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A PARTITION-GRAPH RESTRICTED COOPERATIVE GAME WITH INCOMPLETE INFORMATION

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Abstract. In a partition-graph restricted cooperative game with incomplete information, it is assumed that a prior coalition takes part in the grand coalition N in form of communication structure. Before the grand coalition is formed, all the players participate in the cooperation just by communication structure. Hence, a new cooperative game with communication structure (a restricted game for short) is formed. In this study, a new value of a restricted game is proposed, which is defined by the Myserson value and the Shapley value. It is proved that the added value of grand coalition is distributed by a certain ratio (*i.e.*, the Shapley value). By generalized the distribution ratio, a further extension of the proposed value is gotten, *i.e.*, the weight extended value. This weight extended value is unique based on restricted component efficiency, weight fairness, weight symmetric across coalitions, null coalition property and restricted linearity.

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1. Introduction

A coalition structure is a partition of the set of players representing several coalitions obtained at the end of game. Game with a coalition structure (*i.e.*, coalition structure game) was firstly introduced by Aumann and Drze [8] in 1974, where the coalitions were independent with each other. Hence there should be no side payment between these final coalitions. Myerson [25] improved this approach by introducing communication structure. Communication structure is a graph representing the bilateral cooperation possibilities among the agents. In this case, a final coalition structure is a set of connected components in a graph. In contrast to the above model, Owen [27] introduced a cooperative game with prior coalition that took into account the probability of cooperation between coalitions, called the Owen coalition structure. Owen also provided the axiomatization of the Owen value, which was seen as an extension of the Shapley value. Later, Casajus [12] proposed a modification of Owen model in the Myerson sense. A coalition structure is formed through a connected graph, which explains the existence of intimate relationships between players. Li and Meng [24] analyzed the probabilistic Harsanyi power solutions for probabilistic graph games, which distributed the Harsanyi dividends proportional to weights

Keywords. Coalition structure, graph, weight, incomplete.

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determined by a probabilistic power measure for probabilistic graph structure. According to Slikker and van den Nouweland [31], in a coalition structure no communication took place between players in different components, while any coalition of players in any of the components could effectively communicate with each other. Following this interpretation, Hu, Xu and Li [20] defined a new value named $\overline{AD}$-value on a coalition structure, which first assigned to every player a value based on his or her contribution in inter prior coalition and then distributed the surplus value of the grand coalition among players equally.

For the cooperative game with a coalition structure (coalition structure game for short), the Owen value is often used as profit allocation method. A two-step payoff process is applied to construct the Owen value by the Shapley value [30]. The prior coalitions (*i.e.*, firms) deal with each other to determine how much each one receives, and then the players bargain within a single prior coalition to determine how these receipts are to be divided. A two-step payoff process can be depicted as follows.

For each cooperative game v with a coalition structure $C = \{C_1, C_2, \ldots, C_m\}$ on the players set N, (In general, we denote a cooperative game v with a coalition structure C by (N, v, C)), the two-step distribution process is as follows.

StepO1: Prior coalitions deal with each other to determine how much one receives, *i.e.*, let $M^C = \{1, 2, \ldots, m\}$ be the set of subscript of elements in C, and a game v_C be defined formally for every $R \subseteq M^C$ by $v_C(R) = v(\cup_{p\in R} C_p)$, where v_C is induced by v, and prior coalitions in C are considered as players. Fixing $C_p \in C$ and $S \subseteq C_p$, such that $S \neq \emptyset$, we consider $C(S) = \{C_1, C_2, \ldots, C_{p-1}, S, C_{p+1}, \ldots, C_m\}$, and define the game $v_{C(S)}(R) \triangleq v(\cup_{p\in R} C_p)$, denoted by $\left(M^C, v_{C(S)}\right)$, $\forall R \subseteq M^C$.

Then for "player" S, the allocation (the Shapley value) of S is $f_p^1\left(M^C, v_{C(S)}\right)$.

StepO2: For any $p \in M^C$, the payoff $Sh_p\left(M^C, v_C\right)$ is allocated between players in $C_p \in C$.

A cooperative game w_p on C_p is defined for every nonempty coalition $S \subseteq C_p$, by

$$w_p(S) \triangleq f_p^1\left(M^C, v_{C(S)}\right).$$

The Owen value $\varphi^{Ow}(N, v, C)$ of player $i \in C_p$ is defined by

$$\varphi_i^{Ow}(N, v, C) = f_i^2(C_p, w_p).$$

We see that a value for a coalition structure game is mainly determined by two functions f^1 and f^2. Functions f^1 and f^2 are both the Shapley values in the Owen value. Hence, there are other values in addition to the Owen value based on the above two-step distribution process, such as the Banzhaf-Owen value defined by Owen [28] (that replaces f^1 and f^2 both with the Banzhaf value [9], and the symmetric coalitional Banzhaf value (that replaces f^1 with the Shapley value, and replaces f^2 with the Banzhaf value). The Banzhaf-Owen value was firstly axiomatically characterized on the class of simple games by Albizuri [1]. At the same time, Amer *et al.* [7] axiomatically characterized on the class of all cooperative games. Because the Owen value was defined for the game without the information about the internal union structure, Casajus [12] proposed a modification of the Owen value in Myerson sense, which denoted connected relationship by a graph. The Myerson-Owen value was defined by replacing f^1 with the Shapley value and f^2 with the Myerson value [25] respectively. Gallego Sanchez [16] defined the Banzhaf-Myerson value by replacing f^1 with the Banzhaf value in the Myerson-Owen value.

In addition, the Owen value has been axiomatically studied in many literatures. Hart and Kurz [19] proposed an alternative axiomatization of the Owen value for infinite universe of players, which were carrier, additivity, anonymity and the coalitional inessential game property. Several axiomatic systems of the Owen value were considered by Albizuri [2]. Owen [27] gave another payoff index for a coalition structure game, which was called the Banzhaf-Owen value. Alonso-Meijide *et al.* [6] and Amer *et al.* [7] proposed a comparative axiomatic characterization of the Banzhaf-Owen coalitional value respectively. Alonso-Meijide and Fiestras-Janeiro [6] pointed out that the Banzhaf-Owen value was dissatisfied symmetry in quotient game. They gave another solution for a coalition structure game, which was known as the symmetric Banzhaf value. Sun and Qiang [32] introduced a probabilistic concept for a coalition structure game, and axiomatically characterized the probabilistic Owen value. Yu *et al.* [35] proposed an uncertain coalition structure game with payoff of belief structure based on the belief structures of D–S theory, the uncertain Owen value was axiomatically characterized by Hukuhara difference as the value for this cooperative game. As described above, for a coalition structure game, most of attention is paid on the axiomatization of some values extended by the Owen value, such as Banzhaf-Owen, Banzhaf-Myerson, and the Myerson-Owen value. In addition, for an incomplete cooperative game without a structure coalition, most of researchers focus on the Shapley value by two ways. One way is to define reduced the Shapley value by using known values of a game, and the other is to transfer the incomplete cooperative game into complete game (*i.e.*, a cooperative game with complete information) by evaluating of coalitional values.

On the other hand, the cooperative games under the restrictions are analyzed, in which some coalitions are feasible and other are not. Myerson [25, 26] introduced connected sets of a graph or a hypergraph as feasible coalitions. Later, other restrictions on the cooperation among players have been studied, such as precedence constraints in Faigle and Kern [15], distributive lattices in Grabisch and Xie [18], permission structures in Gilles *et al.* [17] and Van den Brink [33], hierarchical structures in Derks and Gilles [14], convex geometries in Bilbao and Edelmann [11], and in Bilbao *et al.* [3], antimatroids in Algaba *et al.* [4], union stable systems in Algaba *et al.* [5], building sets in Koshevoy and Talman [21], quasi-building system in Koshevoy *et al.* [22], augmenting systems in Bilbao [10], and in Algaba *et al.* [5], set system in Li and Shan [23]. However, all above researches consider only one kind of cooperative restrictions, such as coalition structure, communication structure or other. For the cooperative game with coalition and graph structure (cooperative game with coalition and graph structure for short, Rene, Gerard and Nigel [34]), the graph restricted communication and a prior coalition are both considered in the cooperative game. By applying the Shapley value to an associated cooperative game, two new values are given for a cooperative game with coalition and communication structure. The two values are defined by the associated cooperative game with no restrictions, but in the reality, this payoff information of cooperative game may not be obtained before the formation of cooperative game with coalition and communication structure. In this situation, the value for cooperative game with coalition and communication structure is needed to define based on the part information of restricted information. However, if we could not get all the values of feasible coalition, *i.e.*, a partition-graph restricted cooperative games with incomplete information, then the above values cannot be used. For this kind of partition-graph restricted cooperative games with incomplete information, there is no relevant research so far.

