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SOME NEW RESULTS ON ROUGH INTERVAL LINEAR PROGRAMMING PROBLEMS AND THEIR APPLICATION TO SCHEDULING AND FIXED-CHARGE TRANSPORTATION PROBLEMS

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Abstract. This paper focuses on linear programming problems in a rough interval environment. By introducing four linear programming problems, an attempt is being made to propose some results on optimal value of a linear programming problem with rough interval parameters. To obtain optimal solutions of a linear programming problem with rough interval data, constraints of the four proposed linear problems are applied. In this regard, firstly, the largest and the smallest feasible spaces for a linear constraint set with rough interval coefficients and parameters are introduced. Then, a rough interval for optimal value of such problems is obtained. Further, an upper approximation interval and a lower approximation interval as the optimal solutions of linear programming problems with rough interval parameters are achieved. Moreover, two solution concepts, surely and possibly solutions, are defined. Some numerical examples demonstrate the validity of the results. In particular, a scheduling problem and a fixed-charge transportation problem (FCTP) under rough interval uncertainty are investigated.

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1. Introduction

Linear programming (LP) is an eminent class of mathematical programming that has been attracted many researchers' attention in last decades. Many real-world problems are modeled as LPs. In conventional LP problems, although the model coefficients and parameters should certainly be determined but there are many instances in which the parameters are not known certainly. Due to multiple sources of distinc nature, uncertainty exists in many real-world problems [11, 26]. Uncertainty may be expressed by fuzziness, randomness, interval data, roughness or their hybrids [15]. The newest theory for modeling vagueness is rough set theory suggested by Pawlack in 1982 [24]. In this approach, a rough set (lower an upper bounds) is introduced as a family of subsets of a certain set called universe. A pair of precise concepts, the lower and the upper approximations, presents any imprecise concept. It is noted that this approach applies in some branches of artificial intelligence, such as inductive reasoning, automatic classification, pattern recognition, learning algorithms, etc. [24]. The rough set theory as an excellent mathematical tool has been extremely applied for dealing with uncertainty

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[14, 18, 21]. Rough intervals (RIs), proposed by Rebolledo [25], are applicable and reliable for managing vagueness. A rough interval has two parts, an upper approximation interval (UAI) and a lower approximation interval (LAI). The lower (upper) approximation interval contains values that could be taken in normal (special) cases by the variable. It is noted that the values outside the UAI are never taken [33]. Actually, rough intervals as an extension of rough sets, satisfy the mathematical definition of upper and lower approximation in rough set theory. Applying rough set theory makes the process of decision-making more reliable and flexible. As an instance, in the water resource management problem, when the water source quantity is imprecise, it should be determined according to the past experiences. It is identified that the water quantity usually is between 12 to $18 \times 10^6\,\mathrm{m}^3$ per year. Further, it ranges from 10 to $20 \times 10^6\,\mathrm{m}^3$ in several special years. Hence, it is possible to propose such an information by the rough interval $([12 \times 10^6, 18 \times 10^6], [10 \times 10^6, 20 \times 10^6])$ [33]. As an another real-world example, a transportation problem (TP) could be considered. In such problems, all the parameters (demand, supply, etc.) may not known precisely, as they are uncertain in nature due to insufficient information, lack of evidence, weather condition and other uncontrollable factors. In this regard, consider D "the demand of petroleum" of a city, which constantly changes between 10 and 15 barrels per day. Generally, it varies within 11 to 13 barrels per day. Other values outside of the interval $[11, 13]$ occur in particular situations such as tourist season, festive seasons, etc. Therefore, the daily demand of petroleum can be described by a rough interval as $D = ([11, 13], [10, 15])$ [28, 29].

Rough set theory has been investigated and applied by several researchers [5, 12, 19, 20, 30, 32]. A method for solving fuzzy integer LP problems with RI coefficients was proposed by Ammar and Emsimir [4]. A new method for obtaining the solutions of fuzzy rough LP problems was suggested in [31]. Garg and Rizk-Allah presented a solution method for dealing with rough multi-objective TPs [12]. They obtained surely and possibly Pareto optimal solutions by introducing four crisp transportation problems. Roy and Midya solved multi-objective fixed-charge TPs (MOFCTPs) with rough and random rough parameters by using an expected value operator and converting the uncertain MOFCTP into a deterministic form [28]. Ghosh and Roy formulated multi-objective product blending FCTP with truck load constraints under fuzzy-rough uncertainties by using transfer station [13]. Midya *et al.* presented a multi-objective fractional FCTP in a rough environment [22]. Some researchers worked on MOFCTPs under uncertainty [23, 29]. Parametric rough linear goal programming (PRLGP) problems were considered in [10]. A PRLGP model can be converted to some approximation models where the lexicographical goal programming may be used for solving them. The concept of rough convex function which is applied for solving rough polynomial geometric programming models, was studied by Cao [7]. In [8], the authors developed a decision-making method in a rough fuzzy environment. To build a crisp programming problem, a new approach for normalizing the rough fuzzy numbers was given in [8].

It should be noted that many investigations have been proposed on LPs under an uncertain environment until now. However, there are some real-world situations which occur in industrial problems where rough set theory could manage the parameters and data more appropriate. In this regard, this paper focuses on LPs with RI coefficients and surveys two important discussions.

Surely and possibly optimal values of the objective function (OVOF) are first obtained. Then, under some assumptions, the UAI and LAI of optimal solutions (OSs) are determined. The main contribution of our proposed study are summarized as:

- Investigate rough interval linear programming (RILP) problems.
- Propose some results on optimal value of RILP problems.
- Present some results on optimal solutions of RILP problems.
- Define two solution concepts for RILPs.
- Discuss the obtained results in some numerical examples.
- A comparison with an existed solution method is drawn in an example.

The reminder of the paper is organized as follows. In Section 2, some preliminaries of intervals and RI arithmetic are reviewed. In Section 3, some results on the OVOF and OSs of RILP problems are given. In this section, we survey three important subjects about a RILP problem. Firstly, the largest and the smallest feasible spaces for a RI linear constraint set are introduced. Then, a RI for OVOF of the RILP problem is obtained by proposing four LP models. Further, a UAI and a LAI as the OSs of the RILP problem are achieved. In general case, obtaining OSs of an ILP problem is a challenging problem. Due to the fact that a RILP may be converted into ILPs, therefore obtaining OSs of a RILP problem is not easy. To obtain the OSs of a RILP problem, feasible spaces of four LP problems are used. It is noted that all feasible solution components of one of the LPs should be positive. In Section 4, the proposed method is stated for a special real-world problem. Actually, a rough interval fixed-charge transportation problem (RIFCTP) is generally investigated in this section. Some numerical examples are presented in Section 5. In this regard, a scheduling problem under RI uncertainty that allocate guards to shifts and a RIFCTP are numerically illustrated. At last, Section 6 concludes this research and gives some future research directions.

2. Intervals and RIs

This section is devoted to some preliminaries of intervals and RIs [1–3, 6, 9, 16, 17, 25, 27].

