RAIRO-Oper. Res. 58 (2024) 3871–3893
https://doi.org/10.1051/ro/2024080

RAIRO Operations Research
www.rairo-ro.org

ALLOCATING THE FIXED COST AS A COMPLEMENTARY INPUT IN TWO-STAGE SYSTEM: A DEA APPROACH

Xi Jin[1], Feng Li[2], Qiwei Xie[3] and Yongjun Li[1,*]

Abstract. The existing research on fixed cost allocation (FCA) in two-stage system data envelopment analysis (DEA) models typically regards the fixed cost as an additional input for each decision-making unit (DMU). However, these models overlook the prevalent real-world scenario where fixed costs act as complementary inputs in production processes. This study proposes a general two-stage network DEA model that incorporates the fixed cost as a complementary input to optimize the allocation scheme. First, we construct a functional relationship between the efficiency scores of DMUs and their allocated fixed costs using a modified super-efficiency DEA model, which effectively classifies DMUs and solves infeasible solutions within the variable returns to scale (VRS) framework. Then, we propose a fair and unique allocation model based on fairness and efficiency maximization principles. Specifically, we allocate fixed costs based on the operational scale for inelastic DMUs while equitably increasing the efficiency scores for elastic DMUs. Finally, a numerical example and an empirical study on subsidy allocation among 30 provinces demonstrate the rationality and acceptability of our approach.

Mathematics Subject Classification. 90B30.

Received September 13, 2023. Accepted April 2, 2024.

1. Introduction

Fixed cost allocation (FCA) is a prevalent concern in business decision-making and operational management [17]. Once an organization establishes a shared platform, it is necessary to equitably allocate the incurred fixed cost across all departments [2]. For example, a manufacturer allocates advertising expenses among retailers [5], while a central bank charges its branches for platform management fees [40]. An appropriate FCA strategy not only encourages inter-departmental cooperation but also enhances overall operational efficiency [12]. Hence, it is crucial to propose a suitable approach to address the FCA problem.

Data envelopment analysis (DEA), a non-parametric mathematical programming technique proposed by Charnes *et al.* [8], has been widely used in addressing the FCA problem [49]. DEA's strength lies in its ability to evaluate the relative efficiency of homogeneous decision-making units (DMUs) with multiple input-output indicators [8], without subjective weight setting or any assumptions about production functions. Since Cook and Kress [16] first applied the DEA approach to solve the FCA problem, it has catalyzed the development of

Keywords. Data envelopment analysis, fixed cost allocation, two-stage network structures, complementary input.

[1] School of Management, University of Science and Technology of China, Hefei, Anhui 230026, P.R. China.

[2] School of Business Administration, Southwestern University of Finance and Economics, Chengdu, Sichuan 611130, P.R. China.

[3] School of Economics and Management, Beijing University of Technology, Beijing 100124, P.R. China.

[*]Corresponding author: lionli@ustc.edu.cn

various DEA-based allocation models [37]. Current DEA-based FCA research can be broadly categorized into two types based on two main principles. The first category is the efficiency invariance principle proposed by Cook and Kress [16], which means that the efficiency of all DMUs remains unchanged after allocating fixed costs. Building upon this principle, Jahanshahloo *et al.* [29] proposed a model to address the computational challenges of Cook and Kress [16] method. Cook and Zhu [17] extended the work of Cook and Kress [16] and proposed a fair FCA model. Lin [45] contributed theoretical amendments to Cook and Zhu [17] method and provided an approach for setting cost targets based on the fixed cost shared by each DMU. Jahanshahloo *et al.* [30] proposed a FCA method based on the principles of efficiency invariance and common weights. Other works following this principle include Mostafaee [50], Lin and Chen [46], and Li *et al.* [43].

The second category is the efficiency maximization principle, which means that all DMUs achieve DEA efficient after the allocation. Beasley [5] first introduced this principle and used a set of common weights to maximize efficiency across all DMUs. However, the optimal allocation scheme based on this principle may not be unique. Consequently, some scholars have added additional constraints to ensure a unique and optimal allocation scheme. For instance, Si *et al.* [53] proposed a FCA model by minimizing differences in allocation costs among DMUs. Li *et al.* [37] introduced the concept of satisfaction and proposed a max-min method to determine the final FCA scheme. Du *et al.* [24] adopted the cross-efficiency method to allocate fixed costs by progressively improving DMUs' efficiency. Li *et al.* [41] proposed a DEA cooperative game cross-efficiency evaluation method to allocate fixed costs. Dai *et al.* [21] allocated the fixed cost using a DEA non-cooperative game cross-efficiency evaluation method. Chu *et al.* [13] proposed a fair FCA scheme based on the principles of efficiency maximization and individual rationality. Yang *et al.* [59] proposed a FCA model that considers both individual efficiency guarantee and collective preference objectives. Xu *et al.* [57] introduced the concept of fairness concern disutility and allocated the fixed cost from optimistic, pessimistic, and mixed perspectives.

Note that the aforementioned studies focus on single-stage systems, treating DMUs as black boxes, which makes identifying inefficiency sources challenging. In response to this limitation, scholars have extended traditional DEA models to network structures, particularly two-stage systems, to model the internal production processes of DMUs [23]. This two-stage network structure enables the assessment of both the overall system efficiency and the efficiency of each stage [27], making it more applicable in real-life production processes. Several studies have explored the FCA problem in two-stage systems. Yu *et al.* [61] proposed an approach based on a two-stage network DEA model and the concept of cross-efficiency. Zhu *et al.* [65] discussed the FCA problem using a two-stage network DEA model and proposed three procedures to obtain a fair cost allocation scheme based on different objectives. Ding *et al.* [22] proposed a FCA method based on the two-stage model and the concepts of satisfaction and fairness. Nevertheless, the allocation schemes developed in these studies are not unique. To tackle this issue, Li *et al.* [40] considered the operation size of each DMU, while Chu *et al.* [11] proposed a leader-follower model to generate a unique and optimal allocation scheme. An *et al.* [1] proposed a unique FCA plan based on the principle of efficiency invariance for both cooperative and noncooperative game relationships between two sub-stages. Xu *et al.* [56] introduced the concept of fairness into both cooperative and noncooperative modes and explored the FCA problem in a two-stage network structure by maximizing fairness utility through a maximum-minimum model.

In the above FCA studies, the fixed cost is typically treated as an additional input. However, when DMUs already have a cost measure in their performance evaluation system, and the allocated cost complements this cost measure, these methods may become inapplicable [35]. Indeed, the scenario where fixed costs as complementary inputs is common in actual production mechanisms, especially in mature organizations or departments [35]. For example, consider an automobile manufacturer with a two-stage production process. The first stage involves purchasing and distributing parts and materials from suppliers, while the second stage entails assembling these parts into finished vehicles. Both sub-stages require substantial investments in machinery and equipment. Suppose the company invests in new machinery for production, the fixed cost associated with this investment complements the initial equipment expenditure as both expenses impact output equally. Similarly, suppose a cloud computing service headquarters builds a data service center for its affiliated units, the construction cost and management fee will be borne by these units after the platform is established. Obviously, these fixed costs complement the expenses that subsidiaries pay for daily business activities. Moreover, combining these two variables can reduce the number of indicators, thereby improving the discrimination of the DEA model [31].

Currently, some researchers have treated the fixed cost as a supplementary input in a single-stage system. For example, Li *et al.* [35] used a DEA super-efficiency model to discuss the relationship between allocated fixed costs and efficiency scores for each DMU, and proposed a FCA model that maximizes DMUs' efficiency improvement. Dai *et al.* [20] extended Li *et al.* [35]'s work to the assumption of variable scale returns (VRS). Lin and Chen [48] also explored the FCA problem in this situation, and proposed a modified additive DEA model to generate a unique allocation scheme. More recently, Chu *et al.* [12] proposed a dual-objective model aiming to maximize DMUs' efficiency improvement and enhance the balance of FCA results. However, these methods are not applicable to the FCA problem of DMUs with a two-stage network structure in actual production activities.

To bridge these research gaps, we propose a DEA-based allocation approach that considers the fixed cost as a complementary input in a general two-stage network structure. We first investigate the functional relationship between DMUs' efficiency values and their allocated fixed costs under the VRS framework by proposing a modified super-BCC model. We categorize DMUs as either elastic or inelastic based on whether their efficiency values vary with the allocated fixed costs. We then design a reasonable and unique FCA scheme based on fairness and efficiency maximization principles. Specifically, for inelastic DMUs, we allocate the fixed cost of each stage based on their operational scale. However, for all elastic DMUs, we formulate an allocation scheme to equitably increase their efficiency values. Finally, we elucidate and validate our approach through a numerical example and an empirical study.

This study makes three contributions to the FCA research. First, we consider the scenario where fixed costs serve as supplementary inputs in a general two-stage system, which enriches the existing FCA literature. Second, we construct the functional relationship between DMUs' efficiency values and their allocated fixed costs and solve the infeasibility problem of the super-DEA model under the VRS assumption using a modified super-BCC model. Third, we differentiate DMUs and propose a fair and unique FCA model based on their types, which offers decision-makers an appropriate strategy for addressing FCA issues.

The remainder of this paper is organized as follows. In Section 2, we present our proposed FCA models. In Section 3, we elucidate our method through a numerical example. Section 4 conducts an empirical study to verify our approach. Section 5 concludes our study.

2. Materials and methods

In this section, we first introduce a two-stage network DEA model that incorporates the fixed cost as a complementary input. Then, we construct the relationship between DEA efficiency scores of DMUs and their allocated costs. Finally, we propose a fair and unique FCA model. To enhance clarity, Table 1 provides an overview of notations used in this study.

2.1. Efficiency evaluation with allocated costs and two-stage structures

Suppose there are a set of homogeneous DMUs, each $DMU_j (\forall j \in J = \{1, 2, \ldots, n\})$ consumes m inputs $x_{ij} (\forall i \in I = \{1, 2, \ldots, m\})$ to produce s outputs $y_{rj} (\forall r \in S = \{1, 2, \ldots, s\})$.The total fixed cost R is allocated among DMUs, with R_j being allocated to each DMU_j. The relative efficiency score after allocation for any given DMU_d (subscript d denotes the focal DMU) is calculated by solving the following BCC model [4]:

$$E_d^{\star} = \mathrm{Max} \frac{\sum_{r=1}^{s} u_r y_{rd} + u_0}{\sum_{i=1}^{m} v_i x_{id} + v_0 R_d}$$

TABLE 1. An overview of notations.

