RAIRO-Oper. Res. 58 (2024) 4607–4619
https://doi.org/10.1051/ro/2024178

RAIRO Operations Research
www.rairo-ro.org

A NOTE ON THE P_3-ISOLATION NUMBER OF A GRAPH

Xiaohua Wei[1], Gang Zhang[2,*] and Biao Zhao[1]

Abstract. For any graph G, a subset $D \subseteq V(G)$ is called a P_3-isolating set of G if $G - N[D]$ contains no P_3 as a subgraph, that is, consists of isolated vertices and isolated edges only. The P_3-isolation number of G, denoted by $\iota(G, P_3)$, is the cardinality of a smallest P_3-isolating set of G. Zhang and Wu [*Discrete Appl. Math.* **304** (2021) 365–374] investigated the parameter $\iota(G, P_3)$ of a graph, and they proved that if $G \notin \{P_3, C_3, C_6\}$ is a connected graph of order n, then $\iota(G, P_3) \leq \frac{2}{7}n$. In this paper, we shall prove that if $G \notin \{P_3, C_7, C_{11}\}$ is a connected graph of order n without triangles and induced 6-cycles, then $\iota(G, P_3) \leq \frac{n}{4}$, and the upper bound is sharp. This extends a result on $\iota(T, P_3)$ of a tree T by Caro and Hansberg [*Filomat* **31** (2017) 3925–3944].

Mathematics Subject Classification. 05C69.

Received December 17, 2023. Accepted September 11, 2024.

1. Introduction

All the graphs considered in this paper are finite, simple and undirected. Let $G = (V, E)$ be a such graph, where $V = V(G)$ is the *vertex set* and $E = E(G)$ is the *edge set* of G. As usual, we call $|V(G)|$ and $|E(G)|$ the *order* and *size* of G. For a vertex $v \in V(G)$, the set $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$ is said to be the *open neighborhood of* v in G, and the set $N_G[v] = N_G(v) \cup \{v\}$ is said to be the *closed neighborhood of* v in G. As usual, we would like to abbreviate $N_G(v)$ and $N_G[v]$ to $N(v)$ and $N[v]$, the same below. For a vertex subset $S \subseteq V(G)$, the set $N(S) = \bigcup_{v\in S} N(v) \setminus S$ is the *open neighborhood of* S in G, and the set $N[S] = N(S) \cup S$ is the *closed neighborhood of* S in G. Let H be another graph. We say that H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, and write $H \subseteq G$ simply. For a subgraph $H \subseteq G$, the *open neighborhood of* H in G is the set $N(H) = N(V(H))$ and the *closed neighborhood of* H is the set $N[H] = N[V(H)]$. Moreover, for any vertex or edge subset S of G, let $G[S]$ denote the subgraph of G induced by S, and $G - S$ denote the subgraph of G obtained by deleting S from G. For any two vertex subsets S_1 and S_2 of G with $S_1 \cap S_2 = \emptyset$, the edge subset $E(S_1, S_2)$ of G is called the *edge cut* of G associated with S_1 and S_2, where each edge of $E(S_1, S_2)$ has one end in S_1 and the other end in S_2. Some graph theory notations and terminologies may be used in this paper but not explicitly defined, and we refer the readers to [2, 4].

A subset $D \subseteq V(G)$ is a *dominating set* of G if $N[D] = V(G)$. The *domination number* of G, denoted by $\gamma(G)$, is the cardinality of a smallest dominating set of G. Let $\mathcal{H}$ be a family of connected graphs. A subset

Keywords. Isolation number, partial domination, induced 6-cycles, triangle-free.

[1] College of Mathematics and System Sciences, Xinjiang University, Urumqi, Xinjiang 830046, P.R. China.

[2] School of Mathematical Sciences, Xiamen University, Xiamen, Fujian 361005, P.R. China.

*Corresponding author: gzh_ang@163.com

$D \subseteq V(G)$ is an $\mathcal{H}$*-isolating set* of G if $G - N[D]$ contains no graph in $\mathcal{H}$. The $\mathcal{H}$*-isolation number* of G, denoted by $\iota(G, \mathcal{H})$, is the cardinality of a smallest $\mathcal{H}$-isolating set of G. In particular, taking $\mathcal{H} = \{H\}$, we call the $\mathcal{H}$-isolation number of G simply the H*-isolation number* of G, denoted by $\iota(G, H)$ instead of $\iota(G, \mathcal{H})$. It is easy to see that, D is a dominating set of G if and only if D is a P_1-isolating set of G, and thus, $\gamma(G) = \iota(G, P_1)$. Moreover, if D is a dominating set of G, then D must be an $\mathcal{H}$-isolating set of G, implying that $\iota(G, \mathcal{H}) \leq \gamma(G)$.

In the last few decades, the problems on dominating sets received numerous attention and have been studied extensively in the literatures (see, for example, [1, 3, 11, 16, 19, 22, 27–30]). Some nice surveys and treatises on various domination parameters in graphs can be found in [13, 14, 18, 21, 24–26]. A well-known result is due to Ore [31], which states that every graph without isolated vertices has the domination number at most half of its order (see [23]). Since $\gamma(G) = \iota(G, P_1)$ for any graph G, Ore's result implies that if $G \ncong P_1$ is a connected graph of order n, then $\iota(G, P_1) \leq \frac{n}{2}$. Note that other types of isolation are also very interesting; while a P_1*-isolating set* of a graph G is a subset $D \subseteq V(G)$ such that $G - N[D]$ is a graph with no vertices, a P_2*-isolating set* of G is a subset $D \subseteq V(G)$ such that $G - N[D]$ is a graph with no edges. In 2017, Caro and Hansberg [12] introduced the conception of $\mathcal{H}$-isolation, and they also called it *partial domination.* It is clear that, the research on $\mathcal{H}$-isolating sets and partial domination is a natural extension of classical domination theory. In this paper, we shall further study this generalized domination number of graphs.

Caro and Hansberg [12] proved that if $G \notin \{P_2, C_5\}$ is a connected graph of order n, then $\iota(G, P_2) \leq \frac{n}{3}$. Moreover, they also showed that if G is a connected graph of order n, then $\iota(G, K_{1,k}) \leq \frac{n}{k+1}$, and if $T \ncong K_{1,k}$ is a tree of order n, then $\iota(G, K_{1,k}) \leq \frac{n}{k+2}$. For maximal outerplanar graphs G of order n, they proved if $G \ncong C_3$, then $\iota(G, P_2) \leq \frac{n}{4}$. Tokunaga, Jiarasuksakun and Kaemawichanurat [32] improved this result by the numbers of vertices of degree 2, and they proved if $G \notin \{C_3, K_4^-\}$ is a maximal outerplanar graph of order n with n_2 vertices of degree 2, then

$$\iota(G, P_2) \leq \begin{cases} \frac{n+n_2}{5}, & \text{if } n_2 \leq \frac{n}{4}, \\ \frac{n-n_2}{3}, & \text{otherwise,} \end{cases}$$

where K_4^- is the *diamond graph* that is obtained from a K_4 by deleting an edge. Borg and Kaemawichanurat [7] proved that if $G \notin \{C_3, K_4^-\}$ is a maximal outerplanar graph of order n with n_2 vertices of degree 2, then $\iota(G, P_3) \leq \frac{n}{5}$, and

$$\iota(G, P_3) \leq \begin{cases} \frac{n+n_2}{6}, & \text{if } n_2 \leq \frac{n}{3}, \\ \frac{n-n_2}{3}, & \text{otherwise.} \end{cases}$$