An interval number $x^{\pm}$ is presented by $[x^-, x^+]$, where $x^- \le x^+$. Obviously, $x^{\pm} \ge 0$ iff $x^- \ge 0$, and $x^{\pm} \le 0$ iff $x^+ \le 0$. The center and radius of $x^{\pm}$ are defined as follows, respectively:

$$(x^{\pm})^c = \frac{1}{2}(x^- + x^+), \quad (x^{\pm})^{\Delta} = \frac{1}{2}(x^+ - x^-).$$

Let $A^- = (a_{ij}^-)_{m\times n}$ and $A^+ = (a_{ij}^+)_{m\times n}$ be two matrices in which $m, n \in \mathbb{N}$ and $a_{ij}^+ \ge a_{ij}^-$, $\forall\ i, j$, then the interval matrix $\mathbf{A}^{\pm}$ is defined as

$$\mathbf{A}^{\pm} = \left[\mathbf{A}^-, \mathbf{A}^+\right] = \left\{\mathbf{A} |\ \mathbf{A}^- \le \mathbf{A} \le \mathbf{A}^+\right\}.$$

Moreover, an interval vector $\mathbf{x}^{\pm}$ is defined as $\mathbf{x}^{\pm} = \{\mathbf{x}|\ \mathbf{x}^- \le \mathbf{x} \le \mathbf{x}^+\}$, where $\mathbf{x}^-, \mathbf{x}^+ \in \mathbb{R}^n$.

A RI $A_R^{\pm}$ is defined as $({}_*A^{\pm}, {}^*A^{\pm})$ where ${}_*A^{\pm}$ and ${}^*A^{\pm}$ are LAI and UAI, respectively and ${}_*A^{\pm} \subseteq {}^*A^{\pm}$ (*i.e.* ${}_*A^- \ge {}^*A^-$ and ${}_*A^+ \le {}^*A^+$). Further,

- $x \in {}_*A^{\pm} \to x \in_s A_R^{\pm}$ (*i.e.* x surely belongs to $A_R^{\pm}$);
- $x \in {}^*A^{\pm} \to x \in_p A_R^{\pm}$ (*i.e.* x possibly belongs to $A_R^{\pm}$);
- $x \notin {}^*A^{\pm} \to x \notin_s A_R^{\pm}$ (*i.e.* x does not surely belong to $A_R^{\pm}$).

Example 2.1. Let $A_R^{\pm}$ be normal height for men as,

$$A_R^{\pm} = \left({}_*A^{\pm}, {}^*A^{\pm}\right) = \left([165, 175], [155, 185]\right),$$

which is shown in Figure 1. According to $A_R^{\pm}$, normal height for men is surely between 165 cm and 175 cm and it is possibly between 155 cm and 185 cm. Therefore, it could be said that a man with the height less than 155 cm or more than 185 cm is not surely normal.

RI arithmetic is usually based on interval arithmetic [1]. Suppose that $A_R^{\pm} = ({}_*A^{\pm}, {}^*A^{\pm})$ and $B_R^{\pm} = ({}_*B^{\pm}, {}^*B^{\pm})$ are two RIs. Then,

- $A_R^{\pm} + B_R^{\pm} = \left(\left[{}_*A^- + {}_*B^-, {}_*A^+ + {}_*B^+\right], \left[{}^*A^- + {}^*B^-, {}^*A^+ + {}^*B^+\right]\right).$
- $-A_R^{\pm} = \left(\left[-{}_*A^+, -{}_*A^-\right], \left[-{}^*A^+, -{}^*A^-\right]\right).$
- $A_R^{\pm} - B_R^{\pm} = \left(\left[{}_*A^- - {}_*B^+, {}_*A^+ - {}_*B^-\right], \left[{}^*A^- - {}^*B^+, {}^*A^+ - {}^*B^-\right]\right).$

As an instance, consider $A_R^{\pm} = \left([5, 8], [3, 10]\right)$ and $B_R^{\pm} = \left([1, 4], [0, 5]\right)$. Then,

$$\begin{aligned} A_R^{\pm} + B_R^{\pm} &= \left([6, 12], [3, 15]\right) \\ A_R^{\pm} - B_R^{\pm} &= \left([1, 7], [-2, 10]\right) \\ -A_R^{\pm} &= \left([-8, -5], [-10, -3]\right). \end{aligned}$$

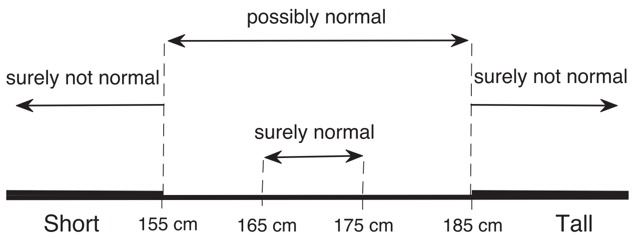

FIGURE 1. Normal height for men represented as a RI.

3. MAIN RESULTS

Generally, a rough interval linear programming (RILP) problem is considered as,

$$\begin{aligned} \min \quad & \sum_{j=1}^{n} \left({}_*c_j^{\pm}, {}^*c_j^{\pm}\right) x_j \\ \text{s.t.} \quad & \sum_{j=1}^{n} \left({}_*a_{ij}^{\pm}, {}^*a_{ij}^{\pm}\right) x_j \geq \left({}_*b_i^{\pm}, {}^*b_i^{\pm}\right) \qquad i = 1, 2, \ldots, m \\ & x_j \geq 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad j = 1, 2, \ldots, n. \end{aligned} \tag{1}$$

The characteristic problem of RILP problem (1) is as,

$$\begin{aligned} \min \quad & \sum_{j=1}^{n} \left({}_*c_j^{\circ}, {}^*c_j^{\circ}\right) x_j \\ \text{s.t.} \quad & \sum_{j=1}^{n} \left({}_*a_{ij}^{\circ}, {}^*a_{ij}^{\circ}\right) x_j \geq \left({}_*b_i^{\circ}, {}^*b_i^{\circ}\right) \qquad i = 1, 2, \ldots, m \\ & x_j \geq 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad j = 1, 2, \ldots, n, \end{aligned} \tag{2}$$

where ${}_*c_j^{\circ} \in {}_*c_j^{\pm}$, ${}^*c_j^{\circ} \in {}^*c_j^{\pm}$, ${}_*a_{ij}^{\circ} \in {}_*a_{ij}^{\pm}$, ${}^*a_{ij}^{\circ} \in {}^*a_{ij}^{\pm}$, ${}_*b_i^{\circ} \in {}_*b_i^{\pm}$, and ${}^*b_i^{\circ} \in {}^*b_i^{\pm}$.

Definition 3.1. Let a_1, a_2, b_1, and b_2 be real numbers. Then,

$$\left(a_1, a_2\right) \geq \left(b_1, b_2\right) \Longleftrightarrow a_1 \geq b_1, a_2 \geq b_2.$$

Theorem 3.1. *For the RI linear constraint set,*

$$\begin{cases} \sum_{j=1}^{n} \left({}_*a_{ij}^{\pm}, {}^*a_{ij}^{\pm}\right) x_j \geq \left({}_*b_i^{\pm}, {}^*b_i^{\pm}\right), & i = 1, 2, \ldots, m, \\ x_j \geq 0, & j = 1, 2, \ldots, n, \end{cases}$$

the largest feasible space (LFS) and the smallest feasible space (SFS) are as follows, respectively:

$$\begin{cases} \sum_{j=1}^{n} {}^*a_{ij}^{+} x_j \geq {}^*b_i^{-}, & i = 1, 2, \ldots, m \\ x_j \geq 0, & j = 1, 2, \ldots, n \end{cases} \tag{3}$$

and

$$\begin{cases} \sum_{j=1}^{n} {}^*a_{ij}^{-} x_j \geq {}^*b_i^{+}, & i = 1, 2, \ldots, m \\ x_j \geq 0, & j = 1, 2, \ldots, n. \end{cases} \tag{4}$$

