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FIRST AND SECOND ORDER NECESSARY OPTIMALITY CONDITIONS FOR MULTIOBJECTIVE PROGRAMMING WITH INTERVAL-VALUED OBJECTIVE FUNCTIONS ON RIEMANNIAN MANIFOLDS

Hilal Ahmad Bhat, Akhlad Iqbal* and Mahwash Aftab

Abstract. The growing dependence on optimization models in decision-making has created a demand for tools that can facilitate the formulation and resolution of a broader range of real-world processes and systems associated with human activity. These situations often involve assumptions that diverge from traditional optimization methodologies. One viable approach for addressing optimization problems in real-life scenarios with uncertainty is interval-valued optimization. Taking into account the significance of interval-valued optimization, in this paper, we derive first and second order necessary optimality conditions for a multi-objective programming problem with interval-valued objective functions defined on a Riemannian manifold. To establish these conditions, we consider the objective functions to be weakly differentiable and twice weakly differentiable for first and second order, respectively. Additionally, we assume that the constraints, both equality and inequality constraints, are differentiable and twice differentiable for first and second order conditions respectively. The first order as well as second order necessary conditions are derived under two types of constraint qualifications. Furthermore, we provide illustrative examples to demonstrate the application of the established results.

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1. Introduction

In the contemporary world, extremum problems inherently involve uncertainty and imprecision, leading to numerous advancements in the study of uncertain optimization problems. One notable development in addressing the inexact nature of quantities is Interval-Valued Optimization Problem (IVOP). Unlike classical optimization problems, where the coefficients of functions are assumed to be real numbers, IVOP allows for the representation of these coefficients as intervals in $\mathbb{R}$. This adjustment proves beneficial for efficient decision-making in uncertain environments, as it offers a more practical and straightforward approach by accounting for uncertainty through intervals in $\mathbb{R}$. Moreover, IVOP eliminates the need for subjective methods such as probability distribution functions or fuzzy numbers with known membership functions when tackling uncertain extremum programming problems.

Keywords. Multi-objective programming problem, interval-valued function, first and second order necessary optimality conditions, constraint qualifications, KKT optimality conditions, Riemannian manifold.

Department of Mathematics, Aligarh Muslim University, Aligarh 202002, Uttar Pradesh, India.
*Corresponding author: akhlad6star@gmail.com; akhlad.mm@amu.ac.in

Due to its vast applications, the field of IVOP has undergone significant developments. Moore [24,25] and Alefeld and Herzberger [2] provide a fundamental overview of interval analysis. Ishibuchi and Tanaka [20] introduced an ordering relation for two closed and bounded intervals in $\mathbb{R}$ based on their center and half-width, offering solution concepts for multi-objective IVOP. For the latest advancements in Karush–Kuhn–Tucker (KKT) optimality conditions for IVOP, interested readers can see [16,27,31,32,37,39].

Moreover, there has been a recent surge in interest in addressing optimization problems on Riemannian manifolds. Such structures naturally arise as constraint sets in many optimization problems which allows to treat constrained optimization problems as unconstrained optimization problems leading to handle such problems more accurately. Many problems in machine learning and computer vision involve optimization over Riemannian manifolds, such as in shape analysis, image analysis, and manifold learning. Extending KKT conditions to these domains enables the development of specialized optimization techniques tailored to the underlying manifold structure. For the foundational development of optimization problems on Riemannian manifolds, one can consult the books [1,15,28,34]. Gabriel [19] presented optimality conditions and duality results on Riemannian manifolds. Li [21] deals with the study of weak sharp minima for constrained optimization problems on Riemannian manifolds. Bento [9,10] presented the proximal point method for finding minima of a special class of nonconvex function on a Hadamard manifold, and proposed a subgradient type algorithm for solving convex feasibility problem on Riemannian manifold.

Examining IVOP on Riemannian manifolds offers unique advantages, such as a non-monotone vector field can become monotone when extended to a suitable Riemannian manifold, as highlighted in [18,28,29]. Additionally, a non-convex optimization problem can be transformed into a convex one when introduced to a Riemannian manifold under a suitable Riemannian metric. For instance, in [13,17], the authors provide examples of an IVOP on Riemannian manifold that are unsolvable with techniques designed for Euclidean spaces but can be addressed using methods developed for Riemannian manifolds. In [12,17] KKT sufficient optimality conditions for a convex IVOP on a Hadamard manifold are presented. Hilal and Iqbal [11,13] introduced the generalized Hukuhara (gH) directional differentiability of Interval-valued functions on Riemannian manifolds and derived KKT sufficient optimality conditions for an IVOP defined on a Riemannian manifold by employing the gH-directional differentiability.

Furthermore, the literature extensively covers second-order optimality conditions for Real-Valued Optimization Problem (RVOP) in Euclidean spaces, as discussed by various authors, including [7,14,23,26,35]. Antczak [8] stands out as the first author to present both the first and second-order necessary conditions for multi-objective programming problems with interval-valued objective functions in Euclidean spaces. However, to the best of the author's knowledge, there are currently no established first or second-order necessary optimality results for an Interval-Valued Optimization problem (IVOP) on Riemannian manifolds. Therefore, addressing this gap in IVOP represents a significant advancement in the field.

This paper aims to establish both first and second-order necessary optimality conditions for multi-objective programming with interval-valued objective functions that are twice weakly differentiable and defined on Riemannian manifolds. We consider both equality and inequality constraints that are twice-differentiable and real-valued. The first-order KKT conditions are derived in two versions: the first version considers the First-Order Kuhn–Tucker Constraint Qualifications (FOKTCQ), while the second version incorporates the Linear Independence Constraint Qualification (LICQ). The second-order KKT necessary conditions are established in two versions. The first version is under the McCormick Second-Order Constraint Qualification (SOMCCQ) and the second version is established under the LICQ constraint qualification. The results presented in this paper are demonstrated through illustrative examples.

2. Preliminaries

In this segment, we revisit essential definitions and notations related to Riemannian manifolds, which will be employed consistently in this paper. Further information can be found in [1,15,29,30,34].

Let M be a smooth n-dimensional manifold, and let $p \in \mathsf{M}$. We denote the set of all smooth real-valued functions (f) defined on a neighborhood of p as $C_p^\infty(\mathsf{M})$. A tangent vector X_p is a mapping from $C_p^\infty(\mathsf{M})$ to $\mathbb{R}$ such that there exists a curve γ on M with $\gamma(0) = p$ and satisfies

$$X_p(f) = \dot\gamma(0)f := \frac{\mathrm{d}}{\mathrm{d}t}(f \circ \gamma)(t)\bigg|_{t=0}, \qquad \forall f \in C_p^\infty(\mathsf{M}).$$

Such a curve γ is said to realize the tangent vector X_p [1].

The tangent space to M at p is represented as $T_p(\mathsf{M})$, and the set of all tangent vectors to a subset $E \subseteq \mathsf{M}$ at p is denoted by $T_p(E)$. A vector field X on M is a smooth mapping from M to the tangent bundle $T\mathsf{M} := \bigcup_{p\in\mathsf{M}} T_p(\mathsf{M})$, assigning to each $p \in \mathsf{M}$ a tangent vector $X_p \in T_p(\mathsf{M})$. The collection of all vector fields on M is represented by $\mathfrak{X}(\mathsf{M})$.

Consider a chart (U, ϕ) that contains the point p, and if we introduce the following notation (see [1])

$$\hat p := \phi(p),\ \ \hat X_{\hat p} := D\phi(p)[X_p] \ \text{ and } \hat f := f \circ \phi^{-1}, \tag{1}$$

then, with this notation, for any $X_p \in T_p(\mathsf{M})$ and $f \in C_p^\infty(\mathsf{M})$,

$$X_p f = \left\langle \hat X_{\hat p}, \nabla \hat f(\hat p) \right\rangle.$$

A differentiable manifold whose tangent spaces are equipped with a smoothly varying inner product concerning $p \in \mathsf{M}$ is termed as a Riemannian manifold. The smoothly varying inner product, denoted by g_p, is referred to as the Riemannian metric.

If we introduce the notation $G : \hat p \to G_{\hat p}$ to denote the matrix-valued function such that the (i, j)th element of $G_{\hat p}$ is $g_{ij}(p)$, then, according to [1],

$$g_p(X_p, Y_p) = \hat\xi_{\hat p}^T G_{\hat p} \hat\eta_{\hat p}. \tag{2}$$

Throughout our discussion, we will consider (M, g) as the Riemannian manifold with the Riemannian metric g.

For $f \in C_p^\infty(\mathsf{M})$, the gradient of f at p, denoted by grad $f(p)$, is defined as the unique tangent vector in $T_p(\mathsf{M})$ satisfying the following condition (see [1]):

$$g_p(\text{grad } f(p), X_p) = X_p f = \left\langle \nabla \hat f(\hat p), \hat X_{\hat p} \right\rangle, \qquad \forall X_p \in T_p(\mathsf{M}). \tag{3}$$

In matrix notation, the coordinate expression of grad $f(p)$ is given by (see [1]).

$$D\phi(p)[\text{grad } f(p)] = G_{\hat p}^{-1} \nabla \hat f(\hat p). \tag{4}$$

The velocity of a smooth curve, $\gamma : I \to \mathsf{M}$, is represented by the vector field $\dot\gamma \in \mathfrak{X}(\gamma)$. The acceleration of γ is the smooth field $\ddot\gamma \in \mathfrak{X}(\gamma)$ defined as (see [15]):

$$\ddot\gamma = \frac{D}{\mathrm{d}t}\dot\gamma,$$

where $\frac{D}{\mathrm{d}t}$ is the covariant derivative induced by the Riemannian connection ∇.

A geodesic is a smooth curve $\gamma : I \to \mathsf{M}$ such that $\ddot\gamma(s) = 0\ \forall s \in I$, where I is an open interval of $\mathbb{R}$. A geodesic γ locally minimizes the arc length. For every $X_p \in T_p(\mathsf{M})$, there always exists a unique geodesic $\gamma : I \to \mathsf{M}, 0 \in I$, such that $\gamma(0) = p$ & $\dot\gamma(0) = X_p$.