Symbol	Meaning
Parameters:	
n	Number of DMUs
m, q, l, s	Number of first-stage inputs, intermediate outputs, second-stage inputs and outputs
s_1, s_2	Number of outputs in the first and second stage, respectively
i, r, j	Indexes of inputs, outputs, and DMUs, respectively
p, t, f, g	Indexes of intermediate outputs, second-stage inputs, first and second-stage outputs
R	The total sum of allocated costs
x_{ij}	The i^{th} observed input of DMU_j
y_{rj}	The r^{th} observed output of DMU_j
z_{pj}	The p^{th} observed intermediate output of DMU_j
h_{tj}	The t^{th} observed input of DMU_j in the second stage
y^1_{fj}	The f^{th} observed output of DMU_j in the first stage
y^2_{gj}	The g^{th} observed output of DMU_j in the second stage
R_j	Cost allocated to DMU_j
R_{1j}, R_{2j}	Cost allocated to the first and second stages of DMU_j, respectively
M	A relatively large number
α_j, β_j	The operation scale parameters for the first and second stages, respectively
Decision variables:	
u_r, v_i	The weights assigned to the r^{th} output and i^{th} input, respectively
φ_p, ω_t	The weights assigned to the p^{th} intermediate output and t^{th} second-stage input
u^1_f, u^2_g	The weights assigned to the f^{th} first-stage output and g^{th} second-stage output
$\sigma, \eta, \lambda_j, \gamma_j$	The variables of super-BCC model
$\overline{E}^{TSB}_j, \underline{E}^{TSB}_j$	The maximum and minimum original super-BCC efficiency values, respectively
$\overline{E}^{\star}_j, \underline{E}^{\star}_j$	The maximum and minimum efficiency values, respectively
δ_j	The satisfaction degree of DMU_j
s^+_j, s^-_j	Slack variables

$$\text{s.t.} \quad \frac{\sum\limits_{r=1}^{s} u_r y_{rj} + u_0}{\sum\limits_{i=1}^{m} v_i x_{ij} + v_0 R_j} \leq 1, \forall j \in J \tag{1}$$

$$\sum_{j=1}^{n} R_j = R$$

$$u_r, v_i \geq 0, v_0 > 0, R_j \geq 0, \forall r \in S, \forall i \in I, \forall j \in J, u_0 for free$$

where u_r and v_i are unknown weights of the r^{th} output and i^{th} input, respectively. v_0 is a positive weight assigned to the fixed cost R_j for DMU_j. Note that $v_0 > 0$ means the fixed cost R_j should be considered in efficiency evaluation. The constraint $\sum_{j=1}^{n} R_j = R$ with $R_j \geq 0$ implies that the total fixed costs borne by all DMUs must equal R, and the fixed cost amount R_j allocated to each DMU_j is a non-negative number. The optimal objective function $E^{\star}_d$ is defined as the BCC efficiency score of DMU_d after allocating the fixed cost $R^{\star}_d$. $E^{\star}_d$ ranges from 0 to 1, with $E^{\star}_d = 1$ indicating that DMU_d is efficient. The parameter u_0 is used to disclose the returns to scale property [20]. Specifically, $u_0 > 0$, $u_0 = 0$ and $u_0 < 0$ represent increasing, constant and decreasing returns to scale, respectively. We assume variable returns to scale (VRS) for all models.

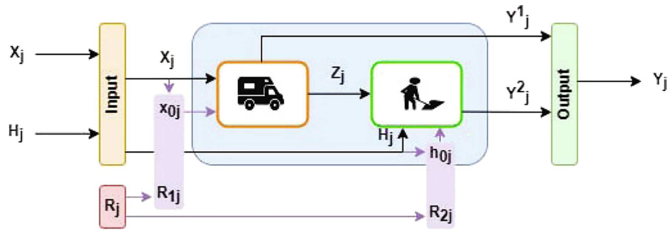

FIGURE 1. Two-stage network process of DMU_j.

Model (1) regards DMUs as black boxes without modeling their internal systems. Various network structures have been proposed to model DMUs' internal operational processes, including series, parallel and hybrid structures [6]. Yet, the two-stage system is most commonly used [19]. In this structure, inputs are processed by the first subsystem to produce intermediate outputs, which are then fed into the second subsystem to generate the final outputs of the whole system, Herein, we adopt a general two-stage network structure as described in Figure 1.

In Figure 1, the allocation cost R_j of each DMU_j is assigned between the first stage with R_{1j} and the second stage with R_{2j}. In the first stage, the original inputs of DMU_j are x_{ij} and x_{0j} that needs to be put together with the fixed cost R_{1j}. The outputs of the first stage include intermediate variables $z_{pj}(\forall p \in P = \{1, \ldots, q\})$ and outputs $y^1_{fj}(\forall f \in F = \{1, \ldots, s_1\})$, where outputs y^1_{fj} are no longer added to the production process. In the second stage, all intermediate outputs z_{pj} and additional inputs $h_{tj}(\forall t \in T = \{1, \ldots, l\})$ are used to produce final outputs $y^2_{gj}(\forall g \in G = \{1, \ldots, s_2\})$, with input h_{0j} being combined with allocated cost R_{2j}.

According to Chen *et al.* [10], the overall efficiency scores E_d of DMU_d can be obtained by weighting the efficiency of each sub-stage, defined as $\pi_1 E_{1d} + \pi_2 E_{2d}$. Here, E_{1d} and E_{2d} are the efficiency values of each stage of DMU_d, π_1 and π_2 represent the proportion of resources consumed in each sub-stage [9]. Specifically:

$$\pi_1 = \frac{\sum_{i=1}^{m} v_i x_{id} + v_0(x_{0d} + R_{1d})}{\sum_{i=1}^{m} v_i x_{id} + \sum_{p=1}^{q} \varphi_p z_{pd} + \sum_{t=1}^{l} \omega_t h_{td} + v_0(x_{0d} + h_{0d} + R_{1d} + R_{2d})}$$

$$\pi_2 = \frac{\sum_{p=1}^{q} \varphi_p z_{pd} + \sum_{t=1}^{l} \omega_t h_{td} + v_0(h_{0d} + R_{2d})}{\sum_{i=1}^{m} v_i x_{id} + \sum_{p=1}^{q} \varphi_p z_{pd} + \sum_{t=1}^{l} \omega_t h_{td} + v_0(x_{0d} + h_{0d} + R_{1d} + R_{2d})}.$$

Therefore, we present the following model to calculate the overall efficiency of DMU_d.

$$E^{\star}_d = \text{Max} \frac{\sum_{p=1}^{q} \varphi_p z_{pd} + \sum_{f=1}^{s_1} u^1_f y^1_{fd} + \sum_{g=1}^{s_2} u^2_g y^2_{gd} + \varphi_0 + u_0}{\sum_{i=1}^{m} v_i x_{id} + \sum_{p=1}^{q} \varphi_p z_{pd} + \sum_{t=1}^{l} \omega_t h_{td} + v_0(x_{0d} + h_{0d} + R_{1d} + R_{2d})}$$

$$\text{s.t.} \quad E_{1j} = \frac{\sum_{p=1}^{q} \varphi_p z_{pj} + \sum_{f=1}^{s_1} u^1_f y^1_{fj} + \varphi_0}{\sum_{i=1}^{m} v_i x_{ij} + v_0(x_{0j} + R_{1j})} \leq 1, \forall j \in J$$

$$
\begin{aligned}
& E_{2j} = \frac{\sum_{g=1}^{s_2} u_g^2 y_{gj}^2 + u_0}{\sum_{p=1}^{q} \varphi_p z_{pj} + \sum_{t=1}^{l} \omega_t h_{tj} + v_0(h_{0j} + R_{2j})} \leq 1, \forall j \in J \\
& \sum_{j=1}^{n} (R_{1j} + R_{2j}) = R \\
& u_f^1, u_g^2, \varphi_p, v_i, \omega_t \geq 0, \forall f \in F, \forall g \in G, \forall p \in P, \forall i \in I, \forall t \in T \\
& R_{1j}, R_{2j} \geq 0, v_0 > 0, \forall j \in J, \varphi_0, u_0 for free.
\end{aligned}
\tag{2}
$$

In Model (2), v_i, φ_p, u_f^1, ω_t, u_g^2 are unknown weights of each input and output, where φ_p links each stage together. Additionally, the allocated costs of each stage R_{1j} and R_{2j} should be combined with the inputs x_{0j} and h_{0j} to participate in production activities, respectively. v_0 is a common weight for allocated costs and inputs since they play similar roles in production activities. E_{1j} and E_{2j} are the efficiency values of DMU_j in the first and second stage, correspondingly. $E_d^\star$ is the overall efficiency of the DMU_d being evaluated.

Model (2) is a fractional programming that can be transformed into a linear Model (3) by C-C transformation [7]. Let $\tau = 1/[\sum_{i=1}^{m} v_i x_{id} + \sum_{p=1}^{q} \varphi_p z_{pd} + \sum_{t=1}^{l} \omega_t h_{td} + v_0(x_{0d} + h_{0d} + R_{1d} + R_{2d})]$, $\mu_f^1 = \tau u_f^1$, $\mu_g^2 = \tau u_g^2$, $\phi_p = \tau \varphi_p$, $\upsilon_i = \tau v_i$, $\upsilon_0 = \tau v_0$, $w_t = \tau \omega_t$, $\mu_0 = \tau u_0$, $\phi_0 = \tau \varphi_0$. Besides, let $r_{1j} = \upsilon_0 R_{1j}$ and $r_{2j} = \upsilon_0 R_{2j}$, then Model (2) can be rewritten as Model (3).

$$
\begin{aligned}
E_d^\star = & \mathrm{Max} \sum_{p=1}^{q} \phi_p z_{pd} + \sum_{f=1}^{s_1} \mu_f^1 y_{fd}^1 + \sum_{g=1}^{s_2} \mu_g^2 y_{gd}^2 + \phi_0 + \mu_0 \\
\text{s.t.} \quad & \sum_{p=1}^{q} \phi_p z_{pj} + \sum_{f=1}^{s_1} \mu_f^1 y_{fj}^1 - \sum_{i=1}^{m} \upsilon_i x_{ij} - \upsilon_0 x_{0j} + \phi_0 - r_{1j} \leq 0, \forall j \in J \\
& \sum_{g=1}^{s_2} \mu_g^2 y_{gj}^2 - \sum_{p=1}^{q} \phi_p z_{pj} - \sum_{t=1}^{l} w_t h_{tj} - \upsilon_0 h_{0j} + \mu_0 - r_{2j} \leq 0, \forall j \in J \\
& \sum_{i=1}^{m} \upsilon_i x_{id} + \sum_{p=1}^{q} \phi_p z_{pd} + \sum_{t=1}^{l} w_t h_{td} + \upsilon_0(x_{0d} + h_{td}) + r_{1d} + r_{2d} = 1 \\
& \sum_{j=1}^{n} (r_{1j} + r_{2j}) = \upsilon_0 R \\
& \mu_f^1, \mu_g^2, \phi_p, \upsilon_i, w_t \geq 0, \forall f \in F, \forall g \in G, \forall p \in P, \forall i \in I, \forall t \in T \\
& r_{1j}, r_{2j} \geq 0, \upsilon_0 > 0, \forall j \in J, \phi_0, \mu_0 for free.
\end{aligned}
\tag{3}
$$

For DMU_d, the optimal solution $(\mu_f^{1\star}, \mu_g^{2\star}, \phi_p^\star, \upsilon_i^\star, w_t^\star, r_{1j}^\star, r_{2j}^\star, \upsilon_0^\star, \phi_0^\star, \mu_0^\star)$ can be obtained by solving Model (3). The post-allocation relative efficiency $E_d^\star$ equals $\sum_{p=1}^{q} \phi_p^\star z_{pd} + \sum_{f=1}^{s_1} \mu_f^{1\star} y_{fd}^1 + \sum_{g=1}^{s_2} \mu_g^{2\star} y_{gd}^2 + \phi_0^\star + \mu_0^\star$. Then a feasible allocation scheme $(R_{1j}^\star = \frac{r_{1j}^\star}{\upsilon_0^\star}, R_{2j}^\star = \frac{r_{2j}^\star}{\upsilon_0^\star}, R_j^\star = R_{1j}^\star + R_{2j}^\star, \forall j \in J)$ by maximizing the efficiency value of DMU_d can be obtained from DMU_d's perspective.