Inspired by Rene, Gerard and Nigel [34], the purpose of this paper is to study a new value for the cooperative game with communication structure in a coalition structure (a restricted game for short), where a prior coalition takes part in the grand coalition N by an inter communication structure. There are many restricted games in the real-life situations. As we all know, airline companies belong to different aviation alliances (or coalitions) and come from different countries. Airline companies in the same aviation alliance have code sharing and cooperation relations. Therefore, the aviation alliance here can be regarded as a communication structure. Meanwhile, airline companies in different countries belong to the coalition structure due to airport public relations. The cooperative relationship composed of airline companies from different countries and different aviation alliances just construct a restricted game. Similarly, there are cross-border e-commerce alliances, communication alliances, etc. In the above cooperative coalitions, all the players participate in the cooperation just by communication structure before the new grand coalition is formed, so the profit of grand coalition is no less than the profit sum of prior coalitions. In this paper, we propose a new cooperative game with communication structure (*i.e.*, a restricted game for short). New values are also given, which are defined by both the Myerson value and the Shapley value. So the added value of coalition structure is allocated by taking the Myerson value as the allocation weights. If no added value is produced by the formation of coalition structure, then it is proved that the Owen value is equal to the Myerson value. Following this method, another weight extended value is introduced and characterized as a general value for the restricted game in this study. This weight extended value is the unique value based on restricted component efficiency, weight fairness, weight symmetric across coalitions, null coalition property and restricted linearity.

The reminder of this paper is organized as follows. In Section 2, a review of cooperative game, coalition structure and communication structure. In Section 3, a restricted game is defined based on D–S theory. In Section 4, a research is carried out for the extended value of the restricted game, and the weight extended value is characterized. Finally, some conclusions are provided in Section 5.

2. Preliminaries

A cooperative game is a pair (N, v), where $N = \{1, 2, \dots, n\}$ is a set of players and v (the characteristic function) is a real set function defined on all subsets of N which satisfies $v(\emptyset) = 0$. A crisp coalition S is a subset of N, and the class of all crisp coalitions of S is denoted by 2^S. For the convenience of the following, we write $S \cup i$ for $S \cup \{i\}$, and $S\backslash i$ for $S\backslash \{i\}$ etc. For any coalition $S \in 2^N$, the cardinality of S is denoted by s.

When there is no ambiguity with respect to N, a game (N, v) is abbreviated as v. The set of all supperadditive games on N is denoted by $G(N)$, *i.e.*, $v \in G(N)$ satisfies the following condition,

$$v(S \cup T) \geq v(S) + v(T), \ \ \forall S, T \in 2^N, \ \ S \cap T = \emptyset.$$

Definition 2.1. *(Shapley [30]) The Shapley value is a linear mapping $Sh : G(N) \to \mathbb{R}^n$ defined for each $v \in G(N)$ and $i \in N$ by*

$$Sh_i(N, v) = \sum_{S \in 2^N, i \notin S} \frac{s!\,(n-s-1)!}{n!} [v(S \cup i) - v(S)], \tag{1}$$

where $\mathbb{R}$ is real set, s and n are the cardinalities of S and N, respectively.

It is well known that the Shapley value is the unique value based on the following axioms.

Efficiency, E. Let any $v \in G(N)$. $\sum_{i \in N} Sh_i(N, v) = v(N)$. If a player $i \in N$ satisfies that $v(S \cup i) = v(S)$, $\forall S \subseteq N\backslash \{i, j\}$, then $Sh_i(N, v) = 0$.

Symmetry, S. Let any $v \in G(N)$, and $i, j \in N$. If $v(S \cup i) = v(S \cup j)$ holds for any $S \subseteq N$, then $Sh_i(N, v) = Sh_j(N, v)$.

Linearity, L. For any $v_1, v_2 \in G(N)$, and $\alpha, \beta \in \mathbb{R}$, define $v_1 + v_2$ by $(\alpha v_1 + \beta v_2)(S) = \alpha v_1(S) + \beta v_2(S)$, $\forall S \subseteq N$. If $v_1 + v_2 \in G(N)$, then $Sh_i(N, \alpha v_1 + \beta v_2) = \alpha Sh_i(N, v_1) + \beta Sh_i(N, v_2)$, $\forall i \in N$.

In the reality, there are some cooperative restrictions on the formation of grand coalition, of which coalition structure and communication structure are two usual cooperation restrictions.

Firstly, a coalition structure C is a finite partition of N, denoted by $\{C_1, C_2, \dots, C_m\}$, *i.e.*, $\cup_{p=1}^m C_p = N$ and $C_p \cap C_k = \emptyset$ for $p \neq k$. We call $C_p\,(1 \leq p \leq m)$ the prior coalition p.

Let $M^C = \{1, 2, \dots, m\}$ be the set of subscript of elements in C, and every $C_p \in C$ is called a prior coalition, $\forall p \in M^C$. The set of all the coalition structures of N is denoted by C^N. For any $C \in C^N$, given any $p \in M^C$, $R \subseteq M^C \backslash p$, and $S \subseteq C_p$, we set

$$T_p^R(S) \triangleq \cup_{k \in R} C_k \cup S.$$

Then we call $T_p^R(S)$ a feasible coalition on C. Especially, if $S = \emptyset$, we denote $T_p^R(\emptyset)$ by T_p^R for short. The set of all feasible coalitions is denoted by $P(C)$, *i.e.*,

$$P(C) \triangleq \left\{ T_p^R(S) \,\middle|\, \forall S \subseteq C_p, \ \ \forall p \in M^C, \ \ \forall R \subseteq M^C \backslash p \right\}.$$

We write CG^N by $CG^N = C^N \times G(N)$. Any $(N, v, C) \in CG^N$ is called cooperative game v with a coalition structure C (a coalition structure cooperative game (N, v, C) for short). Because $v \in G(N)$ is supperadditive, so (N, v, C) is also supperadditive, *i.e.*,

$$v(S \cup T) \geq v(S) + v(T), \ \ \forall S, T \in P(C), \ \ S \cap T = \emptyset. \tag{2}$$

Let ϕ be an operator on CG^N, so $\phi(N,v,C) \in \mathbb{R}^n$ for each $(N,v,C) \in CG^N$. There are some properties that ϕ should satisfy as follows.

Efficiency, E. $\sum_{i\in N} \phi_i(N,v,C) = v(N)$.

Symmetric Within Coalitions, SWC. For any $i,j \in C_p \in C \left(p \in M^C\right)$ and $S \subseteq N\backslash\{i,j\}$, if $v(S\cup i) = v(S\cup j)$, then $\phi_i(N,v,C) = \phi_j(N,v,C)$.

Symmetric Across Coalitions, SAC. For any $C_s, C_t \in C$ and $K \subseteq M^C\backslash\{s,t\}$, if $v(\cup_{k\in K} C_k \cup C_s) = v(\cup_{k\in K} C_k \cup C_t)$, then $\sum_{i\in C_s} \phi_i(N,v,C) = \sum_{i\in C_t} \phi_i(N,v,C)$.

Null Player Property, NP. Giving $i \in N$, for any $S \subseteq N\backslash i$, if $v(S\cup i) = v(S)$ (*i.e.*, i is a null player), then $\phi_i(N,v,C) = 0$.

Linearity, L. For any $(C,v),(C,w) \in CG^N$, and $\alpha,\beta \in \mathbb{R}$, define a game $(\alpha v + \beta w)(S) = \alpha v(S) + \beta w(S)$, $\forall S \in P(C)$, then $\phi_i(N,\alpha v + \beta w, C) = \alpha\phi_i(N,v,C) + \beta\phi_i(N,w,C)$, $\forall i \in N$.

Owen [27] has proven that there is an unique operator on CG^N that satisfies **E, SWC, SAC, NP** and **L**, which is called the Owen value, defined by

$$\begin{aligned}\varphi_i^{Ow}(N,v,C) = &\sum_{R\subseteq M^C,\ \forall p\notin R}\ \sum_{S\subseteq C_p,\ i\notin S} \frac{r!\,(m-r-1)!}{m!} \times \frac{s!\,(c_p-s-1)!}{c_p!} \\ &\times \left[v\left(Q_R\cup S\cup i\right) - v\left(Q_R\cup S\right)\right],\end{aligned} \tag{3}$$

where $Q_R = \cup_{k\in R} C_k$.

It is seen that the Owen value is an extension of the Shapley value to a coalition structure game.

Remark 2.2. The Owen value is considered as a two-step distribution process, *i.e.*, StepO1 and StepO2 in this study.

Secondly, for another cooperation restriction, communication structure is represented by graph.

Let $LN = \{\{i,j\} : i,j \in N \text{ and } i \neq j\}$ be the set of unordered pairs of elements infinite set N. A communication structure for N is a graph (N,L), where the set of vertices is N and the set of edges (or links) $L \subseteq LN$ is the set of feasible communications among them. Hence we identify a communication structure for N with the set of links L. A communication structure game is a triple (N,v,L), where $(N,v) \in G(N)$ and L is a feasible communication set for N. We denote by $G_C(N)$ all the communication structure games.

Let any $(N,v,L) \in G_C(N)$. A coalition $S \subseteq N$ whose vertices are connected by the links in L is called connected. The maximal connected coalitions correspond to the sets of vertices of the connected components in the graph (N,L) are denoted as N/L. This family N/L is actually a partition of N. If $S \subseteq N$ is a coalition, then $L(S) = \{\{i,j\} \in L : i,j \in S\}$, and $(S,v,L(S)) \in G_C(N)$ represents a restriction to S of communication structure game (N,v,L). We use $S/L = S/L(S)$. Following Myerson [25], a final coalition structure is formed by the connected components of graph in a communication structure game, and they cannot get beneficial collaborations among them.

The Myerson value is an allocation rule for the communication structure game. Given $(N,v,L) \in G_C(N)$, Myerson defines a new game $(N,v/L) \in G(N)$ incorporating the information of communication structure, *i.e.*,

$$(v/L)(S) = \sum_{T\in S/L} v(T),\ \ \forall S \subseteq N.$$

The Myerson value $\varphi^M(N,v,L)$ for game (N,v,L) is defined as

$$\varphi^M(N,v,L) = Sh(N,v/L), \tag{4}$$

where $Sh(N,v/L)$ is the Shapley value for the game $(N,v/L)$. The Myerson value satisfies component efficiency and fairness as follows.