For any $X_p \in T_p(\mathsf{M})$, the exponential map $\exp_p : T_p(\mathsf{M}) \to \mathsf{M}$ at p is defined as

$$\exp_p(X_p) = \gamma(1);$$

where $\gamma(s)$, for $s \in I$ with $0 \in I$, is a geodesic emanating from $\gamma(0) = p$ in the direction $X_p = \dot{\gamma}(0)$, and I is an interval in $\mathbb{R}$. The exponential map is differentiable at p, and its differential is an identity map.

The Riemannian Hessian of $f \in C_p^{\infty}(\mathsf{M})$ at a point p in M is a (symmetric) linear mapping Hess $f(p) : T_p(\mathsf{M}) \to T_p(\mathsf{M})$ defined as (see [1, 15])

$$\text{Hess } f(p)[X_p] = \nabla_{X_p} \text{grad } f, \qquad \forall X_p \in T_p(\mathsf{M}), \tag{5}$$

where ∇ is a Riemannian connection on M. Moreover, we have (see [15]):

$$\text{Hess } f(p)[X_p] = \nabla_{X_p} \text{grad } f = \frac{D}{\mathrm{d}t} \text{grad } f(\gamma(t))\bigg|_{t=0}, \tag{6}$$

where $\gamma : I \to \mathsf{M}$ is a smooth curve such that $\gamma(0) = p$ and $\dot{\gamma}(0) = X_p$ and $\frac{D}{\mathrm{d}t}$ is the induced covariant derivative induced by the Riemannian connection ∇.

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space. We use the following convention for the equalities and inequalities throughout this paper.

For any vectors $p = (p_1, \dots, p_n)^T$ and $q = (q_1, \dots, q_n)^T$ in $\mathbb{R}^n$, define

- $p = q \Leftrightarrow p_s = q_s \quad \forall s = 1, 2, \dots, n$;
- $p \geqq q \Leftrightarrow p_s \geqq q_s \quad \forall s = 1, 2, \dots, n$;
- $p > q \Leftrightarrow p_s > q_s \quad \forall s = 1, 2, \dots, n$;
- $p \geq q \Leftrightarrow p \geqq q$ and $p \neq q$.

Let $\mathbb{R}^n_+ = \{p \in \mathbb{R}^n : p \geq 0\}$ and $\mathbb{R}^n_{++} = \{p \in \mathbb{R}^n : p > 0\}$ be the sets representing the non-negative orthant and the interior of the non-negative orthant, respectively.

Let $\mathbb{I}$ denote the class of all closed and bounded intervals in $\mathbb{R}$. In this paper, when we refer to A as a closed interval, we imply that A is bounded in $\mathbb{R}$. For any $A \in \mathbb{I}$, we represent $A = [a^L, a^U]$, where a^L and a^U are the lower and upper bounds of A, respectively.

Let $A_1 = [a_1^L, a_1^U] \in \mathbb{I}$, $A_2 = [a_2^L, a_2^U] \in \mathbb{I}$, and $k \in \mathbb{R}$, then we have

(i) $A_1 + A_2 = \{a_1 + a_2 : a_1 \in A_1 \text{ and } a_2 \in A_2\} = [a_1^L + a_2^L, a_1^U + a_2^U]$,
(ii) $-A_1 = \{-a_1 : a_1 \in A_1\} = [-a_1^U, -a_1^L]$
(iv) $k + A_1 = \{k + a_1 : a_1 \in A_1\} = [k + a_1^L, k + a_1^U]$,
(v) $kA_1 = \begin{cases} [ka_1^L, ka_1^U]; & k \geq 0 \\ [ka_1^U, ka_1^L]; & k < 0 \end{cases}$.

For a more in-depth exploration of interval analysis, we recommend consulting Moore [25] and Alefeld and Herzberger [2]. To establish the ordering relation between intervals in $\mathbb{I}$, we adopt the following convention:

Let $A_1 = [a_1^L, a_1^U]$ and $A_2 = [a_2^L, a_2^U] \in \mathbb{I}$, we write

(i) $A_1 \leqq_{LU} A_2$ if and only if $a_1^L \leqq a_2^L$ & $a_1^U \leqq a_2^U$;
(ii) $A_1 <_{LU} A_2$ if and only if $A_1 \leqq_{LU} A_2$ & $A_1 \neq A_2$;
(iii) $A_1 \prec_{LU} A_2$ if and only if $a_1^L < a_2^L$ & $a_1^U < a_2^U$.

We say that $A = (A_1, A_2, \dots, A_r)$ is an interval r-tuple if each component $A_k = [a_k^L, a_k^U]$ is a closed interval for $k \in \{1, 2, 3, \dots, r\}$. Let $A = (A_1, A_2, \dots, A_r)$ and $B = (B_1, B_2, \dots, B_r)$ be two interval r-tuples.

We write

- $A \leqq_{LU} B$ if and only if $A_k \leqq_{LU} B_k$ for each $k \in \{1, 2, 3, \dots, r\}$.
- $A <_{LU} B$ if and only if $A \leqq_{LU} B$ for each $k \in \{1, 2, 3, \dots, r\}$ and $A \neq B$.

Consider a non-empty subset $\mathcal{A}$ of M. A function $f : \mathcal{A} \to \mathbb{I}$ is termed an interval-valued function (IVF) if $f(p) = \left[f^L(p), f^U(p)\right]$ with $f^L, f^U : \mathcal{A} \to \mathbb{R}$ such that $f^L(p) \leq f^U(p)$ for each $p \in \mathcal{A}$.

Next, we present the notion of differentiating an Interval-Valued Function (IVF) on Riemannian Manifolds, employing a straightforward differentiation concept originally introduced by Wu [37] on Euclidean spaces.

Definition 2.1. Consider a nonempty set $\mathcal{A}$ in M, and let $f : \mathcal{A} \to \mathbb{I}$ be an (IVF) with $f(p) = \left[f^L(p), f^U(p)\right]$. We say that f is weakly differentiable at $p \in \mathcal{A}$ if the end point functions f^L and f^U are differentiable at p.

Next, we introduce the notion of twice weak differentiability of an interval-valued function.

Definition 2.2. Consider a nonempty open set $\mathcal{A}$ in M, and let $f : \mathcal{A} \to \mathbb{I}$ be an (IVF) with $f(p) = \left[f^L(p), f^U(p)\right]$. Suppose $\bar{p} \in \mathcal{A}$ is fixed. We say that f is twice weakly differentiable IVF at $\bar{p}$ if the end points functions f^L and f^U are twice differentiable at $\bar{p}$.

3. First order conditions

Consider the Multiobjective Interval-Valued Optimization Problem (MIVOP) outlined below:

$$\begin{aligned} \text{minimize} \quad & f(p) = (f_1(p), f_2(p), \ldots, f_n(p)) & & \\ \text{subject to} \quad & c_i(p) \leqq 0, & & i \in I = \{1, 2, \ldots, m\}, \\ & d_j(p) = 0, & & j \in J = \{1, 2, \ldots, r\}, \end{aligned} \tag{P}$$

where the objective functions $f_k : \mathsf{M} \to \mathbb{I}$, $f_k(p) = [f_k^L(p),\ f_k^U(p)]$, $k \in K = \{1, 2, \ldots, n\}$ are interval-valued, each $c_i, d_j : \mathsf{M} \to \mathbb{R}, i \in I, j \in J$, are real-valued functions.

In addressing first-order optimality conditions, it is assumed that each (IVF) $f_k, \forall k \in K$ is weakly differentiable on M, and each constraint $c_i, \forall i \in I$, and $d_j, \forall j \in J$ is differentiable on M. Similarly, when addressing second-order optimality conditions, the assumption is made that each IVF $f_k, \forall k \in K$ is twice weakly differentiable on M, and each constraint $c_i, \forall i \in I$, and $d_j, \forall j \in J$ is twice differentiable on M.

For the sake of convenience, we write $c := (c_1, c_2, \ldots, c_m) : \mathsf{M} \to \mathbb{R}^m$ and $d := (d_1, d_2, \ldots, d_r) : \mathsf{M} \to \mathbb{R}^r$.

Furthermore, we denote by $\mathcal{X}$ the set of all feasible solutions of MIVOP (P), *i.e.*,

$$\mathcal{X} = \{p \in \mathsf{M} : c_i(p) \leqq 0, i \in I, d_j(p) = 0, j \in J\},$$

and we denote by $I(\bar{p})$, the collection of indices of all those inequality constraints that are active at $\bar{p} \in \mathcal{X}$, *i.e.*,

$$I(\bar{p}) = \{i \in I : c_i(\bar{p}) = 0\}.$$

Wu [39] introduced various notions of Pareto optimal solutions for the Multiobjective Interval-Valued Optimization problems. Consequently, drawing from the solution concepts outlined by Wu [39], we introduce the notion of local (weak) LU-Pareto optimal solutions for the specified MIVOP (P).

Definition 3.1. A feasible point $\bar{p} \in \mathcal{X}$ is considered as a local weak LU-Pareto optimal solution for the problem (P) if there exists a neighborhood $\mathcal{B}(\bar{p}, \epsilon)$ around $\bar{p}$ such that no other $p \in \mathcal{X} \cap \mathcal{B}(\bar{p}, \epsilon)$ satisfies $f_k(p) <_{LU} f_k(\bar{p})$, for each $k \in K$.

Definition 3.2. A feasible point $\bar{p} \in \mathcal{X}$ is considered as a local LU-Pareto optimal solution if there exists a neighborhood $\mathcal{B}(\bar{p}, \epsilon)$ around $\bar{p}$ such that no $p \in \mathcal{X} \cap \mathcal{B}(\bar{p}, \epsilon)$ satisfies $f(p) <_{LU} f(\bar{p})$.

