geq 0,\quad t\in I,\forall\ \nu\in C_1. \quad (40)$$

Suppose $\nu \in C_1$ so that $\nu + \bar{\nu} \in C_1$. If we substitute $\nu + \bar{\nu}$ for ν in (40), we obtain

$$\nu^T \sum_{i=1}^{k} \bar{\lambda}_i \big((\Phi_{i\nu} + \bar{w}_i) - \bar{\Gamma}_i(\Psi_{i\nu} - \bar{s}_i) - D\big(\Phi_{i\dot{\nu}} - \bar{\Gamma}_i \Psi_{i\dot{\nu}}\big)\big) \geq 0, \quad t \in I. \tag{41}$$

This, due to polar cone's characteristics, provides

$$\sum_{i=1}^{k} \bar{\lambda}_i \Big((\Phi_{i\nu} + \bar{w}_i) - \bar{\Gamma}_i(\Psi_{i\nu} - \bar{s}_i) - D(\Phi_{i\dot{\nu}} - \bar{\Gamma}_i \Psi_{i\dot{\nu}})\Big) \in C_1^*, \quad t \in I. \tag{42}$$

Again, in equation (40), if we simultaneously choose $\nu = 0$ and $\nu = 2\bar{\nu}$, we get

$$\bar{\nu}^T \sum_{i=1}^{k} \bar{\lambda}_i \big((\Phi_{i\nu} + \bar{w}_i) - \bar{\Gamma}_i(\Psi_{i\nu} - \bar{s}_i) - D\big(\Phi_{i\dot{\nu}} - \bar{\Gamma}_i \Psi_{i\dot{\nu}}\big)\big) = 0, \quad t \in I. \tag{43}$$

However, based on relation (3) and the knowledge that $\bar{p}_i = 0,\ i \in \tilde{N}$, we have

$$\int_c^d \big(\Phi_i(t, \nu, \dot{\nu}, \kappa, \dot{\kappa}) + s(\bar{\nu}|E_i) - \bar{\kappa}^T \bar{z}_i\big)\, \mathrm{d}t - \Gamma_i \int_c^d \big(\Psi_i(t, \nu, \dot{\nu}, \kappa, \dot{\kappa}) - s(\bar{\nu}|F_i) + \bar{\kappa}^T \bar{r}_i\big)\, \mathrm{d}t = 0. \tag{44}$$

Due to the fact that $\beta > 0$, by (31) and $\gamma = \xi\bar{\kappa}$, we obtain $\bar{\kappa} \in N_{J_i}(\bar{z}_i), i \in \tilde{N}$. This denotes

$$\bar{\kappa}^T \bar{z}_i = s(\bar{\kappa}|J_i), \quad i \in \tilde{N}. \tag{45}$$

Similarly by (32), we have $\bar{\kappa} \in N_{K_i}(\bar{r}_i), i \in \tilde{N}$. This implies

$$\bar{\kappa}^T \bar{r}_i = s(\bar{\kappa}|K_i), \quad i \in \tilde{N}. \tag{46}$$

Combining (29), (30), (44)–(46), we get

$$\int_c^d (\Phi_i(t, \nu, \dot{\nu}, \kappa, \dot{\kappa}) + \bar{\nu}^T \bar{w}_i - s(\bar{\kappa}|J_i))\, \mathrm{d}t - \Gamma_i \int_c^d (\Psi_i(t, \nu, \dot{\nu}, \kappa, \dot{\kappa}) - \bar{\nu}^T \bar{t}_i + s(\bar{\kappa}|K_i))\, \mathrm{d}t = 0. \tag{47}$$