Component efficiency, CE. For any $(N,v,L) \in G_C(N)$,

$$\sum_{i\in S} \varphi_i^M(N,v,L) = v(S),\ \ \forall S \in N/L. \tag{5}$$

Fairness, F. For any $(i,j) \in L$,

$$\varphi_i^M (N,v,L) - \varphi_i^M (N,v,L \backslash \{i,j\}) = \varphi_j^M (N,v,L) - \varphi_j^M (N,v,L \backslash \{i,j\}), \tag{6}$$

where $L_{-ij} = L \backslash \{i,j\}$.

3. A partition-graph restricted cooperative game with incomplete information

A coalition structure is a partition of players, while a communication structure is a cooperation limitation that only players in the same communication structure can increase the coalition profits. The above two cooperative games are two independent research directions, until the formation of the cooperative game with coalition and communication structure (cooperative game with coalition and communication structure for short). The graph restricted communication and a prior coalition are both considered in the above cooperative game. In reality, this payoff information of cooperative game may not be obtained before the formation of cooperative game with coalition and communication structure. In this situation, the value for cooperative game with coalition and communication structure is needed to define based on only incomplete information. For this kind of partition-graph restricted cooperative games with incomplete information, we could not get all the values of feasible coalitions, and there is no relevant research so far.

As we all know, airline companies belong to different aviation alliances (or coalitions) and come from different countries. Airline companies in the same aviation alliance have code sharing and cooperation relations. Therefore, the aviation alliance here can be regarded as a communication structure. Meanwhile, airline companies in different countries belong to the coalition structure due to airport public relations. The cooperative relationship composed of airline companies from different countries and different aviation alliances is just a partition-graph restricted cooperative games with incomplete information. Similarly, there are cross-border e-commerce alliances, communication alliances, etc.

Next, we attempt to define a new cooperative game to describe this kind of incomplete information during cooperation form process.

For any $C \in C^N$, $L \subseteq LN$, any $p \in M^C$, we let

$$LC = \{C_p/L\}_{p=1}^{p=m},$$

then LC is the limited coalition structure C on L. The set of all the limited coalition structures on L is denoted by $\mathbb{C}L$.

In a limited coalition structure, there are communication structures within a prior coalition C_p, $\forall p \in M^C$. Given a communication structure, the payoff of coalitions may take a certain modification through the formation of coalition structure. Hence, a new cooperative game (named communication game) is introduced based on communication structure game as follows.

Definition 3.1. *Let any $C \in C^N, L \subseteq LN$ and $(N,v,L) \in G_C(N)$. A communication game (N,v^L,C) is defined for any $p \in M^C$,*

$$v^L\left(T_p^R(S)\right) = \sum_{T \in T_p^R/L} v\left(T_p^R\right) + \sum_{T \in S/L} v(T), \ \forall S \subseteq C_p.$$

It is seen that values of communication structure game are retained after coalition structure is formed. It is mean that a coalition structure does not take any change for the communication structure game. This is not very realistic as seen in the following example.

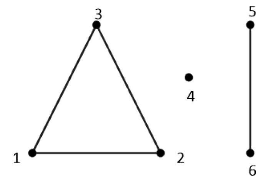

FIGURE 1. The communication structure L.

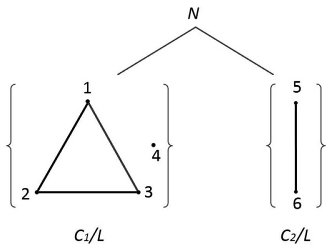

FIGURE 2. The restricted communication structure game.

Example 3.2. *We let $N = \{1, 2, 3, 4, 5, 6\}$ be a grand coalition, where $i \in N$ represents the ith project investor. We also let $L = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{4\}, \{5, 6\}\}$ denote the link among the players, as shown by Figure 1, for example, sharing of resources such as code or protocols. Hence, we have $N/L = \{\{1, 2, 3\}, \{4\}, \{5, 6\}\}$. Now, they want to construct a new project program based their former cooperative history, such as players coming from the same countries or provinces, so a coalition structure is formed, i.e., $C = \{\{1, 2, 3, 4\}, \{5, 6\}\}$. It is gotten that*

$$C_1/L = \{\{1, 2, 3\}, \{4\}\}, C_2/L = \{\{5, 6\}\}.$$

As shown by Figure 2, a limited coalition structure is formed,

$$\begin{aligned} v^L(N) &= v(1, 2, 3, 4) + v(5, 6), \\ v^L(1, 2, 3, 4) &= v(1, 2, 3) + v(4), \\ v^L(S) &= v(S), \ \ \forall S \subseteq \{1, 2, 3\} \text{ or } \{4\}, \{5, 6\}. \end{aligned}$$

As we see, there are two cases for the non-empty coalition S in C_p in this communication game,

$$v^{LC}\left(T_p^R(S)\right) = \begin{cases} \sum\limits_{T \in T_p^R/L} v\left(T_p^R\right) + \sum\limits_{T \in S/L} v(T) & S \neq C_p \\ \sum\limits_{k \in R \cup \{p\}} v(C_k/L) & S = C_p. \end{cases} \tag{7}$$

Obviously, the coalition value is either additive or the same with communication structure game. However, coalition structure game is based on the assumption that the players in the same prior coalition could cooperate with each other. However, there may exist some cooperative limitations between players even in the same prior coalition. Rene, Gerard and Nigel [34] considered the graph restricted communication and a prior coalition in the cooperative game, *i.e.*, a cooperative game with coalition and graph structure. Based on the associated cooperative game with no restriction, we define the cooperative game with coalition and graph structure as follows.

A cooperative game with coalition and graph structure is quadruple (N, v, L, C), where $(N, v) \in G(N)$, $L \subseteq LN$ is a feasible communication set for N and $C \in C^N$, *i.e.* C is a coalition structure on N. The game $(N, v) \in G(N)$ is called the associated cooperative game of (N, v, L, C).

As we see, a cooperative game with coalition and graph structure is defined on the associated cooperative game $(N, v) \in G(N)$. If a cooperative game is constructed based on a communication structure game, then the payoff of grand coalition by constructing coalition structure would be much improved than the one based on a communication structure. For this kind of cooperative games with comminution structure in prior coalitions (partition-graph restricted cooperative game with incomplete information, a restricted game for short), the profit (or payoff) for the grand coalition N, may be more than the sum of their uncooperative value $\sum_{k\in R\cup\{p\}} v(C_k/L)$. Hence, only based on $\sum_{k\in R\cup\{p\}} v(C_k/L)$, we need to evaluate the allocation value for players. The impassive cooperative value is $\sum_{k\in R\cup\{p\}} v(C_k/L)$ as shown in equation (7). In order to represent the restricted game, Dempster–Shafer evidence theory (D–S theory) is introduced, which is proposed by Dempster [13] and developed by Shafer [29].

In D–S theory, a problem domain denoted by a finite nonempty set Ω of mutually exclusive and exhaustive hypotheses is called the frame of discernment. Let 2^Ω denote the power set of Ω. The elements of 2^Ω are called propositions. Given a frame of discernment Ω , a belief structure is a mapping $m : 2^\Omega \to [0, 1]$, such that

$$m(\emptyset) = 0 \quad \text{and} \sum_{A\in 2^\Omega} m(A) = 1.$$

A belief structure is also called a mass function or a basic probability assignment (BPA). Associated with each belief structure is the belief measure and plausibility measure, *Bel* function and *Pl* function, respectively. The belief function *Bel*: $2^\Omega \to [0, 1]$ is defined as

$$Bel(A) = \sum_{B\subseteq A} m(B). \tag{8}$$

The plausibility function $Pl : 2^\Omega \to [0, 1]$ is defined as

$$Pl(A) = 1 - m(\overline{A}) = \sum_{B\cup A\neq\emptyset} m(B), \tag{9}$$

where $\overline{A} = N - A$. Obviously, $Pl(A) \geq Bel(A)$, these functions *Bel* and *Pl* are the lower limit function and upper limit function of the probability to which proposition A is supported, respectively. The difference between the belief and the plausibility of proposition A expresses the ignorance of the assessment for proposition A.

Let $N = \{1, 2, \ldots, n\}$ be the set of players and $C \in C^N$. A belief structure is defined on the coalition structure C , where m^S $(\forall S \in P(C))$ is a belief structure defined on the frame of discernment $\Omega = \{h_1, h_2, \ldots, h_l\}$. The gain of restricted game is represented by a belief structure on Ω, and the frame of discernment Ω contains possible gain assessment grades, for example from h_1 to h_l, to express such a gain. In order to represent the uncertainty of payoffs, the gain assessment grades $h_1, \ldots, h_l$ can be ordered linguistic variables so that fuzziness can also be included in this game. Here, $m^S(A)$ stands for the probability of the gain assessment grade A $(\forall A \in \Omega)$ for the coalition S. The value $m^S(A)$ represents the probability of gain assessment grade A, *e.g.*, $m^N(H)$ expresses the probability of High profit for the coalition N.