Lemma 3.1. *Suppose $\bar{p} \in \mathcal{X}$ is a local weak LU-Pareto optimal solution to problem* (P). *Then there exists $\lambda^L, \lambda^U \in \mathbb{R}^n_+$ such that $\bar{p}$ is a local minimizer*[1] *of the problem* (WOP_λ) *associated to* (P). *The problem* (WOP_λ) *is outlined below:*

$$\begin{aligned} \text{minimize} \quad & \bar{f}(p) = \sum_{k=1}^{n} \lambda_k^L f_k^L(p) + \sum_{k=1}^{n} \lambda_k^U f_k^U(p) \\ \text{subject to} \quad & p \in \mathcal{X}. \end{aligned} \tag{WOP_λ}$$

[1] In the context of a real-valued function $f : \mathsf{M} \to \mathbb{R}$ and a subset $\mathcal{A} \subseteq \mathsf{M}$, a point $p \in \mathcal{A}$ is considered a local minimizer of f over $\mathcal{A}$ if there exists a neighborhood $\mathcal{B}(p; \epsilon)$ around p such that $f(p) \leqq f(q)$ for all $q \in \mathcal{A} \cap \mathcal{B}(p; \epsilon)$.

Proof. It is given that $\bar{p} \in \mathcal{X}$ is a local weak LU-Pareto optimal solution to problem (P). By Definition 3.1, $\exists$ a neighborhood $\mathcal{B}(\bar{p};\epsilon)$ around $\bar{p}$ such that no other $p \in \mathcal{X} \cap \mathcal{B}(\bar{p};\epsilon)$ satisfies $f_k(p) <_{LU} f_k(\bar{p})$, for each $k \in K$

$$\implies \begin{cases} f_k^L(p) < f_k^L(\bar{p}) \\ f_k^U(p) \leqq f_k^U(\bar{p}) \end{cases} \text{or} \quad \begin{cases} f_k^L(p) \leqq f_k^L(\bar{p}) \\ f_k^U(p) < f_k^U(\bar{p}) \end{cases} \text{or} \quad \begin{cases} f_k^L(p) < f_k^L(\bar{p}) \\ f_k^U(p) < f_k^U(\bar{p}). \end{cases}$$

This implies for each $p \in \mathcal{X} \cap \mathcal{B}(\bar{p},\epsilon)$, we have for each $k \in K$

$$\begin{cases} f_k^L(\bar{p}) \leqq f_k^L(p) \\ f_k^U(\bar{p}) < f_k^U(p) \end{cases} \text{or} \quad \begin{cases} f_k^L(\bar{p}) < f_k^L(p) \\ f_k^U(\bar{p}) \leqq f_k^U(p) \end{cases} \text{or} \quad \begin{cases} f_k^L(\bar{p}) \leqq f_k^L(p) \\ f_k^U(\bar{p}) \leqq f_k^U(p). \end{cases}$$

For $\lambda_k^L, \lambda_k^U \in \mathbb{R}_+^n$, by combining the above three cases, we have that for each $p \in \mathcal{X} \cap \mathcal{B}(\bar{p},\epsilon)$

$$\sum_{k=1}^{n} \lambda_k^L f_k^L(\bar{p}) + \sum_{k=1}^{n} \lambda_k^U f_k^U(\bar{p}) \leqq \sum_{k=1}^{n} \lambda_k^L f_k^L(p) + \sum_{k=1}^{n} \lambda_k^U f_k^U(p)$$
$$\implies \bar{f}(\bar{p}) \leqq \bar{f}(p), \quad \forall p \in \mathcal{X} \cap \mathcal{B}(\bar{p},\epsilon).$$

Hence, $\bar{p}$ is a local minimizer of $\bar{f}$ over $\mathcal{X}$. This completes the proof □

Now, we introduce the first-order Kuhn–Tucker constraint qualification for the problem (P). For this, we first define the following set

$$\mathbb{Y}^1(\bar{p}) := \{X_{\bar{p}} \in T_{\bar{p}}(\mathsf{M}) : g_{\bar{p}}(\text{grad } c_i(\bar{p}) \ , X_{\bar{p}}) \leqq 0, i \in I(\bar{p}), g_{\bar{p}}(\text{grad } d_j(\bar{p}), X_{\bar{p}}) = 0, j \in J\}.$$

Definition 3.3. For a given $\bar{p} \in \mathcal{X}$ and assuming that the constraint functions $c = (c_1, c_2, \ldots, c_m)$ and $d = (d_1, d_2, \ldots, d_r)$ are differentiable at $\bar{p}$, the FOKTCQ is considered satisfied at $\bar{p}$ if, for any $X_{\bar{p}} \in \mathbb{Y}^1(\bar{p})$, $X_{\bar{p}} \neq 0$, there exists a function $h : [0,1] \to \mathsf{M}$ that is continuously differentiable at 0 and a real scalar $\beta > 0$ such that

$$h(0) = \bar{p}, \quad h(s) \in \mathcal{X}, \quad \forall s \in [0,1], \quad \text{and} \quad \dot{h}(0) = \beta X_{\bar{p}}.$$

Assuming the Kuhn–Tucker constraint qualification, Wu [38] established KKT necessary conditions for a scalar differentiable optimization problem in which the objective function is interval-valued. Antczak [8] later generalized these conditions to a differentiable vector optimization problem involving multiple interval-valued functions. In this context, we further extend these conditions to encompass a differentiable vector optimization problem with multiple interval-valued functions defined on a Riemannian manifold.

Theorem 3.1 (First order KKT- necessary conditions)**.** *Consider $\bar{p}$ as a local weak LU-Pareto optimal solution to problem* (P)*. If (FOKTCQ) condition holds at $\bar{p}$, then there exist Lagrange multipliers $\lambda^L, \lambda^U \in \mathbb{R}^n$, $\mu \in \mathbb{R}^m$, and $\vartheta \in \mathbb{R}^r$ such that*

(i) $\sum_{k=1}^{n} \lambda_k^L \text{grad } f_k^L(\bar{p}) \sum_{k=1}^{n} \lambda_k^U \text{grad } f_k^U(\bar{p}) + \sum_{i=1}^{m} \mu_i \text{grad } c_i(\bar{p}) + \sum_{j=1}^{r} \vartheta_j \text{grad } d_j(\bar{p}) = 0;$
(ii) $\mu_i c_i(\bar{p}) = 0, \quad i \in I;$
(iii) $\left(\lambda^L, \lambda^U\right) \geq 0, \quad \mu \geqq 0.$

Proof. We assume $\bar{p} \in \mathcal{X}$ is a local weak LU-Pareto optimal solution for the problem (P) and the (FOKTCQ) condition holds at $\bar{p}$. We prove that there exists no vector $X_{\bar{p}} \in \mathbb{Y}^1(\bar{p}), X_{\bar{p}} \neq 0$, such that $g_{\bar{p}}(\text{grad } f_k^L(\bar{p}), X_{\bar{p}}) < 0$ and $g_{\bar{p}}(\text{grad } f_k^U(\bar{p}), X_{\bar{p}}) < 0, k \in K$, *i.e.*, there exists no vector $X_{\bar{p}} \in T_{\bar{p}}(\mathsf{M}), X_{\bar{p}} \neq 0$, satisfying the following system of inequalities:

$$g_{\bar{p}}\big(\text{grad } f_k^L(\bar{p}), X_{\bar{p}}\big) < 0, \quad g_{\bar{p}}\big(\text{grad } f_k^U(\bar{p}), X_{\bar{p}}\big) < 0, \qquad k \in K; \tag{7}$$
$$g_{\bar{p}}(\text{grad } c_i(\bar{p}), X_{\bar{p}}) \leqq 0, \qquad i \in I; \tag{8}$$

$$g_{\bar{p}}(\text{grad } d_j(\bar{p}), X_{\bar{p}}) = 0, \qquad j \in J. \tag{9}$$

Suppose, on contrary, there exists $X_{\bar{p}} \in T_{\bar{p}}(\mathsf{M}), X_{\bar{p}} \neq 0$, satisfying (7)–(9). It is easy to see $X_{\bar{p}} \in \mathbb{Y}^1(\bar{p})$. Given the assumption that the (FOKTCQ) condition is fulfilled at $\bar{p}$, there exists a function $h : [0,1] \to \mathsf{M}$ that is continuously differentiable at 0, along with a positive real number β, such that

$$h(0) = \bar{p},\ h(s) \in \mathcal{X},\ \forall s \in [0,1] \text{ and } \dot{h}(0) = \beta X_{\bar{p}}.$$

Also, by hypothesis, $f_k, k \in K$ is weakly differentiable at $\bar{p}$, it follows from Definition 2.1 that all f_k^L and $f_k^U, k \in K$ are differentiable at $\bar{p}$. Therefore, the functions $f_k^L \circ h, f_k^U \circ h : [0,1] \to \mathbb{R}, k \in K$ are differentiable at 0 and by Taylor series expansion, it follows:

$$\begin{aligned} &\big(f_k^L \circ h\big)(s) = \big(f_k^L \circ h\big)(0) + s\big(f_k^L \circ h\big)'(0) + O_k^L\big(s^2\big), \quad k \in K;\\ \text{and } &\big(f_k^U \circ h\big)(s) = \big(f_k^U \circ h\big)(0) + s\big(f_k^U \circ h\big)'(0) + O_k^U\big(s^2\big), \quad k \in K. \end{aligned}$$

Since [15]

$$\big(f_k^L \circ h\big)'(0) = \dot{h}(0)\big(f_k^L\big) = df_k^L(h(0))\Big[\dot{h}(0)\Big] = \beta g_{\bar{p}}\big(\text{grad } f_k^L(\bar{p}), X_{\bar{p}}\big),$$

we have:

$$f_k^L(h(s)) = f_k^L(\bar{p}) + s\beta g_{\bar{p}}\big(\text{grad } f_k^L(\bar{p}), X_{\bar{p}}\big) + O_k^L\big(s^2\big), \quad k \in K; \tag{10}$$

$$\text{and } f_k^U(h(s)) = f_k^U(\bar{p}) + s\beta g_{\bar{p}}\big(\text{grad } f_k^U(\bar{p}), X_{\bar{p}}\big) + O_k^U\big(s^2\big), \quad k \in K. \tag{11}$$