Recently, the same authors of [7] extended the results above to $\iota(G, K_{1,k})$ $(k \geq 2)$, and they in [8] proved that if G a maximal outerplanar graph of order $n \geq 2k+1$ with n_2 vertices of degree 2, then $\iota(G, K_{1,k}) \leq \frac{n}{k+3}$, and

$$\iota(G, K_{1,k}) \leq \begin{cases} \frac{n+n_2}{k+4}, & \text{if } n_2 \leq \frac{3n}{2k+5}, \\ \frac{n-n_2}{k+1}, & \text{otherwise.} \end{cases}$$

For more research and results on the $\mathcal{H}$-isolation number of a graph, the readers can be referred to [5, 6, 8–10, 17, 20, 33]. Among the previous work, it is worth noting that the study of P_k-isolating sets is a pretty interesting and important problem in this topic, where k is a positive integer. Particularly, a P_3*-isolating set* of a graph G is a subset $D \subseteq V(G)$ such that $G - N[D]$ contains no P_3 as a subgraph if and only if $\Delta(G - N[D]) \leq 1$ if and only if $G - N[D]$ consists of isolated vertices and isolated edges only. Zhang and Wu [34] proved that if $G \notin \{P_3, C_3, C_6\}$ is a connected graph of order n, then $\iota(G, P_3) \leq \frac{2}{7}n$. Borg [6] proved a stronger result that if G is a connected graph of order $n \geq 8$ with n_1 vertices of degree 1, then $\iota(G, P_3) \leq \frac{4n-n_1}{14}$. The *girth* of a graph G, denoted by $g(G)$, is the length of a shortest cycle of G. In [34], the authors also proved that if $G \notin \{P_3, C_7, C_{11}\}$ is a connected graph of order n with girth at least 7, then $\iota(G, P_3) \leq \frac{n}{4}$. Moreover, they proposed the following problem in the closing.

Problem 1.1 ([35])**.** Let k be a positive integer. Determine the exact upper bound on $\iota(G, P_k)$ for connected graphs G of order n.

Towards Problem 1.1, Zhang and Wu [35], Borg [6] independently, proved that if $G \ncong C_7$ is a connected graph of order n, then $\iota(G, P_4) \le \frac{n}{4}$. Chen and Xu [15] proved that if $G \ncong C_8$ is a connected graph of order n, then $\iota(G, P_5) \le \frac{2}{9}n$. More explicitly, we make a summary of known results on $\iota(G, P_k)$ of a connected graph G for small k.

Theorem 1.2. *Let G be a connected graph of order n. Then the following holds:*

(i) [31] *If $G \ncong P_1$, then $\iota(G, P_1) \le \frac{n}{2}$.*
(ii) [12] *If $G \notin \{P_2, C_5\}$, then $\iota(G, P_2) \le \frac{n}{3}$.*
(iii) [6, 34] *If $G \notin \{P_3, C_3, C_6\}$, then $\iota(G, P_3) \le \frac{2}{7}n$.*
(iv) [34] *If $G \notin \{P_3, C_7, C_{11}\}$ and $g(G) \ge 7$, then $\iota(G, P_3) \le \frac{n}{4}$.*
(v) [6, 35] *If $G \ncong C_7$, then $\iota(G, P_4) \le \frac{n}{4}$.*
(vi) [15] *If $G \ncong C_8$, then $\iota(G, P_5) \le \frac{2}{9}n$.*

Inspired by Theorem 1.2, it is natural for us to propose a conjecture as follows, which is also mentioned in [15] as an open problem.

Conjecture 1.3. Let $k \ge 4$ be an integer. If $G \ncong C_{k+3}$ is a connected graph of order n, then $\iota(G, P_k) \le \frac{2n}{k+4}$.

If Conjecture 1.3 is true, then the upper bound above is tight, since we can easily check that C_{k+4} is a graph attaining the bound. Instead of attacking this conjecture, we shall in this paper continue the study on the P_3-isolation number of a graph. The main result of this paper is the following.

Theorem 1.4. *If $G \notin \{P_3, C_7, C_{11}\}$ is a connected triangle-free graph of order n without induced 6-cycles, then $\iota(G, P_3) \le \left\lfloor \frac{n}{4} \right\rfloor$.*

Note that Theorem 1.2(iii) and (iv) improve the results of Caro and Hansberg [12] on the $K_{1,k}$-isolation number in connected graphs and trees for the case $k = 2$, respectively. Our main result Theorem 1.4 improves the result of Theorem 1.2(iv), and also the Caro–Hansberg result [12] on $\iota(T, K_{1,2}) = \iota(T, P_3)$ in trees T.

In the rest of this section, we will construct some extremal graphs that attain the bound in Theorem 1.4. Firstly, it is easy to check that for each graph $G \in \{P_4, P_8, C_4, C_8, C_{12}, C_{16}\}$, $\iota(G, P_3) = \frac{|V(G)|}{4}$. For an infinite set of extremal graphs, let F be a connected triangle-free graph of order $k \ge 1$ without induced 6-cycles. For any vertex $u \in V(F)$, let H_u be a copy of P_3, C_7 or C_{11}. A graph G is obtained from F and $\bigcup_{u \in V(F)} H_u$, in which each vertex u of F is joined to a vertex of H_u by exactly one edge. Let v_d^u and $(v_d^u)'$ (if they exist) be the vertices of H_u at distance d from u in G, where $1 \le d \le 6$. As an illustration in Figure 1, we take

$$D_u = \begin{cases} \{u\}, & \text{if } H_u \cong P_3, \\ \{u, v_4^u\}, & \text{if } H_u \cong C_7, \\ \{u, v_4^u, (v_4^u)'\}, & \text{if } H_u \cong C_{11}. \end{cases}$$

It is easy to see that G is a connected graph without triangles and induced 6-cycles, and the set $\bigcup_{u \in V(F)} D_u$ is a smallest P_3-isolating set of G. Therefore,

$$\iota(G, P_3) = \left| \bigcup_{u \in V(F)} D_u \right| = \sum_{u \in V(F)} |D_u| = \sum_{u \in V(F)} \frac{|\{u\} \cup V(H_u)|}{4} = \frac{|V(G)|}{4} = \frac{n}{4}.$$

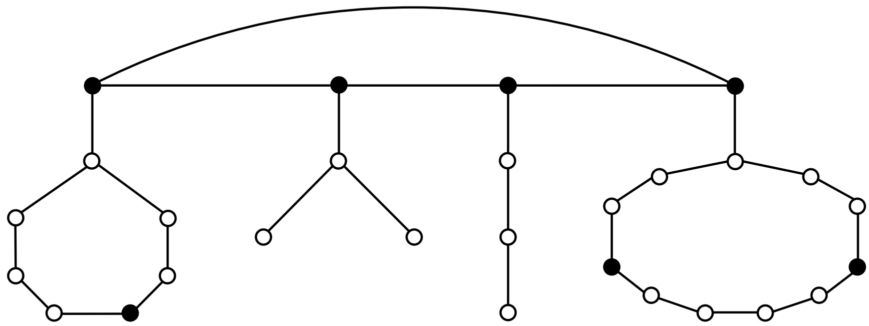

FIGURE 1. A graph G of order $n = 28$ with $\iota(G, P_3) = 7 = \frac{n}{4}$.