2.2. Relationship between DEA efficiency and cost allocation

Before proposing our allocation method, we examine the relationship between DEA efficiency and allocated cost. Suppose a fixed cost k_d for DMU_d, with k_{1d} and k_{2d} for each stage, where k_d ranges from 0 to R and $k_d = k_{1d} + k_{2d}$. We introduce these parameters into Model (3) to calculate efficiency scores at different k_d levels. Clearly, DMU_d exhibits both maximum and minimum efficiency values within the interval $[0, R]$, neither of which exceeds 1. This implies that if the minimum efficiency value of DMU_d equals 1, its maximum efficiency value is also 1. In this case, one may infer that the efficiency of DMU_d remains constant despite cost allocation variations. Consequently, the BCC model fails to distinguish the impact of allocated cost on efficiency scores for efficient DMUs. To address this issue, we employ the super-BCC model, which yields efficiency scores over 1 by excluding the evaluated DMU_d from the reference set. For inefficient DMUs, the super-BCC model and the BCC model evaluate identical efficiency values. The original super-BCC model [3] is expressed as follows:

$$
\begin{aligned}
E_d^{TSB}(k_d) = \text{Max} & \sum_{p=1}^{q}\phi_p z_{pd} + \sum_{f=1}^{s_1}\mu_f^1 y_{fd}^1 + \sum_{g=1}^{s_2}\mu_g^2 y_{gd}^2 + \phi_0 + \mu_0 \\
\text{s.t.} \quad & \sum_{p=1}^{q}\phi_p z_{pj} + \sum_{f=1}^{s_1}\mu_f^1 y_{fj}^1 - \sum_{i=1}^{m}\upsilon_i x_{ij} - \upsilon_0 x_{0j} + \phi_0 - r_{1j} \le 0, \forall j \in J \setminus \{d\} \\
& \sum_{g=1}^{s_2}\mu_g^2 y_{gj}^2 - \sum_{p=1}^{q}\phi_p z_{pj} - \sum_{t=1}^{l} w_t h_{tj} - \upsilon_0 h_{0j} + \mu_0 - r_{2j} \le 0, \forall j \in J \setminus \{d\} \\
& \sum_{i=1}^{m}\upsilon_i x_{id} + \sum_{p=1}^{q}\phi_p z_{pd} + \sum_{t=1}^{l} w_t h_{td} + \upsilon_0(x_{0d} + h_{0d}) + r_{1d} + r_{2d} = 1 \\
& \sum_{j=1}^{n}(r_{1j} + r_{2j}) = \upsilon_0 R \\
& r_{1d} = \upsilon_0 \times k_{1d}, r_{2d} = \upsilon_0 \times k_{2d} \\
& \mu_f^1, \mu_g^2, \phi_p, \upsilon_i, w_t \ge 0, \forall f \in F, \forall g \in G, \forall p \in P, \forall i \in I, \forall t \in T \\
& r_{1j}, r_{2j} \ge 0, \upsilon_0 > 0, \forall j \in J, \phi_0, \mu_0 for free.
\end{aligned} \tag{4}
$$

The distinction between Models (4) and (3) is the exclusion of the evaluated DMU_d from the constraint conditions, denoted as $DMU_j(\forall j \in J \setminus \{d\})$. By setting k_d as a parameter within the interval $[0, R]$ and incorporating it into Model (4), we can obtain the efficiency value $E_d^{TSB}(k_d)$ for each k_d.

However, Model (4) may face infeasibility issues when evaluating efficient DMUs [52]. In order to investigate the relationship between allocated costs and efficiency values, we must address the infeasibility problem of Model (4). To deal with the potential infeasibility issues of the super-efficiency DEA model under the VRS assumption, Cook *et al.* [18] modified the original super-efficiency DEA model by switching all extreme points causing infeasibility into non-extreme projections. Subsequently, Dai *et al.* [20] extended this method to the single-stage FCA model. Drawing inspiration from their approaches, we propose the following Model (5) based on a two-stage system by treating allocated cost as a complementary input.

$$
\begin{aligned}
\text{Min} \quad & \sigma + M \times \eta \\
\text{s.t.} \quad & \sum_{\substack{j=1 \\ j\neq d}}^{n}\lambda_j y_{fj}^1 \ge (1-\eta) y_{fd}^1, \forall f \in F \\
& \sum_{\substack{j=1 \\ j\neq d}}^{n}\gamma_j y_{gj}^2 \ge (1-\eta) y_{gd}^2, \forall g \in G \\
& \sum_{\substack{j=1 \\ j\neq d}}^{n}(\gamma_j - \lambda_j) z_{pj} \le \sigma z_{pd}, \forall p \in P \\
& \sum_{\substack{j=1 \\ j\neq d}}^{n}\lambda_j x_{ij} \le (1+\sigma) x_{id}, \forall i \in I \\
& \sum_{\substack{j=1 \\ j\neq d}}^{n}\lambda_j (x_{0j} + R_{1j}) + \sum_{\substack{j=1 \\ j\neq d}}^{n}\gamma_j (h_{0j} + R_{2j}) \le (1+\sigma)(x_{0d} + h_{0d} + R_d)
\end{aligned} \tag{5}
$$

$$\sum_{\substack{j=1\\j\neq d}}^{n}\gamma_j h_{tj} \leq (1+\sigma)h_{td}, \forall t \in T$$
$$\sum_{\substack{j=1\\j\neq d}}^{n}\lambda_j = 1, \sum_{\substack{j=1\\j\neq d}}^{n}\gamma_j = 1$$
$$\sum_{j=1}^{n}(R_{1j}+R_{2j}) = R$$
$$\lambda_j, \gamma_j, \eta \geq 0, \forall j \in J \setminus \{d\}$$
$$R_{1j},\ R_{2j} \geq 0, \forall j \in J, \sigma for free.$$

In Model (5), σ, η, λ_j and γ_j are unknown variables, and M is a relatively large number usually set to 10^5. The optimal solution is denoted as $(\sigma^\star, \eta^\star, \lambda_j^\star, \gamma_j^\star)$. The super-BCC efficiency score of DMU_d is defined as $E_d^\star(k_d) = \sigma^\star + \frac{1}{1-\eta^\star}$ when the allocated cost is k_d.

Model (5) is a nonlinear programming due to the product of two variables (*i.e.*, $\lambda_j R_{1j}$, $\gamma_j R_{2j}$). To linearize Model (5), we transform the variables R_{1j} and R_{2j} into a set of parameters. Consequently, we formulate Model (6) to identify a feasible allocation plan when DMU_d bears a cost allocation of k_d. Model (6) is presented as follows (see Appendix A for the derivation process).

$$\begin{aligned}
\text{Max}\quad & \sum_{f=1}^{s_1}\mu_f^1 y_{fd}^1 + \sum_{g=1}^{s_2}\mu_g^2 y_{gd}^2 - \sum_{i=1}^{m}\upsilon_i x_{id} - \sum_{t=1}^{l}\omega_t h_{td} - \upsilon_0(x_{0d}+h_{0d}) - r_d + \phi_0 + \mu_0 \\
\text{s.t.}\quad & \sum_{f=1}^{s_1}\mu_f^1 y_{fj}^1 + \sum_{p=1}^{q}\phi_p z_{pj} - \sum_{i=1}^{m}\upsilon_i x_{ij} - \upsilon_0 x_{0j} - r_{1j} + \phi_0 \leq 0, \forall j \in J \setminus \{d\} \\
& \sum_{g=1}^{s_2}\mu_g^2 y_{gj}^2 - \sum_{p=1}^{q}\phi_p z_{pj} - \sum_{t=1}^{l}\omega_t h_{tj} - \upsilon_0 h_{0j} - r_{2j} + \mu_0 \leq 0, \forall j \in J \setminus \{d\} \\
& \sum_{p=1}^{q}\phi_p z_{pd} + \sum_{i=1}^{m}\upsilon_i x_{id} + \sum_{t=1}^{l}\omega_t h_{td} + \upsilon_0(x_{0d}+h_{0d}) + r_d = 1 \\
& \sum_{f=1}^{s_1}\mu_f^1 y_{fd}^1 + \sum_{g=1}^{s_2}\mu_g^2 y_{gd}^2 \leq M \\
& \sum_{j=1}^{n}(r_{1j}+r_{2j}) = \upsilon_0 R \\
& r_{1d} = \upsilon_0 \times k_{1d}, r_{2d} = \upsilon_0 \times k_{2d} \\
& \mu_f^1, \mu_g^2, \phi_p, \upsilon_i, \omega_t \geq 0, \forall f \in F, \forall g \in G, \forall p \in P, \forall i \in I, \forall t \in T \\
& r_{1j}, r_{2j} \geq 0, \upsilon_0 > 0, \forall j \in J, \phi_0, \mu_0 for free.
\end{aligned} \tag{6}$$

When the fixed cost k_d is allocated to DMU_d, we can obtain an optimal allocation scheme $(R_{1d} = k_{1d}, R_{2d} = k_{2d}, R_{1j}^\star = \frac{r_{1j}^\star}{\upsilon_0^\star}, R_{2j}^\star = \frac{r_{2j}^\star}{\upsilon_0^\star}, \forall j \in J \setminus \{d\})$ from the perspective of DMU_d by solving Model (6). Then we substitute this allocation scheme into Model (5) to get the optimal solution $(\sigma^\star, \eta^\star, \lambda_j^\star, \gamma_j^\star)$. Based on the definition of efficiency values $E_d^\star(k_d)$, we can obtain the efficiency value corresponding to the fixed cost k_d. The maximum and minimum efficiency values are denoted as $\overline{E}_j^\star$ and $\underline{E}_j^\star$. This leads us to derive Theorem 1.

Theorem 1. *$E_d^\star(k_d)$ is a monotone non-increasing function of k_d, $k_d \in [0, R]$.*

Proof. See Appendix B for details. □

Theorem 1 establishes a monotone non-increasing relationship between DMUs' allocated costs and efficiency scores. This suggests DMUs must compete with each other to obtain as little cost as possible for maximizing their efficiency values. To describe this relationship, we introduce the following two definitions.

Definition 1. *If the super-BCC efficiency value of DMU_d remains constant as the allocated cost k_d increases, then DMU_d is considered inelastic. The set of inelastic DMUs is defined as* $J_{\text{inelastic}} = \{j|\underline{E}_j^\star = \overline{E}_j^\star, \forall j \in J\}$.

Definition 2. *If the super-BCC efficiency value of DMU_d varies with the allocated cost k_d, then DMU_d is considered elastic. The set of elastic DMUs is defined as* $J_{\text{elastic}} = \{j|\underline{E}_j^\star < \overline{E}_j^\star, \forall j \in J\}$.