Using $s \to 0$ in (10) and (11), we get for sufficiently small s that

$$f_k^L(h(s)) = f_k^L(\bar{p}) + s\beta g_{\bar{p}}\big(\text{grad } f_k^L(\bar{p}), X_{\bar{p}}\big), \quad k \in K; \tag{12}$$

$$\text{and } f_k^U(h(s)) = f_k^U(\bar{p}) + s\beta g_{\bar{p}}\big(\text{grad } f_k^U(\bar{p}), X_{\bar{p}}\big), \quad k \in K. \tag{13}$$

By assumption, $X_{\bar{p}} \in T_{\bar{p}}(\mathsf{M}), X_{\bar{p}} \neq 0$, satisfies (7). Hence, by $s > 0$ and $\beta > 0$ (12) and (13) respectively yields, for sufficiently small s,

$$f_k^L(h(s)) < f_k^L(\bar{p}), \quad k \in K; \tag{14}$$

$$f_k^U(h(s)) < f_k^U(\bar{p}), \quad k \in K, \tag{15}$$

which is a contradiction to the assumption that $\bar{p} \in \mathcal{X}$ is local weak LU-Pareto optimal solution for problem (P). Therefore, there exists no vector $X_{\bar{p}} \in T_{\bar{p}}(\mathsf{M}), X_{\bar{p}} \neq 0$, that satisfies the system of inequalities (7)–(9).

Suppose that (U, ϕ) is a chart containing $\bar{p}$, then using (1) and (3) in the system of inequalities (7), (8), and (9), we have the following inconsistent system of inequalities

$$\Big\langle \nabla \hat{f}_k^L(\hat{\bar{p}}), \hat{X}_{\hat{\bar{p}}} \Big\rangle < 0, \quad \Big\langle \nabla \hat{f}_k^U(\hat{\bar{p}}), \hat{X}_{\hat{\bar{p}}} \Big\rangle < 0, \quad k \in K; \tag{16}$$

$$\Big\langle \nabla \hat{c}_i(\hat{\bar{p}}), \hat{X}_{\hat{\bar{p}}} \Big\rangle \leqq 0, \qquad i \in I; \tag{17}$$

$$\Big\langle \nabla \hat{d}_j(\hat{\bar{p}}), \hat{X}_{\hat{\bar{p}}} \Big\rangle = 0, \qquad j \in J; \tag{18}$$

where $\langle ., . \rangle$ is the usual inner product in $\mathbb{R}^n$. As the system of inequalities (16)–(18) is inconsistent, Motzkin's theorem of alternatives (refer to [22]) guarantees the existence of Lagrange multipliers $\lambda^L, \lambda^U \in \mathbb{R}^n$, $\mu \in \mathbb{R}^{I(\bar{p})}$, $\vartheta \in \mathbb{R}^r$, satisfying $(\lambda^L, \lambda^U) \geq 0$ and $\mu \geq 0$,

$$\sum_{k=1}^{n} \lambda_k^L \nabla \hat{f}_k^L(\hat{\bar{p}}) + \sum_{k=1}^{n} \lambda_k^U \nabla \hat{f}_k^U(\hat{\bar{p}}) + \sum_{i \in I(\bar{p})} \mu_i \nabla \hat{c}_i(\hat{\bar{p}}) + \sum_{j=1}^{r} \vartheta_j \nabla \hat{d}_j(\hat{\bar{p}}) = 0. \tag{19}$$

Setting $\mu_i = 0$ for all $i \in I \setminus I(\bar{p})$, then (19) yields the following

$$\sum_{k=1}^{n} \lambda_k^L \nabla \hat{f}_k^L(\hat{\bar{p}}) + \sum_{k=1}^{n} \lambda_k^U \nabla \hat{f}_k^U(\hat{\bar{p}}) + \sum_{i=1}^{m} \mu_i \nabla \hat{c}_i(\hat{\bar{p}}) + \sum_{j=1}^{r} \vartheta_j \nabla \hat{d}_j(\hat{\bar{p}}) = 0; \tag{20}$$

$$\mu_i c_i(\bar{p}) = 0, \quad i \in I; \tag{21}$$

$$\left(\lambda^L, \lambda^U\right) \geq 0, \quad \mu \geqq 0. \tag{22}$$

By linearity of $(D\phi(p))^{-1}$ and $G_{\hat{\bar{p}}}^{-1}$, the system (20)–(22), together with (4) yields,

$$\sum_{k=1}^{n} \lambda_k^L \text{grad } f_k^L(\bar{p}) + \sum_{k=1}^{n} \lambda_k^U \text{grad } f_k^U(\bar{p}) + \sum_{i=1}^{m} \mu_i \text{grad } c_i(\bar{p}) + \sum_{j=1}^{r} \vartheta_j \text{grad } d_j(\bar{p}) = 0;$$
$$\mu_i c_i(\bar{p}) = 0, \quad i \in I;$$
$$\left(\lambda^L, \lambda^U\right) \geq 0, \quad \mu \geqq 0.$$

□

Now, let's consider the optimization problem with a single real-valued objective function outlined as follows:

$$\begin{aligned} &\text{minimize} \quad f(p) \\ &\text{subject to} \quad p \in \mathcal{X}, \end{aligned} \tag{P*}$$

where $f : \mathsf{M} \to \mathbb{R}$ is a real-valued function.

Next, we turn our attention to the LICQ on M, which was introduced by Yang *et al.* [40]. This condition is defined as follows:

Definition 3.4 ([40])**.** Let $\bar{p} \in \mathcal{X}$, and suppose the constraints c_i for $i \in I(\bar{p})$ and d_j for $j \in J$ are differentiable at $\bar{p}$. The LICQ is considered to hold at $\bar{p}$ if the set

$$\{\text{grad } c_i(\bar{p}); i \in I(\bar{p})\} \bigcup \{\text{grad } d_j(\bar{p}); j \in J\},$$

is linearly independent on $T_{\bar{p}}(\mathsf{M})$.

The subsequent theorem provides the necessary optimality conditions for problem (P*) which directly follows from Theorem 4.4 and Remark 4.6.1 presented by Yang *et al.* [40]

Theorem 3.2. *Let $\bar{p} \in \mathcal{X}$ be a local solution to problem (P*) and that the LICQ holds at $\bar{p}$, then there exists Lagrange multiplier vectors $\mu \in \mathbb{R}^m$ and $\vartheta \in \mathbb{R}^r$ such that*

(i) $\text{grad } f(\bar{p}) + \sum_{i=1}^{m} \mu_i \text{grad } c_i(\bar{p}) + \sum_{j=1}^{r} \vartheta_j \text{grad } d_j(\bar{p}) = 0$;
(ii) $\mu_i c_i(\bar{p}) = 0, \quad i \in I$;
(iii) $\mu_i \geqq 0, \quad i \in I$.

We will now present the necessary optimality conditions for problem (P) in the scenario where the constraints of the problem satisfy the LICQ.

Theorem 3.3 (First order necessary conditions)**.** *Assuming $\bar{p} \in \mathcal{X}$ as a local weak LU-Pareto optimal solution to problem (P), and that the (LICQ) condition holds at $\bar{p} \in \mathcal{X}$. Then, there exist Lagrange multiplier vectors $\mu \in \mathbb{R}^m$ and $\vartheta \in \mathbb{R}^r$ such that*

(i) $\sum_{k=1}^{n} \left(\text{grad } f_k^L(\bar{p}) + \text{grad } f_k^U(\bar{p})\right) + \sum_{i=1}^{m} \mu_i \text{grad } c_i(\bar{p}) + \sum_{j=1}^{r} \vartheta_j \text{grad } d_j(\bar{p}) = 0$;
(ii) $\mu_i c_i(\bar{p}) = 0, \quad i \in I$;
(iii) $\mu_i \geqq 0, \quad i \in I$.

Proof. We define a real-valued function $\bar{f}$ as follows:

$$\bar{f}(p) = \sum_{k=1}^{n} \big(f_k^L(p) + f_k^U(p)\big). \tag{23}$$

From Lemma 3.1 together with $\lambda_k^L = \lambda_k^U = 1$ and the fact that $\bar{p} \in \mathcal{X}$ is a local weak LU-Pareto optimal solution to problem (P), $\bar{p}$ is a local minimizer (solution) to the problem outlined below.

$$\begin{aligned} &\text{minimize} \;\; \bar{f}(p) = \sum_{k=1}^{n} \big(f_k^L(p) + f_k^U(p)\big) \\ &\text{subject to} \;\; p \in \mathcal{X}. \end{aligned}$$

Hence, from Theorem 3.2, there exists Lagrange multiplier vectors $\mu \in \mathbb{R}^m$and $\vartheta \in \mathbb{R}^r$ such that

(a) grad $\bar{f}(\bar{p}) + \sum_{i=1}^{m} \mu_i \text{grad } c_i(\bar{p}) + \sum_{j=1}^{r} \vartheta_j \text{grad } d_j(\bar{p}) = 0$;
(b) $\mu_i c_i(\bar{p}) = 0, i \in I$;
(c) $\mu_i \geqq 0, i \in I$.

By linearity of vector field grad f and from equation (23), the above conditions (a), (b) and (c) yield the necessary optimality conditions (i), (ii) and (iii) of this theorem. This completes the proof. □

4. Second order conditions

Here, we provide the second order necessary conditions for the problem (P). The multiple interval-valued objective functions are presumed to be twice weakly differentiable, while the inequality and equality constraints are considered to be twice differentiable.

First and foremost, we introduce SOMCCQ that we will use to establish second-order necessary optimality conditions for the problem (P). For more information regarding SOMCCQ one can see [23] for the case of Euclidean space.