2. Proof of Theorem 1.4

In this section, we prove Theorem 1.4. The following lemmas are useful.

Lemma 2.1 ([12, 34]).

(i) *If $G \ncong P_3$ is a path of order n, then $\iota(G, P_3) \le \frac{n}{4}$.*
(ii) *If $G \notin \{C_3, C_6, C_7, C_{11}\}$ is a cycle of order n, then $\iota(G, P_3) \le \frac{n}{4}$.*

Lemma 2.2 ([34]). *Let G be a graph and $S \subseteq V(G)$ be a subset. If D is a P_3-isolating set of $G[S]$ with $E(S \setminus N[D], V \setminus S) = \emptyset$, then $\iota(G, P_3) \le |D| + \iota(G - S, P_3)$.*

Lemma 2.3 ([5, 34]). *If $G_1, G_2, \cdots, G_k$ are the distinct components of a graph G, then $\iota(G, P_3) = \sum_{i=1}^{k} \iota(G_i, P_3)$.*

Proof of Theorem 1.4. Let G be a connected graph of order n without triangles and induced 6-cycles. Then $G \notin \{C_3, C_6\}$. Suppose $G \notin \{P_3, C_7, C_{11}\}$. Apply induction on n. Since $\iota(G, P_3)$ is an integer, it suffices to show $\iota(G, P_3) \le \frac{n}{4}$. The result is trivial for $n \le 3$. So, we let $n \ge 4$.

If $\Delta(G) \le 2$, then G is a path or cycle. By Lemma 2.1, $\iota(G, P_3) \le \frac{n}{4}$. Let $v \in V(G)$ with $d(v) = \Delta(G)$. If $\Delta(G) = n - 1$, then $\iota(G, P_3) = |\{v\}| = 1 \le \frac{n}{4}$. So, we may assume that $3 \le \Delta(G) \le n - 2$.

Let $G' = G - N[v]$ with order $n' = |V(G')|$. Let $\mathcal{H}$ be the set of components of G', and $\mathcal{H}^*$ the set of components of G' isomorphic to a copy of P_3, C_7 or C_{11}. Let $\mathcal{H}' = \mathcal{H} \setminus \mathcal{H}^*$. By the induction hypothesis, $\iota(H, P_3) \le \frac{|V(H)|}{4}$ for any $H \in \mathcal{H}'$.

If $\mathcal{H}^* = \emptyset$, then since $\Delta(G) \le n-2$, $\mathcal{H}' \neq \emptyset$. The set $\{v\}$ is a P_3-isolating set of $G[N[v]]$. By Lemmas 2.2 and 2.3, and the induction hypothesis, we have

$$\iota(G, P_3) \le |\{v\}| + \iota(G', P_3) = 1 + \sum_{H \in \mathcal{H}'} \iota(H, P_3) \le 1 + \frac{1}{4}(n - \Delta(G) - 1) \le \frac{n}{4}.$$

Hence, we may assume that $\mathcal{H}^* \neq \emptyset$. Choose a vertex $x \in N(v)$ such that there exists some $H \in \mathcal{H}^*$ with $x \in N(H)$. Let $xy \in E(G)$ for some $y \in V(H)$. Recall $H \in \{P_3, C_7, C_{11}\}$. Let y_i and y_i' (if they exist) be the vertices at distance i from y in H, where $1 \le i \le 5$. Moreover, let $\mathcal{H}_x^*$ be the set of components H of $\mathcal{H}^*$ such that $N(H) = \{x\}$, and $\mathcal{H}_x'$ the set of components H of $\mathcal{H}'$ such that $N(H) = \{x\}$. Clearly, $\mathcal{H}_x^* \subseteq \mathcal{H}^*$ and $\mathcal{H}_x' \subseteq \mathcal{H}'$. We divide the following proof into two cases.

Case 1: $\mathcal{H}_x^* \neq \emptyset$.

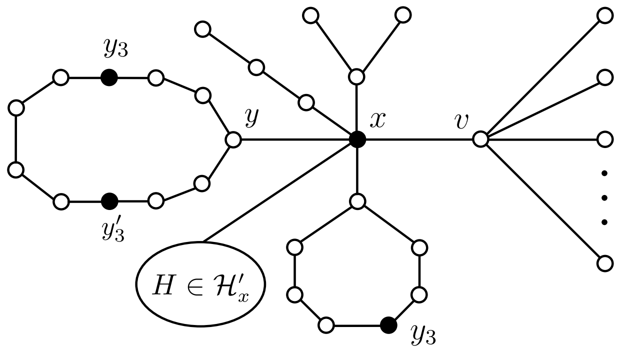

FIGURE 2. For the case of $\mathcal{H}_x^* \neq \emptyset$.

Let c_3, c_7 and c_{11} be the numbers of components of $\mathcal{H}_x^*$ that are isomorphic to P_3, C_7 and C_{11}, respectively. By the present assumption, $c_3 + c_7 + c_{11} \geq 1$. Let $X = \{x\} \cup \bigcup_{H \in \mathcal{H}_x^*} V(H)$ and $(G - X)_v$ be the component of $G - X$ containing v. Then $G - X = (G - X)_v \cup \bigcup_{H \in \mathcal{H}_x'} H$. For each $H \in \mathcal{H}_x^*$, let

$$D_H = \begin{cases} \{x\}, & \text{if } H \cong P_3, \\ \{x, y_3\}, & \text{if } H \cong C_7, \\ \{x, y_3, y_3'\}, & \text{if } H \cong C_{11}, \end{cases}$$

as shown in Figure 2. We can see that $\bigcup_{H \in \mathcal{H}_x^*} D_H$ is a P_3-isolating set of $G[X]$, and there are two subcases to consider in the following.

Subcase 1.1: $(G - X)_v \notin \{P_3, C_7, C_{11}\}$.

It is clear that each component of $G - X$ contains no triangles and induced 6-cycles, and $E(X \setminus N[D_X], V \setminus X) = \emptyset$ where $D_X = \bigcup_{H \in \mathcal{H}_x^*} D_H$. By Lemmas 2.2 and 2.3, and by the induction hypothesis, we have

$$\begin{aligned} \iota(G, P_3) &\leq |D_X| + \iota(G - X, P_3) \\ &= 1 + c_7 + 2c_{11} + \iota((G - X)_v, P_3) + \sum_{H \in \mathcal{H}_x'} \iota(H, P_3) \\ &\leq 1 + c_7 + 2c_{11} + \frac{|V((G - X)_v)|}{4} + \sum_{H \in \mathcal{H}_x'} \frac{|V(H)|}{4} \\ &\leq 1 + c_7 + 2c_{11} + \frac{1}{4}(n - 1 - 3c_3 - 7c_7 - 11c_{11}) \\ &= \frac{n}{4} + \frac{3}{4}(1 - (c_3 + c_7 + c_{11})) \leq \frac{n}{4}. \end{aligned}$$

Subcase 1.2: $(G - X)_v \in \{P_3, C_7, C_{11}\}$.

Let $Y = X \cup V((G - X)_v)$ and $G - Y = \bigcup_{H \in \mathcal{H}_x'} H$. Let v_i and v_i' (if they exist) be the vertices at distance i from v in $(G - X)_v$, where $1 \leq i \leq 5$.

Subcase 1.2.1: $(G - X)_v \cong P_3$.