2.3. Models for fixed cost allocation of two-stage network structures

By examining how efficiency values relate to allocated fixed cost, we can classify DMUs based on the aforementioned definitions. In this sub-section, we propose a FCA approach to obtain a reasonable and unique allocation scheme. In the case of inelastic DMUs, their efficiency values keep constant regardless of the allocated fixed cost. While these DMUs can bear more fixed costs without affecting their own efficiency values, it is important to consider their operation scale to ensure fairness. Logically, DMUs are more likely to endorse a scheme that allocates higher costs to those with larger operation sizes and lower costs to those with smaller scales. The operation scale parameters [40], measured by current input usages and output productions in each stage, are formulated as follows:

$$\alpha_j = \frac{(\sum\limits_{p=1}^{q} \hat{z}_{pj} + \sum\limits_{f=1}^{s_1} \hat{y}_{fj}^1) \cdot \sum\limits_{i=1}^{m} \hat{x}_{ij}}{\sum\limits_{j=1}^{n} (\sum\limits_{p=1}^{q} \hat{z}_{pj} + \sum\limits_{f=1}^{s_1} \hat{y}_{fj}^1) \cdot \sum\limits_{i=1}^{m} \hat{x}_{ij} + \sum\limits_{j=1}^{n} (\sum\limits_{p=1}^{q} \hat{z}_{pj} + \sum\limits_{t=1}^{l} \hat{h}_{tj}) \cdot \sum\limits_{g=1}^{s_2} \hat{y}_{gj}^2} \tag{7}$$

$$\beta_j = \frac{(\sum\limits_{p=1}^{q} \hat{z}_{pj} + \sum\limits_{t=1}^{l} \hat{h}_{tj}) \cdot \sum\limits_{g=1}^{s_2} \hat{y}_{gj}^2}{\sum\limits_{j=1}^{n} (\sum\limits_{p=1}^{q} \hat{z}_{pj} + \sum\limits_{f=1}^{s_1} \hat{y}_{fj}^1) \cdot \sum\limits_{i=1}^{m} \hat{x}_{ij} + \sum\limits_{j=1}^{n} (\sum\limits_{p=1}^{q} \hat{z}_{pj} + \sum\limits_{t=1}^{l} \hat{h}_{tj}) \cdot \sum\limits_{g=1}^{s_2} \hat{y}_{gj}^2} \tag{8}$$

where $\sum_{j=1}^{n}(\alpha_j + \beta_j) = 1$. The normalized input-output variables are denoted as $\hat{x}_{ij} = \frac{x_{ij}}{\sum_{j=1}^{n} x_{ij}}$, $\hat{z}_{pj} = \frac{z_{pj}}{\sum_{j=1}^{n} z_{pj}}$, $\hat{y}_{fj}^1 = \frac{y_{fj}^1}{\sum_{j=1}^{n} y_{fj}^1}$, $\hat{h}_{tj} = \frac{h_{tj}}{\sum_{j=1}^{n} h_{tj}}$, $\hat{y}_{gj}^2 = \frac{y_{gj}^2}{\sum_{j=1}^{n} y_{gj}^2}$, $\forall i, p, f, t, g, j$.

For elastic DMUs, there exists a one-to-one correspondence between their efficiency and allocated fixed cost, and the function curves are monotonically decreasing. These DMUs aim to minimize their cost allocation to maximize their own efficiency. However, it is unattainable due to the fixed total cost allocation. Hence, an allocation principle is required to find the optimal efficiency value within the efficiency range. Khodabakhshi and Aryavash [34] suggested that a fair allocation scheme should ensure equal satisfaction for each DMU with the final allocation plan. We adopt the concept of satisfaction proposed by Li *et al.* [37] and define the satisfaction of elastic DMUs with the optimal scheme as:

$$\delta_j = \frac{E_j^\star(R_j) - \underline{E}_j^\star}{\overline{E}_j^\star - \underline{E}_j^\star}, j \in J_{\text{elastic}} \tag{9}$$

where $\delta_j \in [0, 1]$, $E_j^\star(R_j)$ is the efficiency function of DMU_j, $\overline{E}_j^\star$ and $\underline{E}_j^\star$ are the maximum and minimum efficiency values, respectively. Following the fairness principle, each elastic DMU is equally satisfied with the optimal scheme. The FCA model is formulated as follows:

$$
\begin{aligned}
&\text{Max} \quad \delta \\
\text{s.t.} \quad &\delta = \frac{E_j^\star(R_j) - \underline{E}_j^\star}{\overline{E}_j^\star - \underline{E}_j^\star}, j \in J_{\text{elastic}} \\
&R_{1j} = \alpha_j R, R_{2j} = \beta_j R, j \in J_{\text{inelastic}} \\
&\sum_{j=1}^{n} (R_{1j} + R_{2j}) = R \\
&0 \le \delta \le 1, R_{1j}, R_{2j} \ge 0, \forall j \in J
\end{aligned} \tag{10}
$$

where α_j and β_j are the operation scale parameters for the first and second stages, respectively. δ is an unknown parameter within the $[0, 1]$ interval. The maximum super-BCC efficiency $\overline{E}_j^\star$ and minimum super-BCC efficiency $\underline{E}_j^\star$ are obtained by solving Model (5) and Model (6). $E_j^\star(R_j)$ is the super-BCC efficiency function corresponding to R_j. In Model (10), each inelastic DMU is allocated a fixed cost of $\alpha_j R$ in the first stage, and $\beta_j R$ in the second stage. Meanwhile, elastic DMUs efficiency scores are equitably maximized.

Next, we outline the process for formulating the optimal FCA scheme in Algorithm 1.

Step 1. Initialize $l = 0, J_{\text{elastic}}^0 = J = \{1, 2, \ldots, n\}, J_{\text{inelastic}}^0 = \emptyset$, and $R^0 = R$.

Step 2. Suppose the allocated cost of the evaluated $DMU_d(d \in J_{\text{elastic}}^1)$ at each stage is k_{1d} and k_{2d}. By solving Model (6), we can obtain an optimal allocation scheme $(R_{1d} = k_{1d}, R_{2d} = k_{2d}, R_{1j}^\star = \frac{r_{1j}^\star}{v_0^\star}, R_{2j}^\star = \frac{r_{2j}^\star}{v_0^\star}, \forall j \in J_{\text{elastic}}^l \setminus \{d\})$. We then substitute it into Model (5) to obtain the efficiency value $E_d^\star(k_d)$, the maximum efficiency value $\overline{E}_j^\star$ and minimum efficiency value $\underline{E}_j^\star$ for all DMUs. Based on Definitions 1 and 2, we divide the set J_{elastic}^l into J_{elastic} and $J_{\text{inelastic}}$. If $J_{\text{inelastic}} = \emptyset$, then go to Step 3. Otherwise, let $l = l + 1, J_{\text{elastic}}^l = J_{\text{elastic}}, J_{\text{inelastic}}^l = J_{\text{inelastic}}^{l-1} + J_{\text{inelastic}}, R^l = R - \sum_{j \in J_{\text{inelastic}}^l} R_j$, then go back to Step 2 with renewed J_{elastic}^l and R^l.

Step 3. Solving Model (11) to obtain the optimal solution $R_j, \forall j \in J$.

$$
\begin{aligned}
&\text{Max} \quad \delta \\
\text{s.t.} \quad &\delta = \frac{E_j^\star(R_j) - \underline{E}_j^\star}{\overline{E}_j^\star - \underline{E}_j^\star}, \forall j \in J_{\text{elastic}}^l \\
&R_{1j} = \alpha_j R, R_{2j} = \beta_j R, \forall j \in J_{\text{inelastic}}^l \\
&\sum_{j \in J_{\text{elastic}}^l} (R_{1j} + R_{2j}) = R^l \\
&0 \le \delta \le 1, R_{1j}, R_{2j} \ge 0, \forall j \in J.
\end{aligned} \tag{11}
$$

The flowchart for Algorithm 1 is depicted in Figure 2.

By searching δ from 0 to 1 with a step size, we can obtain an allocation plan. To quickly solve Model (11), we employ the binary search algorithm. This algorithm, a commonly used rapid search technique, continually partitions the solution space and excludes half of the space with each iteration until a solution is found or the solution space is sufficiently reduced. Its time complexity is $O(logn)$, where n represents the number of elements [51]. The procedural steps of Algorithm 2 are outlined as follows:

Step 1. Initialize $\delta_1 = 0, \delta_2 = 1$.

Step 2. Let $\delta = \frac{\delta_1 + \delta_2}{2}$. The efficiency value $E_j^\star(R_j) = \delta(\overline{E}_j^\star - \underline{E}_j^\star) + \underline{E}_j^\star$ can be calculated for $DMU_j(j \in J_{\text{elastic}}^l)$. We can obtain $R_j(j \in J_{\text{elastic}}^l)$ based on the correspondence between the cost allocation R_j and the efficiency value $E_j^\star(R_j)$.

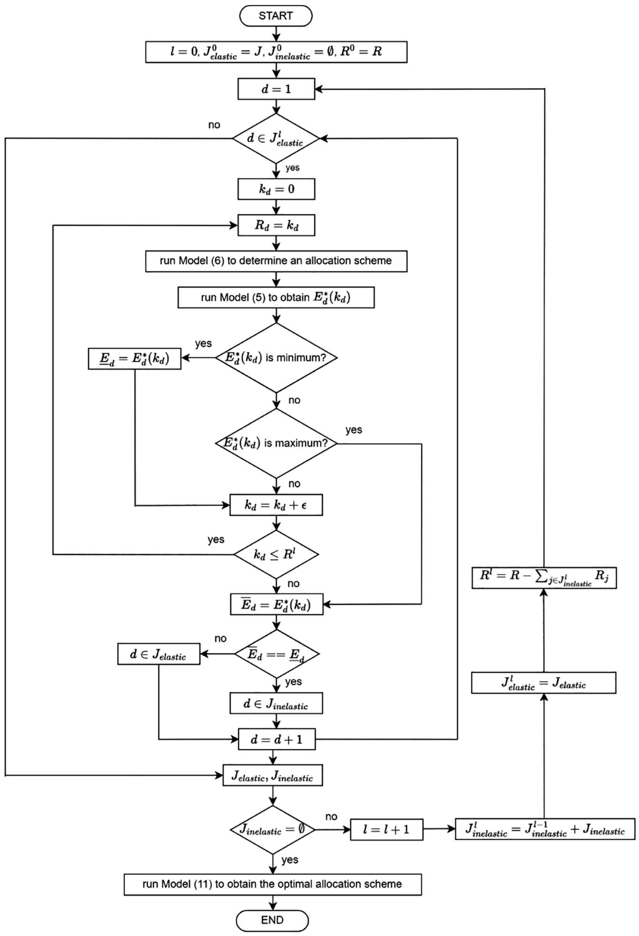

FIGURE 2. Flowchart of Algorithm 1.

Step 3. If $|\sum_{j\in J^l_{\text{elastic}}} R_j - R^l| \le \varepsilon$ (ε is a small positive number, we set $\varepsilon = 10^{-5}$ in this study), then algorithm terminates. Otherwise, if $\sum_{j\in J^l_{\text{elastic}}} R_j > R^l$, let $\delta_1 = \delta$, then go back to Step 2, if $\sum_{j\in J^l_{\text{elastic}}} R_j < R^l$, let $\delta_2 = \delta$, then go back to Step 2.