Given any coalition S in $P(C)$, if we obtain the specify payoff value $v^S(h_i)$ for any h_i in the frame of discernment Ω, $\forall 1 \leq i \leq l$, then the other possible payoff value for the coalition S can be computed. However, for the belief structure $m^S(h_1, \ldots, h_l)$, the uncertain payoff value, denoted as $v^S\left(m^S(h_1, \ldots, h_l)\right)$, could not be represented by $v^S(h_i)\,(1 \leq i \leq l)$ without the value of $m^S(A)$, $\forall A \subseteq \Omega$, and the cardinality of A is more than one, *i.e.*, $a \neq 1$. For example, element m^S may possess term $m^S(h_1, \ldots, h_l) = \alpha$, , it is difficult to determine how to assign the basic probability a among $h_1, \ldots, h_l$. There two extreme situation occurs: in the best case, $v^S\left(m^S(h_1, \ldots, h_l)\right) = \alpha \times v^S(h_k)$, where $v^S(h_k) = \max\left\{v^S(h_1), \ldots, v^S(h_l)\right\}$; in the worst case, $v^S\left(m^S(h_1, \ldots, h_l)\right) = \alpha \times v^S(h_k)$, where $v^S(h_k) = \min\left\{v^S(h_1), \ldots, v^S(h_l)\right\}$.

Example 3.3. *With more than half of all Chinese living in cities, fog and haze are becoming big political issues in Beijing-Tianjin-Hebei Region. Air is flowing, so fog and haze pollution need to be done in joint governance.*

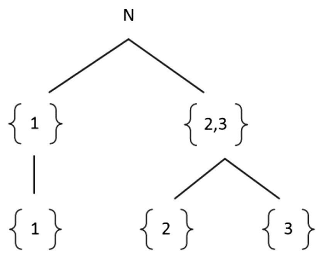

FIGURE 3. The coalition structure.

Hence, the fog and haze in Beijing is not a problem for Beijing itself, and it need the efforts of Beijing surrounding areas, such as Tianjin and Hebei. Therefore, the fog and haze governance of Beijing forms a cooperative coalition $N = \{1, 2, 3\}$ *, where* $1, 2$ *and* 3 *represents Beijing, Tianjin and Hebei respectively.*

Let the coalition structure be $C = \{C_1, C_2\}$, where $C_1 = \{1\}$, and $C_2 = \{2, 3\}$. Hence, the set of feasible coalitions of N is

$$P(C) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}.$$

Before the coalition structure is formed, there is no cooperative history among prior coalitions, so we cannot get the values of coalitions among multiple prior coalitions, such as the value $v(1, 2, 3)$. In most of situations, we estimate the value of the grand coalition $v(N)$ is at least equal to the value $v(1) + v(2, 3)$.

Based on belief structure, we define a new game as follows.

Definition 3.4. *Let any* $C \in C^N$*,* $L \subseteq LN$ *and* $(N, v, L) \in G_C(N)$*. For any* $C_p \in C$*,* $R \subseteq M^C \backslash p$*, and* $S \subseteq C_p$*,* $\forall p \in M^C$*, a* α *- partition - graph restricted cooperative game with incomplete information (*α *- restricted game for short) for* LC *is a cooperative game* $\left(N, v_\alpha^{LC}, C\right)$ *with*

$$v_\alpha^{LC}\left(T_p^R(S)\right) = \begin{cases} v^L\left(T_p^R(S)\right) & S \neq C_p,\ \ R \neq M^C \backslash p \\ \alpha \cdot \left[\sum_{k \in R \cup \{p\}} v^L(C_k/L)\right] & \textit{otherwise}, \end{cases} \tag{10}$$

where $\alpha \geq 1$. It is seen that a communication game is a special kind of α - restricted games when $\alpha = 1$. In the next subsection, we attempt to give a general value for the α - restricted game. Without ambiguity, we write $v^L(C_k/L)$ simply as $v(C_k/L)$.

The coefficient α can be seen as the risk attitude or tolerance of players, which is inversely proportional to risk aversion coefficient. The more risk tolerance players have, the more prospective payoff they get. When the payers are risk averse and want to avoid potential considerable losses, they could adopt the optimistic value criterion. Thus, we can adopt a risk coefficient α to represent the risk tolerance of players, by brife structure meaning.

4. Two allocation values for a restricted game

As we know, the Owen value is a value of the coalition structure game, while the Myerson value is the value for a communication structure game. As the α - restricted game is a connection between the coalition structure game and the communication structure game, there may be some relationships between two values. Indeed, the Owen value for the communication game (*i.e.*, a special kind of α - restricted games) is just the Myerson value, which is proved as follows.

Theorem 4.1. *For any* $C \in C^N$*,* $L \subseteq LN$ *and* $(N, v, L) \in G_C(N)$*, a communication game* (N, v^L, C) *satisfies that*

$$\varphi^M(N, v, L) = \varphi^{Ow}\left(N, v^L, C\right).$$

Proof of Theorem 4.1. See Appendix A.1.

It is gotten that the Myerson value for the communication structure game (N, v, L) is equal to the Owen value of communication game (N, v^L, C) . Following the Owen value, we define a new value for the α - restricted game by two-step allocation method. Let any $C \in C^N$ and $L \subseteq LN$, and the α - restricted game (N, v_a^{LC}, C).

StepO1: Prior coalitions deal with each other to determine how much one receives, *i.e.*, let $M^C = \{1, 2, \ldots, m\}$ be the set of subscript of elements in C, and a new cooperative game $v_{\alpha,C}^{LC}$, be defined formally for every $R \subseteq M^C$ by $v_{\alpha,C}^{LC}(R) = v_{\alpha}^{LC}\left(T_p^R\right)$, where $v_{\alpha,C}^{LC}$ is induced by v_{α}^{LC}.

Then for "player" $p \in M^C$, the allocation (the Shapley value) of p is $Sh_p(M^C, v_{\alpha,C}^{LC})$.

StepO2: For any $p \in M^C$, the payoff $Sh_p(M^C, v_{\alpha,C}^{LC})$ is allocated between players in $C_p \in C$.

A game w_p on C_p is defined for every nonempty coalition $S \subseteq C_p$ by

$$w_p\left(S\right) \triangleq \begin{cases} Sh_p(M^C, v_{\alpha,C}^{LC}) & S = C_p \\ v^L\left(S\right) & S \neq C_p. \end{cases} \tag{11}$$

The extended value of player $i \in C_p$ is defined by

$$\varphi_i\left(N, v_{\alpha}^{LC}, C\right) = Sh_i\left(C_p, w_p/L\right). \tag{12}$$

Proposition 4.2. *For any $C \in C^N$, the extended value $\varphi\left(N, v_{\alpha}^{LC}, C\right)$ for a restricted game (N, v_{α}^{LC}, C) satisfies that,*

$$\varphi_i\left(N, v_{\alpha}^{LC}, C\right) = \varphi_i^M\left(N, v, L\right) + \frac{(\alpha - 1) \cdot Sh_i\left(C_p, v/L\right) \cdot \sum_{k \in M^C} v(C_k/L)}{m \cdot v\left(C_p/L\right)}, \ \ \forall i \in C_p, \forall p \in M^c. \tag{13}$$

Proof of Proposition 4.2. See Appendix A.2.

By Proposition 4.2, when $\alpha = 1$, $\varphi\left(N, v_{\alpha}^{LC}, C\right) = \varphi^M\left(N, v, L\right)$. This means the Myerson value is a special case of the extended value in equation (13).

Let

$$\lambda_i\left(C_p, w_p, L\right) = \frac{Sh_i\left(C_p, v/L\right)}{m \cdot v\left(C_p/L\right)}, \quad \varphi_i^{\lambda}\left(N, v_{\alpha}^{LC}, C\right) = \varphi_i\left(N, v_{\alpha}^{LC}, C\right), \ \ \forall i \in C_p, \forall p \in M^c..$$

then the extended value is

$$\begin{aligned} \varphi_i^{\lambda}\left(N, v_{\alpha}^{LC}, C\right) &= \varphi_i^M\left(N, v, L\right) + \frac{(\alpha - 1) \cdot Sh_i\left(C_p, v/L\right)}{m \cdot v\left(C_p/L\right)} \cdot \sum\nolimits_{k \in M^C} v(C_k/L) \\ &= \varphi_i^M\left(N, v, L\right) + \lambda_i \cdot (\alpha - 1) \cdot \sum\nolimits_{k \in M^C} v(C_k/L). \end{aligned}$$

Following this method, a new weight extended value is defined for the restricted game (N, v^{LC}, C),

$$\varphi_i^{\theta}\left(N, v_{\alpha}^{LC}, C\right) = \varphi_i^M\left(N, v, L\right) + \theta_i\left(\alpha - 1\right) \cdot \sum\nolimits_{k \in M^C} v(C_k/L), \tag{14}$$

where $0 \leq \theta_i \leq 1$ such that

$$\sum_{p \in M^C} \sum_{i \in C_p} \theta_i = 1.$$

θ_i called the allocation weight of player $i \in N$, and the set of all the allocation weights θ on N is denoted by N_θ.

Furthermore, we let $[\theta]_p$ be the allocation weight for prior coalition C_p, *i.e.*,

$$[\theta]_p = \sum_{i \in C_p} \theta_i, \ \ \forall p \in M^C.$$

Especially, if $[\theta]_p = 1/m$, then it is average allocation between coalition structure, *i.e.*, if

$$\theta_i = \lambda_i = \frac{Sh_i\left(C_p, v/L\right)}{v\left(C_p/L\right) \cdot m}, \ \ \forall i \in C_p, \ \ \forall p \in M^C,$$

then the weight extended value is just the extended value. Hence, the weight extended value is a further extension of the extended value. Next, we can also denote the extended value by $\varphi^{\lambda}\left(N, v_{\alpha}^{LC}, C\right)$.