It is easy to observe that $\bigcup_{H\in\mathcal{H}_x^*} D_H$ is also a P_3-isolating set of $G[Y]$. By Lemmas 2.2 and 2.3, and by the induction hypothesis, we have

$$\begin{aligned}
\iota(G,P_3) &\le |\bigcup_{H\in\mathcal{H}_x^*} D_H| + \iota(G-Y,P_3)\\
&= 1 + c_7 + 2c_{11} + \sum_{H\in\mathcal{H}'_x} \iota(H,P_3)\\
&\le 1 + c_7 + 2c_{11} + \sum_{H\in\mathcal{H}'_x} \frac{|V(H)|}{4}\\
&\le 1 + c_7 + 2c_{11} + \frac{1}{4}(n-1-3c_3-7c_7-11c_{11}-3)\\
&= \frac{n}{4} - \frac{3}{4}(c_3+c_7+c_{11}) < \frac{n}{4}.
\end{aligned}$$

Subcase 1.2.2: $(G-X)_v \cong C_7$.

Observe that $\{v_3\}\cup\bigcup_{H\in\mathcal{H}_x^*} D_H$ is a P_3-isolating set of $G[Y]$. By Lemmas 2.2 and 2.3, and by the induction hypothesis, we have

$$\begin{aligned}
\iota(G,P_3) &\le |\{v_3\}\cup\bigcup_{H\in\mathcal{H}_x^*} D_H| + \iota(G-Y,P_3)\\
&= 2 + c_7 + 2c_{11} + \sum_{H\in\mathcal{H}'_x} \iota(H,P_3)\\
&\le 2 + c_7 + 2c_{11} + \sum_{H\in\mathcal{H}'_x} \frac{|V(H)|}{4}\\
&\le 2 + c_7 + 2c_{11} + \frac{1}{4}(n-1-3c_3-7c_7-11c_{11}-7)\\
&= \frac{n}{4} - \frac{3}{4}(c_3+c_7+c_{11}) < \frac{n}{4}.
\end{aligned}$$

Subcase 1.2.3: $(G-X)_v \cong C_{11}$.

We see that $\{v_3,v'_3\}\cup\bigcup_{H\in\mathcal{H}_x^*} D_H$ is a P_3-isolating set of $G[Y]$. By Lemmas 2.2 and 2.3, and by the induction hypothesis, we have

$$\begin{aligned}
\iota(G,P_3) &\le |\{v_3,v'_3\}\cup\bigcup_{H\in\mathcal{H}_x^*} D_H| + \iota(G-Y,P_3)\\
&= 3 + c_7 + 2c_{11} + \sum_{H\in\mathcal{H}'_x} \iota(H,P_3)\\
&\le 3 + c_7 + 2c_{11} + \sum_{H\in\mathcal{H}'_x} \frac{|V(H)|}{4}\\
&\le 3 + c_7 + 2c_{11} + \frac{1}{4}(n-1-3c_3-7c_7-11c_{11}-11)\\
&= \frac{n}{4} - \frac{3}{4}(c_3+c_7+c_{11}) < \frac{n}{4}.
\end{aligned}$$

Case 2: $\mathcal{H}_x^* = \emptyset$.

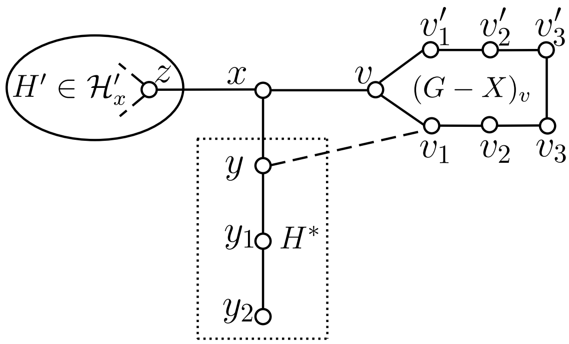

FIGURE 3. For the case of $\mathcal{H}'_x \neq \emptyset$, where $H^* \cong P_3$ and $(G-X)_v \cong C_7$.

By the arbitrary choice of x, we know that $|N(H)| \geq 2$ for any $H \in \mathcal{H}^*$. Recall that $\mathcal{H}^* \neq \emptyset$. Take $H^* \in \mathcal{H}^*$. Let $x \in N(H^*)$, $xy \in E(G)$ for some $y \in V(H^*)$, and y_i, y'_i defined as previously. Let $X = \{x\} \cup V(H^*)$ and $(G-X)_v$ be the component of $G-X$ containing v. Clearly, $G - X = (G-X)_v \cup \bigcup_{H \in \mathcal{H}'_x} H$.

Subcase 2.1: $(G-X)_v \in \{P_3, C_7, C_{11}\}$.

It is easy to see that $\Delta(G) = d(v) = 3$ throughout this subcase. Let $Y = X \cup V((G-X)_v)$. Then $G - Y = \bigcup_{H \in \mathcal{H}'_x} H$. Since $xy, xv \in E(G)$, $|\mathcal{H}'_x| \in \{0, 1\}$.

We first suppose $|\mathcal{H}'_x| = |\{H'\}| = 1$. Since $|N(H^*)| \geq 2$, $G[Y] - x = G[V(H^*) \cup V((G-X)_v)]$ is a connected graph, and since $H^*, (G-X)_v \in \{P_3, C_7, C_{11}\}$, we have $|V(H^*)|, |V((G-X)_v)| \in \{3, 7, 11\}$. Hence, $|V(G[Y] - x)| = |G[V(H^*) \cup V((G-X)_v)]| = |V(H^*)| + |V((G-X)_v)| \in \{6, 10, 14, 18, 22\}$, and $G[Y] - x \notin \{P_3, C_7, C_{11}\}$. As shown in Figure 3, let $xz \in E(G)$ for some $z \in V(H')$. Let $Z = \{z\} \cup Y$. By the induction hypothesis, we have

$$\iota(G[Y] - x, P_3) \leq \left\lfloor \frac{1}{4}(|V(G[Y] - x)|) \right\rfloor = \frac{1}{4}(|V(G[Y] - x)| - 2) = \frac{1}{4}(|Z| - 4).$$

Let $D_{Y \setminus \{x\}}$ be a smallest P_3-isolating set of $G[Y] - x$. Then $D_Z = \{x\} \cup D_{Y \setminus \{x\}}$ is a P_3-isolating set of $G[Z]$ with $E(Z \setminus D_Z, V \setminus Z) = \emptyset$, and we have

$$|D_Z| = |\{x\} \cup D_{Y \setminus \{x\}}| = |\{x\}| + |D_{Y \setminus \{x\}}| = 1 + \iota(G[Y] - x, P_3) \leq 1 + \frac{1}{4}(|Z| - 4) = \frac{1}{4}|Z|.$$

Note that $d(z) \leq \Delta(G) = 3$. Then, $G - Z$ is a graph with at most two components. If there is a component of $G - Z$ isomorphic to a member of $\{P_3, C_7, C_{11}\}$, then it is clear that $d(x) = 3 = \Delta(G)$ and $z \in N(x)$. By regarding x and z as v and x separately, we can see $\mathcal{H}^*_z \neq \emptyset$, and this subcase is covered by Case 1. If each component of $G - Z$ is not a member of $\{P_3, C_7, C_{11}\}$, then by Lemmas 2.2 and 2.3, and by the induction hypothesis, we have

$$\iota(G, P_3) \leq |D_Z| + \iota(G - Z, P_3) \leq \frac{|Z|}{4} + \frac{|V(G-Z)|}{4} = \frac{|V(G)|}{4} = \frac{n}{4}.$$

Now suppose $|\mathcal{H}'_x| = 0$. Hence, $\mathcal{H}'_x = \emptyset$ and $G = G[\{x\} \cup V(H^*) \cup V((G-X)_v)]$. Recall that $d(v) = \Delta(G) = 3$. Clearly, $d_{(G-X)_v}(v) = 2$. Let v_i, v'_i be the two vertices at distance i from v in $(G-X)_v$, where $1 \leq i \leq 5$. Based on the structure of H^*, we further distinguish Subcase 2.1 into the following three subcases.