Theorem 2. *The optimal solution of Model (11) used in Algorithm 1 is unique.*

Proof. See Appendix C for details. □

After allocating a unique allocation plan R_j for each elastic DMU, we proceed to allocate the fixed cost for their two subsystems. This is because DMUs' efficiency scores remain consistent regardless of how the cost is allocated between its two stages. Considering differences between the two subsystems, we introduce the concept of leader-follower [11]. For instance, suppose the first stage is the leader, decision-makers may prioritize its efficiency by maximizing its efficiency value while limiting that of the second stage. With this assumption, we propose the following model to allocate the fixed cost in each stage.

$$
\begin{aligned}
&\text{Min} \quad \sum_{j \in J^l_{\text{elastic}}} s^+_j + s^-_j \\
&\text{s.t.} \quad \frac{\sum_{p=1}^{q} \varphi_p z_{pj} + \sum_{f=1}^{s_1} \mu^1_f y^1_{fj} + \sum_{g=1}^{s_2} \mu^2_g y^2_{gj} + \varphi_0 + \mu_0}{\sum_{i=1}^{m} \upsilon_i x_{ij} + \sum_{p=1}^{q} \varphi_p z_{pj} + \sum_{t=1}^{l} \omega_t h_{tj} + \upsilon_0(x_{0j} + h_{0j} + R_j)} \leq E^\star_j, \forall j \in J^l_{\text{elastic}} \\
&E_{1j} = \frac{\sum_{p=1}^{q} \varphi_p z_{pj} + \sum_{f=1}^{s_1} \mu^1_f y^1_{fj} + \varphi_0 + s^+_j}{\sum_{i=1}^{m} \upsilon_i x_{ij} + \upsilon_0(x_{0j} + R_{1j}) - s^-_j} = 1, \forall j \in J^l_{\text{elastic}} \\
&E_{2j} = \frac{\sum_{g=1}^{s_2} \mu^2_g y^2_{gj} + \mu_0}{\sum_{p=1}^{q} \varphi_p z_{pj} + \sum_{t=1}^{l} \omega_t h_{tj} + \upsilon_0(h_{0j} + R_{2j})} \leq 1, \forall j \in J^l_{\text{elastic}} \\
&R_{1j} + R_{2j} = R^\star_j, \forall j \in J^l_{\text{elastic}} \\
&0 \leq R_{1j} \leq R_{2j}, \forall j \in J^l_{\text{elastic}} \\
&\mu^1_f, \mu^2_g, \varphi_p, \upsilon_i, \omega_t, s^+_j, s^-_j \geq 0, \forall f \in F, \forall g \in G, \forall p \in P, \forall i \in I, \forall t \in T, \forall j \in J^l_{\text{elastic}} \\
&\upsilon_0 > 0, \varphi_0, \mu_0 for free.
\end{aligned}
\tag{12}
$$

In Model (12), s^+_j and s^-_j are two slack variables introduced to maximize the efficiency of the first stage $E_{1j}(\forall j \in J^l_{\text{elastic}})$. $R^\star_j$ is the cost allocation of DMU_j, and $E^\star_j$ is the efficiency value corresponding to $R^\star_j$. The constraint $0 \leq R_{1j} \leq R_{2j}$ ensures that the cost allocation R_{1j} assigned to the first stage should not exceed the cost allocation R_{2j} of the second stage when decision-makers prioritize enhancing the first stage's efficiency. By solving Model (12), we obtain the cost allocation scheme of each stage for elastic DMUs.

3. Numerical illustration

In this section, we use a numerical analysis to clarify our model. Table 2 presents a numerical example wherein the production activities of DMUs are modeled as a two-stage system. In the first sub-stage, two inputs (X_1, X_2) are consumed to produce an intermediate product (Z) and a final output (Y^1). In the second sub-stage, the final output (Y^2) is produced by consuming the intermediate output (Z) and an additional input (H). Specifically, both X_2 in the first stage and H in the second stage should be combined with allocated cost in each stage. The total amount of fixed cost allocation is $R = 10$. The post-allocation BCC efficiency values for each DMU are presented in the last column of Table 2.

As shown in the last column of Table 2, the BCC model identifies DMU_3, DMU_4, DMU_7 and DMU_8 as efficient when maximizing their own efficiency. It is remarkable that the efficiency values of these DMUs may not necessarily remain constant with changes in allocated fixed cost, as the BCC efficiency value is capped at 1. This constraint limits its ability to distinguish between DMUs types. Thus, we use the super-BCC model to investigate the relationship between cost allocation and efficiency values.

We take DMU_3 as an example for the sake of illustration. In Table 3, we set $k_3 = \epsilon \times t$, where t ranges from 0 to 200, and $\epsilon = 0.05$. By substituting various values of k_3 into Model (4), the corresponding efficiency values are attained as shown in Table 3. Apparently, the efficiency value $E^{TSB}_3(k_3)$ decreases as k_3 increases.

In line with the aforementioned approach, we set the cost allocation of $DMU_d(d = 1, \ldots, 8)$ as parameters varying within the interval $[0, 10]$. These parameters are then incorporated into Model (4) to obtain optimal efficiency values for the given cost allocation. The maximum traditional super-BCC efficiency values $\overline{E}^{TSB}_j$ and

TABLE 2. Sample dataset and BCC efficiency score after fixed cost allocation.

DMU	X_1	X_2	Z	Y^1	H	Y^2	BCC efficiency
1	205	35	28	81	5	651	0.7769
2	142	28	41	90	21	243	0.7937
3	208	14	29	12	7	227	1.0000
4	113	16	26	91	3	512	1.0000
5	147	28	43	63	9	212	0.7910
6	246	28	13	9	19	583	0.5993
7	196	21	36	27	14	883	1.0000
8	102	27	13	54	4	292	1.0000

TABLE 3. Relationship between DEA efficiency of DMU_3 and cost allocation.

t	$k_3 = 0.05 \times t$	$E_3^{TSB}(k_3)$
0	0	1.2857
1–20	0.05–1	1.2815–1.2045
21–40	1.05–2	1.2007–1.1304
41–60	2.05–3	1.1269–1.0625
61–80	3.05–4	1.0593–1.0000
81–100	4.05–5	0.9960–0.9266
101–120	5.05–6	0.9233–0.8632
121–140	6.05–7	0.8600–0.8015
141–160	7.05–8	0.7986–0.7438
161–180	8.05–9	0.7411–0.6897
181–199	9.05–9.95	0.6871–0.6414
200	10	0.6389

TABLE 4. Efficiency scores based on different DEA models.

DMUs	$\overline{E}_j^{TSB}$	$\underline{E}_j^{TSB}$	$\overline{E}_j^{\star}$	$\underline{E}_j^{\star}$
1	0.8427	0.6649	0.8427	0.6649
2	0.7937	0.7937	0.7937	0.7937
3	1.2857	0.6389	1.2857	0.6389
4	Infeasible	Infeasible	2.2531	1.2677
5	0.7910	0.7546	0.7910	0.7546
6	0.5993	0.4982	0.5993	0.4982
7	Infeasible	Infeasible	1.2706	1.0584
8	1.1078	1.1078	1.1078	1.1078

the minimum values $\underline{E}_j^{TSB}$ are presented in the second and third columns of Table 4. We find that DMU_3 and DMU_8, deemed efficient under the BCC model, have maximum efficiency values exceeding 1 in the super-BCC model. Moreover, these two DMUs exhibit distinct characteristics. Specifically, the efficiency of DMU_3 varies with changes in cost allocation, while the efficiency of DMU_8 remains unaffected by such changes. Notably, the efficiency of DMU_4 and DMU_7 exhibit infeasible solutions.

To discuss the impact of allocated cost k_d on the super-BCC efficiency value $E_d^{TSB}(k_d)$, we depict the efficiency function curves in Figure 3.

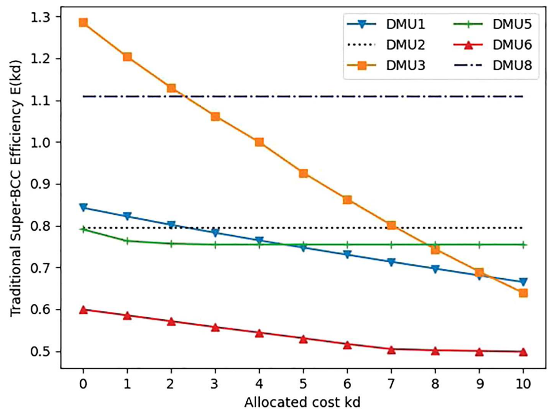

FIGURE 3. Traditional super-BCC efficiency function curves of DMUs.

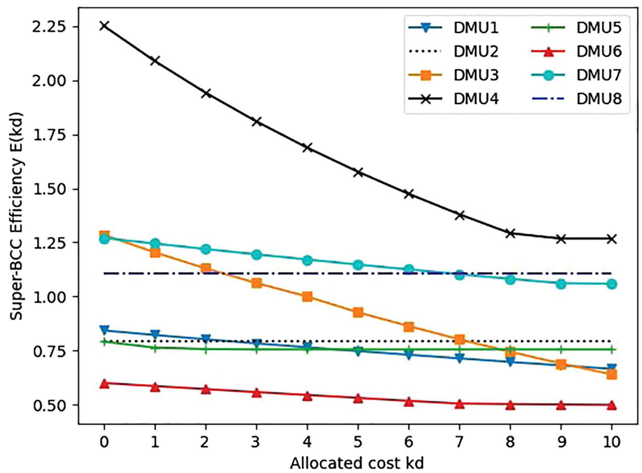

FIGURE 4. Super-BCC efficiency function curves of DMUs.

As evident from Figure 3, the efficiency values of DMU_1, DMU_3 and DMU_6 decrease monotonically as their allocated cost k_d increases, while DMU_5's efficiency initially decreases but stabilizes when the fixed cost exceeds 2.4. In contrast, the efficiency values of DMU_2 and DMU_8 remain constant. Despite both DMU_3 and DMU_8 having efficiency values of 1 in the traditional BCC model, the function curves indicate a significant difference between these two DMUs. Obviously, the efficiency value of DMU_3 is highly influenced by the allocated cost. Note that the efficiency function curves of DMU_4 and DMU_7 are omitted from Figure 3 due to infeasible solutions.

To calculate the modified super-BCC efficiency value of DMU_d, we substitute the optimal allocation plan into Model (5) to get the optimal value $\sigma^\star$ and $\eta^\star$. Following the same calculation steps highlighted earlier, we obtain the maximum super-BCC efficiency score $\overline{E}_j^\star$ as well as the minimum score $\underline{E}_j^\star$ for each DMU. The results are presented in columns four and five of Table 4. Notably, except for DMU_4 and DMU_7 which were previously infeasible in the traditional super-BCC model, the results for other DMUs remain consistent. This suggests that our proposed modified super-BCC model effectively addresses the issue of infeasibility.

To visually exhibit the relationship between the allocated cost k_d and the super-BCC efficiency value, we depict the super-BCC efficiency function curves for all DMUs in Figure 4.

TABLE 5. The optimal allocation scheme for DMUs.

DMUs	$l = 0, R^0 = 10,$		$l = 1, R^1 = 7.6650,$				
	$\underline{E}_j^\star$	$\overline{E}_j^\star$	$\underline{E}_j^\star$	$\overline{E}_j^\star$	R_{1j}	R_{2j}	R_j
1	0.6649	0.8427	0.6960	0.8512	0.3472	1.4547	1.8019
2	0.7937	0.7937	0.7937	0.7937	1.2983	0.3963	1.6946
3	0.6389	1.2857	0.7792	1.2539	0.3576	1.1770	1.5346
4	1.2677	2.2531	1.9386	2.5511	0.4429	1.3209	1.7637
5	0.7546	0.7910	0.7687	0.8586	0.2470	1.1348	1.3818
6	0.4982	0.5993	0.5185	0.5862	0.1739	0.7113	0.8852
7	1.0584	1.2706	1.3431	1.4610	0.0655	0.2468	0.3123
8	1.1078	1.1078	1.1078	1.1078	0.5249	0.1155	0.6404
Total					3.4572	6.5573	10

Compared with Figure 3, we observe that Figure 4 includes the efficiency function curves of DMU_4 and DMU_7, both of which have efficiency values exceeding 1 and decrease as the allocated fixed cost k_d increases. The remaining DMUs are consistent with Figure 3. We also find that DMUs identified as efficient by the BCC model (*i.e.*, DMU_3, DMU_4, DMU_7 and DMU_8) have initial efficiency values exceeding 1 in Figure 4. The efficiency values for DMU_3, DMU_4 and DMU_7 decrease with an increase in the allocated fixed cost, while the efficiency value for DMU_8 remains constant. This suggests our method effectively distinguish between DMUs types.