By Proposition 4.2, we have known the Myerson value $\varphi^M\left(N, v, L\right)$ is a special case of the extended value $\varphi^{\lambda}\left(N, v_{\alpha}^{LC}, C\right)$. Next, we give two examples to show the difference between the weight extended value $\varphi^{\theta}\left(N, v_{\alpha}^{LC}, C\right)$, the extended value $\varphi^{\lambda}\left(N, v_{\alpha}^{LC}, C\right)$, the Myerson value $\varphi^M\left(N, v, L\right)$ and the Owen value $\varphi^{Ow}\left(N, v, C\right)$.

Example 4.3. *c.f. Example 3.2 Let* $N = \{1,2,3,4,5,6\}$ *be a grand coalition, where* $L = \{\{1,2\},\{1,3\},\{2,3\},\{4\},\{5,6\}\}$, *which denotes the link of airline companies, such as code sharing or such as code or member points sharing, and* $C = \{\{1,2,3,4\},\{5,6\}\}$. *Hence, we have* $N/L = \{\{1,2,3\},\{4\},\{5,6\}\}$, *which means coming from the same airline companies, such as code sharing or such as Star Alliance, One World, Sky Team or WOW Cargo Alliance. Now, they want to construct a new project program based their former cooperative history, such as the same country, so a coalition structure is formed.*

$$v\left(T\right) = 0\left(|T| = 1\right), v\left(T\right) = 1\left(T \in \{\{1,2\},\{1,3\},\{2,3\}\}\right), v\left(1,2,3\right) = 3, v\left(5,6\right) = 3.$$

By equation (10), we compute the α-restricted game, *i.e.*,

$$v_{\alpha}^{LC}\left(N\right) = \alpha \cdot \left[\left(v\left(1,2,3\right) + v\left(4\right) + v\left(5,6\right)\right)\right] = 6\alpha.$$

By equation (13), the extended value $\varphi^{\lambda}\left(N, v_{\alpha}^{LC}, C\right)$ is gotten,

$$\begin{aligned}
&\varphi_1^{\lambda}\left(N, v_{\alpha}^{LC}, C\right) = \varphi_2^{\lambda}\left(N, v_{\alpha}^{LC}, C\right) = \varphi_3^{\lambda}\left(N, v_{\alpha}^{LC}, C\right) = \alpha, \\
&\varphi_5^{\lambda}\left(N, v_{\alpha}^{LC}, C\right) = \varphi_6^{\lambda}\left(N, v_{\alpha}^{LC}, C\right) = 3\alpha/2.
\end{aligned}$$

By equation (14), we have that the weight extended value $\varphi^{\theta}\left(N, v_{\alpha}^{LC}, C\right)$ is

$$\begin{aligned}
&\varphi_i^{\theta}\left(N, v_{\alpha}^{LC}, C\right) = 1 + 6\theta_i\left(\alpha - 1\right), i = 1,2,3, \\
&\varphi_i^{\theta}\left(N, v_{\alpha}^{LC}, C\right) = 3/2 + 6\theta_i\left(\alpha - 1\right), i = 5,6.
\end{aligned}$$

Also the Myerson value $\varphi^M\left(N, v, L\right)$ is

$$\begin{aligned}
&\varphi_1^M\left(N, v, L\right) = \varphi_2^M\left(N, v, L\right) = \varphi_3^M\left(N, v, L\right) = 1, \\
&\varphi_5^M\left(N, v, L\right) = \varphi_6^M\left(N, v, L\right) = 3/2.
\end{aligned}$$

By Figure 4, when $\theta_i = 1/6$, $i = 1,2,3$; $\theta_i = 1/4$, $i = 5,6$, the extended value $\varphi^{\lambda}\left(N, v_{\alpha}^{LC}, C\right) = \varphi^{\theta}\left(N, v_{\alpha}^{LC}, C\right)$. Thus, the weight extended value φ^{θ} is a generalized form of φ^{λ}.

By Proposition 4.2 and Figure 5, the Myerson value is included in the extended value, *i.e.*, when $\alpha = 1$, the extended value is just the Myerson value. In other words, $\varphi^{\lambda}\left(N, v_{\alpha}^{LC}, C\right) = \varphi^M\left(N, v, L\right)$. We consider the added value after prior coalition cooperation, which is proportional by the Myerson value. In a word, both $\varphi^{\theta}\left(N, v_{\alpha}^{LC}, C\right)$ and $\varphi^{\lambda}\left(N, v_{\alpha}^{LC}, C\right)$ are different from the Myerson value $\varphi^M\left(N, v, L\right)$.

Next, we show that the value $\varphi^{\theta}\left(N, v_{\alpha}^{LC}, C\right)$ and $\varphi^{\lambda}\left(N, v_{\alpha}^{LC}, C\right)$ are also different from the Owen value $\varphi^{Ow}\left(N, v, C\right)$.

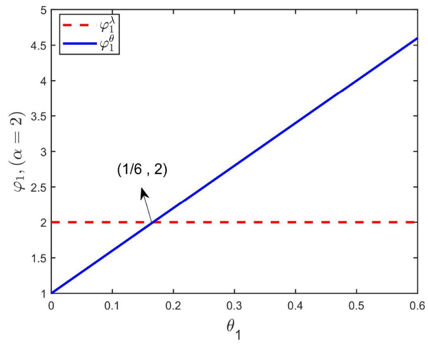

FIGURE 4. The value of player 1 relative to his weight in the weight extended value.

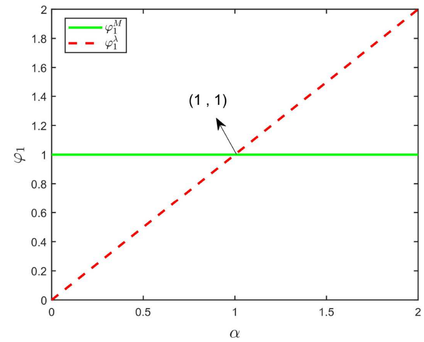

FIGURE 5. The extended value and the Myerson value of player 1.

Example 4.4. *(c.f. Example 4.3) Let* $N = \{1, 2, 3, 4, 5, 6\}$ *be a grand coalition, where* $L = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{4\}, \{5, 6\}\}$*, and* $C = \{\{1, 2, 3, 4\}, \{5, 6\}\}$*.*

First, the cooperative game $\left(N, v_\alpha^{LC}, C\right)$ is defined by equation (10), and $\left(N, v_\alpha^{LC}, C\right)$ is an incomplete information game, *i.e.*, the α-restricted game, in which players first start with a prior coalition (*i.e.*, a small coalition) in the coalition structure C, and do not form the grand coalition N. Thus, the payoff of $v(N)$ is unknown, where $v_\alpha^{LC}(N)$ in $\left(N, v_\alpha^{LC}, C\right)$ is evaluated based on the sum of the payoff of the prior coalition in C, as shown in Example 4.3,

$$v_\alpha^{LC}(N) = \alpha \cdot [(v(1, 2, 3) + v(4) + v(5, 6))] = 6\alpha.$$

The new values φ^λ and φ^θ can solve not only cooperative game with complete information, but also some with incomplete information. However, the Owen value need all the values of feasible coalitions, so can only be applicated to coalition structure game under complete information.

Next, we will show the difference between the Owen value $\varphi^{Ow}(N, v, C)$, the extended value $\varphi^\lambda\left(N, v_\alpha^{LC}, C\right)$ and the weight extended value $\varphi^\theta\left(N, v_\alpha^{LC}, C\right)$. We assume that $v^o(S) = v_\alpha^{LC}(S)$, $\forall S \in P(C)$,

$$\begin{aligned} &v^o(N) = v_\alpha^{LC}(N) = 6\alpha, \\ &v^o(\{1, 2, 3, 4\}) = v(\{1, 2, 3\}) + v(\{4\}). \end{aligned}$$

Then (N, v^o, C) is a coalition structure game, and the Owen value for (N, v^o, C) is

$$\begin{aligned} &\varphi_1^{Ow}(N, v^o, C) = \varphi_2^{Ow}(N, v^o, C) = \varphi_3^{Ow}(N, v^o, C) = \alpha, \\ &\varphi_5^{Ow}(N, v^o, C) = \varphi_6^{Ow}(N, v^o, C) = 3\alpha/2. \end{aligned}$$

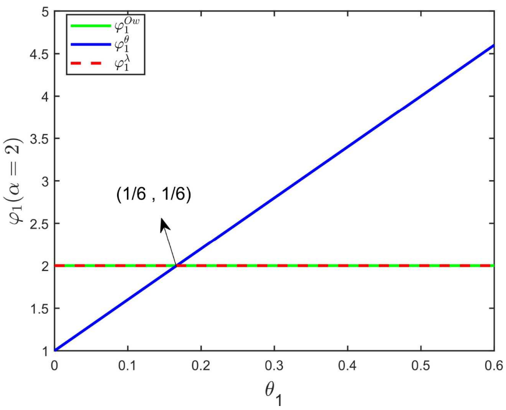

FIGURE 6. The extended value, the weight extended value and the Owen value of player 1.