Subcase 2.1.1: $H^* \cong P_3$.

(I) If $d_{H^*}(y) = 2$, then $d(y) = 3 = \Delta(G) = d(v)$. By the present assumption, $N_{H^*}(y) = \{y_1, y'_1\}$. Since $|N(H^*)| \geq 2$, we may assume that $y_1 v_1 \in E(G)$. Since G is a triangle-free graph, $xv_1, xv'_1, xy_1, xy'_1, v_1 v'_1, y_1 y'_1 \notin E(G)$.

(i) Suppose that $(G-X)_v \cong P_3$. If $y_1v_1' \in E(G)$, then $\{y_1\}$ is a P_3-isolating set of G, implying that $\iota(G,P_3) \le |\{y_1\}| = 1 < \frac{7}{4} = \frac{n}{4}$. If $y_1'v_1 \in E(G)$, then $\{v_1\}$ is a P_3-isolating set of G, implying that $\iota(G,P_3) \le |\{v_1\}| = 1 < \frac{7}{4} = \frac{n}{4}$. If $y_1v_1', y_1'v_1 \notin E(G)$, then $y_1'v_1' \notin E(G)$; otherwise, $y_1'yy_1v_1vv_1'y_1'$ is an induced 6-cycle of G. Now the set $\{x\}$ is a P_3-isolating set of G, and $\iota(G,P_3) \le |\{x\}| = 1 < \frac{7}{4} = \frac{n}{4}$.

(ii) Suppose that $(G-X)_v \cong C_7$. Clearly, $d(v_1) = 3 = \Delta(G)$. Since $y_1'yy_1v_1vv_1'y_1'$ can not be an induced 6-cycle of G, $y_1'v_1' \notin E(G)$. Thus, $d(y_1') = 1$. Take

$$D = \begin{cases} \{x, v_3\}, & \text{if } y_1v_1' \notin E(G), \\ \{y_1, v_3\}, & \text{if } y_1v_1' \in E(G) \text{ and } xv_2' \notin E(G), \\ \{v_1, v_2'\}, & \text{if } y_1v_1' \in E(G) \text{ and } xv_2' \in E(G). \end{cases}$$

For any subcase, it can be easy to check that D is a P_3-isolating set of G. Hence, we have $\iota(G,P_3) \le |D| = 2 < \frac{11}{4} = \frac{n}{4}$.

(iii) Suppose that $(G-X)_v \cong C_{11}$. As noted in (ii), $d(y_1') = 1$ also. Take

$$D = \begin{cases} \{x, v_3, v_3'\}, & \text{if } y_1v_1' \notin E(G), \\ \{y_1, v_3, v_3'\}, & \text{if } y_1v_1' \in E(G) \text{ and } xv_5 \notin E(G) \text{ and } xv_5' \notin E(G), \\ \{x, v_2, v_2'\}, & \text{if } y_1v_1' \in E(G) \text{ and } xv_5 \in E(G) \text{ or } xv_5' \in E(G). \end{cases}$$

It is easy to check that D is a P_3-isolating set of G, and we have $\iota(G,P_3) \le |D| = 3 < \frac{15}{4} = \frac{n}{4}$.

(II) If $d_{H^*}(y) = 1$, then $N_{H^*}(y_1) = \{y, y_2\}$. Clearly, $xv_1, xv_1', xy_1 \notin E(G)$.

(i) Suppose that $yv_1 \in E(G)$ or $yv_1' \in E(G)$. By the symmetry of v_1 and v_1', we may let $yv_1 \in E(G)$. Then $y_1v_1 \notin E(G)$; otherwise, yy_1v_1y is a triangle of G. If $y_1v_1' \in E(G)$, then $y_2v_1' \notin E(G)$; otherwise, $y_1y_2v_1'y_1$ is a triangle of G. If $y_1v_1' \notin E(G)$ and $y_2v_1' \in E(G)$, then since $y_2v_1'vv_1yy_1y_2$ or $y_2v_1'vxyy_1y_2$ can not be an induced 6-cycle of G, $y_2v_1 \in E(G)$ and $xy_2 \in E(G)$. However, now $x, y_1, v_1, v_1' \in N(y_2)$ and $d(y_2) \ge 4$, a contradiction to $\Delta(G) = 3$. Thus, no matter whether $y_1v_1' \notin E(G)$ or not, we determine $y_2v_1' \notin E(G)$. Take

$$D = \begin{cases} \{y\}, & \text{if } (G-X)_v \cong P_3, \\ \{y, v_3'\}, & \text{if } (G-X)_v \cong C_7, \\ \{y, v_3, v_3'\}, & \text{if } (G-X)_v \cong C_{11}. \end{cases}$$

It is easy to see that D is a P_3-isolating set of G, and we have

$$\iota(G,P_3) \le |D| = \frac{1 + |V((G-X)_v)|}{4} = \frac{|\{x\} \cup V(H^*)| + |V((G-X)_v)| - 3}{4} = \frac{n-3}{4} < \frac{n}{4}.$$

(ii) Suppose that $yv_1, yv_1' \notin E(G)$. Further, we may assume that $y_1v_1 \in E(G)$ or $y_1v_1' \in E(G)$. By the symmetry of v_1 and v_1', we may let $y_1v_1 \in E(G)$. Then $y_2v_1 \notin E(G)$; otherwise, $y_1y_2v_1y_1$ is a triangle of G. For the subcase of $(G-X)_v \in \{C_7, C_{11}\}$, $xv_3 \notin E(G)$; otherwise, $xyy_1v_1v_2v_3x$ is an induced 6-cycle of G. If $xy_2 \notin E(G)$, then $y_2v_1' \notin E(G)$; otherwise, $y_2v_1'vxyy_1y_2$ is an induced 6-cycle of G. Take

$$D = \begin{cases} \{v_1\}, & \text{if } (G-X)_v \cong P_3, \\ \{v_1, v_2'\}, & \text{if } (G-X)_v \cong C_7, \\ \{v_1, v_3', v_5\}, & \text{if } (G-X)_v \cong C_{11}. \end{cases}$$

If $xy_2 \in E(G)$, then we take

$$D = \begin{cases} \{x\}, & \text{if } (G-X)_v \cong P_3, \\ \{x, v_3\}, & \text{if } (G-X)_v \cong C_7, \\ \{x, v_3, v_3'\}, & \text{if } (G-X)_v \cong C_{11}. \end{cases}$$

For any subcase above, we can easily check that D is a P_3-isolating set of G. Thus, we have $\iota(G,P_3) \le |D| = \frac{n-3}{4} < \frac{n}{4}$.