Based on Definitions 1 and 2, DMU_2 and DMU_8 are inelastic while the remaining DMUs are elastic. Thus, we have $J_{\text{inelastic}} = \{2, 8\}$, $J_{\text{elastic}} = \{1, 3, 4, 5, 6, 7\}$. We now illustrate the entire cost allocation process with the dataset given in Table 2. First, we initialize $J_{\text{elastic}} = \{1, 2, \ldots, 8\}$, $J_{\text{inelastic}} = \emptyset$, and $R^0 = 10$. The maximum and minimum super-BCC efficiency values $\overline{E}_j^\star$ and $\underline{E}_j^\star$ for each DMU_j can be obtained by solving Models (5) and (6). The results are shown in the second and third columns of Table 5. Next, we divide the DMUs into $J^l_{\text{elastic}} = \{1, 3, 4, 5, 6, 7\}$, $J^l_{\text{inelastic}} = \{2, 8\}$. The renewed value R^1 is calculated as $R^1 = R - \sum_{j \in J^l_{\text{inelastic}}} R_j = 7.6650$. We then solve Models (5) and (6) again with the updated values J^l_{elastic} and R^1 to obtain new $\overline{E}_j^\star$ and $\underline{E}_j^\star$ for each DMU_j. The results are shown in the fourth and fifth columns of Table 5. For the remaining elastic DMUs, we obtain their allocation plan for each stage by solving Models (11) and (12). The optimal allocation scheme is presented in columns 6 to 8 of Table 5. The satisfaction is $\delta = 0.6984$.

4. Empirical study

In this section, we apply our approach to allocate subsidies for high-tech industries. Our research models the innovation process as a two-stage network structure. The first stage is the technology research and development process, which reflects the transformation of R&D inputs into R&D outputs. The second stage represents the economic transformation process, which reflects the conversion of R&D outputs into economic outputs. In 2014, the government provided a subsidy of 1785786 units (1 unit = 10 thousand Chinese Yuan (CNY)) to encourage R&D innovation and technological advancement in high-tech industries. We assume a one-year lag period for new outputs due to the inherent delay in the technology innovation process [28]. Following previous literature on performance evaluation of high-technology industry [26], we use data from 2014 on R&D personnel, funding and fixed assets as inputs, and data from 2015 on patents and new product development projects as first-stage R&D outputs. For the second stage, we use data from 2015 on non-R&D investment and new product development funding along with first-stage R&D outputs as inputs. Finally, we use data from 2016 on sales revenue from new products and export loan value as final outputs. Given that national subsidies are intended to support R&D innovation and technological advancement, it is necessary to combine them with R&D funds and new product

TABLE 6. Variables declarations.

Category	Variable	Units
Input	R & D personnel (x_1)	Man-year
	Fixed assets (x_2)	100-million CNY
	R&D funding (x_3)	10-thousand CNY
Intermediate	Patents (z_1)	Piece
	Number of new product development projects (z_2)	Item
Additional Input	Non-R&D investment (h_1)	10-thousand CNY
	New development funding (h_2)	10-thousand CNY
Output	Sales revenue of new products (y_1)	10-thousand CNY
	Export loan value (y_2)	100-million CNY

TABLE 7. Efficiency evaluation results among 30 provinces.

Province	$\underline{E}_j^\star$	$\overline{E}_j^\star$	Province	$\underline{E}_j^\star$	$\overline{E}_j^\star$
Beijing	0.7243	0.7243	Henan	3.6528	3.6528
Tianjin	0.6859	0.6859	Hubei	0.4615	0.4615
Hebei	2.0111	2.0111	Hunan	2.2521	2.2521
Shanxi	12.5593	17.0078	Guangdong	1.9349	1.9349
Inner Mongolia	1.6235	5.2743	Guangxi	3.6514	3.6514
Liaoning	0.7206	0.7206	Hainan	0.7045	2.2235
Jilin	2.0033	4.2945	Chongqing	2.8849	2.8849
Heilongjiang	0.5560	0.6077	Sichuan	0.6320	0.6320
Shanghai	1.5438	1.5438	Guizhou	0.8002	1.0168
Jiangsu	3.9961	3.9961	Yunnan	0.5874	1.1385
Zhejiang	0.8254	0.8254	Shaanxi	0.4750	0.4750
Anhui	2.3485	2.3485	Gansu	1.5257	1.5257
Fujian	0.8084	0.8084	Qinghai	7.1961	9.5259
Jiangxi	2.5875	2.5875	Ningxia	1.1594	4.6962
Shandong	0.5896	0.5896	Xinjiang	3.0366	21.2037

development funds, respectively. The data used in this study are collected from the China Statistic Yearbook on High-Technology Industry from 2015 to 2017. Variable declarations are listed in the Table 6.

We first solve Models (5) and (6) to get the efficiency function curve for each province. After 10 iterations, we obtain the maximum and minimum efficiency values for all provinces, as shown in Table 7. Based on Definitions 1 and 2, we divide them into elastic and inelastic groups. The elastic group includes 10 provinces, such as Shanxi, Inner Mongolia, Jilin, Heilongjiang, Hainan, Guizhou, Yunnan, Qinghai, Ningxia and Xinjiang. The remaining provinces are inelastic. In inelastic group, we note that Heilongjiang consistently exhibits inefficient, while Hainan, Guizhou and Yunnan can achieve efficient through a rational allocation of subsidies.

We then obtain the subsidy allocation schemes for each province by running Models (11) and (12). The results are displayed in the first three columns of Table 8. We observe that in inelastic provinces, those with larger operational scales (*e.g.*, Guangdong and Jiangsu) receive more subsidies compared to provinces with smaller scales (*e.g.*, Gansu, Guangxi, Hebei, Liaoning and Jiangxi). This is due to the fact that the efficiency values of these provinces do not increase with the subsidies. To maximize overall efficiency, decision-makers prioritize provinces with the potential to enhance efficiency scores. To maintain fairness, subsidies are allocated to these inelastic provinces based on their operational scales. On the other hand, we find that some elastic provinces, such as Jilin, Yunnan, Hainan, Guizhou, Ningxia, Shanxi, Inner Mongolia and Xinjiang, also receive substantial

TABLE 8. Comparative analysis results based on different methods.

Province	Our approach			Li *et al.* [40]			Xu *et al.* [56]		
	R_{1j}	R_{2j}	R_j	R_{1j}	R_{2j}	R_j	R_{1j}	R_{2j}	R_j
Beijing	3441.52	2582.49	6024.01	34275.23	35495.75	69770.98	0.00	48137.73	48137.73
Tianjin	621.03	1070.27	1691.30	12645.18	55269.11	67914.29	0.00	72698.95	72698.95
Hebei	362.42	125.31	487.72	3885.71	8598.53	12484.24	0.00	55730.07	55730.07
Shanxi	14982.79	23851.55	38834.35	1266.03	4344.53	5610.55	0.00	63739.17	63739.17
Inner Mongolia	1069.94	31034.23	32104.18	0.01	6634.91	6634.92	0.00	62209.43	62209.43
Liaoning	431.76	199.00	32104.18	5515.42	8496.04	14011.47	0.00	53631.72	53631.72
Jilin	32267.31	68810.69	101077.99	0.00	5925.91	5925.91	0.00	59988.11	59988.11
Heilongjiang	208.91	1003.36	1212.27	5887.46	627.40	6514.87	0.00	55630.19	55630.19
Shanghai	9369.25	17293.12	26662.38	32008.27	12720.99	44729.26	0.00	41282.22	41282.22
Jiangsu	42141.76	85440.73	127582.50	67672.66	239406.91	307079.57	0.00	115695.50	115695.50
Zhejiang	42141.76	6822.15	14860.18	53201.63	44304.99	97506.62	0.00	48249.69	48249.69
Anhui	1078.23	767.61	1845.83	13854.37	24279.28	38133.65	0.00	60015.00	60015.00
Fujian	1415.58	2147.30	3562.88	15359.28	42557.02	57916.29	0.00	62593.70	62593.70
Jiangxi	485.55	174.03	659.59	3508.95	16260.48	19769.43	0.00	64359.67	64359.67
Shandong	9211.34	8240.07	17451.42	45510.10	47697.94	93208.04	0.00	44851.52	44851.52
Henan	923.38	1629.62	2553.00	0.58	122120.18	122120.76	0.00	114473.52	114473.52
Hubei	1583.43	845.24	2428.67	11937.77	6266.81	18204.58	0.00	39973.04	39973.04
Hunan	730.10	734.31	1464.41	7682.77	29818.33	37501.10	0.00	59689.09	59689.09
Guangdong	406409.97	747532.43	1153942.40	194329.02	425379.87	619708.89	0.00	7489.33	7489.33
Guangxi	36.17	18. 80	54.96	694.11	6543.65	7237.76	0.00	63528.94	63528.94
Hainan	20534.08	40748.11	61282.19	2936.03	2508.48	5444.51	0.00	60512.81	60512.81
Chongqing	288.19	812.37	1100.55	7742.20	38450.61	46192.81	0.00	80834.46	80834.46
Sichuan	1723.39	1999.18	3722.57	19393.60	14199.39	33593.00	0.00	51817.97	51817.97
Guizhou	28276.42	30581.24	58857.66	7074.81	0.01	7074.82	0.00	55801.41	55801.41
Yunnan	20110.51	43094.58	63205.10	3133.98	0.00	3133.98	0.00	59053.13	59053.13
Shaanxi	606.38	422.28	1028.65	8902.19	0.03	8902.21	0.00	35362.49	35362.49
Gansu	7.02	2.06	9.08	1525.24	6267.21	7792.45	0.00	62164.24	62164.24
Qinghai	444.66	2230.69	2675.35	1508.48	5031.27	6539.75	0.00	61968.69	61968.69
Ningxia	19484.36	21189.29	40673.65	2309.11	5560.19	7869.30	0.00	61988.90	61988.90
Xinjiang	6589.94	11510.46	18100.40	1802.36	5457.64	7260.00	0.00	62315.33	62315.33

subsidies, especially Yunnan, Hainan, and Guizhou. Despite their smaller operational scales, the significant difference between their maximum and minimum efficiency values indicates a pronounced impact of subsidies on these provinces. As a result, they obtain relatively higher subsidies.

Next, we compare our allocation scheme with the actual subsidies received by each province. The findings reveal that the government provides more substantial subsidies to inelastic provinces (*e.g.*, Shanghai, Guangdong, Beijing, Shaanxi, and Liaoning), while offering fewer subsidies to elastic provinces (*e.g.*, Qinghai, Xinjiang, Inner Mongolia, Ningxia, and Shanxi). Our research emphasizes the principles of fairness and efficiency maximization when designing a subsidy allocation scheme. We suggest that government policies should favor elastic provinces with the potential for efficiency improvement. Therefore, it is crucial to evaluate the performance of each province before allocating subsidies. By investigating the relationship between subsidy allocation and efficiency scores, government can distinguish which provinces have greater potential for further improvement and which are unlikely to be affected by additional subsidies. This approach enables the government to maximize overall efficiency through more effective subsidy allocation. From this perspective, our experimental results demonstrate that the proposed approach is reasonable and acceptable.

Finally, we compare our allocation scheme with the existing fixed cost allocation methods based on two-stage systems. Table 8 presents the results of Li *et al.* [40] and Xu *et al.* [56]. Although our method is the only one that considers the fixed cost as a complementary input in a two-stage network structure, we gain insights by comparing it with other models. Previous research mainly obtains a unique and optimal FCA scheme on an efficient allocation set by proposing principles, such as operational scale [40] and fairness concern [56]. In columns 5 to 10 of Table 8, we observe significant differences in the subsidies obtained by each province at different stages. For example, in the method of Li *et al.* [40], provinces like Inner Mongolia, Jilin, and Henan receive almost zero subsidies in the first stage, while provinces such as Guizhou, Yunnan, and Shaanxi receive almost zero subsidies in the second stage. In the method of Xu *et al.* [56], we find that when the first stage acts as the leader, the first-stage subsidies allocated to all provinces are zero. Furthermore, these methods tend to allocate more subsidies to provinces less sensitive to subsidies. In Li *et al.* [40] results, inelastic provinces such as Chongqing, Jiangxi, Liaoning, Hebei, and Shaanxi receive more subsidies. The same problem also exists in the method of Xu *et al.* [56]. Provinces such as Chongqing, Jiangxi, Guangxi, and Gansu receive more subsidies while those with larger scales (*e.g.*, Guangdong and Zhejiang) receive fewer subsidies. Our method distinguishes types of provinces and adheres to the fairness principle by considering both operational scale and equal efficiency improvement among provinces.