The above equation together with (42) and (43) shows that $(\bar{\nu}, \bar{\kappa}, \bar{\Gamma}, \bar{\lambda}, \bar{w}_1, \bar{w}_2, \ldots, \bar{w}_k, \bar{t}_1, \bar{t}_2, \ldots, \bar{t}_k, \bar{q} = 0)$ is a feasible solution to (NFVD′). Under the assumptions of Theorem 1, if $(\bar{\nu}, \bar{\kappa}, \bar{\Gamma}, \bar{\lambda}, \bar{w}_1, \bar{w}_2, \ldots, \bar{w}_k, \bar{t}_1, \bar{t}_2, \ldots, \bar{t}_k, \bar{q} = 0)$ is not an efficient solution to (NFVD′), then there exists other feasible solution $(\bar{\mu}, \bar{\varsigma}, \bar{\Upsilon}, \bar{\lambda}, \bar{w}_1, \bar{w}_2, \ldots, \bar{w}_k, \bar{t}_1, \bar{t}_2, \ldots, \bar{t}_k, \bar{q})$ to (NFVD′) such that $(\bar{\Upsilon} - \bar{\Gamma}) \in K\backslash\{0\}$. Since $(\bar{\nu}, \bar{\kappa}, \bar{\Gamma}, \bar{\lambda}, \bar{z}_1, \bar{z}_2, \ldots, \bar{z}_k, \bar{r}_1, \bar{r}_2, \ldots, \bar{r}_k, \bar{p})$ is an feasible solution of (NFVP′), by Theorem 1, we have $(\bar{\Upsilon} - \bar{\Gamma}) \notin K\backslash\{0\}$. Therefore, the contradiction suggests that $(\bar{\nu}, \bar{\kappa}, \bar{\Gamma}, \bar{\lambda}, \bar{w}_1, \bar{w}_2, \ldots, \bar{w}_k, \bar{t}_1, \bar{t}_2, \ldots, \bar{t}_k, \bar{q} = 0)$ is an efficient solution of (NFVD′). The proof is now complete. □

A converse duality theorem may be formulated in light of the previous theorem, and its proof will proceed in same manner as the proof of Theorem 4. because the primal-dual pair is symmetric programming problems.

Theorem 5 (Converse Duality). *Consider that all feasible solutions to* (NFVP′) *and* (NFVD′) *satisfy the Weak duality theorem's preconditions. Fix* $\lambda = \bar{\lambda}$. *Let*

(i) $(\bar{\mu}, \bar{\varsigma}, \bar{\Upsilon}, \bar{\lambda}, \bar{w}_1, \bar{w}_2, \ldots, \bar{w}_k, \bar{s}_1, \bar{s}_2, \ldots, \bar{s}_k, \bar{q})$ *be an efficient solution of* (NFVP′),
(ii) *the matrices* $Y_i - \bar{\Upsilon}_i Z_i$ *be non-singular,*
(iii) *the set* $\{(\Phi_{i\nu} - \bar{w}_i) - \bar{\Upsilon}_i(\Psi_{i\nu} + \bar{s}_i) - D\big(\Phi_{i\dot{\nu}} - \bar{\Upsilon}_i \Psi_{i\dot{\nu}}\big) + (Y_i - \bar{\Upsilon}_i Z_i)\bar{q}_i,\ t \in I\}$ *be linearly independent, and*
(iv) *the matrix* $\sum_{i=1}^{k} \lambda_i^0 [(Y_i\bar{q}_i)_\nu - \bar{\Upsilon}_i(Z_i\bar{q}_i)_\nu - D(Y_i\bar{q}_i)_{\dot{\nu}} + \bar{\Upsilon}_i D(Z_i\bar{q}_i)_{\dot{\nu}} + D^2(Y_i\bar{q}_i)_{\ddot{\nu}} - \Upsilon_i D^2(Z_i\bar{q}_i)_{\ddot{\nu}} - D^3(Y_i\bar{q}_i)_{\dddot{\nu}} + \bar{\Upsilon} D^3(Z_i\bar{q}_i)_{\dddot{\nu}} + D^4(Y_i\bar{q}_i)_{\ddddot{\nu}} - \bar{\Upsilon} D^4(Z_i\bar{q}_i)_{\ddddot{\nu}}],\ t \in I,$ *be positive or negative definite.*

Then, $(\bar{\mu}, \bar{\varsigma}, \Upsilon, \bar{\lambda}, \bar{w}_1, \bar{w}_2, \ldots, \bar{w}_k, \bar{s}_1, \bar{s}_2, \ldots, \bar{s}_k, \bar{q} = 0)$ *is an efficient solution of* $(\text{NFVD}')_\lambda$.

4. Conclusion

In this paper, we construct a pair of non-differentiable second-order symmetric multiobjective fractional variational programming with cone constraints, where each component of the objective function contain support function of a compact convex set. The (C, ρ, θ)-convexity/(C, ρ, θ)-pseudo-convexity functions are defined. This research may also be used to variational control over cones and higher order symmetric fractional programming problems. This will be feature task for the researcher.

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