We define the following set that will be used frequently in the subsequent results.

$$\mathbb{Y}^2(\bar{p}) := \{X_{\bar{p}} \in T_{\bar{p}}(\mathsf{M}) : g_{\bar{p}}(\text{grad } c_i(\bar{p})\ , X_{\bar{p}}) = 0, i \in I(\bar{p}),\;\; g_{\bar{p}}(\text{grad } d_j(\bar{p}), X_{\bar{p}}) = 0,\;\; j \in J\}.$$

Definition 4.1. Assume that the functions involved in the problem (P) are twice differentiable at a feasible point $\bar{p} \in \mathcal{X}$. The SOMCCQ is said to hold at $\bar{p} \in \mathcal{X}$ if and only if, for every tangent vector $X_{\bar{p}} \in \mathbb{Y}^2(\bar{p})$ where $X_{\bar{p}} \neq 0$, there exists a twice-differentiable function $h : [0, s_0] \to \mathcal{X}$ for $s_0 > 0$ such that for any $s \in [0, s_0]$

$$h(0) = \bar{p},\;\; h(s) \in \mathcal{X},\;\; \forall s \in [0, s_0],\;\; \dot{h}(0) = X_{\bar{p}},\;\; \ddot{h}(0) = Z_{\bar{p}}. \tag{24}$$

$$g_{\bar{p}}(\text{Hess } c_i(\bar{p})[X_{\bar{p}}], X_{\bar{p}}) + g_{\bar{p}}(\text{grad } c_i(\bar{p}), Z_{\bar{p}}) = 0,\;\; i \in I(\bar{p}), \tag{25}$$

$$g_{\bar{p}}(\text{Hess } d_j(\bar{p})[X_{\bar{p}}], X_{\bar{p}}) + g_{\bar{p}}(\text{grad } d_j(\bar{p}), Z_{\bar{p}}) = 0,\;\; j \in J. \tag{26}$$

Theorem 4.1 (Second order necessary conditions)**.** *Consider $\bar{p} \in \mathcal{X}$ as a local weak LU-Pareto optimal solution for the problem* (P)*. Assuming that the objective functions $f_k, k \in K$ are twice weakly differentiable at $\bar{p}$, as well as the constraints $c_i, i \in I$ and $d_j, j \in J$, are twice differentiable at $\bar{p}$. Furthermore, assume that the (FOKTCQ) condition is met at $\bar{p}$. Let $\lambda^L, \lambda^U \in \mathbb{R}^n, \mu \in \mathbb{R}^m$ and $\vartheta \in \mathbb{R}^r$ be the Lagrange multipliers such that*

(i) $\sum_{k=1}^{n} \lambda_k^L \text{grad } f_k^L(\bar{p}) + \sum_{k=1}^{n} \lambda_k^U \text{grad } f_k^U(\bar{p}) + \sum_{i=1}^{m} \mu_i \text{grad } c_i(\bar{p}) + \sum_{j=1}^{r} \vartheta_j \text{grad } d_j(\bar{p}) = 0$;
(ii) $\mu_i c_i(\bar{p}) = 0,\;\; i \in I$;
(iii) $\big(\lambda^L, \lambda^U\big) \geq 0,\;\; \mu \geqq 0.$
Further, suppose that the (SOMCCQ) condition holds at $\bar{p}$. Then
(iv) $\sum_{k=1}^{n} \lambda_k^L g_{\bar{p}}\big(\text{Hess } f_k^L(\bar{p})[X_{\bar{p}}], X_{\bar{p}}\big) + \sum_{k=1}^{n} \lambda_k^U g_{\bar{p}}\big(\text{Hess } f_k^U(\bar{p})[X_{\bar{p}}], X_{\bar{p}}\big)$
$+ \sum_{i=1}^{m} \mu_i g_{\bar{p}}(\text{Hess } c_i(\bar{p})[X_{\bar{p}}], X_{\bar{p}}) + \sum_{j=1}^{r} \vartheta_j g_{\bar{p}}(\text{Hess } d_j(\bar{p})[X_{\bar{p}}], X_{\bar{p}}) \geqq 0 \quad \forall X_{\bar{p}} \in \mathbb{Y}^2(\bar{p}).$

Proof. Since, the (SOMCCQ) condition holds at $\bar{p}$. Multiplying (25) and (26) by the corresponding multipliers $\mu_i \in I(\bar{p})$ and $\vartheta_j, j \in J$ respectively, and taking into account $\mu_i = 0, i \in I \setminus I(\bar{p})$, and then summing the resulting equations, we have for any $X_{\bar{p}} \in \mathbb{Y}^2(\bar{p})$

$$\sum_{i=1}^{m} \mu_i g_{\bar{p}}(\text{Hess } c_i(\bar{p})[X_{\bar{p}}], X_{\bar{p}}) + \sum_{i=1}^{m} \mu_i g_{\bar{p}}(\text{grad } c_i(\bar{p}), Z_{\bar{p}}) = 0; \tag{27}$$

$$\sum_{j=1}^{r} \vartheta_j g_{\bar{p}}(\text{Hess } d_j(\bar{p})[X_{\bar{p}}], X_{\bar{p}}) + \sum_{j=1}^{r} \vartheta_j g_{\bar{p}}(\text{grad } d_j(\bar{p}), Z_{\bar{p}}) = 0. \tag{28}$$

From condition (i) of this theorem, we have

$$\sum_{k=1}^{n} \lambda_k^L \text{grad } f_k^L(\bar{p}) + \sum_{k=1}^{n} \lambda_k^U \text{grad } f_k^U(\bar{p}) = -\left(\sum_{i=1}^{m} \mu_i \text{grad } c_i(\bar{p}) + \sum_{j=1}^{r} \vartheta_j \text{grad } d_j(\bar{p}) \right). \tag{29}$$

Choosing $X_{\bar{p}} \in \mathbb{Y}^2(\bar{p})$, and taking into account the linearity of $g_{\bar{p}}$, (29) yields

$$\begin{aligned} &\sum_{k=1}^{n} \lambda_k^L g_{\bar{p}}\big(\text{grad } f_k^L(\bar{p}), X_{\bar{p}}\big) + \sum_{k=1}^{n} \lambda_k^U g_{\bar{p}}\big(\text{grad } f_k^U(\bar{p}), X_{\bar{p}}\big) \\ &\quad = -\left(\sum_{i=1}^{m} \mu_i g_{\bar{p}}(\text{grad } c_i(\bar{p}), X_{\bar{p}}) + \sum_{j=1}^{r} \vartheta_j g_{\bar{p}}(\text{grad } d_j(\bar{p}), X_{\bar{p}}) \right). \end{aligned} \tag{30}$$

Since, $X_{\bar{p}} \in \mathbb{Y}^2(\bar{p})$ and taking into account $\mu_i = 0, i \in I \setminus I(\bar{p})$, from the formulation of $\mathbb{Y}^2(\bar{p})$, (30) yields

$$\sum_{k=1}^{n} \lambda_k^L g_{\bar{p}}\big(\text{grad } f_k^L(\bar{p}), X_{\bar{p}}\big) + \sum_{k=1}^{n} \lambda_k^U g_{\bar{p}}\big(\text{grad } f_k^U(\bar{p}), X_{\bar{p}}\big) = 0. \tag{31}$$

Since, $\bar{p} \in \mathcal{X}$ is a local weak LU-Pareto optimal solution of the problem (P), from Lemma 3.1, $\bar{p}$ is a local minimizer of the Weighting Optimization Problem (WOP_λ) associated to problem (P). As a result, we have

$$\sum_{k=1}^{n} \lambda_k^L f_k^L(p) + \sum_{k=1}^{n} \lambda_k^U f_k^U(p) \geqq \sum_{k=1}^{n} \lambda_k^L f_k^L(\bar{p}) + \sum_{k=1}^{n} \lambda_k^U f_k^U(\bar{p}) \tag{32}$$

for all $p \in \mathcal{X} \cap \mathcal{B}(\bar{p}, \epsilon)$, where $\mathcal{B}(\bar{p}, \epsilon)$ is a neighbourhood about $\bar{p}$. Since, SOMCCQ condition holds at $\bar{p}$, by Definition 4.1, there exists a twice differentiable function $h : [0, s_0] \to \mathcal{X}$ for $s_0 > 0$ such that for any $s \in [0, s_0]$

$$h(0) = \bar{p}, \ h(s) \in \mathcal{X}, \ \forall s \in [0, s_0], \ \dot{h}(0) = X_{\bar{p}}, \ \ddot{h}(0) = Z_{\bar{p}}.$$

By second order Taylor's expansion, we have

$$\big(f_k^L \circ h\big)(s) = \big(f_k^L \circ h\big)(0) + s\big(f_k^L \circ h\big)'(0) + \frac{s^2}{2!}\big(f_k^L \circ h\big)''(0) + O_k^L\big(s^3\big) \tag{33}$$

but

$$\big(f_k^L \circ h\big)'(0) = \dot{h}(0)(f_k^L) = X_{\bar{p}}(f_k^L) = g_{\bar{p}}\big(\text{grad } f_k^L(\bar{p}), X_{\bar{p}}\big) \tag{34}$$

and,

$$(f_k^L \circ h)''(0) = \frac{\mathrm{d}}{\mathrm{d}t} g_{\bar{p}}(\text{grad } f_k^L(\bar{p}), X_{\bar{p}})$$

$$\begin{aligned}
&= g_{\bar{p}}\left(\frac{D}{\mathrm{dt}}\text{grad } f_k^L(\bar{p}), X_{\bar{p}}\right) + g_{\bar{p}}\left(\text{grad } f_k^L(\bar{p}), \frac{D}{\mathrm{dt}}\dot{h}(0)\right) \\
&= g_{\bar{p}}\left(\frac{D}{\mathrm{dt}}\text{grad } \left(f_k^L \circ h\right)(0), X_{\bar{p}}\right) + g_{\bar{p}}\left(\text{grad } f_k^L(\bar{p}), \ddot{h}(0)\right) \\
&= g_{\bar{p}}\left(\text{Hess } f_k^L(h(0))\left[\dot{h}(0)\right], X_{\bar{p}}\right) + g_{\bar{p}}\left(\text{grad } f_k^L(\bar{p}), Z_{\bar{p}}\right) \ \ (\text{using } (6)) \\
&= g_{\bar{p}}\left(\text{Hess } f_k^L(\bar{p})[X_{\bar{p}}], X_{\bar{p}}\right) + g_{\bar{p}}\left(\text{grad } f_k^L(\bar{p}), Z_{\bar{p}}\right). \qquad (35)
\end{aligned}$$