Proof. Consider the following characteristic constraint:

$$\begin{cases}\sum_{j=1}^{n}\big({}_*a_{ij}^{\circ},{}^*a_{ij}^{\circ}\big)x_j \geq ({}_*b_i^{\circ},{}^*b_i^{\circ}), & i=1,2,\ldots,m\\ x_j \geq 0, & j=1,2,\ldots,n,\end{cases}$$

where for each $i=1,2,\ldots,m$ and $j=1,2,\ldots,n$, ${}_*a_{ij}^{\circ}\in{}_*a_{ij}^{\pm}$, ${}^*a_{ij}^{\circ}\in{}^*a_{ij}^{\pm}$, ${}_*b_i^{\circ}\in{}_*b_i^{\pm}$, and ${}^*b_i^{\circ}\in{}^*b_i^{\pm}$. Since ${}_*a_{ij}^{\circ}\in{}_*a_{ij}^{\pm}$, ${}^*a_{ij}^{\circ}\in{}^*a_{ij}^{\pm}$, and $x_j\geq 0$, then

$$\sum_{j=1}^{n}\big({}_*a_{ij}^{+},{}^*a_{ij}^{+}\big)x_j \geq \sum_{j=1}^{n}\big({}_*a_{ij}^{\circ},{}^*a_{ij}^{\circ}\big)x_j \geq ({}_*b_i^{\circ},{}^*b_i^{\circ}) \geq \big({}_*b_i^{-},{}^*b_i^{-}\big), \quad \forall\, i=1,\ldots,m.$$

According to Definition 3.1, it is concluded that

$$\sum_{j=1}^{n}{}_*a_{ij}^{+}x_j \geq {}_*b_i^{-}\ ,\quad \sum_{j=1}^{n}{}^*a_{ij}^{+}x_j \geq {}^*b_i^{-}.$$

Obviously, $\sum_{j=1}^{n}{}^*a_{ij}^{+}x_j \geq \sum_{j=1}^{n}{}_*a_{ij}^{+}x_j$ and ${}_*b_i^{-}\geq{}^*b_i^{-}$, hence

$$\sum_{j=1}^{n}{}^*a_{ij}^{+}x_j \geq \sum_{j=1}^{n}{}_*a_{ij}^{+}x_j \geq {}_*b_i^{-} \geq {}^*b_i^{-}.$$

Finally, the LFS could be written as:

$$\begin{cases}\sum_{j=1}^{n}{}^*a_{ij}^{+}x_j \geq {}^*b_i^{-}, & i=1,2,\ldots,m\\ x_j\geq 0, & j=1,2,\ldots,n.\end{cases}$$

Similarly, for the SFS, we obtain

$$\begin{cases}\sum_{j=1}^{n}{}^*a_{ij}^{-}x_j \geq {}^*b_i^{+}, & i=1,2,\ldots,m\\ x_j\geq 0, & j=1,2,\ldots,n.\end{cases}$$

□

Example 3.1. For the RI linear inequality

$$\begin{cases}\big([3,4],[1,5]\big)x_1+\big([2,3],[1,6]\big)x_2 \geq \big([4,5],[2,6]\big)\\ x_1,x_2\geq 0,\end{cases}$$

the LFS is obtained by solving the following system:

$$\begin{cases}5x_1+6x_2\geq 2\\ x_1,x_2\geq 0,\end{cases}$$

and the SFS is achieved by solving the following system:

$$\begin{cases}x_1+x_2\geq 6\\ x_1,x_2\geq 0.\end{cases}$$

Two spaces have been shown in Figures 2 and 3. This can be seen that the SFS lies in the LFS.

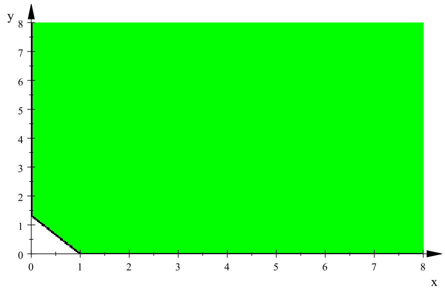

FIGURE 2. The LFS of Example 3.1.

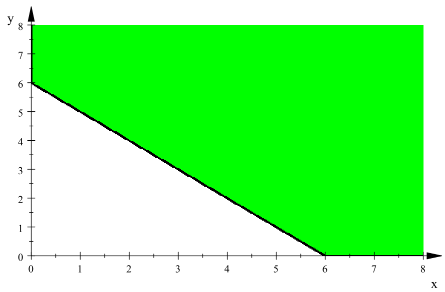

FIGURE 3. The SFS of Example 3.1.

In what follows, it is tried to determine surely and possibly OVOF of the RILP problem.

Theorem 3.2. *The RI for OVOF of RILP problem* (1) *is given by* $({}_*z^{\pm}, {}^*z^{\pm})$, *where*

$$\begin{aligned} {}^*z^- = \min\ & \sum_{j=1}^{n} {}^*c_j^- x_j \\ \text{s.t.}\ & \sum_{j=1}^{n} {}^*a_{ij}^+ x_j \geq {}^*b_i^-, \quad i = 1, \ldots, m \\ & x_j \geq 0 \qquad\qquad j = 1, \ldots, n, \end{aligned} \tag{5}$$

$$\begin{aligned} {}^*z^+ = \min\ & \sum_{j=1}^{n} {}^*c_j^+ x_j \\ \text{s.t.}\ & \sum_{j=1}^{n} {}^*a_{ij}^- x_j \geq {}^*b_i^+, \quad i = 1, \ldots, m \\ & x_j \geq 0 \qquad\qquad j = 1, \ldots, n, \end{aligned} \tag{6}$$

$$\begin{aligned} {}_*z^- = \min\ & \sum_{j=1}^{n} {}_*c_j^- x_j \\ \text{s.t.}\ & \sum_{j=1}^{n} {}_*a_{ij}^+ x_j \geq {}_*b_i^-, \quad i = 1, \ldots, m \\ & x_j \geq 0 \qquad\qquad j = 1, \ldots, n, \end{aligned} \tag{7}$$

$$\begin{aligned} {}_*z^+ = \min \ & \sum_{j=1}^{n} {}_*c_j^+ x_j \\ \text{s.t. } & \sum_{j=1}^{n} {}_*a_{ij}^- x_j \geq {}_*b_i^+, \quad i = 1, \ldots, m \\ & x_j \geq 0 \qquad\qquad j = 1, \ldots, n. \end{aligned} \tag{8}$$

Proof. Suppose $({}_*z^\circ, {}^*z^\circ)$ and $({}_*\mathbf{x}^\circ, {}^*\mathbf{x}^\circ)$ are the OVOF and OSs of problem (2), respectively. Therefore ${}^*\mathbf{x}^\circ$ is the OS of the following problem

$$\begin{aligned} \min \ & \sum_{j=1}^{n} {}^*c_j^\circ x_j \\ \text{s.t. } & \sum_{j=1}^{n} {}^*a_{ij}^\circ x_j \geq {}^*b_i^\circ, \quad i = 1, \ldots, m \\ & x_j \geq 0 \qquad\qquad j = 1, \ldots, n \end{aligned} \tag{9}$$

and ${}_*\mathbf{x}^\circ$ is the OS of the following problem

$$\begin{aligned} \min \ & \sum_{j=1}^{n} {}_*c_j^\circ x_j \\ \text{s.t. } & \sum_{j=1}^{n} {}_*a_{ij}^\circ x_j \geq {}_*b_i^\circ, \quad i = 1, \ldots, m \\ & x_j \geq 0 \qquad\qquad j = 1, \ldots, n. \end{aligned} \tag{10}$$

It should be proved that ${}^*z^\circ \in {}^*z^\pm$ and ${}_*z^\circ \in {}_*z^\pm$.