5. Conclusions

Our study addresses the FCA problem of fixed costs as supplementary cost input in a general two-stage system. Specifically, we first propose a modified super-BCC model to explore the functional relationship between allocated fixed costs and efficiency values under a VRS framework. This model not only distinguishes between different types of DMUs but also solves infeasible solutions of super-BCC model under the VRS assumption. We then divide DMUs into elastic and inelastic groups based on their relationship between allocated costs and efficiency scores, and propose a fair and unique allocation scheme for each DMU based on the principles of fairness and efficiency maximization. Finally, we verify the rationality and acceptability of our proposed method through a numerical analysis and an empirical study.

This study contributes to the literature on the FCA problem in two-stage network structure DEA models by proposing a fair and unique allocation model for DMUs with fixed cost as supplementary inputs. However, our research has some limitations. First, our model involves a considerable computational burden, especially when handling a large number of DMUs or allocating huge fixed costs. Second, the results are heavily affected by the operational scale. While we use a widely adopted approach of measuring the operational size based on input usages and output productions, this method may not be optimal. These two points will be addressed in our future research.

Appendix A.

The input-oriented two-stage network super-BCC model without the allocated cost variable is as follows:

$$
\begin{aligned}
\text{Min}\quad & \sigma + M \times \eta \\
\text{s.t.}\quad & \sum_{\substack{j=1\\ j\neq d}}^{n} \lambda_j y_{fj}^1 \geq (1-\eta) y_{fd}^1, \forall f \in F \\
& \sum_{\substack{j=1\\ j\neq d}}^{n} \gamma_j y_{gj}^2 \geq (1-\eta) y_{gd}^2, \forall g \in G \\
& \sum_{\substack{j=1\\ j\neq d}}^{n} (\gamma_j - \lambda_j) z_{pj} \leq \sigma z_{pd}, \forall p \in P \\
& \sum_{\substack{j=1\\ j\neq d}}^{n} \lambda_j x_{ij} \leq (1+\sigma) x_{id}, \forall i \in I \cup \{0\} \qquad \text{(A.1)} \\
& \sum_{\substack{j=1\\ j\neq d}}^{n} \gamma_j h_{tj} \leq (1+\sigma) h_{td}, \forall t \in T \cup \{0\}
\end{aligned}
$$

$$\sum_{\substack{j=1\\j\neq d}}^{n}\lambda_j=1,\sum_{\substack{j=1\\j\neq d}}^{n}\gamma_j=1$$
$$\lambda_j,\gamma_j,\eta\geq 0,\forall j\in J\setminus\{d\},\sigma for free.$$

The dual model of Model (A.1) is as follows:

$$\begin{aligned}
\text{Max}\quad & \sum_{f=1}^{s_1}\mu_f^1y_{fd}^1+\sum_{g=1}^{s_2}\mu_g^2y_{gd}^2-\sum_{i=1}^{m}\upsilon_ix_{id}-\sum_{t=1}^{l}\omega_th_{td}-\upsilon_0(x_{0d}+h_{0d})+\phi_0+\mu_0\\
\text{s.t.}\quad & \sum_{f=1}^{s_1}\mu_f^1y_{fj}^1+\sum_{p=1}^{q}\phi_pz_{pj}-\sum_{i=1}^{m}\upsilon_ix_{ij}-\upsilon_0x_{0j}+\phi_0\leq 0,\forall j\in J\setminus\{d\}\\
& \sum_{g=1}^{s_2}\mu_g^2y_{gj}^2-\sum_{p=1}^{q}\phi_pz_{pj}-\sum_{t=1}^{l}\omega_th_{tj}-\upsilon_0h_{0j}+\mu_0\leq 0,\forall j\in J\setminus\{d\} \qquad \text{(A.2)}\\
& \sum_{p=1}^{q}\phi_pz_{pd}+\sum_{i=1}^{m}\upsilon_ix_{id}+\sum_{t=1}^{l}\omega_th_{td}+\upsilon_0(x_{0d}+h_{0d})=1\\
& \sum_{f=1}^{s_1}\mu_f^1y_{fd}^1+\sum_{g=1}^{s_2}\mu_g^2y_{gd}^2\leq M\\
& u_f^1,u_g^2,\phi_p,\upsilon_i,\omega_t\geq 0,\upsilon_0>0,\forall f\in F,\forall g\in G,\forall p\in P,\forall i\in I,\forall t\in T,\phi_0,\mu_0 for free.
\end{aligned}$$

(A.3)

The Model (A.2) can be transformed to the following Model (A.4) after considering the fixed cost.

$$\begin{aligned}
\text{Max}\quad & \sum_{f=1}^{s_1}\mu_f^1y_{fd}^1+\sum_{g=1}^{s_2}\mu_g^2y_{gd}^2-\sum_{i=1}^{m}\upsilon_ix_{id}-\sum_{t=1}^{l}\omega_th_{td}-\upsilon_0(x_{0d}+h_{0d}+R_d)+\phi_0+\mu_0\\
\text{s.t.}\quad & \sum_{f=1}^{s_1}\mu_f^1y_{fj}^1+\sum_{p=1}^{q}\phi_pz_{pj}-\sum_{i=1}^{m}\upsilon_ix_{ij}-\upsilon_0(x_{0j}+R_{1j})+\phi_0\leq 0,\forall j\in J\setminus\{d\}\\
& \sum_{g=1}^{s_2}\mu_g^2y_{gj}^2-\sum_{p=1}^{q}\phi_pz_{pj}-\sum_{t=1}^{l}\omega_th_{tj}-\upsilon_0(h_{0j}+R_{2j})+\mu_0\leq 0,\forall j\in J\setminus\{d\}\\
& \sum_{p=1}^{q}\phi_pz_{pd}+\sum_{i=1}^{m}\upsilon_ix_{id}+\sum_{t=1}^{l}\omega_th_{td}+\upsilon_0(x_{0d}+h_{0d}+R_d)=1 \qquad \text{(A.4)}\\
& \sum_{f=1}^{s_1}\mu_f^1y_{fd}^1+\sum_{g=1}^{s_2}\mu_g^2y_{gd}^2\leq M\\
& \sum_{j=1}^{n}(R_{1j}+R_{2j})=R\\
& \mu_f^1,\mu_g^2,\phi_p,\upsilon_i,\omega_t\geq 0,\forall f\in F,\forall g\in G,\forall p\in P,\forall i\in I,\forall t\in T\\
& R_{1j},R_{2j}\geq 0,\upsilon_0>0,\forall j\in J,\phi_0,\mu_0 for free.
\end{aligned}$$

Then Model (A.4) can be transformed to a linear Model (6) by setting $r_{1j}=\upsilon_0R_{1j}$ and $r_{2j}=\upsilon_0R_{2j}$.

Appendix B.

Theorem 1. *$E^{\star}_d(k_d)$ is a monotone non-increasing function of k_d, $k_d \in [0, R]$.*

Proof. Given $R_{1d} = k_{1d}, R_{2d} = k_{2d}, R_d = k_d = k_{1d} + k_{2d}$, we can obtain the optimal allocation scheme $(R_{1d}, R_{2d}, R^{\star}_{1j}, R^{\star}_{2j}, \forall j \in J \setminus \{d\})$ by solving Model (6). Substituting this into Model (5), we obtain the optimal solution $(\lambda^{\star}_j, \gamma^{\star}_j, \sigma^{\star}, \eta^{\star})$ and define the corresponding super-BCC efficiency score as $E^{\star}_d(k_d) = \sigma^{\star} + \frac{1}{1-\eta^{\star}}$.

Suppose there exists another allocation scheme $(R'_{1d} = k_{1d} + \Delta k_1(\Delta k_1 \geq 0), R'_{2d} = k_{2d}, R'_{1j}, R'_{2j}, \forall j \in J \setminus \{d\})$, where $k'_d \geq k_d$. The solution $(\lambda^{\star}_j, \gamma^{\star}_j, \sigma^{\star}, \eta^{\star})$ is still feasible for Model (5) since it satisfies all constraints. Thus, the super-BCC efficiency scores for both allocation schemes are equal, *i.e.*, $E_d(k'_d) = E^{\star}_d(k_d)$. However, this solution may not be optimal as it only satisfies the constraints without minimizing the objective function.

Suppose the optimal solution of Model (5) with allocation plan $(R'_{1d}, R'_{2d}, R'_{1j}, R'_{2j}, \forall j \in J \setminus \{d\})$ is given by $(\lambda'_j, \gamma'_j, \sigma', \eta')$. The corresponding optimal super-BCC efficiency score that minimizes the objective function is denoted by $E^{\star}_d(k'_d)$, which is less than $E_d(k'_d)$. Thus, we have $E^{\star}_d(k'_d) \leq E^{\star}_d(k_d)$ since $E_d(k'_d) = E^{\star}_d(k_d)$.

In summary, $\forall k_d \in [0, R], E^{\star}_d(k_d)$ is monotone non-increasing. □

Appendix C.

Theorem 2. *The optimal solution of Model (11) used in Algorithm 1 is unique.*

Proof. Algorithm 1 terminates until the condition $J_{\text{inelastic}} = \emptyset$ is satisfied. Since each inelastic DMU is allocated with $\alpha_j R$ and $\beta_j R$, we only need to prove that the allocation among elastic DMUs is unique. In Model (11), the constraint $E^{\star}_j = \underline{E}^{\star}_j + \delta(\overline{E}^{\star}_j - \underline{E}^{\star}_j), j \in J^l_{\text{elastic}}$ indicates that the efficiency score $E^{\star}_j$ is a linear function of δ. Since the inverse function $R_j = f^{-1}_j(E^{\star}_j) = g_j(\delta)$ is strictly monotone decreasing in δ, there exists only one δ that satisfies the constraint $\sum_{j \in J^l_{\text{elastic}}} R_j = R^l$. Therefore, the optimal allocation scheme for each DMU is unique. □

Acknowledgements

This work was supported by the National Natural Science Foundation of China (No.72071192 and No.72188101), the Humanities and Social Science Fund of Ministry of Education (No.23XJC630007), the Sichuan Science and Technology Program (No.24NSFSC2816), the National Natural Science Foundation of China (No.71901178), the Beijing Municipal Social Science Foundation (No.SZ202210005004) and Beijing Natural Science Foundation (No.9242004).

References

[1] Q. An, P. Wang and S. Shi, Fixed cost allocation for two-stage systems with cooperative relationship using data envelopment analysis. *Comput. Ind. Eng.* **145** (2020) 106534.

[2] Q. An, P. Wang, H. Yang and Z. Wang, Fixed cost allocation in two-stage system using DEA from a noncooperative view. *OR Spectr.* **43** (2021) 1077–1102.

[3] P. Andersen and N.C. Petersen, A procedure for ranking efficient units in data envelopment analysis. *Manage. Sci.* **39** (1993) 1261–1264.

[4] R.D. Banker, A. Charnes and W.W. Cooper, Some models for estimating technical and scale inefficiencies in data envelopment analysis. *Manage. Sci.* **30** (1984) 1078–1092.