Using (34) and (35) in (33), we have

$$\begin{aligned}
f_k^L(h(s)) = f_k^L(\bar{p}) + s g_{\bar{p}}\left(\text{grad } f_k^L(\bar{p}), X_{\bar{p}}\right) + \frac{s^2}{2!}\left[g_{\bar{p}}\left(\text{Hess } f_k^L(\bar{p})[X_{\bar{p}}], X_{\bar{p}}\right)\right. \\
\left. + g_{\bar{p}}\left(\text{grad } f_k^L(\bar{p}), Z_{\bar{p}}\right)\right] + O_k^L(s^3). \qquad (36)
\end{aligned}$$

Similarly, the second order Taylor's expansion of $f_k^U, k \in K$ is as follows

$$\begin{aligned}
f_k^U(h(s)) = f_k^U(\bar{p}) + s g_{\bar{p}}\left(\text{grad } f_k^U(\bar{p}), X_{\bar{p}}\right) + \frac{s^2}{2!}\left[g_{\bar{p}}\left(\text{Hess } f_k^U(\bar{p})[X_{\bar{p}}], X_{\bar{p}}\right)\right. \\
\left. + g_{\bar{p}}\left(\text{grad } f_k^U(\bar{p}), Z_{\bar{p}}\right)\right] + O_k^U(s^3). \qquad (37)
\end{aligned}$$

Multiplying (36) and (37) by corresponding Lagrange multipliers λ_k^L and $\lambda_k^U, k \in K$ respectively, and then summing the resulting equations together, we have

$$\begin{aligned}
&\sum_{k=1}^{n} \lambda_k^L f_k^L(h(s)) + \sum_{k=1}^{n} \lambda_k^U f_k^U(h(s)) - \sum_{k=1}^{n} \lambda_k^L f_k^L(\bar{p}) - \sum_{k=1}^{n} \lambda_k^U f_k^U(\bar{p}) \\
&\quad = s\left(\sum_{k=1}^{n} \lambda_k^L g_{\bar{p}}\left(\text{grad } f_k^L(\bar{p}), X_{\bar{p}}\right) + \sum_{k=1}^{n} \lambda_k^U g_{\bar{p}}\left(\text{grad } f_k^U(\bar{p}), X_{\bar{p}}\right)\right) \\
&\qquad + \frac{s^2}{2!}\left(\sum_{k=1}^{n} \lambda_k^L g_{\bar{p}}\left(\text{Hess } f_k^L(\bar{p})[X_{\bar{p}}], X_{\bar{p}}\right) + \sum_{k=1}^{n} \lambda_k^U g_{\bar{p}}\left(\text{Hess } f_k^U(\bar{p})[X_{\bar{p}}], X_{\bar{p}}\right)\right) \\
&\qquad + \frac{s^2}{2!}\left(\sum_{k=1}^{n} \lambda_k^L g_{\bar{p}}\left(\text{grad } f_k^L(\bar{p}), Z_{\bar{p}}\right) + \sum_{k=1}^{n} \lambda_k^U g_{\bar{p}}\left(\text{grad } f_k^U(\bar{p}), Z_{\bar{p}}\right)\right) + \sum_{k=1}^{n} \lambda_k^L O_k^L(s^3) + \sum_{k=1}^{n} \lambda_k^U O_k^U(s^3). \qquad (38)
\end{aligned}$$

For sufficiently small, $s > 0$ using inequality (32), equation(38) yields

$$\begin{aligned}
0 &\leqq s\left(\sum_{k=1}^{n} \lambda_k^L g_{\bar{p}}\left(\text{grad } f_k^L(\bar{p}), X_{\bar{p}}\right) + \sum_{k=1}^{n} \lambda_k^U g_{\bar{p}}\left(\text{grad } f_k^U(\bar{p}), X_{\bar{p}}\right)\right) \\
&\quad + \frac{s^2}{2!}\left(\sum_{k=1}^{n} \lambda_k^L g_{\bar{p}}\left(\text{Hess } f_k^L(\bar{p})[X_{\bar{p}}], X_{\bar{p}}\right) + \sum_{k=1}^{n} \lambda_k^U g_{\bar{p}}\left(\text{Hess } f_k^U(\bar{p})[X_{\bar{p}}], X_{\bar{p}}\right)\right) \\
&\quad + \frac{s^2}{2!}\left(\sum_{k=1}^{n} \lambda_k^L g_{\bar{p}}\left(\text{grad } f_k^L(\bar{p}), Z_{\bar{p}}\right) + \sum_{k=1}^{n} \lambda_k^U g_{\bar{p}}\left(\text{grad } f_k^U(\bar{p}), Z_{\bar{p}}\right)\right) + \sum_{k=1}^{n} \lambda_k^L O_k^L(s^3) + \sum_{k=1}^{n} \lambda_k^U O_k^U(s^3). \qquad (39)
\end{aligned}$$

Using (31) in (39), we have

$$0 \leqq \sum_{k=1}^{n} \lambda_k^L g_{\bar{p}}\left(\text{Hess } f_k^L(\bar{p})[X_{\bar{p}}], X_{\bar{p}}\right) + \sum_{k=1}^{n} \lambda_k^U g_{\bar{p}}\left(\text{Hess } f_k^U(\bar{p})[X_{\bar{p}}], X_{\bar{p}}\right)$$

$$+\sum_{k=1}^{n}\lambda_k^L g_{\bar{p}}\big(\text{grad } f_k^L(\bar{p}), Z_{\bar{p}}\big)+\sum_{k=1}^{n}\lambda_k^U g_{\bar{p}}\big(\text{grad } f_k^U(\bar{p}), Z_{\bar{p}}\big)+\sum_{k=1}^{n}\lambda_k^L O_k^L(s)+\sum_{k=1}^{n}\lambda_k^U O_k^U(s). \tag{40}$$

From (27), (28) and (40), we have for sufficiently small $s > 0$

$$\begin{aligned}0 \leqq &\sum_{k=1}^{n}\lambda_k^L g_{\bar{p}}\big(\text{Hess } f_k^L(\bar{p})[X_{\bar{p}}], X_{\bar{p}}\big)+\sum_{k=1}^{n}\lambda_k^U g_{\bar{p}}\big(\text{Hess } f_k^U(\bar{p})[X_{\bar{p}}], X_{\bar{p}}\big)\\ &+\sum_{i=1}^{m}\mu_i g_{\bar{p}}(\text{Hess } c_i(\bar{p})[X_{\bar{p}}], X_{\bar{p}})+\sum_{j=1}^{r}\vartheta_j g_{\bar{p}}(\text{Hess } d_j(\bar{p})[X_{\bar{p}}], X_{\bar{p}})\\ &+\sum_{k=1}^{n}\lambda_k^L g_{\bar{p}}\big(\text{grad } f_k^L(\bar{p}), Z_{\bar{p}}\big)+\sum_{k=1}^{n}\lambda_k^U g_{\bar{p}}\big(\text{grad } f_k^U(\bar{p}), Z_{\bar{p}}\big)\\ &+\sum_{i=1}^{m}\mu_i g_{\bar{p}}(\text{grad } c_i(\bar{p}), Z_{\bar{p}})+\sum_{j=1}^{r}\vartheta_j g_{\bar{p}}(\text{grad } d_j(\bar{p}), Z_{\bar{p}})+\sum_{k=1}^{n}\lambda_k^L O_k^L(s)+\sum_{k=1}^{n}\lambda_k^U O_k^U(s).\end{aligned} \tag{41}$$

By utilizing the linearity of the metric $g_{\bar{p}}$ and the condition (i) of this theorem, the inequality (41) yields

$$\begin{aligned}0 \leqq &\sum_{k=1}^{n}\lambda_k^L g_{\bar{p}}\big(\text{Hess } f_k^L(\bar{p})[X_{\bar{p}}], X_{\bar{p}}\big)+\sum_{k=1}^{n}\lambda_k^U g_{\bar{p}}\big(\text{Hess } f_k^U(\bar{p})[X_{\bar{p}}], X_{\bar{p}}\big)\\ &+\sum_{i=1}^{m}\mu_i g_{\bar{p}}(\text{Hess } c_i(\bar{p})[X_{\bar{p}}], X_{\bar{p}})+\sum_{j=1}^{r}\vartheta_j g_{\bar{p}}(\text{Hess } d_j(\bar{p})[X_{\bar{p}}], X_{\bar{p}})+\sum_{k=1}^{n}\lambda_k^L O_k^L(s)+\sum_{k=1}^{n}\lambda_k^U O_k^U(s).\end{aligned} \tag{42}$$

Since, $O_k^L(s) \to 0$ and $O_k^U(s) \to 0, k \in K$, as $s \to 0$, therefore (42) yields the condition (iv) of this theorem. This completes the proof. □

Now we present the second-order necessary conditions for the problem (P) that involve the concept of LICQ which we have presented in Definition 3.4.