In order to do that, let ${}^*\mathbf{x}^1$ and ${}^*\mathbf{x}^2$ be the OSs of problems (5) and (6), respectively. Since ${}^*\mathbf{x}^\circ$ is the OS of problem (9), then it is a feasible solution of problem (9). Problem (5) has the LFS, therefore ${}^*\mathbf{x}^\circ$ is a feasible solution of problem (5). Since ${}^*\mathbf{x}^1$ is the OS of problem (5), then

$${}^*z^\circ = \sum_{j=1}^{n} {}^*c_j^\circ {}^*x_j^\circ \geq \sum_{j=1}^{n} {}^*c_j^- {}^*x_j^\circ \geq \sum_{j=1}^{n} {}^*c_j^- {}^*x_j^1 = {}^*z^-.$$

It is known that ${}^*\mathbf{x}^2$ is the OS of problem (6), hence it could be concluded that it is a feasible solution of problem (6). Problem (6) has the SFS, therefore ${}^*\mathbf{x}^2$ is a feasible solution of problem (9). Since ${}^*\mathbf{x}^\circ$ is the OS of problem (9), then

$${}^*z^+ = \sum_{j=1}^{n} {}^*c_j^+ {}^*x_j^2 \geq \sum_{j=1}^{n} {}^*c_j^\circ {}^*x_j^2 \geq \sum_{j=1}^{n} {}^*c_j^\circ {}^*x_j^\circ = {}^*z^\circ.$$

Therefore ${}^*z^+ \geq {}^*z^\circ \geq {}^*z^-$, and ${}^*z^\circ \in {}^*z^\pm$ are obtained.

According to the fact that feasible space of problem (7) (problem (8)) is larger (smaller) than the feasible space of problem (10), similarly, we can prove that ${}_*z^\circ \in {}_*z^\pm$. Thus, $({}_*z^\circ, {}^*z^\circ) \in ({}_*z^\pm, {}^*z^\pm)$.

To obtain UAI and LAI for the OSs of RILP problem (1), the following sets, $\forall\ i = 1, 2, \ldots, m$, are defined as,

$$\begin{aligned} {}^*S_i^+ &= \left\{ (x_1, x_2, \ldots, x_n) : \ \textstyle\sum_{j=1}^{n} {}^*a_{ij}^+ x_j \geq {}^*b_i^-, \ x_j \geq 0, \ j = 1, 2, \ldots, n \right\}, \\ {}^*S_i^- &= \left\{ (x_1, x_2, \ldots, x_n) : \ \textstyle\sum_{j=1}^{n} {}^*a_{ij}^- x_j \leq {}^*b_i^+, \ x_j \geq 0, \ j = 1, 2, \ldots, n \right\}, \\ {}_*S_i^+ &= \left\{ (x_1, x_2, \ldots, x_n) : \ \textstyle\sum_{j=1}^{n} {}_*a_{ij}^+ x_j \geq {}_*b_i^-, \ x_j \geq 0, \ j = 1, 2, \ldots, n \right\}, \\ {}_*S_i^- &= \left\{ (x_1, x_2, \ldots, x_n) : \ \textstyle\sum_{j=1}^{n} {}_*a_{ij}^- x_j \leq {}_*b_i^+, \ x_j \geq 0, \ j = 1, 2, \ldots, n \right\}. \end{aligned}$$

Note that ${}^*S_i^+$ and ${}_*S_i^+$ are the feasible spaces of problems (5) and (7), respectively. Also, ${}^*S_i^-$ and ${}_*S_i^-$ are the feasible spaces of problems (6) and (8) with inverse sign, respectively. □

Theorem 3.3. *Suppose that in RILP problem (1), $m = n$ and all feasible solution components of ${}^*S_i^+$ (for each i) are positive and also the OVOF of problem (5) is finite. Then $({}_*\mathbf{x}^\pm, {}^*\mathbf{x}^\pm)$ is the RI for OSs of problem (1), in which*

$${}^*\mathbf{x}^\pm = \left(\cap_{i=1}^m {}^*S_i^+\right) \cap \left(\cap_{i=1}^m {}^*S_i^-\right), \;\; {}_*\mathbf{x}^\pm = \left(\cap_{i=1}^m {}_*S_i^+\right) \cap \left(\cap_{i=1}^m {}_*S_i^-\right).$$

Proof. Firstly, we prove ${}^*\mathbf{x}^\pm = \left(\cap_{i=1}^m {}^*S_i^+\right) \cap \left(\cap_{i=1}^m {}^*S_i^-\right)$.

($\Longrightarrow$) Suppose that $\left({}_*\mathbf{x}^\circ, {}^*\mathbf{x}^\circ\right)$ is the OS of problem (2). Therefore ${}^*\mathbf{x}^\circ$ is the OS of the problem

$$\begin{aligned} \min \; & \sum_{j=1}^n {}^*c_j^\circ x_j \\ \text{s.t.} \; & \sum_{j=1}^n {}^*a_{ij}^\circ x_j \geq {}^*b_i^\circ, \quad i = 1, \dots, m \\ & x_j \geq 0 \qquad\qquad\quad j = 1, \dots, n, \end{aligned} \tag{11}$$

and ${}_*\mathbf{x}^\circ$ is the OS of the problem

$$\begin{aligned} \min \; & \sum_{j=1}^n {}_*c_j^\circ x_j \\ \text{s.t.} \; & \sum_{j=1}^n {}_*a_{ij}^\circ x_j \geq {}_*b_i^\circ, \quad i = 1, \dots, m \\ & x_j \geq 0 \qquad\qquad\quad j = 1, \dots, n. \end{aligned} \tag{12}$$

□

It should be shown that ${}^*\mathbf{x}^\circ \in {}^*\mathbf{x}$ and ${}_*\mathbf{x}^\circ \in {}_*\mathbf{x}$, or equivalently ${}^*\mathbf{x}^\circ \in \left(\cap_{i=1}^m {}^*S_i^+\right) \cap \left(\cap_{i=1}^m {}^*S_i^-\right)$ and ${}_*\mathbf{x}^\circ \in \left(\cap_{i=1}^m {}_*S_i^+\right) \cap \left(\cap_{i=1}^m {}_*S_i^-\right)$. The proof of the first one is proposed in the following. The second can be proved similarly.

Due to $m = n$ and all feasible solution components of ${}^*S_i^+$ (for each i) are positive and also the OVOF of problem (5) is finite, then all constraints of problem (11) are binding in ${}^*\mathbf{x}^\circ$, so for each $i = 1, 2, \dots, m$, $\sum_{j=1}^n {}^*a_{ij}^\circ {}^*x_j^\circ = {}^*b_i^\circ$. Since ${}^*x_j^\circ \geq 0$, then

$$\sum_{j=1}^n {}^*a_{ij}^+ {}^*x_j^\circ \geq \sum_{j=1}^n {}^*a_{ij}^\circ {}^*x_j^\circ = {}^*b_i^\circ \geq {}^*b_i^-, \quad i = 1, 2, \dots, m$$

$$\sum_{j=1}^n {}^*a_{ij}^- {}^*x_j^\circ \leq \sum_{j=1}^n {}^*a_{ij}^\circ {}^*x_j^\circ = {}^*b_i^\circ \leq {}^*b_i^+, \quad i = 1, 2, \dots, m$$

and ${}^*\mathbf{x}^\circ \in {}^*S_i^+$ and ${}^*\mathbf{x}^\circ \in {}^*S_i^-$, $i = 1, 2, \dots, m$. Hence ${}^*\mathbf{x}^\circ \in \left(\cap_{i=1}^m {}^*S_i^+\right) \cap \left(\cap_{i=1}^m {}^*S_i^-\right)$.