As shown by Figure 6, the extended value $\varphi^{\lambda}\left(N, v_{\alpha}^{LC}, C\right)$, the weight extended value $\varphi^{\theta}\left(N, v_{\alpha}^{LC}, C\right)$ and the Owen value $\varphi^{Ow}\left(N, v, C\right)$ are different for (N, v^{o}, C). In a world, the extended value $\varphi^{\lambda}\left(N, v_{\alpha}^{LC}, C\right)$ and the weight extended value $\varphi^{\theta}\left(N, v_{\alpha}^{LC}, C\right)$ are different from the Owen value $\varphi^{Ow}\left(N, v, C\right)$. Because the first two values can be applied to games under incomplete information, *i.e.*, the α-restricted game, but the Owen value $\varphi^{Ow}\left(N, v, C\right)$ is only for coalition structure game. Secondly, even for the coalition structure game with complete information, such as the three values are also different. As we shown by Figure 6, the Owen value $\varphi^{Ow}\left(N, v^{o}, C\right)$ is included in the weight extended value $\varphi^{\theta}\left(N, v_{\alpha}^{LC}, C\right)$. From Examples 4.3 and 4.4, we can get that, the extended value $\varphi^{\lambda}\left(N, v_{\alpha}^{LC}, C\right)$, the weight extended value $\varphi^{\theta}\left(N, v_{\alpha}^{LC}, C\right)$, the Myerson value $\varphi^{M}\left(N, v, L\right)$ and the Owen value $\varphi^{Ow}\left(N, v, C\right)$ are different. The extended value $\varphi^{\lambda}\left(N, v_{\alpha}^{LC}, C\right)$ and the weight extended value $\varphi^{\theta}\left(N, v_{\alpha}^{LC}, C\right)$ are all extensions of the Myerson value $\varphi^{M}\left(N, v, L\right)$ and the Owen value $\varphi^{Ow}\left(N, v, C\right)$.

We introduce the following properties to characterize the new weight extended value φ on the restricted game (N, v_{α}^{LC}, C).

Restricted Component efficiency, RCE . For any $C \in C^{N}$, a restricted game (N, v_{a}^{LC}, C),

$$\sum_{i \in N} \varphi_i\left(N, v_{\alpha}^{LC}, C\right) = v_{\alpha}^{LC}\left(N\right).$$

Weight Fairness, WF. For any $C \in C^{N}$, a restricted game (N, v_{α}^{LC}, C) and any $\{i, j\} \in LC_p$, $\theta_i = \theta_j$, $\forall p \in M^{C}$, $\theta \in N_{\theta}$, we have that,

$$\varphi_i\left(N, v_{\alpha}^{LC}, C\right) - \varphi_i\left(N, v_{\alpha}^{L-ijC}, C\right) = \varphi_j\left(N, v_{\alpha}^{LC}, C\right) - \varphi_j\left(N, v_{\alpha}^{L-ijC}, C\right).$$

Weight Symmetric Across Coalitions, WSAC. Let any $C \in C^{N}$, a restricted game (N, v_{α}^{LC}, C) , and $LC = \{LC_1, \ldots, LC_m\}$. For any $LC_s, LC_t \in LC$ and $K \subseteq M^{C} \setminus \{s, t\}$, if

$$[\theta]_t = [\theta]_s,\ \ v_{\alpha}^{LC}\left(\cup_{k \in K} LC_k \cup LC_s\right) = v_{\alpha}^{LC}\left(\cup_{k \in K} LC_k \cup LC_t\right),$$

then

$$\sum_{i \in LC_s} \varphi_i\left(N, v_{\alpha}^{LC}, C\right) = \sum_{i \in LC_t} \varphi_i\left(N, v_{\alpha}^{LC}, C\right).$$

If

$$v_{\alpha}^{LC}\left(\cup_{k \in K} LC_k \cup LC_s\right) = v_{\alpha}^{LC}\left(\cup_{k \in K} LC_k\right),$$

then

$$\sum_{i \in LC_s} \varphi_i\left(N, v_{\alpha}^{LC}, C\right) = [\theta]_s\left(1 - 1/\alpha\right) v_{\alpha}^{LC}.$$

TABLE 1. The compartment between two values.

	RCE	WPP	RL	F	SAC	SWC	WF	WSAC
The extended value φ^{λ}	✓	✓	✓	✓	✓	✓	✓	✓
The weight extended value φ^{θ}	✓	✓	✓	×	×	×	✓	✓

Weight Player Property, WPP. Let any $C \in C^N$, a restricted game (N, v_α^{LC}, C), and $LC = \{LC_1, \ldots, LC_m\}$. For any $i, j \in LC_p$ and $p \in M^C$, if

$$\theta_i = \theta_j,\ \ v_\alpha^{LC}(S \cup i) = v_\alpha^{LC}(S \cup j),\ \ \forall S \in P(C),$$

then

$$\varphi_i^{\theta}(N, v_\alpha^{LC}, C) = \varphi_j^{\theta}(N, v_\alpha^{LC}, C).$$

Restricted Linearity, RL. For any $C \in C^N$, two any restricted games (N, v_α^{LC}, C), (N, w_α^{LC}, C) and $\beta, \gamma \in \mathbb{R}$, define a new game

$$(\beta v + \gamma w)(S) = \beta v(S) + \gamma w(S),\ \ \forall S \in P(C \backslash L),$$

then

$$\varphi_i\left(N, (\beta v + \gamma w)_\alpha^{LC}, C\right) = \beta \cdot \varphi_i\left(N, v_\alpha^{LC}, C\right) + \gamma \cdot \varphi_i\left(N, w_\alpha^{LC}, C\right),\ \ \forall i \in N.$$

It is not hard to see that the extended value $\varphi^{\lambda}(N, v_\alpha^{LC}, C)$ and the weight extended value $\varphi^{\theta}(N, v_\alpha^{LC}, C)$ satisfy **RL**.

RCE, **WF**, **WSAC**, **WPP** and **RL** are extensions of **E**, **F**, **SAC**, **SWC** and **L**, respectively. **RCE** means that the sum of allocated values in S is $v_\alpha^{LC}(S)$. **WF** requires that player's leaving effect is proportional to his weight. **WSAC** means that if two prior coalitions' contributions are the same, then their values are the same based on the same weight. **WPP** means that if two players have the same effect on the coalition value and this weight is the same, then their values are the same. **RL** means the α-restricted game is linear.

Proposition 4.5. *The weight extended value $\varphi^{\theta}(N, v_\alpha^{LC}, C)$ satisfies **RCE, WF, WSAC, WPP** and **RL**.*

Proof of Proposition 4.5. See Appendix A.3.

As a special case of weight extended value $\varphi^{\theta}(N, v_\alpha^{LC}, C)$, the extended value $\varphi^{\lambda}(N, v_\alpha^{LC}, C)$ also have some properties.

Corollary 4.6. *The extended value $\varphi^{\lambda}(N, v_\alpha^{LC}, C)$ satisfies **RCE,F,WSAC,SWC** and **RL**.*

Proof of Corollary 4.6. It is similar to Proof of Proposition 4.5. We do not repeat it.

Both $\varphi^{\theta}(N, v_\alpha^{LC}, C)$ and $\varphi^{\lambda}(N, v_\alpha^{LC}, C)$ satisfy **RCE**, **WSAC** and **RL**. The difference between them lies in that $\varphi^{\theta}(N, v_\alpha^{LC}, C)$ has **WSAC**, **WF** and **WPP**, while $\varphi^{\lambda}(N, v_\alpha^{LC}, C)$ has **SAC**, **F** and **WSAC**, is just as shown in Table 1.

Next, we check the uniqueness of two values.

Proposition 4.7. *The weigh extended value $\varphi^{\theta}(N, v_\alpha^{LC}, C)$ is the unique value satisfying **RCE, WF, WSAC, WPP** and **RL**.*

Proof of Proposition 4.7. See Appendix A.4.

Remark 4.8. The axioms used in Proposition 4.7 are independent of each other. Let $\varphi\left(N, v_{\alpha}^{LC}, C\right)$ be a value on any restricted game $\left(N, v_{\alpha}^{LC}, C\right)$.

(i). The value defined by

$$\varphi_i\left(N, v_{\alpha}^{LC}, C\right) = k_0 \cdot \varphi_i^M \cdot (N, v, L) + \theta_i \cdot (1 - 1/\alpha) \cdot v_{\alpha}^{LC}(N), \ \ \forall i \in N, k_0 \neq 1,$$

satisfies **WF**, **WSAC**, **WPP** and **RL**, except **RCE**.
(ii). The value defined by

$$\varphi_i\left(N, v_{\alpha}^{LC}, C\right) = \varphi_i^M \cdot (N, v, L) + \frac{[\theta]_p}{c_p} \cdot (1 - 1/\alpha) \cdot v_{\alpha}^{LC}(N), \ \ \forall i \in C_p, \forall p \in M^c,$$

satisfies **RCE**, **WSAC**, **WPP** and **RL**, except **WF**.
(iii). The value defined by

$$\varphi_i\left(N, v_{\alpha}^{LC}, C\right) = \begin{cases} \varphi_i^M(N, v, L) + \theta_i (1 - 1/\alpha) v_{\alpha}^{LC}(N) - 1 & i \in C_p \\ \varphi_i^M(N, v, L) + \theta_i (1 - 1/\alpha) v_{\alpha}^{LC}(N) + \frac{c_p}{n - c_p} & otherwise, \end{cases}$$

satisfies **RCE**, **WF**, **WPP** and **RL**, except **WSAC**.
(iv). The value defined by

$$\varphi_i\left(N, v_{\alpha}^{LC}, C\right) = \begin{cases} \varphi_i^M(N, v, L) + \theta_i (1 - 1/\alpha) v_{\alpha}^{LC}(N) - 1 & i = k, i \in C_p \\ \varphi_i^M(N, v, L) + \theta_i (1 - 1/\alpha) v_{\alpha}^{LC}(N) + \frac{1}{c_p - 1} & i \neq k, i \in C_p \\ \varphi_i^M(N, v, L) + \theta_i (1 - 1/\alpha) v_{\alpha}^{LC}(N) & otherwise, \end{cases}$$

such that $c_p \geq 2$, $k \in C_p$ then $\phi(C, v)$ satisfies **RCE**, **WF**, **WSAC** and **RL**, except **WPP**.
(v). The value defined by

$$\varphi_i\left(N, v_{\alpha}^{LC}, C\right) = \frac{v(C_p/L)}{c_p} + \theta_i \left(1 - \frac{1}{\alpha}\right) v_{\alpha}^{LC}(N), \ \ \forall i \in c_p, \ \ p \in M^C,$$

satisfies **RCE**, **WF**, **WSAC** and **WPP**, except **RL**.