Based on the supposition that $yv_1, yv_1' \notin E(G)$, it remains to consider the case that $y_1v_1, y_1v_1' \notin E(G)$. Since $|N(H^*)| \geq 2$, $y_2v_1 \in E(G)$ or $y_2v_1' \in E(G)$. By the symmetry of v_1 and v_1', we may let $y_2v_1 \in E(G)$. Since $y_2v_1vxyy_1y_2$ can not be an induced 6-cycle of G, we know $xy_2 \in E(G)$. Take

$$D = \begin{cases} \{y_2\}, & \text{if } (G-X)_v \cong P_3, \\ \{y_2, v_3'\}, & \text{if } (G-X)_v \cong C_7, \\ \{y_2, v_3, v_3'\}, & \text{if } (G-X)_v \cong C_{11}. \end{cases}$$

We can check that D is a P_3-isolating set of G, and thus, $\iota(G, P_3) \leq |D| = \frac{n-3}{4} < \frac{n}{4}$.

Subcase 2.1.2: $H^* \cong C_7$.

(i) Suppose that $(G-X)_v \cong P_3$. Clearly, $d(y) = 3 = d(v) = \Delta(G)$. Since G is triangle-free, $xv_1, xv_1', xy_1, xy_1' \notin E(G)$. Since $y_2v_1vxyy_1y_2$ or $y_2v_1'vxyy_1y_2$ can not be an induced 6-cycle of G, $y_2v_1 \notin E(G)$ and $y_2v_1' \notin E(G)$. Similarly, $y_2'v_1 \notin E(G)$ and $y_2'v_1' \notin E(G)$; otherwise, $y_2'v_1vxyy_1'y_2'$ or $y_2'v_1'vxyy_1'y_2'$ is an induced 6-cycle of G. Hence, we know $E(\{y_2, y_2'\}, \{v_1, v_1'\}) = \emptyset$.
If $E(\{y_1, y_1'\}, \{v_1, v_1'\}) = \emptyset$, then by $|N(H^*)| \geq 2$, $E(\{y_3, y_3'\}, \{v_1, v_1'\}) \neq \emptyset$. By the symmetries of y_3 and y_3', and v_1 and v_1', we may assume that $y_3v_1 \in E(G)$. It is clear that $\{x, y_3\}$ is a P_3-isolating set of G, and $\iota(G, P_3) \leq |\{x, y_3\}| = 2 < \frac{11}{4} = \frac{n}{4}$.
If $E(\{y_1, y_1'\}, \{v_1, v_1'\}) \neq \emptyset$, then by the symmetries of y_1 and y_1', and v_1 and v_1', we may assume that $y_1v_1 \in E(G)$. Clearly, $d(y_1) = 3 = \Delta(G)$ and $y_1'v_1' \notin E(G)$; otherwise, $d(y_1') = 3 = \Delta(G)$ and $y_1'v_1'vv_1y_1yy_1'$ is an induced 6-cycle of G. Moreover, $y_3v_1' \notin E(G)$; otherwise, $d(y_3) = 3 = \Delta(G)$ and $y_3v_1'vv_1y_1y_2y_3$ is an induced 6-cycle of G. Take

$$D = \begin{cases} \{x, y_3\}, & \text{if } y_1'v_1 \notin E(G), \\ \{v_1, y_3'\}, & \text{if } y_1'v_1 \in E(G) \text{ and } xy_2 \notin E(G), \\ \{y_1', y_2\}, & \text{if } y_1'v_1 \in E(G) \text{ and } xy_2 \in E(G). \end{cases}$$

In particular, for the subcase of $y_1'v_1 \in E(G)$ and $xy_2 \in E(G)$, we know $y_3'v_1' \notin E(G)$ since $y_3'v_1'vxy_2y_3y_3'$ can not be an induced 6-cycle of G. Therefore, for any subcase above, D is a P_3-isolating set of G, and $\iota(G, P_3) \leq |D| = 2 < \frac{11}{4} = \frac{n}{4}$.

(ii) Suppose that $(G-X)_v \cong C_7$. Clearly, $d(y) = 3 = d(v) = \Delta(G)$ and $xv_1, xv_1', xy_1, xy_1' \notin E(G)$. In the same manner of proof in (i), we may first assume that $E(\{y_2, y_2'\}, \{v_1, v_1'\}) = \emptyset$. Moreover, if $E(\{y_1, y_1'\}, \{v_1, v_1'\}) = \emptyset$, then $E(\{y_3, y_3'\}, \{v_1, v_1'\}) \neq \emptyset$ and we further assume that $y_3v_1 \in E(G)$. It is easy to see that $D = \{x, y_3, v_3\}$ is a P_3-isolating set of G. If $E(\{y_1, y_1'\}, \{v_1, v_1'\}) \neq \emptyset$, then we may assume that $y_1v_1 \in E(G)$. Since $d(y_1) = d(v_1) = 3 = \Delta(G)$ and $y_1'v_1'vv_1y_1yy_1'$ is not an induced 6-cycle of G, $y_1v_1', y_1'v_1 \notin E(G)$ and $y_1'v_1' \notin E(G)$. It is clear that $D = \{x, y_3, v_3\}$ is also a P_3-isolating set of G. For these two subcases, we have $\iota(G, P_3) \leq |D| = |\{x, y_3, v_3\}| = 3 < \frac{15}{4} = \frac{n}{4}$.

(iii) Suppose that $(G-X)_v \cong C_{11}$. By the same proof of (ii), now $D = \{x, y_3, v_3, v_3'\}$ is a P_3-isolating set of G, and we have $\iota(G, P_3) \leq |D| = 4 < \frac{19}{4} = \frac{n}{4}$.

Subcase 2.1.3: $H^* \cong C_{11}$.

(i) Suppose that $(G-X)_v \cong P_3$. For the same reason in Subcase 2.1.2(i), we have $E(\{y_2, y_2'\}, \{v_1, v_1'\}) = \emptyset$. If $E(\{y_1, y_1'\}, \{v_1, v_1'\}) = \emptyset$, then it is easy to check that $D = \{x, y_4, y_4'\}$ is a P_3-isolating set of G, and $\iota(G, P_3) \leq |D| = 3 < \frac{15}{4} = \frac{n}{4}$.
If $E(\{y_1, y_1'\}, \{v_1, v_1'\}) \neq \emptyset$, then we may assume that $y_1v_1 \in E(G)$. Clearly, $xv_1, xv_1', xy_1, xy_1', y_2v_1 \notin E(G)$. Then, $y_1'v_1' \notin E(G)$; otherwise, $y_1'v_1'vv_1y_1yy_1'$ is an induced 6-cycle of G. Moreover, $y_5v_1 \notin E(G)$; otherwise, $d(v_1) = 3 = \Delta(G)$ and $y_5v_1y_1y_2y_3y_4y_5$ is an induced 6-cycle of G. Take

$$D = \begin{cases} \{x, y_3, y_4'\}, & \text{if } y_1'v_1 \notin E(G), \\ \{v_1, y_4, y_4'\}, & \text{if } y_1'v_1 \in E(G) \text{ and } xy_2 \notin E(G) \text{ and } xy_2' \notin E(G), \\ \{v_1, y_3, y_4'\}, & \text{if } y_1'v_1 \in E(G) \text{ and } xy_2 \in E(G), \\ \{v_1, y_3', y_4\}, & \text{if } y_1'v_1 \in E(G) \text{ and } xy_2' \in E(G). \end{cases}$$

We can check that D is a P_3-isolating set of G, and $\iota(G, P_3) \leq |D| = 3 < \frac{15}{4} = \frac{n}{4}$.