($\Longleftarrow$) Suppose that ${}^*\mathbf{x}^\circ \in \left(\cap_{i=1}^m {}^*S_i^+\right) \cap \left(\cap_{i=1}^m {}^*S_i^-\right)$ and ${}_*\mathbf{x}^\circ \in \left(\cap_{i=1}^m {}_*S_i^+\right) \cap \left(\cap_{i=1}^m {}_*S_i^-\right)$. It is necessary to prove that there is a characteristic problem as

$$\begin{aligned} \min \quad & \sum_{j=1}^n ({}_*c_j^\diamond, {}^*c_j^\diamond) x_j \\ \text{s.t.} \quad & \sum_{j=1}^n ({}_*a_{ij}^\diamond, {}^*a_{ij}^\diamond) x_j \geq ({}_*b_i^\diamond, {}^*b_i^\diamond) \quad i = 1, 2, \dots, m \\ & x_j \geq 0 \qquad\qquad\qquad\qquad\qquad\quad j = 1, 2, \dots, n, \end{aligned}$$

such that ${}^*\mathbf{x}^\circ$ is the OS of the problem

$$\begin{aligned} \min \ & \sum_{j=1}^{n} {}^*c_j^\diamond x_j \\ \text{s.t.} \ & \sum_{j=1}^{n} {}^*a_{ij}^\diamond x_j \geq {}^*b_i^\diamond, \quad i = 1, \dots, m \\ & x_j \geq 0 \qquad\qquad\qquad j = 1, \dots, n, \end{aligned} \tag{13}$$

and ${}_*\mathbf{x}^\circ$ is the OS of the problem

$$\begin{aligned} \min \ & \sum_{j=1}^{n} {}_*c_j^\diamond x_j \\ \text{s.t.} \ & \sum_{j=1}^{n} {}_*a_{ij}^\diamond x_j \geq {}_*b_i^\diamond, \quad i = 1, \dots, m \\ & x_j \geq 0 \qquad\qquad\qquad j = 1, \dots, n. \end{aligned} \tag{14}$$

The first one is proved in the following. The second can be proved similarly.

Due to ${}^*\mathbf{x}^\circ \in \left(\cap_{i=1}^m {}^*S_i^+\right) \cap \left(\cap_{i=1}^m {}^*S_i^-\right)$, we have

$$\begin{aligned} & \sum_{j=1}^{n} {}^*a_{ij}^{+}\,{}^*x_j^\circ \geq {}^*b_i^-, \quad i = 1, 2, \dots, m, \\ & \sum_{j=1}^{n} {}^*a_{ij}^{-}\,{}^*x_j^\circ \leq {}^*b_i^+, \quad i = 1, 2, \dots, m, \\ & {}^*x_j^\circ \geq 0, \qquad\qquad\quad j = 1, 2, \dots, n. \end{aligned}$$

By considering the definition of radius and center and rewriting the above inequalities,

$$\begin{aligned} & \sum_{j=1}^{n} \left(\left({}^*a_{ij}^\pm\right)^c + \left({}^*a_{ij}^\pm\right)^\Delta \right) {}^*x_j^\circ \geq \left({}^*b_i^\pm\right)^c - \left({}^*b_i^\pm\right)^\Delta, \quad i = 1, 2, \dots, m \\ & \sum_{j=1}^{n} \left(\left({}^*a_{ij}^\pm\right)^c - \left({}^*a_{ij}^\pm\right)^\Delta \right) {}^*x_j^\circ \leq \left({}^*b_i^\pm\right)^c + \left({}^*b_i^\pm\right)^\Delta, \quad i = 1, 2, \dots, m \\ & {}^*x_j^\circ \geq 0, \qquad\qquad\qquad\qquad\qquad\qquad\qquad j = 1, 2, \dots, n, \end{aligned}$$

are obtained. Therefore, $\forall i = 1, \dots, m$,

$$-\left(\sum_{j=1}^{n} \left({}^*a_{ij}^\pm\right)^\Delta {}^*x_j^\circ + \left({}^*b_i^\pm\right)^\Delta \right) \leq \sum_{j=1}^{n} \left({}^*a_{ij}^\pm\right)^c {}^*x_j^\circ - \left({}^*b_i^\pm\right)^c \leq \sum_{j=1}^{n} \left({}^*a_{ij}^\pm\right)^\Delta {}^*x_j^\circ + \left({}^*b_i^\pm\right)^\Delta,$$

and

$$\left| \sum_{j=1}^{n} \left({}^*a_{ij}^\pm\right)^c {}^*x_j^\circ - \left({}^*b_i^\pm\right)^c \right| \leq \sum_{j=1}^{n} \left({}^*a_{ij}^\pm\right)^\Delta {}^*x_j^\circ + \left({}^*b_i^\pm\right)^\Delta, \quad i = 1, 2, \dots, m.$$

TABLE 1. The process of solving RILP problem (1).

– Obtaining RI for OVOF
1) Solve four LP problems (5)–(8).
2) Save the OVOF of these problems as ${}^*z^-, {}^*z^+, {}_*z^-$, and ${}_*z^+$.
3) $({}_*z^\pm, {}^*z^\pm)$ is RI for OVOF.

– Obtaining RI for OSs
1) Check $m = n$, all feasible solution components of ${}^*S_i^+$ are positive and ${}^*z^-$ is finite.
2) Obtain ${}^*S_i^+, {}^*S_i^-, {}_*S_i^+, {}_*S_i^+$.
3) Obtain ${}^*\mathbf{x}^\pm$ and ${}_*\mathbf{x}^\pm$ as follows:
$${}^*\mathbf{x}^\pm = \left(\cap_{i=1}^m {}^*S_i^+\right) \cap \left(\cap_{i=1}^m {}^*S_i^-\right),\ {}_*\mathbf{x}^\pm = \left(\cap_{i=1}^m {}_*S_i^+\right) \cap \left(\cap_{i=1}^m {}_*S_i^-\right).$$
4) $\left({}_*\mathbf{x}^\pm, {}^*\mathbf{x}^\pm\right)$ is RI for OSs.