As a special kind of weight extended value, it is seen that the extended value has the following properties.

Corollary 4.9. *The extended value $\varphi^{\lambda}\left(N, v_{\alpha}^{LC}, C\right)$ is the unique value satisfying **RCE**, **F**, **SAC**, **SWC** and **RL**, while **RCE**, **F**, **SAC**, **SWC** and **RL** are independent of each other.*

The properties of two values is displayed in Table 1. As a special kind of weight extended value, the extend value is unique based on **RCE**, **F**, **SAC**, **SWC** and **RL**.

In a word, the extended value $\varphi^{\lambda}\left(N, v_{\alpha}^{LC}, C\right)$, the weight extended value $\varphi^{\theta}\left(N, v_{\alpha}^{LC}, C\right)$, the Myerson value $\varphi^M(N, v, L)$ and the Owen value $\varphi^{Ow}(N, v, C)$ are different. From Examples 4.3 and 4.4, with the risk coefficient α increasing, the payoff value of α - restricted game is increasing. Thus, the player could estimate their allocation profit based on their risk tolerance. As discussed above, the risk coefficient α which provides by every player in the game are equal. Then the weight extended value is equal to the total payoff value of the coalition which is composed by all the players of the game. But in the real world, different players may have different risk tolerances. In most cases, the risk coefficient α of every player is unequal. All the players in the prior coalition know that the payoff which gained in the coalition will be greater than the payoff which acts alone. Before forming the grand coalition, the players will determine the uncertain distribution of the payoff of the coalition according with their cooperation history and expertise. Thus, we suppose the uncertain distribution of the payoff of any coalition is common knowledge. Also, the risk coefficient α of every player has reached a consensus.

5. Conclusions

A cooperative game with comminution structure in a prior coalition (α - restricted game for short) is defined based on the risk coefficient α. In a α - restricted game, it is assumed that prior coalition takes part in big coalition N by an inter communication structure. Before the grand coalition is formed, all the players participate in the cooperation just by communication structure. Hence, a feasible coalition in communication structure is defined by the Myerson value in the inter prior coalition. By this means, the profits of whole cooperation are more the profit sum of individual coalitions. The distributed value for the restricted game is defined by the Myerson value. It is seen the added value by coalition structure is allocated by taking the Myerson value as the allocation weights. Following this method, a weight extended value is introduced and characterized as a general value for the restricted game. This weight extended value is the unique value based on Component efficiency, Fairness, Symmetric across coalitions, Null coalition property and Linearity. The extended value, the weight extended value, the Myerson value and the Owen value are different. The extended value and the weight extended value in this paper are all extentions of the Myerson value and the Owen value.

Appendix A.

A.1. Proof of Theorem 4.1.

For any $p \in M^C$ and $i \in C_p$, it is gotten that

$$\begin{aligned}
&\varphi_i^{Ow}\left(N, v^L, C\right) \\
&= \sum_{R \subseteq M^C,\ p \notin R} \ \sum_{S \subseteq C_p,\ i \notin S} \frac{r!\,(m-r-1)!}{m!} \times \frac{s!\,(c_p-s-1)!}{c_p!} \times \left[v^L\left(Q_R \cup S \cup i\right) - v^L\left(Q_R \cup S\right)\right] \\
&= \sum_{R \subseteq M^C,\ p \notin R} \ \sum_{S \subseteq C_p,\ i \notin S} \frac{r!\,(m-r-1)!}{m!} \times \frac{s!\,(c_p-s-1)!}{c_p!} \times \left[v^L\left(S \cup i\right) - v^L\left(S\right)\right] \\
&= \sum_{R \subseteq M^C,\ p \notin R} \frac{r!\,(m-r-1)!}{m!} \times \sum_{S \subseteq C_p, i \notin S} \frac{s!\,(c_p-s-1)!}{c_p!} \times \left[v^L\left(S \cup i\right) - v^L\left(S\right)\right] \\
&\quad = \sum_{r=0}^{m-1} \frac{r!\,(m-r-1)!}{m!} \times C_{m-1}^r \times \sum_{S \subseteq C_p, i \notin S} \frac{s!\,(c_p-s-1)!}{c_p!} \times \left[v^L\left(S \cup i\right) - v^L\left(S\right)\right] \\
&\quad = \sum_{r=0}^{m-1} \frac{1}{m} \times \sum_{S \subseteq C_p, i \notin S} \frac{s!\,(c_p-s-1)!}{c_p!} \times \left[v^L\left(S \cup i\right) - v^L\left(S\right)\right] \\
&\quad = \sum_{S \subseteq C_p,\ i \notin S} \frac{s!\,(c_p-s-1)!}{c_p!} \times \left[v^L\left(S \cup i\right) - v^L\left(S\right)\right] = \varphi_i^M\left(N, v, L\right).
\end{aligned}$$

A.2. Proof of Proposition 4.2.

By equation (10), it is gotten that

$$Sh_p\left(M^C, v_{\alpha,C}^{LC}\right) = v^L\left(C_p\right) + \frac{\alpha-1}{m} \cdot \sum\nolimits_{k \in M^C} v(C_k/L), \ \ \forall p \in M^C.$$

By equation (11), we have that

$$\begin{aligned}
&w_p\left(C_p\right) = Sh_p\left(M^C, v_{\alpha,C}^{LC}\right) = v^L\left(C_p\right) + \frac{\alpha-1}{m} \cdot \sum\nolimits_{k \in M^C} v(C_k/L), \ \ \forall p \in M^C, \\
&w_p\left(S\right) = v^L\left(S\right), \ \ \forall S \subset C_p.
\end{aligned}$$

Hence, it is obtained that

$$Sh_i\left(C_p, w_p/L\right) = \varphi_i^M\left(N, v, L\right) + \frac{(\alpha-1)\cdot Sh_i\left(C_p, v/L\right)\cdot \sum_{k\in M^C} v\left(C_k/L\right)}{m\cdot v\left(C_p/L\right)}, \quad \forall i \in C_p.$$

A.3. Proof of Proposition 4.5.

(i). RCE. As the Myerson value $\varphi^M\left(N, v_\alpha^{LC}\right)$ satisfies **CE**, so it is gotten that

$$\begin{aligned}\sum_{i\in C_p}\varphi_i^\theta\left(N, v_\alpha^{LC}, C\right) &= \sum_{i\in C_p}\varphi_i^M\left(N, v, L\right) + \sum_{i\in C_p}\theta_i\cdot(\alpha-1)\cdot\sum\nolimits_{k\in M^C} v(C_k/L)\\ &= v\left(LC_p\right) + \sum_{i\in C_p}\theta_i\cdot(\alpha-1)\cdot\sum\nolimits_{k\in M^C} v(C_k/L).\end{aligned}$$

Furthermore, we get that

$$\begin{aligned}&\sum_{p\in M^C}\sum_{i\in C_p}\varphi_i^\theta\left(N, v_\alpha^{LC}, C\right)\\ &= \sum\nolimits_{i\in C_p}\phi_i(N, v, L) + \sum\nolimits_{i\in C_p}\theta_i\cdot(\alpha-1)\cdot\sum\nolimits_{k\in M^C} v(C_k/L)\\ &= \alpha\sum\nolimits_{k\in M^C} v(C_k/L)\\ &= v_\alpha^{LC}\left(N\right).\end{aligned}$$

(ii). WF. Because the Myerson value $\varphi^M\left(N, v_\alpha^{LC}, C\right)$ satisfies **F**, it is obtained that

$$\begin{aligned}&\varphi_i\left(N, v_\alpha^{LC}, C\right) - \varphi_i\left(N, v_\alpha^{L_{-ij}C}, C\right)\\ &= \varphi_i^M\left(N, v, L\right) + \theta_i\cdot(1-1/\alpha)\, v_\alpha^{LC}\left(N\right) - \varphi_i^M\left(N, v, L_{-ij}\right) - \theta_i\cdot(1-1/\alpha)\, v_\alpha^{L_{-ij}C}\left(N\right)\\ &= \varphi_i^M\left(N, v, L\right) - \varphi_i^M\left(N, v, L_{-ij}\right) + \theta_i\cdot(1-1/\alpha)\cdot\left(v_\alpha^{LC}\left(N\right) - v_\alpha^{L_{-ij}C}\right)\\ &= \varphi_j^M\left(N, v, L\right) - \varphi_j^M\left(N, v, L_{-ij}\right) + \theta_j\left(1-1/\alpha\right)\cdot v_\alpha^{LC}\left(N\right) - \theta_j\cdot(1-1/\alpha)\cdot v_\alpha^{L_{-ij}C}\left(N\right)\\ &= \varphi_j\left(N, v_\alpha^{LC}, C\right) - \varphi_j\left(N, v_\alpha^{L_{-ij}C}, C\right).\end{aligned}$$