We assume that the functions comprising the problem (P) are twice differentiable. Suppose $\bar{p}$ is a local weak LU-Pareto optimal solution of problem (P) and $\mu \in \mathbb{R}^m$ and $\vartheta \in \mathbb{R}^r$ are some Lagrange multipliers that satisfy the necessary conditions (i)–(iii), of Theorem 3.3. Then we define the *Critical cone* $\mathfrak{C}(\bar{p}, \mu, \vartheta)$ associated with $(\bar{p}, \mu, \vartheta)$ as follows

$$\mathfrak{C}(\bar{p}, \mu, \vartheta) = \big\{X_{\bar{p}} \in \mathbb{Y}^1(\bar{p}) : g_{\bar{p}}(\text{grad } c_i(\bar{p})\, X_{\bar{p}}) = 0, \quad \forall i \in I(\bar{p}) \text{ with } \mu_i > 0\big\}.$$

Equivalently, one can easily verify the following

$$X_{\bar{p}} \in \mathfrak{C}(\bar{p}, \mu, \vartheta) \Leftrightarrow \begin{cases} X_{\bar{p}} \in T_{\bar{p}}(\mathsf{M}); \\ g_{\bar{p}}(\text{grad } c_i(\bar{p})\, X_{\bar{p}}) \leqq 0, \quad \forall i \in I(\bar{p}) \text{ with } \mu_i = 0; \\ g_{\bar{p}}(\text{grad } c_i(\bar{p})\, X_{\bar{p}}) = 0, \quad \forall i \in I(\bar{p}) \text{ with } \mu_i > 0; \\ g_{\bar{p}}(\text{grad } d_j(\bar{p})\, X_{\bar{p}}) = 0, \quad \forall j \in J. \end{cases}$$

The next theorem presents the necessary second order conditions for the problem (P^*) which follows directly from Theorem 4.4, Remark 4.6.1 and Theorem 4.7 established by Yang *et al.* [40].

Theorem 4.2. *Let $\bar{p} \in \mathcal{X}$ be a local solution to problem (P^*) and that the LICQ holds at $\bar{p}$. Let $\mu \in \mathbb{R}^m$ and $\vartheta \in \mathbb{R}^r$ be the Lagrange multiplier vectors for which the conditions (i)–(iii) of Theorem 3.2 are satisfied. Then*

$$g_{\bar{p}}(\text{Hess } f(\bar{p})[X_{\bar{p}}], X_{\bar{p}}) + \sum_{i=1}^{m}\mu_i g_{\bar{p}}(\text{Hess } c_i(\bar{p})[X_{\bar{p}}], X_{\bar{p}})$$

$$+\sum_{j=1}^{r}\vartheta_j g_{\bar{p}}(\text{Hess } d_j(\bar{p})[X_{\bar{p}}], X_{\bar{p}}) \geqq 0, \qquad \forall X_{\bar{p}} \in \mathfrak{C}(\bar{p}, \mu, \vartheta).$$

Based on Theorem 4.2, we now present the second order necessary conditions for the problem (P) in which the objective functions satisfy the LICQ.

Theorem 4.3 (Second order necessary conditions)**.** *Assume that $\bar{p} \in \mathcal{X}$ is a local weak LU-Pareto optimal solution of the problem (P), and the LICQ condition is satisfied at $\bar{p}$. Let $\mu \in \mathbb{R}^m$ and $\vartheta \in \mathbb{R}^r$ be the Lagrange multiplier vectors for which the conditions (i)–(iii) of Theorem 3.3 are satisfied. Then*

$$\sum_{k=1}^{n} g_{\bar{p}}\big(\text{Hess } f_k^L(\bar{p})[X_{\bar{p}}], X_{\bar{p}}\big) + \sum_{k=1}^{n} g_{\bar{p}}\big(\text{Hess } f_k^U(\bar{p})[X_{\bar{p}}], X_{\bar{p}}\big)$$
$$+\sum_{i=1}^{m}\mu_i g_{\bar{p}}(\text{Hess } c_i(\bar{p})[X_{\bar{p}}], X_{\bar{p}}) + \sum_{j=1}^{r}\vartheta_j g_{\bar{p}}(\text{Hess } d_j(\bar{p})[X_{\bar{p}}], X_{\bar{p}}) \geqq 0, \qquad \forall X_{\bar{p}} \in \mathfrak{C}(\bar{p}, \mu, \vartheta). \tag{43}$$

Proof. We define a real-valued function $\bar{f}$ as follows:

$$\bar{f}(p) = \sum_{k=1}^{n} f_k^L(p) + \sum_{k=1}^{n} f_k^U(p). \tag{44}$$

Since, $\bar{p}$ is a local weak LU-Pareto optimal solution to (P), we have from Lemma 3.1 with $\lambda_k^L = \lambda_k^U = 1, k \in K$ that $\bar{p}$ is a local minimizer to the following problem

$$\text{minimize} \quad \bar{f}(p) = \sum_{k=1}^{n} f_k^L(p) + \sum_{k=1}^{n} f_k^U(p) \tag{P$_1^*$}$$
$$\text{subject to} \quad p \in \mathcal{X}.$$

By hypothesis of the theorem, all the conditions of Theorem 4.2 are satisfied for the above problem (P$_1^*$). Therefore, by Theorem 4.2 we conclude that

$$g_{\bar{p}}\big(\text{Hess } \bar{f}(\bar{p})[X_{\bar{p}}], X_{\bar{p}}\big) + \sum_{i=1}^{m}\mu_i g_{\bar{p}}(\text{Hess } c_i(\bar{p})[X_{\bar{p}}], X_{\bar{p}})$$
$$+\sum_{j=1}^{r}\vartheta_j g_{\bar{p}}(\text{Hess } d_j(\bar{p})[X_{\bar{p}}], X_{\bar{p}}) \geqq 0, \qquad \forall X_{\bar{p}} \in \mathfrak{C}(\bar{p}, \mu, \vartheta). \tag{45}$$

By the linearity of vector field Hess f, equation (44) and inequality (45), yield the 2nd order condition (43). □

Now, Define the Lagrange function $L : \mathsf{M} \times \mathbb{R}^{2n} \times \mathbb{R}^m \times \mathbb{R}^r \to \mathbb{R}$ for the problem (P) by

$$L(p, \lambda, \mu, \vartheta) = \sum_{k=1}^{n}\lambda_k^L f_k^L(p) + \sum_{k=1}^{n}\lambda_k^U f_k^U(p) + \sum_{i=1}^{m}\mu_i c_i(p) + \sum_{j=1}^{r}\vartheta_j d_j(p).$$

Utilizing the Lagrange function as defined above, the second-order conditions formulated in Theorem 4.1 can be expressed in the dual form, as presented in the following theorem.

Theorem 4.4 (Second order conditions in dual form)**.** *Consider $\bar{p} \in \mathcal{X}$ as a local weak LU-Pareto optimal solution for problem (P). Assuming that the objective functions $f_k, k \in K$ are twice weakly differentiable at $\bar{p}$, as well as the constraints $c_i, i \in I$ and $d_j, j \in J$, are twice differentiable at $\bar{p}$. Furthermore, assume that the (FOKTCQ) condition is met at $\bar{p}$. Let $\lambda^L, \lambda^U \in \mathbb{R}^n, \mu \in \mathbb{R}^m$ and $\vartheta \in \mathbb{R}^r$ be the Lagrange multiplier vectors such that*

(i) $\text{grad L}(\bar{\text{p}}, \lambda, \mu, \vartheta) = 0$;
(ii) $\mu_i c_i(\bar{p}) = 0,\ i \in I$;
(iii) $\left(\lambda^L, \lambda^U\right) \geq 0,\ \mu \geqq 0$.

Further, suppose that the (SOMCCQ) holds at $\bar{p}$. Then

(iv) $g_{\bar{p}}(\text{Hess L}(\bar{p}, \lambda, \mu, \vartheta)[X_{\bar{p}}], X_{\bar{p}}) \geqq 0, \quad \forall X_{\bar{p}} \in \mathbb{Y}^2(\bar{p})$.

Remark 4.1. Considering the dual form established in Theorem 4.4, the dual forms of the other results described in the preceding sections can be derived straightforwardly. We omit the details here.

We now illustrate the second order necessary conditions established in Theorems 4.1 and 4.3 with the help of examples. At first, we illustrate the necessary conditions formulated in Theorem 4.1 (involving FOKTCQ and SOMCCQ conditions) in the following example.

Example 4.1. Let $\mathsf{M} = \{(p_1, p_2) \in \mathbb{R}^2 : p_1, p_2 > 0\}$ be a Riemannian manifold with Riemannian metric given by $(g_{ij})_p = \left(\frac{\delta_{ij}}{p_i p_j}\right)$. The geodesic $\gamma : \mathbb{R} \to \mathsf{M}$ emanating from $\gamma(0) = p$ in the direction $\dot{\gamma}(0) = v \in \mathbb{R}^2$ is given by

$$\gamma(s) = \left(p_1 e^{\frac{v_1}{p_1}s}, p_2 e^{\frac{v_2}{p_2}s}\right).$$

Consider an optimization problem defined on M as outlined below:

$$\begin{aligned} \text{minimize} \quad & f(p) = (f_1(p), f_2(p)); \\ \text{where} \quad & f_1(p) = \left[\frac{p_1}{p_2}, \frac{p_1}{p_2} + 1\right], \\ \text{and} \quad & f_2(p) = \left[e^{(\ln p_1)^2 + (\ln(\frac{p_2}{2}))^2}, e^{(\ln p_1)^2 + (\ln(\frac{p_2}{2}))^2} + 1\right]; \\ \text{subject to} \quad & c(p) = \frac{p_2}{p_1} - 2 \ \leqq 0; \\ & d(p) = (\ln 2p_1)^2 - (\ln p_2)^2 = 0. \end{aligned} \tag{MIVP 1}$$

Note that the feasible region is $\mathcal{X} = \{(p_1, p_2) \in \mathsf{M} : p_2 = 2p_1\}$, and by Definition 3.1, $\bar{p} = (1, 2)$ is a local weak LU-Pareto optimal solution of (MIVP 1). At first, we show first order necessary conditions are satisfied at $\bar{p} = (1, 2)$ under FOKTCQ condition. By simple calculations, we have

$$\mathbb{Y}^1(\bar{p}) = \left\{(v_1, v_2) \in \mathbb{R}^2 : v_2 = 2v_1\right\}, \bar{p} = (1, 2).$$

For any non-zero $X_{\bar{p}} = (v, 2v) \in \mathbb{Y}^1(\bar{p})$, we define $h : [0, 1] \to \mathsf{M}$ by

$$h(s) = (e^{vs}, 2e^{vs}). \tag{46}$$

It is now easy to verify that the FOKTCQ condition is fulfilled at $\bar{p} = (1, 2)$ with $\beta = 1$. Furthermore, the gradient of the functions constituting (MIVP 1) at $\bar{p} = (1, 2)$ is given as follows:

- grad $f_1^L(\bar{p}) = \text{grad } f_1^U(\bar{p}) = (\frac{1}{2}, -1)$;
- grad $f_2^L(\bar{p}) = \text{grad } f_2^U(\bar{p}) = (0, 0)$;
- grad $c(\bar{p}) = (-2, 4)$;
- grad $d(\bar{p}) = (\ln 4, -\ln 16)$.