By defining the vector $\mathbf{y} \in \mathbb{R}^n$ as

$$y_i = \begin{cases} \frac{\sum_{j=1}^n \left({}^*a_{ij}^\pm\right)^c {}^*x_j^\circ - \left({}^*b_i^\pm\right)^c}{\sum_{j=1}^n \left({}^*a_{ij}^\pm\right)^\Delta {}^*x_j^\circ + \left({}^*b_i^\pm\right)^\Delta} & \sum_{j=1}^n \left({}^*a_{ij}^\pm\right)^\Delta {}^*x_j^\circ + \left({}^*b_i^\pm\right)^\Delta > 0 \\ 1 & \sum_{j=1}^n \left({}^*a_{ij}^\pm\right)^\Delta {}^*x_j^\circ + \left({}^*b_i^\pm\right)^\Delta = 0, \end{cases}$$

it is concluded that $|y_i| \leq 1$, and

$$\sum_{j=1}^n \left({}^*a_{ij}^\pm\right)^c {}^*x_j^\circ - \left({}^*b_i^\pm\right)^c = y_i \left(\sum_{j=1}^n \left({}^*a_{ij}^\pm\right)^\Delta {}^*x_j^\circ + \left({}^*b_i^\pm\right)^\Delta\right), \quad i = 1, 2, \ldots, m.$$

Therefore,

$$\sum_{j=1}^n \left(\left({}^*a_{ij}^\pm\right)^c - y_i \left({}^*a_{ij}^\pm\right)^\Delta\right) {}^*x_j^\circ = \left({}^*b_i^\pm\right)^c + y_i \left({}^*b_i^\pm\right)^\Delta, \quad i = 1, 2, \ldots, m.$$

Considering $|y_i| \leq 1$,

$${}^*a_{ij}^\diamond = \left({}^*a_{ij}^\pm\right)^c - y_i \left({}^*a_{ij}^\pm\right)^\Delta \in {}^*a_{ij}^\pm,$$

and

$${}^*b_i^\diamond = \left({}^*b_i^\pm\right)^c + y_i \left({}^*b_i^\pm\right)^\Delta \in {}^*b_i^\pm,$$

are achieved. Hence ${}^*\mathbf{x}^\circ$ is a feasible solution of problem (13) such that its constraints are binding in ${}^*\mathbf{x}^\circ$.

According to the assumption, ${}^*z^-$ (the OVOF of problem (5)) is finite. Since ${}^*z^- \leq {}_*z^- \leq {}_*z^+ \leq {}^*z^+$, and the problem is minimization, then all characteristic problems are finite. Therefore the OVOF of problem (13) is finite. Due to the facts that $m = n$ and all feasible solution components of ${}^*S_i^+$ (for each i) are positive, therefore ${}^*\mathbf{x}^\circ$ is the only extreme point and hence ${}^*\mathbf{x}^\circ$ is the OS of problem (13). The proof of ${}_*\mathbf{x}^\circ$ is the OS of problem (14), could be done similarly.

The proposed results are summerized in Table 1. As an application of the results given in Section 3, a rough interval fixed-charge transportation problem is generally investigated in the following section.

4. Solving a rough interval fixed-charge transportation problem

In general, a rough interval fixed-charge transportation problem (RIFCTP) is considered as

$$\begin{aligned} \min\ & \sum_{i=1}^{m}\sum_{j=1}^{n}\left(\left({}_*c_{ij}^{\pm},{}^*c_{ij}^{\pm}\right)x_{ij}+\left({}_*f_{ij}^{\pm},{}^*f_{ij}^{\pm}\right)y_{ij}\right) & \\ \text{s.t.}\ & \sum_{i=1}^{m} x_{ij} \geq \left({}_*b_j^{\pm},{}^*b_j^{\pm}\right), & j=1,2,\ldots,n, \\ & \sum_{j=1}^{n} x_{ij} \leq \left({}_*a_i^{\pm},{}^*a_i^{\pm}\right), & i=1,2,\ldots,m, \\ & x_{ij}\geq 0 & \forall i,j, \\ & y_{ij}=\begin{cases}0 & x_{ij}=0\\ 1 & x_{ij}>0,\end{cases} & \end{aligned} \tag{15}$$

in which x_{ij} is the amount transported from the ith source to the jth destination, y_{ij} is the binary variable taking the value of 1 when source i is used and 0 otherwise. Furthermore, $({}_*c_{ij}^{\pm},{}^*c_{ij}^{\pm})$ is the rough shipping cost per unit amount for transporting from the ith source to the jth destination, $({}_*f_{ij}^{\pm},{}^*f_{ij}^{\pm})$ is the rough fixed charge associated with the ith source to the jth destination, $({}_*a_i^{\pm},{}^*a_i^{\pm})$ and $({}_*b_j^{\pm},{}^*b_j^{\pm})$ are the rough capacity of the ith source point and the demand of the jth destination point, respectively.

To obtain a RI for OVOF of the RIFCTP (*i.e.* $({}_*z^{\pm},{}^*z^{\pm})$) by using Theorem 3.2, the following problems should be solved:

$$\begin{aligned} {}^*z^{-} = \min\ & \sum_{i=1}^{m}\sum_{j=1}^{n}\left({}^*c_{ij}^{-}x_{ij}+{}^*f_{ij}^{-}y_{ij}\right) & \\ \text{s.t.}\ & \sum_{i=1}^{m} x_{ij} \geq {}^*b_j^{-}, & j=1,2,\ldots,n, \\ & \sum_{j=1}^{n} x_{ij} \leq {}^*a_i^{+}, & i=1,2,\ldots,m, \\ & x_{ij}\geq 0 & \forall i,j, \\ & y_{ij}=\begin{cases}0 & x_{ij}=0\\ 1 & x_{ij}>0\end{cases} & \\ {}^*z^{+} = \min\ & \sum_{i=1}^{m}\sum_{j=1}^{n}\left({}^*c_{ij}^{+}x_{ij}+{}^*f_{ij}^{+}y_{ij}\right) & \\ \text{s.t.}\ & \sum_{i=1}^{m} x_{ij} \geq {}^*b_j^{+}, & j=1,2,\ldots,n, \\ & \sum_{j=1}^{n} x_{ij} \leq {}^*a_i^{-}, & i=1,2,\ldots,m, \\ & x_{ij}\geq 0 & \forall i,j, \\ & y_{ij}=\begin{cases}0 & x_{ij}=0\\ 1 & x_{ij}>0\end{cases} & \\ {}_*z^{-} = \min\ & \sum_{i=1}^{m}\sum_{j=1}^{n}\left({}_*c_{ij}^{-}x_{ij}+{}_*f_{ij}^{-}y_{ij}\right) & \end{aligned}$$

$$\begin{aligned}
&\text{s.t.}\ \sum_{i=1}^{m} x_{ij} \geq {}_*b_j^-, && j=1,2,\ldots,n,\\
&\sum_{j=1}^{n} x_{ij} \leq {}_*a_i^+, && i=1,2,\ldots,m,\\
&x_{ij} \geq 0 && \forall i,j,\\
&y_{ij} = \begin{cases} 0 & x_{ij}=0\\ 1 & x_{ij}>0 \end{cases}\\
{}_*z^+ = \min\ &\sum_{i=1}^{m}\sum_{j=1}^{n} \left({}_*c_{ij}^+ x_{ij} + {}_*f_{ij}^+ y_{ij}\right)\\
&\text{s.t.}\ \sum_{i=1}^{m} x_{ij} \geq {}_*b_j^+, && j=1,2,\ldots,n,\\
&\sum_{j=1}^{n} x_{ij} \leq {}_*a_i^-, && i=1,2,\ldots,m,\\
&x_{ij} \geq 0 && \forall i,j,\\
&y_{ij} = \begin{cases} 0 & x_{ij}=0\\ 1 & x_{ij}>0. \end{cases}
\end{aligned}$$

5. Numerical examples

In this following, the results are illustrated in some numerical examples.