(ii) Suppose that $(G-X)_v \cong C_7$. For the same reason in Subcase 2.1.2 (i), we have $E(\{y_2, y_2'\}, \{v_1, v_1'\}) = \emptyset$. If $E(\{y_1, y_1'\}, \{v_1, v_1'\}) = \emptyset$, then it is clear that $D = \{x, v_3, y_4, y_4'\}$ is a P_3-isolating set of G, and $\iota(G, P_3) \leq |D| = 4 < \frac{19}{4} = \frac{n}{4}$.

If $E(\{y_1, y_1'\}, \{v_1, v_1'\}) \neq \emptyset$, then we may assume that $y_1 v_1 \in E(G)$. Since $d(y_1) = d(v_1) = 3 = \Delta(G)$ and $y_1' v_1' v v_1 y_1 y y_1'$ is not an induced 6-cycle of G, we know that $y_1 v_1', y_1' v_1 \notin E(G)$ and $y_1' v_1' \notin E(G)$. Take

$$D = \begin{cases} \{x, v_3, y_3, y_4'\}, & \text{if } y_5 v_1' \notin E(G), \\ \{y, v_3, y_5, y_3'\}, & \text{if } y_5 v_1' \in E(G). \end{cases}$$

It is clear that D is a P_3-isolating set of G. Thus, $\iota(G, P_3) \leq |D| = 4 < \frac{19}{4} = \frac{n}{4}$.

(iii) Suppose that $(G-X)_v \cong C_{11}$. Let D be the set defined as in (ii). Corresponding to every subcase in (ii), now $\{v_3'\} \cup D$ is a P_3-isolating set of G. Therefore, we have $\iota(G, P_3) \leq |\{v_3'\} \cup D| = 5 < \frac{23}{4} = \frac{n}{4}$.

Subcase 2.2: $(G-X)_v \notin \{P_3, C_7, C_{11}\}$.

Recall that $X = \{x\} \cup V(H^*)$ and $G - X = (G-X)_v \cup \bigcup_{H \in \mathcal{H}_x'} H$, where $(G-X)_v$ is the component of $G - X$ containing v. There are also three subcases to consider due to the structure of H^*.

Subcase 2.2.1: $H^* \cong P_3$.

If $d_{H^*}(y) = 2$, then $\{y\}$ is a P_3-isolating set of $G[X]$. By Lemmas 2.2 and 2.3, and by the present structure and induction hypothesis, we have

$$\iota(G, P_3) \leq |\{y\}| + \iota((G-X)_v, P_3) + \sum_{H \in \mathcal{H}_x'} \iota(H, P_3) \leq 1 + \frac{n-4}{4} = \frac{n}{4}.$$

Let $d_{H^*}(y) = 1$. By the choice of x, $x' y_1 \notin E(G)$ for any $x' \in N(v)$; otherwise, by regarding x' as x and y_1 as y, this subcase is covered by either the subcase of $d_{H^*}(y) = 2$ if $(G-X')_v \notin \{P_3, C_7, C_{11}\}$, or Subcase 2.1.1 if $(G-X')_v \in \{P_3, C_7, C_{11}\}$, where $X' = \{x'\} \cup V(H^*)$ and $(G-X')_v$ is the component of $G - X'$ containing v. Hence, $d(y_1) = 2$. If $d(y_2) = 1$ or $x y_2 \in E(G)$, then $\{x\}$ is a P_3-isolating set of $G[X]$. By Lemmas 2.2 and 2.3, and by the induction hypothesis, we have

$$\iota(G, P_3) \leq |\{x\}| + \iota((G-X)_v, P_3) + \sum_{H \in \mathcal{H}_x'} \iota(H, P_3) \leq 1 + \frac{n-4}{4} = \frac{n}{4}.$$

Hence, $x' y_2 \in E(G)$ for some $x' \in N(v) \setminus \{x\}$. It is easy to see that now $y_2 x' v x y y_1 y_2$ is a 6-cycle in G, and thus, $y x' \in E(G)$ for avoidance of a triangle or an induced 6-cycle in G. We can regard x' as x, and this subcase will be covered by either the subcase of $x y_2 \in E(G)$ if $(G-X')_v \notin \{P_3, C_7, C_{11}\}$, or Subcase 2.1.1 if $(G-X')_v \in \{P_3, C_7, C_{11}\}$, where $X' = \{x'\} \cup V(H^*)$.

Subcase 2.2.2: $H^* \in \{C_7, C_{11}\}$.

Let $X^- = \{x, y, y_1, y_1'\}$. Clearly, $X^- \subset X$. Since G has no triangles, $G[X^-] \cong K_{1,3}$. As an illustration in Figure 4, we shall consider the graph $G - X^-$. Let $c(G-X)$ and $c(G-X^-)$ be the numbers of components of $G - X$ and $G - X^-$, respectively. Then, $c(G-X) \leq c(G-X^-) \leq c(G-X)+1$. If $c(G-X^-) = c(G-X)+1$, then $G - X^- = (G-X) \cup (G-X^-)_{y_2} = (G-X)_v \cup (G-X^-)_{y_2} \cup \bigcup_{H \in \mathcal{H}_x'} H$, where $(G-X^-)_{y_2}$ is the component of $G - X^-$ containing the vertices of $V(H^*) \setminus \{y, y_1, y_1'\}$. It is clear that $(G-X^-)_{y_2} \in \{P_4, P_8\}$ and $(G-X^-)_{y_2} \notin \{P_3, C_7, C_{11}\}$. Recall that $(G-X)_v \notin \{P_3, C_7, C_{11}\}$. The set $\{y\}$ is a P_3-isolating set of $G[X^-]$.

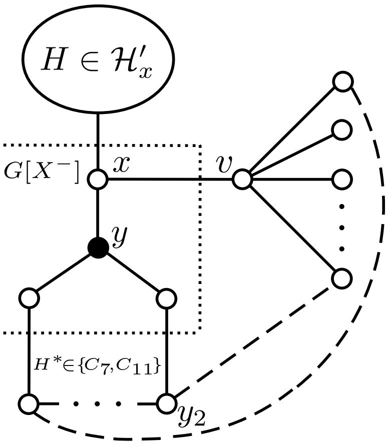

FIGURE 4. The set X^- and the graph $G - X^-$.