(iii). WSAC. It is seen that

$$\sum_{i\in C_p}\varphi_i^\theta\left(N, v_\alpha^{LC}, C\right) = v\left(LC_p\right) + \sum_{i\in C_p}\theta_i\cdot(1-1/\alpha)\cdot v_\alpha^{LC}\left(N\right) = v\left(LC_p\right) + [\theta]_p\cdot(1-1/\alpha)\cdot v_\alpha^{LC}\left(N\right).$$

Hence, if $[\theta]_t = [\theta]_s, K \subseteq M^C\setminus\{s,t\}, v_\alpha^{LC}\left(\cup_{k\in K} LC_k \cup LC_s\right) = v_\alpha^{LC}\left(\cup_{k\in K} LC_k \cup LC_t\right)$, then

$$\sum_{i\in LC_s}\varphi_i\left(N, v_\alpha^{LC}, C\right) = \sum_{i\in LC_t}\varphi_i\left(N, v_\alpha^{LC}, C\right).$$

(iv). WPP. For any $i, j \in LC_p$ and $p \in M^C$, if

$$\theta_i = \theta_j, v_\alpha^{LC}\left(S\cup i\right) = v_\alpha^{LC}\left(S\cup j\right), \quad \forall S \in P\left(C\right),$$

then

$$\begin{aligned}\varphi_i^\theta(N, v_\alpha^{LC}, C) &= \varphi_i^M\left(N, v, L\right) + \theta_i\left(1-1/\alpha\right) v_\alpha^{LC}\left(N\right)\\ &= \varphi_j^M\left(N, v, L\right) + \theta_j\left(1-1/\alpha\right) v_\alpha^{LC}\left(N\right)\\ &= \varphi_j^\theta(N, v_\alpha^{LC}, C).\end{aligned}$$

A.4. Proof of Proposition 4.7.

If the weight extended value $\varphi^\theta(N, v_\alpha^{LC})$ is not the unique value that satisfies **RCE**, **WF**, **WSAC**, **WPP** and **RL**, then we suppose that $f^\theta(N, v_\alpha^{LC})$ be another value that satisfies **RCE**, **WF**, **WSAC**, **WPP** and **RL**. Set

$$F_p\left(M^C, v\right) = \sum_{i \in C_p} f_i^\theta\left(N, v_\alpha^{LC}, C\right), \ \ \forall p \in M^C.$$

By **WPP**, **RCE**, **WSAC** and **RL**, then (N, v_α^{LC}, C) it can be uniquely represented by

$$v_\alpha^{LC} = \sum_{R \subseteq M^C, R \neq \emptyset} \lambda_R u_R, u_R(K) = \begin{cases} 1 & R \subseteq K \\ 0 & otherwise. \end{cases} \tag{A.1}$$

where

$$\lambda_R = \sum_{K \subseteq R} (-1)^{|R|-|K|} v_\alpha^{LC}(K).$$

For any (N, v_α^{LC}, C), (N, w_α^{LC}, C) and $\beta, \gamma \in \mathbb{R}$, define a new game $(\beta v + \gamma w)(S) = \beta v(S) + \gamma w(S), \forall S \in P(C/L)$, then by **RL**, we can get that

$$\sum_{i \in C_p} f_i^\theta\left(N, (\beta v + \gamma w)_\alpha^{LC}, C\right) = \beta \cdot \sum_{i \in C_p} f_i^\theta\left(N, v_\alpha^{LC}, C\right) + \gamma \cdot \sum_{i \in C_p} f_i^\theta\left(N, w_\alpha^{LC}, C\right).$$

This means that

$$F_p\left(M^C, \beta v + \gamma w\right) = \beta F_p\left(M^C, v\right) + \gamma F_p\left(M^C, w\right), \ \ \forall p \in M^C.$$

Moreover, by $v_\alpha^{LC} = \sum_{R \subseteq M^C, R \neq \emptyset} \lambda_R u_R$, if we can show the uniqueness of $F\left(M^C, u_R\right) = \left(F_1\left(M^C, u_R\right), \ldots, F_m\left(M^C, u_R\right)\right)$, for the unanimity game (M^C, u_R), then $F\left(M^C, v\right) = \left(F_1\left(M^C, v\right), \ldots, F_m\left(M^C, v\right)\right)$ is the unique for (M^C, v).

For any $s, t \in R$ and $K \subseteq M^C \backslash \{s, t\}$, $u_R(K \cup s) = u_R(K \cup t)$, then by **WSAC**, we get that $[\theta]_t$ $F_s\left(M^C, u_R\right) = [\theta]_s \cdot F_t\left(M^C, u_R\right)$. Based on **RCE**, we get that

$$\sum_{p \in M^C} F_p\left(M^C, u_R\right) = 1.$$

Moreover, for any $C_s \in C$ and $K \subseteq M^C \backslash s, s \notin R$, it is $u_R(K \cup \{s\}) = u_R(K)$, then by **WSAC**, we have that $F_s\left(M^C, u_R\right) = [\theta]_s(1 - 1/\alpha)$.

Therefore, the uniqueness of $F\left(M^C, u_R\right)$ for the unanimity game (M^C, u_R) is proven. In the conclusion, $F\left(M^C, v\right)$ is the unique for (M^C, v).

Next, we prove that $f^\theta(N, v^{LC})$ is the unique for players in prior coalition C_p, when the value for coalition C_p is fixed as proved above.

For any $p \in M^C$, if any player $i \in C_p$ satisfies that i is isolated in graph LC_p, then by **RCE**, it is gotten that

$$f_i^\theta\left(N, v_\alpha^{LC}\right) = \varphi_i^\theta\left(N, v_\alpha^{LC}\right) = v_\alpha^{LC}(i) = v(i).$$

Moreover, we suppose that if the cardinality of LC_p is less than l, then $f^\theta\left(N, v_\alpha^{LC}, C\right) = \varphi^\theta\left(N, v_\alpha^{LC}, C\right)$. Now we consider (N, v, L, C) with the cardinality of LC_p is equal to l. By

$$f^\theta\left(N, v_\alpha^{L-ijC}, C\right) = \varphi^\theta\left(N, v_\alpha^{L-ijC}, C\right).$$

By **WF**, we get that

$$\begin{aligned}&\theta_j\cdot\left(f_i^\theta\left(N,v_\alpha^{LC},C\right)-f_i^\theta\left(N,v_\alpha^{L_{-ij}C},C\right)\right)=\theta_i\cdot\left(f_j^\theta\left(N,v_\alpha^{LC},C\right)-f_j^\theta\left(N,v_\alpha^{L_{-ij}C},C\right)\right)\\ \Leftrightarrow&\theta_j\cdot\left(f_i^\theta\left(N,v_\alpha^{LC},C\right)-\varphi_i^\theta\left(N,v_\alpha^{L_{-ij}C},C\right)\right)=\theta_i\cdot\left(f_j^\theta\left(N,v_\alpha^{LC},C\right)-\varphi_j^\theta\left(N,v_\alpha^{L_{-ij}C},C\right)\right).\end{aligned}$$

Therefore, we have that

$$\theta_j\cdot\left(f_i^\theta\left(N,v_\alpha^{LC},C\right)-\varphi_i^\theta\left(N,v_\alpha^{LC},C\right)\right)=\theta_i\cdot\left(f_j^\theta\left(N,v_\alpha^{LC},C\right)-\varphi_j^\theta\left(N,v_\alpha^{LC},C\right)\right).$$

Now we let

$$\frac{f_i^\theta\left(N,v_\alpha^{LC},C\right)-\varphi_i^\theta\left(N,v_\alpha^{LC},C\right)}{\theta_i}=\frac{f_j^\theta\left(N,v_\alpha^{LC},C\right)-\varphi_j^\theta\left(N,v_\alpha^{LC},C\right)}{\theta_j}=k_0,$$

then

$$f_i^\theta\left(N,v_\alpha^{LC},C\right)=k_0\theta_i+\varphi_i^\theta\left(N,v_\alpha^{LC},C\right),f_j^\theta\left(N,v_\alpha^{LC},C\right)=k_0\theta_j+\varphi_j^\theta\left(N,v_\alpha^{LC},C\right).$$

Because it is proved that

$$\sum_{i\in C_p}f_i^\theta\left(N,v_\alpha^{LC},C\right)=\sum_{i\in C_p}\varphi_i^\theta\left(N,v_\alpha^{LC},C\right)=F_p\left(M^C,v\right).$$

Hence, the above two equality means that k_0, *i.e.*,

$$k_0\cdot\sum_{i\in LC_p}\theta_i=k_0\cdot[\theta]_p=0.$$

So we get conclusion that $k_0=0$, and $f_i^\theta\left(N,v_\alpha^{LC},C\right)=\varphi_i^\theta\left(N,v_\alpha^{LC},C\right)$ for any $i\in LC_p$.

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Conflict of Interest

The authors Yiding Shi, Xiaohui Yu and Jing Liu declares no conflict of interest.

Author contribution statement

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