Example 5.1. Consider the following RILP:

$$\begin{aligned}
&\min\ \big([1,3],[1,5]\big)x_1 + \big([1,2],[1,3]\big)x_2\\
&\text{s.t.}\ \left(\left[-\frac{1}{4},0\right],\left[-\frac{1}{2},0\right]\right)x_1 + \left(\left[\frac{11}{4},3\right],\left[\frac{5}{2},3\right]\right)x_2 \geq \big([4,5],[3,5]\big)\\
&\left(\left[\frac{7}{2},\frac{7}{2}\right],[3,4]\right)x_1 + \left(\left[-\frac{3}{2},-1\right],[-2,-1]\right)x_2 \geq \big([2,4],[0,6]\big)\\
&x_1,x_2 \geq 0.
\end{aligned} \tag{16}$$

By using Theorem 3.2, surely and possibly OVOF are obtained by solving four problems which have been given in Table 2. Hence, the RI for the OVOF is $\left(\left[\frac{16}{7},10\right],\left[\frac{5}{4},\frac{358}{13}\right]\right)$, where $\left[\frac{16}{7},10\right]$ and $\left[\frac{5}{4},\frac{358}{13}\right]$ are surely and possibly OVOF, respectively.

For obtaining RI of OSs, note that $m=n=2$. Also, all feasible solution components of ${}^*S_i^+$, for $i=1,2$, are positive (see Fig. 4).

$$\begin{aligned}
{}^*S_1^+ &= \left\{ (x_1,x_2):\ 0x_1+3x_2 \geq 3, x_1,x_2 \geq 0 \right\},\\
{}^*S_2^+ &= \left\{ (x_1,x_2):\ 4x_1-x_2 \geq 0, x_1,x_2 \geq 0 \right\},\\
{}^*S_1^- &= \left\{ (x_1,x_2):\ -\tfrac{1}{2}x_1+\tfrac{5}{2}x_2 \leq 5, x_1,x_2 \geq 0 \right\},\\
{}^*S_2^- &= \left\{ (x_1,x_2):\ 3x_1-2x_2 \leq 6, x_1,x_2 \geq 0 \right\},\\
{}_*S_1^+ &= \left\{ (x_1,x_2):\ 0x_1+3x_2 \geq 4, x_1,x_2 \geq 0 \right\},\\
{}_*S_2^+ &= \left\{ (x_1,x_2):\ \tfrac{7}{2}x_1-x_2 \geq 2, x_1,x_2 \geq 0 \right\},\\
{}_*S_1^- &= \left\{ (x_1,x_2):\ -\tfrac{1}{4}x_1+\tfrac{11}{4}x_2 \leq 5, x_1,x_2 \geq 0 \right\},\\
{}_*S_2^- &= \left\{ (x_1,x_2):\ \tfrac{7}{2}x_1-\tfrac{3}{2}x_2 \leq 4, x_1,x_2 \geq 0 \right\}.
\end{aligned}$$

TABLE 2. Problems and solutions for RILP problem (16).

Problems	Solutions
$\min\ x_1+x_2$ s.t. $0x_1+3x_2\geq 3$ $4x_1-x_2\geq 0$ $x_1,x_2\geq 0$	${}^*z^-=\frac{5}{4}$ ${}^*\mathbf{x}^1=(\frac{1}{4},1)^t$
$\min\ 5x_1+3x_2$ s.t. $-\frac{1}{2}x_1+\frac{5}{2}x_2\geq 5$ $3x_1-2x_2\geq 6$ $x_1,x_2\geq 0$	${}^*z^+=\frac{358}{13}$ ${}^*\mathbf{x}^2=(\frac{50}{13},\frac{36}{13})^t$
$\min\ x_1+x_2$ s.t. $0x_1+3x_2\geq 4$ $\frac{7}{2}x_1-x_2\geq 2$ $x_1,x_2\geq 0$	${}_*z^-=\frac{16}{7}$ ${}_*\mathbf{x}^1=(\frac{20}{21},\frac{4}{3})^t$
$\min\ 3x_1+2x_2$ s.t. $-\frac{1}{4}x_1+\frac{11}{4}x_2\geq 5$ $\frac{7}{2}x_1-\frac{3}{2}x_2\geq 4$ $x_1,x_2\geq 0$	${}_*z^+=10$ ${}_*\mathbf{x}^2=(2,2)^t$

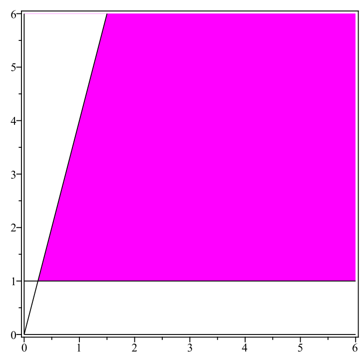

FIGURE 4. ${}^*S_i^+$ for $i=1,2$, for RILP (16).

According to Theorem 3.3, $\left({}_*\mathbf{x}^{\pm},{}^*\mathbf{x}^{\pm}\right)$ is the RI for OSs of problem (16), where

$$ {}^*\mathbf{x}^{\pm}=\left(\cap_{i=1}^{2}\,{}^*S_i^+\right)\cap\left(\cap_{i=1}^{2}\,{}^*S_i^-\right),\ \ {}_*\mathbf{x}^{\pm}=\left(\cap_{i=1}^{2}\,{}_*S_i^+\right)\cap\left(\cap_{i=1}^{2}\,{}_*S_i^-\right). $$

The results are given in Table 2 and Figure 5. The red and blue spaces show UAI and LAI of OSs, respectively. In Figure 5, it can be seen that the red space (UAI) includes the blue space (LAI). Therefore, the RI for OSs is as follows:

$$ \begin{pmatrix} \left(\left[\frac{20}{21},2\right],\left[\frac{1}{4},\frac{50}{13}\right]\right) \\ \left(\left[\frac{4}{3},2\right],\left[1,\frac{36}{13}\right]\right) \end{pmatrix}. $$

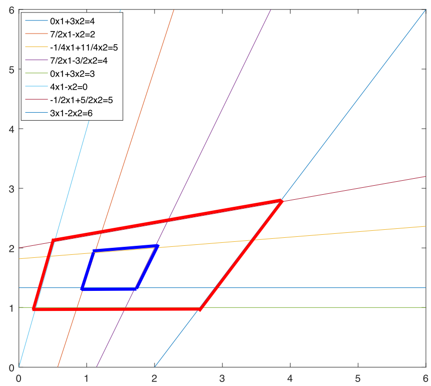

FIGURE 5. UAI and LAI spaces for OSs of RILP (16).

TABLE 3. Possibly and surely OVOF and OSs of Example 5.2 for $m = n = 5$.

	Surely	Possibly
x_1	$[0.0000, 1.2086]$	$[0.0000, 2.1477]$
x_2	$[0.5593, 1.5168]$	$[0.0000, 1.9805]$
x_3	$[0.0000, 0.0001]$	$[0.0000, 0.0001]$
x_4	$[0.0000, 0.0001]$	$[0.0000, 0.0625]$
x_5	$[0.6305, 0.9213]$	$[0.5146, 2.2580]$
OVOF	$[13.9187, 32.3447]$	$[4.4014, 34.3447]$

TABLE 4. Possibly and surely OVOF for Example 5.2.

m	n	Surely	Possibly
10	15	$[12.5920, 17.5358]$	$[4.6849, 52.2690]$
150	200	$[15.5003, 21.0559]$	$[6.0843, 88.8894]$
500	500	$[16.1270, 21.5941]$	$[6.1179, 98.8470]$
600	700	$[16.0185, 19.5426]$	$[6.1218, 95.0829]$

Example 5.2. Let all coefficients ${}^*a_{ij}^{\pm}$, ${}_*a_{ij}^{\pm}$, ${}^*b^{\pm}$, ${}_*b_i^{\pm}$, ${}^*c_j^{\pm}$, and ${}_*c_j^{\pm}$ of RILP problem (1) for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$ be random generated from the continuous uniform distribution with interval specified by $[-1, 2]$, $[-3, 4]$, $[2, 4]$, $[0, 8]$, $[5, 8]$, and $[2, 10]$, respectively. Moreover, let the sign of the lower and upper bounds of the intervals be the same. The computations have been done by using MATLAB R2016b. Firstly, consider $m = n = 5$. As shown in Table 3, surely OVOF lies in possibly OVOF, that is $[13.9187, 32.3447] \subseteq [4.4014, 34.3447]$. Also, UAI of OSs includes LAI of OSs. For given values of m and n, possibly and surely OVOF are presented in Table 4. In all cases, surely OVOF is located in possibly OVOF.