By Lemmas 2.2 and 2.3, and the induction hypothesis, we have

$$\begin{aligned}\iota(G, P_3) &\le |\{y\}| + \iota(G - X^-, P_3)\\ &= 1 + \iota((G-X)_v, P_3) + \iota((G-X^-)_{y_2}, P_3) + \sum_{H \in \mathcal{H}'_x} \iota(H, P_3)\\ &\le 1 + \frac{|V((G-X)_v)|}{4} + \frac{|V((G-X^-)_{y_2})|}{4} + \sum_{H \in \mathcal{H}'_x} \frac{|V(H)|}{4}\\ &= 1 + \frac{n-4}{4} = \frac{n}{4}\cdot\end{aligned}$$

Let $c(G - X^-) = c(G - X)$. Then, $G - X^- = (G - X^-)_v \cup \bigcup_{H \in \mathcal{H}'_x} H$, where $(G - X^-)_v$ is the component of $G - X^-$ containing v. Clearly, $(G - X)_v \subset (G - X^-)_v$ and $V(H^*) \setminus \{y, y_1, y'_1\} \subset V((G - X^-)_v)$. Note that $(G - X^-)_v \ncong P_3$. If $(G - X^-)_v \in \{C_7, C_{11}\}$, then $\Delta(G) = d(v) = 3 = d(y)$. Let $X' = \{x\} \cup V((G - X^-)_v)$. It is easy to see that $G - X' = (G - X')_y \cup \bigcup_{H \in \mathcal{H}'_x} H$, where $(G - X')_y \cong P_3$ is the component of $G - X'$ containing y. We switch the roles of y and v and regard X' as X, and this subcase is covered by Subcases 2.1.2 or 2.1.3 in terms of the structure of $(G - X^-)_v$. If $(G - X^-)_v \notin \{C_7, C_{11}\}$, then the set $\{y\}$ is a P_3-isolating set of $G[X^-]$. By Lemmas 2.2 and 2.3, and the induction hypothesis, we have

$$\begin{aligned}\iota(G, P_3) &\le |\{y\}| + \iota(G - X^-, P_3)\\ &= 1 + \iota((G-X^-)_v, P_3) + \sum_{H \in \mathcal{H}'_x} \iota(H, P_3)\\ &\le 1 + \frac{|V((G-X^-)_v)|}{4} + \sum_{H \in \mathcal{H}'_x} \frac{|V(H)|}{4}\\ &= 1 + \frac{n-4}{4} = \frac{n}{4}\cdot\end{aligned}$$

This completes the proof of Theorem 1.4. □

ACKNOWLEDGEMENTS

The authors would like to thank the reviewers for their valuable comments and suggestions.

References

[1] N.A. Abd Aziz, N. Jafari Rad and H. Kamarulhaili, A note on the double domination number in maximal outerplanar and planar graphs. *RAIRO:RO* **56** (2022) 3367–3371.

[2] B. Bollobás, Modern Graph Theory. Springer, New York (1998).

[3] B. Bollobás and E.J. Cockayne, Graph-theoretic parameters concerning domination, independence and irredundance. *J. Graph Theory* **3** (1979) 241–249.

[4] J.A. Bondy and U.S.R. Murty, Graph Theory. Springer, New York (2008).

[5] P. Borg, Isolation of Cycles. *Graphs Combin.* **36** (2020) 631–637.

[6] P. Borg, Isolation of connected graphs. *Discrete Appl. Math.* **339** (2023) 154–165.

[7] P. Borg and P. Kaemawichanurat, Partial domination of maximal outerplanar graphs. *Discrete Appl. Math.* **283** (2020) 306–314.

[8] P. Borg and P. Kaemawichanurat, Extensions of the art gallery theorem. *Ann. Comb.* **27** (2023) 31–50.

[9] P. Borg, K. Fenech and P. Kaemawichanurat, Isolation of k-cliques. *Discrete Math.* **343** (2020) 111879.

[10] P. Borg, K. Fenech and P. Kaemawichanurat, Isolation of k-cliques II. *Discrete Math.* **345** (2022) 112641.

[11] C.N. Campos and Y. Wakabayashi, On dominating sets of maximal outerplanar graphs. *Discrete Appl. Math.* **161** (2013) 330–335.

[12] Y. Caro and A. Hansberg, Partial domination – the isolation number of a graph. *Filomat* **31** (2017) 3925–3944.

[13] M. Chellali, O. Favaron, A. Hansberg and L. Volkmann, k-domination and k-independence in graphs: a survey. *Graphs Combin.* **28** (2012) 1–55.

[14] T.T. Chelvam and M. Sivagami, Domination in Cayley graphs: a survey. *AKCE Int. J. Graphs Comb.* **16** (2019) 27–40.

[15] J. Chen and S.-J. Xu, P_5-isolation in graphs. *Discrete Appl. Math.* **340** (2023) 331–349.

[16] E.K. Cho, I. Choi, H. Kwon and B. Park, A tight bound for independent domination of cubic graphs without 4-cycles. *J. Graph Theory* **104** (2023) 372–386.

[17] Q. Cui and J. Zhang, A sharp upper bound on the cycle isolation number of graphs. *Graphs Combin.* **39** (2023) 117.

[18] W.J. Desormeaux and M.A. Henning, Paired domination in graphs: A survey and recent results. *Util. Math.* **94** (2014) 101–166.

[19] P. Dorbec, M.A. Henning, M. Montassier and J. Southey, Independent domination in cubic graphs. *J. Graph Theory* **80** (2015) 329–349.

[20] O. Favaron and P. Kaemawichanurat, Inequalities between the K_k-isolation number and the independent K_k-isolation number of a graph. *Discrete Appl. Math.* **289** (2021) 93–97.

[21] W. Goddard and M.A. Henning, Independent domination in graphs: A survey and recent results. *Discrete Math.* **313** (2013) 839–854.

[22] D. Gonçalves, A. Pinlou, M. Rao and S. Thomassé, The domination number of grids. *SIAM J. Discrete Math.* **25** (2011) 1443–1453.

[23] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Fundamentals of Domination in Graphs. Marcel Dekker Inc., New York (1998).

[24] T.W. Haynes, S.T. Hedetniemi and M.A. Henning, editors, Structures of domination in graphs. In Vol. 66 *Developments in Mathematics*. Springer, Cham (2021).

[25] T.W. Haynes, S.T. Hedetniemi and M.A. Henning, Domination in Graphs: Core Concepts. *Springer Monographs in Mathematics*. Springer, Cham (2023).

[26] M.A. Henning, A survey of selected recent results on total domination in graphs. *Discrete Math.* **309** (2009) 32–63.

[27] H. Hua, X. Hua, S. Klavžar and K. Xu, Relating the total domination number and the annihilation number for quasi-trees and some composite graphs. *Discrete Math.* **345** (2022) 112965.

[28] R. Khoeilar, H. Karami, M. Chellali, S.M. Sheikholeslami and L. Volkmann, Nordhaus–Gaddum type results for connected and total domination. *RAIRO:RO* **55** (2021) S853–S862.

[29] W.B. Kinnersley, D.B. West and R. Zamani, Extremal problems for game domination number. *SIAM J. Discrete Math.* **27** (2013) 2090–2107.

[30] S. Kosari, Z. Shao, X. Shi, S.M. Sheikholeslami, M. Chellali, R. Khoeilar and H. Karami, Cubic graphs have paired-domination number at most four-seventh of their orders. *Discrete Math.* **345** (2022) 113086.

[31] O. Ore, Theory of Graphs. In Vol. 38 *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI (1962).

[32] S. Tokunaga, T. Jiarasuksakun and P. Kaemawichanurat, Isolation number on maximal outerplanar graphs. *Discrete Appl. Math.* **267** (2019) 215–218.

[33] J. Yan, Isolation of the diamond graph. *Bull. Malays. Math. Sci. Soc.* **45** (2022) 1169–1181.

[34] G. Zhang and B. Wu, $K_{1,2}$-isolation in graphs. *Discrete Appl. Math.* **304** (2021) 365–374.

[35] G. Zhang and B. Wu, Isolation of cycles and trees in graphs. *J. Xinjiang Univ. (Nat. Sci. Ed. Chin. Eng.)* **39** (2022) 169–175.