[5] J.E. Beasley, Allocating fixed costs and resources via data envelopment analysis. *Eur. J. Oper. Res.* **147** (2003) 198–216.

[6] L. Castelli, R. Pesenti and W. Ukovich, A classification of DEA models when the internal structure of the decision making units is considered. *Ann. Oper. Res.* **173** (2010) 207–235.

[7] A. Charnes and W.W. Cooper, Programming with linear fractional functionals. *Nav. Res. Logist. Q.* **9** (1962) 181–186.

[8] A. Charnes, W.W. Cooper and E. Rhodes, Measuring the efficiency of decision making units. *Eur. J. Oper. Res.* **2** (1978) 429–444.

[9] Y. Chen, W.D. Cook, N. Li and J. Zhu, Additive efficiency decomposition in two-stage DEA. *Eur. J. Oper. Res.* **196** (2009) 1170–1176.

[10] Y. Chen, L. Liang and J. Zhu, Equivalence in two-stage DEA approaches. *Eur. J. Oper. Res.* **193** (2009) 600–604.

[11] J. Chu, J. Wu, C. Chu and T. Zhang, DEA-based fixed cost allocation in two-stage systems: leader-follower and satisfaction degree bargaining game approaches. *Omega* **94** (2020) 102054.

[12] J. Chu, Y. Dong and F. Wei, Efficiency improvement and balance in fixed cost allocation: A trade-off approach based on DEA. *Comput. Ind. Eng.* **183** (2023) 109527.

[13] J. Chu, W. Su, F. Li and Z. Yuan, Individual rationality and overall fairness in fixed cost allocation: an approach under DEA cross-efficiency evaluation mechanism. *J. Oper. Res. Soc.* **74** (2023) 992–1007.

[14] J. Chu, W. Su, J. Wu and Z. Yuan, DEA-based proportional-sharing fixed cost allocation considering bi-objective optimization. *OR Spectr.* (2023) 1–28.

[15] C.S. Cinca and C.M. Molinero, Selecting DEA specifications and ranking units via PCA. *J. Oper. Res. Soc.* **55** (2004) 521–528.

[16] W.D. Cook and M. Kress, Characterizing an equitable allocation of shared costs: a DEA approach. *Eur. J. Oper. Res.* **119** (1999) 652–661.

[17] W.D. Cook and J. Zhu, Allocation of shared costs among decision making units: a DEA approach. *Comput. Oper. Res.* **32** (2005) 2171–2178.

[18] W.D. Cook, L. Liang, Y. Zha and J. Zhu, A modified super-efficiency DEA model for infeasibility. *J. Oper. Res. Soc.* **60** (2009) 276–281.

[19] W.D. Cook, L. Liang and J. Zhu, Measuring performance of two-stage network structures by DEA: a review and future perspective. *Omega* **38** (2010) 423–430.

[20] Q. Dai, Y. Li and L. Liang, Allocating fixed costs with considering the return to scale: a DEA approach. *J. Syst. Sci. Complex.* **29** (2016) 1320–1341.

[21] Q. Dai, Y. Li, X. Lei and D. Wu, A DEA-based incentive approach for allocating common revenues or fixed costs. *Eur. J. Oper. Res.* **292** (2021) 675–686.

[22] T. Ding, Q. Zhu, B. Zhang and L. Liang, Centralized fixed cost allocation for generalized two-stage network DEA. *INFOR: Inf. Syst. Oper. Res.* **57** (2019) 123–140.

[23] J. Du, L. Liang, Y. Chen, W.D. Cook and J. Zhu, A bargaining game model for measuring performance of two-stage network structures. *Eur. J. Oper. Res.* **210** (2011) 390–397.

[24] J. Du, W.D. Cook, L. Liang and J. Zhu, Fixed cost and resource allocation based on DEA cross-efficiency. *Eur. J. Oper. Res.* **235** (2014) 206–214.

[25] A. Emrouznejad and G. Yang, A survey and analysis of the first 40 years of scholarly literature in DEA: 1978–2016. *Soc.-Econ. Plan. Sci.* **61** (2018) 4–8.

[26] F. Feng, B. Wang, Y. Zou and Y. Du, A new Internet DEA structure: Measurement of Chinese R&D innovation efficiency in high technology industry. *Int. J. Bus. Manage.* **8** (2013) 32.

[27] H. Fukuyama and R. Matousek, Modelling bank performance: a network DEA approach. *Eur. J. Oper. Res.* **259** (2017) 721–732.

[28] C. Han, S.R. Thomas, M. Yang, P. Ieromonachou and H. Zhang, Evaluating R&D investment efficiency in China's high-tech industry. *J. High. Technol. Manage. Res.* **28** (2017) 93–109.

[29] G.R. Jahanshahloo, F.H. Lotfi, N. Shoja and M. Sanei, An alternative approach for equitable allocation of shared costs by using DEA. *Appl. Math. Comput.* **153** (2004) 267–274.

[30] G.R. Jahanshahloo, J. Sadeghi and M. Khodabakhshi, Proposing a method for fixed cost allocation using DEA based on the efficiency invariance and common set of weights principles. *Math. Methods Oper. Res.* **85** (2017) 223–240.

[31] L. Jenkins and M. Anderson, A multivariate statistical approach to reducing the number of variables in data envelopment analysis. *Eur. J. Oper. Res.* **147** (2003) 51–61.

[32] C. Kao, Network data envelopment analysis: a review. *Eur. J. Oper. Res.* **239** (2014) 1–16.

[33] C. Kao and S.N. Hwang, Efficiency decomposition in two-stage data envelopment analysis: an application to non-life insurance companies in Taiwan. *Eur. J. Oper. Res.* **185** (2008) 418–429.

[34] M. Khodabakhshi and K. Aryavash, The fair allocation of common fixed cost or revenue using DEA concept. *Ann. Oper. Res.* **214** (2014) 187–194.

[35] Y. Li, F. Yang, L. Liang and Z. Hua, Allocating the fixed cost as a complement of other cost inputs: A DEA approach. *Eur. J. Oper. Res.* **197** (2009) 389–401.

[36] Y. Li, Y. Chen, L. Liang and J. Xie, DEA models for extended two-stage network structures. *Omega* **40** (2012) 611–618.
[37] Y. Li, M. Yang, Y. Chen, Q. Dai and L. Liang, Allocating a fixed cost based on data envelopment analysis and satisfaction degree. *Omega* **41** (2013) 55–60.
[38] Y. Li, X. Lei, Q. Dai and L. Liang, Performance evaluation of participating nations at the 2012 London Summer Olympics by a two-stage data envelopment analysis. *Eur. J. Oper. Res.* **243** (2015) 964–973.
[39] F. Li, J. Song, A. Dolgui and L. Liang, Using common weights and efficiency invariance principles for resource allocation and target setting. *Int. J. Prod. Res.* **55** (2017) 4982–4997.
[40] F. Li, Q. Zhu and Z. Chen, Allocating a fixed cost across the decision making units with two-stage network structures. *Omega* **83** (2019) 139–154.
[41] F. Li, Q. Zhu and L. Liang, A new data envelopment analysis based approach for fixed cost allocation. *Ann. Oper. Res.* **274** (2019) 347–372.
[42] F. Li, Z. Yan, Q. Zhu, M. Yin and G. Kou, Allocating a fixed cost across decision making units with explicitly considering efficiency rankings. *J. Oper. Res. Soc.* **72** (2021) 1432–1446.
[43] F. Li, Y. Wang, A. Emrouznejad, Q. Zhu and G. Kou, Allocating a fixed cost across decision-making units with undesirable outputs: A bargaining game approach. *J. Oper. Res. Soc.* **73** (2022) 2309–2325.
[44] L. Liang, J. Wu, W.D. Cook and J. Zhu, The DEA game cross-efficiency model and its Nash equilibrium. *Oper. Res.* **56** (2008) 1278–1288.
[45] R. Lin, Allocating fixed costs or resources and setting targets via data envelopment analysis. *Appl. Math. Comput.* **217** (2011) 6349–6358.
[46] R. Lin and Z. Chen, Fixed input allocation methods based on super CCR efficiency invariance and practical feasibility. *Appl. Math. Model.* **40** (2016) 5377–5392.
[47] R. Lin and Y. Liu, Super-efficiency based on the directional distance function in the presence of negative data. *Omega* **85** (2019) 26–34.
[48] R. Lin and Z. Chen, A DEA-based method of allocating the fixed cost as a complement to the original input. *Int. Trans. Oper. Res.* **27** (2020) 2230–2250.
[49] S. Lozano, Information sharing in DEA: a cooperative game theory approach. *Eur. J. Oper. Res.* **222** (2012) 558–565.
[50] A. Mostafaee, An equitable method for allocating fixed costs by using data envelopment analysis. *J. Oper. Res. Soc.* **64** (2013) 326–335.
[51] S.V. Nagaraj, Optimal binary search trees. *Theor. Comput. Sci.* **188** (1997) 1–44.
[52] L.M. Seiford and J. Zhu, Infeasibility of super-efficiency data envelopment analysis models. *INFOR: Inf. Syst. Oper. Res.* **37** (1999) 174–187.
[53] X. Si, L. Liang, G. Jia, L. Yang, H. Wu and Y. Li, Proportional sharing and DEA in allocating the fixed cost. *Appl. Math. Comput.* **219** (2013) 6580–6590.
[54] F. Wei, Y. Fu, F. Yang, C. Sun and S. Ang, Closest target setting with minimum improvement costs considering demand and resource mismatches. *Oper. Res. Int. J.* **23** (2023) 42.
[55] J. Wu, Q. Zhu, X. Ji, J. Chu and L. Liang, Two-stage network processes with shared resources and resources recovered from undesirable outputs. *Eur. J. Oper. Res.* **251** (2016) 182–197.
[56] G. Xu, J. Wu and Q. Zhu, Fixed cost allocation in two-stage system: a data-driven approach from the perspective of fairness concern. *Comput. Ind. Eng.* **173** (2022) 108647.
[57] G. Xu, J. Wu, Q. Zhu, and Y. Pan, Fixed cost allocation based on data envelopment analysis from inequality aversion perspectives. *Eur. J. Oper. Res.* **313** (2024) 281–295.
[58] A.H. Yadolladi and R.K. Matin, Centralized resource allocation with the possibility of downsizing in two-stage network production systems. *RAIRO:RO* **55** (2021) 2583–2598.
[59] J. Yang, D. Li and Y. Li, A generalized data envelopment analysis approach for fixed cost allocation with preference information. *Omega* **122** (2024) 102948.
[60] P. Yin, J. Chu, J. Wu, J. Ding, M. Yang and Y. Wang, A DEA-based two-stage network approach for hotel performance analysis: an internal cooperation perspective. *Omega* **93** (2020) 102035.
[61] M.M. Yu, L.H. Chen and B. Hsiao, A fixed cost allocation based on the two-stage network data envelopment approach. *J. Bus. Res.* **69** (2016) 1817–1822.
[62] L. Zhang and K. Chen, Hierarchical network systems: an application to high-technology industry in China. *Omega* **82** (2019) 118–131.
[63] T. Zhao, J. Xie, Y. Chen and L. Liang, Coordination efficiency for general two-stage network system. *RAIRO:RO* **56** (2022) 3801–3815.

[64] J. Zhu, Robustness of the efficient DMUs in data envelopment analysis. *Eur. J. Oper. Res.* **90** (1996) 451–460.

[65] W. Zhu, Q. Zhang and H. Wang, Fixed costs and shared resources allocation in two-stage network DEA. *Ann. Oper. Res.* **278** (2019) 177–194.