Example 5.3. A scheduling problem that allocate guards to shifts is considered. Suppose that the time gaps are as 4 h and each shift is 8 h. Assume that x_j, for $j = 1, 2, \ldots, 6$, is the number of guards in time gap j, and b_j is the smallest number of guards that should be present during time gap j. Note that a guard who begins work at the start of time gap j will then be available during time gaps j and $j + 1$. The necessary information are shown in Table 5.

TABLE 5. The requirements of Example 5.3.

Time gap	Surely number of guards	Possibly number of guards
2 am–6 am	[5, 8]	[4, 10]
6 am–10 am	[8, 14]	[6, 16]
10 am–2 pm	[8, 114]	[6, 16]
2 pm–6 pm	[7, 15]	[6, 17]
6 pm–10 pm	[6, 10]	[4, 12]
10 pm–2 am	[5, 8]	[4, 10]

The problem could be formulated as a RI linear programming problem,

$$\begin{aligned}
\min\ & x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \\
\text{s.t. } & x_1 + x_6 \geq \big([5,8],[4,10]\big) \\
& x_1 + x_2 \geq \big([8,14],[6,16]\big) \\
& x_2 + x_3 \geq \big([8,14],[6,16]\big) \\
& x_3 + x_4 \geq \big([7,15],[6,17]\big) \\
& x_4 + x_5 \geq \big([6,10],[4,12]\big) \\
& x_5 + x_6 \geq \big([5,8],[4,10]\big) \\
& x_j \geq 0 \qquad\qquad j = 1,2,\ldots,6.
\end{aligned}$$

According to the obtained results, we have

$$\begin{aligned}
{}^*z^- &= 16, \quad {}^*\mathbf{x}^1 = \big(4,2,6,0,4,0\big), \\
{}^*z^+ &= 43, \quad {}^*\mathbf{x}^2 = \big(10,6,15,2,10,0\big), \\
{}_*z^- &= 20, \quad {}_*\mathbf{x}^1 = \big(5,3,6,1,5,0\big), \\
{}_*z^+ &= 37, \quad {}_*\mathbf{x}^2 = \big(8,6,13,2,8,0\big).
\end{aligned}$$

Therefore, RI of OVOF is $\big([20,37],[16,43]\big)$, *i.e.* the number of required guards is surely between 20 and 37 and it is possibly between 16 and 43.

Example 5.4 ([20])**.** There are three factories as sources and four cities as destinations. Each factory and consumer represent a potential supply and demand, respectively. Transportation costs and fixed-charge coefficients in term of RI are given in [20]. For minimizing the total transportation cost, by using four problems given in Section 4, surely and possibly OVOF are obtained as

$$\big({}_*z^{\pm}, {}^*z^{\pm}\big) = \big([636.25, 789.75], [616, 1047]\big).$$

As an example, to obtain ${}^*z^-$ the following model is obtained:

$$\begin{aligned}
{}^*z^- = \min\ & 5x_{11} + 9x_{12} + 10x_{13} + 9x_{14} + 11x_{21} + 11x_{22} + 12x_{23} + 6x_{24} \\
& + 8x_{31} + 9x_{32} + 9x_{33} + 14x_{34} + 10y_{11} + 12y_{12} + 14y_{13} + 13y_{14} \\
& + 18y_{21} + 13y_{22} + 14y_{23} + 9y_{24} + 20y_{31} + 14y_{32} + 16y_{33} + 15y_{34} \\
\text{s.t.}\ \ & x_{11} + x_{21} + x_{31} \geq 18 \\
& x_{12} + x_{22} + x_{32} \geq 19 \\
& x_{13} + x_{23} + x_{33} \geq 20 \\
& x_{14} + x_{24} + x_{34} \geq 19 \\
& x_{11} + x_{12} + x_{13} + x_{14} \leq 35 \\
& x_{21} + x_{22} + x_{23} + x_{24} \leq 29 \\
& x_{31} + x_{32} + x_{33} + x_{34} \leq 24 \\
& x_{ij} \geq 0, \quad i = 1,2,3, \quad j = 1,2,3,4 \\
& y_{ij} = \begin{cases} 0 & x_{ij} = 0 \\ 1 & x_{ij} > 0 \end{cases}, \quad i = 1,2,3, \quad j = 1,2,3,4.
\end{aligned}$$

By solving the above problem, the following results are achieved.

$$\begin{aligned}
& x_{11} = 18, \quad x_{21} = 0, \quad x_{31} = 0, \quad x_{12} = 15, \quad x_{22} = 0, \quad x_{32} = 4, \\
& x_{13} = 0, \quad x_{23} = 0, \quad x_{33} = 20, \quad x_{14} = 0, \quad x_{24} = 19, \quad x_{34} = 0, \\
& y_{11} = 1, \quad y_{21} = 0, \quad y_{31} = 0, \quad y_{12} = 1, \quad y_{22} = 0, \quad y_{32} = 1, \\
& y_{13} = 0, \quad y_{23} = 0, \quad y_{33} = 1, \quad y_{14} = 0, \quad y_{24} = 1, \quad y_{34} = 0, \\
& {}^*z^- = 616.
\end{aligned}$$

To compare the results, consider the methods proposed by Midya and Roy [20]. The OVOFs obtained by fuzzy programming approach using $\leq_{HW}$ and $\leq_{RC}$ are $[769.30, 1041.30]$ and $[635, 887]$, respectively. We observe that these solutions lie in $[616, 1047]$ which is possibly OVOF. Also, expected values of the objective function obtained by $\leq_{HW}$, $\leq_{RC}$, and RI in [20] are 905.30, 761, and 749, respectively which also lie in possibly OVOF obtained by our method. Therefore, the interval possibly OVOF includes all solutions presented by the other methods.

6. Conclusion

In this paper, RILP as a particular type of linear programming has been considered. Some definitions and properties of intervals and rough intervals were reviewed. By introducing four linear programming problems, two solution types for the optimal values of the objective function of a RILP problem were defined. Moreover, under some assumptions, the upper approximation interval and the lower approximation interval for optimal solutions of the RILP problem were determined. The results obtained from the numerical examples approved the validity of the theorems. Actually two real-world problems, a scheduling problem under RI uncertainty and a RIFCTP were considered as two numerical examples. A comparison was made in the RIFCTP between the new results and some existed ones. Proposing new efficient approaches for dealing with RILP problems could be considered as a topic for future studies. Further, applying RIs for dealing with uncertain parameters in other mathematical programming types such as multiobjective programming might be interesting for future research.

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