It is now easy to verify that the necessary conditions (i)–(iii) of Theorem 4.1 are satisfied at $\bar{p} = (1, 2)$ with Lagrange multipliers $(\lambda_1^L, \lambda_1^U) = (2, 2), (\lambda_2^L, \lambda_2^U) = (1, 1), \mu = 1,\ \&\ \vartheta = 0$.

Next, we show that the necessary second order conditions for (MIVP 1) are also satisfied at $\bar{p} = (1, 2)$. At first, note that $\mathbb{Y}^2(\bar{p}) = \mathbb{Y}^1(\bar{p}), \bar{p} = (1, 2)$. For each $X_{\bar{p}} = (p, 2p) \in \mathbb{Y}^2(\bar{p}), p \neq 0$, we consider the same twice differentiable function $h : [0, 1] \to \mathsf{M}$ as defined by equation (46). It is now easy to verify that the (SOMCCQ) is fulfilled at $\bar{p} = (1, 2)$. Furthermore, for any $X_{\bar{p}} = (p, 2p) \in \mathbb{Y}^2(\bar{p}), p \neq 0, \bar{p} = (1, 2)$, we have the following:

– $g_{\bar{p}}\big(\text{Hess } f_1^L(\bar{p})[X_{\bar{p}}], X_{\bar{p}}\big) = g_{\bar{p}}\big(\text{Hess } f_1^U(\bar{p})[X_{\bar{p}}], X_{\bar{p}}\big) = 0$;
– $g_{\bar{p}}\big(\text{Hess } f_2^L(\bar{p})[X_{\bar{p}}], X_{\bar{p}}\big) = g_{\bar{p}}\big(\text{Hess } f_2^U(\bar{p})[X_{\bar{p}}], X_{\bar{p}}\big) = 4p^2$;
– $g_{\bar{p}}(\text{Hess } c(\bar{p})[X_{\bar{p}}], X_{\bar{p}}) = g_{\bar{p}}(\text{Hess } d(\bar{p})[X_{\bar{p}}], X_{\bar{p}}) = 0$;

It is now easy to verify that the second order necessary condition (iv) of Theorem 4.1 is satisfied at $\bar{p} = (1, 2)$.

We now illustrate the necessary conditions formulated in Theorem 4.3 (involving LICQ condition) in the following example.

Example 4.2. Let $\mathsf{M} = S^2_{++}$ which is the Riemannian manifold of 2×2 symmetric positive definite matrices with real entries and Riemannian metric defined by

$$g_P(A, B) = \text{Trace}\left(P^{-1}AP^{-1}B\right),\ \forall P \in S^2_{++}, A, B \in T_P\left(S^2_{++}\right).$$

The geodesic emanating from $P \in S^2_{++}$ in any direction $V \in T_P(S^2_{++})$ is given by

$$\gamma(t) = P^{\frac{1}{2}} \exp\Big(tP^{-\frac{1}{2}}VP^{-\frac{1}{2}}\Big)P^{\frac{1}{2}}.$$

Let $\bar{P} = I$, a 2 x 2 identity matrix, then the geodesic emanating from $\bar{P} = I$ in any direction $V \in T_I(S^2_{++})$ is given by

$$\gamma(t) = \exp(tV).$$

Consider the following interval-valued optimization problem which is defined on S^2_{++}.

$$\begin{aligned}
\text{minimize}\quad & f(\mathrm{P}) = (f_1(\mathrm{P}), f_2(\mathrm{P}));\\
\text{where}\quad & f_1(\mathrm{P}) = [\ln\det P,\ \ln\det P + 1];\\
\text{and}\quad & f_2(\mathrm{P}) = [(\ln\det P)^2,\ (\ln\det P)^2 + 1];\\
\text{subject to}\quad & c(\mathrm{P}) = -\ln\det P \leqq 0;\\
& d(\mathrm{P}) = p_{11}^2 + 2p_{22}^2 - 3p_{11}p_{22} = 0;\\
\text{where}\quad & P = \begin{bmatrix} p_{11} & p_{12}\\ p_{21} & p_{22}\end{bmatrix},\ p_{12} = p_{21}.
\end{aligned} \tag{MIVP 2}$$

Note that the feasible region is $\mathcal{X} = \{P \in S^2_{++} : \det P \geqq 1 \text{ and } (q_{11} = q_{22} \text{ or } q_{11} = 2q_{22})\}$ and by Definition 2.2, $\bar{P} = I$ is a local weak LU-Pareto optimal solution of (MIVP 2). The gradient of the functions involved in (MIVP 2) at $\bar{P} = I$ is given by

– grad $f_1^L(I) =$ grad $f_1^U(I) = I$.
– grad $f_2^L(I) =$ grad $f_2^U(I) = 0$.
– grad $c(I) = -I$.
– grad $d(I) = \begin{bmatrix} -1 & 0\\ 0 & 1\end{bmatrix}$.

The set $\{\text{grad } c(I), \text{grad } d(I)\}$ is clearly linearly independent on $T_I(S^2_{++})$. Thus (LICQ) condition is fulfilled at I. Furthermore, the conditions (i)–(iii) of Theorem 3.3 are satisfied with Lagrange multipliers $\mu = 2$ and $\vartheta = 0$.

Now the critical cone associated with $(\bar{p}, \mu, \vartheta) = (I, 2, 0)$ is defined by

$$\mathfrak{C}(\bar{P}, \mu, \vartheta) = \Big\{V \in T_{\bar{P}}(\mathsf{M}) : g_{\bar{P}}(\text{grad } c(\bar{P}), V) = 0, g_{\bar{P}}(\text{grad } d(\bar{P}), V) = 0\Big\}.$$

By simple calculations, we get

$$\mathfrak{C}(\bar{P}, \mu, \vartheta) = \left\{\begin{bmatrix} 0 & v\\ v & 0\end{bmatrix} : v \in \mathbb{R}\right\}.$$

Moreover, for any $V \in T_{\bar{P}}(S^2_{++})$, we can easily calculate the following

– $g_{\bar{P}}\big(\text{Hess } f_1^L(\bar{P})[V], V\big) = g_{\bar{P}}\big(\text{Hess } f_1^U(\bar{P})[V], V\big) = 0;$
– $g_{\bar{P}}\big(\text{Hess } f_2^L(\bar{P})[V], V\big) = g_{\bar{P}}\big(\text{Hess } f_2^U(\bar{P})[V], V\big) = 2(\text{Trace } (V))^2;$
– $g_{\bar{P}}\big(\text{Hess } c(\bar{P})[V], V\big) = 0;$
– $g_{\bar{P}}\big(\text{Hess } d(\bar{P})[V], V\big) = v_{11}^2 + 5v_{22}^2 - 6v_{11}v_{22}.$

It is now easy to verify, for any $V \in \mathfrak{C}(\bar{P}, \mu, \vartheta)$, the second order necessary condition (43) of Theorem 4.3 holds true.

5. Conclusion

In this paper, we have established first and second order necessary optimality conditions for a multi-objective optimization problem with interval-valued objective functions defined on Riemannian manifolds. The interval-valued objective functions are presumed to be weakly differentiable and twice weakly differentiable for the development of first order and second order necessary conditions, respectively. We have proved two versions of first order necessary conditions under two types of constraint qualifications – FOKTCQ and LICQ. Furthermore, we have also established two versions of second order necessary conditions under two types of constraint qualifications – SOMCCQ and LICQ. We have considered both the equality and inequality constraints which are real-valued functions, and are assumed to be differentiable for the development of first order necessary conditions and twice differentiable for the development of second order necessary conditions.

However, it would be of interest to investigate second order sufficient optimality conditions for other classes of interval-valued optimizations problems on Riemannian manifolds. Additional inquiries can be made to investigate the possibility of similar results for non-smooth IVOP on non-linear spaces. The KKT conditions possess a vast applicability in machine learning and artificial intelligence such as support vector machines, neural networks, linear regression, ridge regression, decision trees etc. One can further investigate similar results for other constraint qualifications. Since the results established in this paper are consider under weak differentiability, it remains of interest to investigate the similar results by invoking generalized Hukuhara differentiability.

A logical extension for the theory presented in this paper would be to investigate strict constraint qualifications and develop an augmented Lagrangian method to address problem (P). In [6], the authors apply augmented Lagrangian methods to solve non-convex optimization problems on Riemannian manifolds, adapting advancements from Euclidean nonlinear programming. They introduce weak constraint qualifications (CQs) to ensure the stationarity of limit points in the primal sequences generated by the algorithm and a stronger sequential optimality condition, positive-approximate-KKT (PAKKT), to ensure the boundedness of the dual sequence. Since the set of all closed and bounded intervals in the real numbers forms a metric space under the Hausdorff metric, it is feasible to work extensively with sequences in this metric space. This supports the extension of the augmented Lagrangian method to interval settings. The global convergence of these methods and the convergence of the dual sequence, given specific constraint qualifications, can be analyzed using the Hausdorff metric. The proposed method can be adapted to interval settings either through weak differentiability or by utilizing the generalized Hukuhara differentiability. For additional studies employing the sequential approach, refer to [3–5].

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