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SHARP BOUNDS ON THE GENERALIZED DISTANCE SPECTRAL RADIUS AND GENERALIZED DISTANCE ENERGY OF STRONGLY CONNECTED DIGRAPHS

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Abstract. Let $D(G)$ be the distance matrix of a strongly connected digraph G, $Tr(G)$ be the diagonal matrix with vertex transmissions of G as diagonal entries. The generalized distance matrix $D_\beta(G)$ of the strongly connected digraph G is defined as $D_\beta(G) = \beta Tr(G)+(1-\beta)D(G)$, for any real $0 \leq \beta \leq 1$. The generalized distance spectral radius of G is the spectral radius of $D_\beta(G)$. Let $\mu_1^\beta(G), \mu_2^\beta(G), \ldots, \mu_n^\beta(G)$ be the eigenvalues of $D_\beta(G)$, the generalized distance energy of the digraph G is $E_{D_\beta}(G) = \sum_{i=1}^{n} |\mu_i^\beta(G) - \frac{\beta W(G)}{n}|$, where $W(G)$ is the sum of distances between all ordered pairs of vertices of G. In this paper, we obtain some sharp upper and lower bounds for the generalized distance spectral radius of G and characterize the extremal digraphs. Moreover, we also give some lower bounds on the generalized distance energy of digraphs.

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1. Introduction

Let G be a digraph with vertex set $V(G)$ and arc set $E(G)$. The number of vertices (arcs) in digraph G is called the order (size) of G. For an arc (v_i, v_j) of G, we call v_i out-adjacent to v_j, and v_j in-adjacent to v_i; we also say v_i, v_j are the end vertices of this arc. A digraph G is called strongly connected if for any pair v_i, v_j of distinct vertices in G, there exists a directed path from v_i to v_j and a directed path from v_j to v_i. An empty digraph is one with no arcs. A simple digraph in which any pair v_i, v_j of distinct vertices of G is joined by symmetrical arcs (v_i, v_j) and (v_j, v_i) is called a complete digraph. Let G be a digraph with vertex set $V(G)$, T be a nonempty subset of $V(G)$, we use $G[T]$ denote the subdigraph of G induced by T.

For a vertex v_i of the digraph G, $N_{v_i}^+(G) = \{v_j : (v_i, v_j) \in E(G)\}$ is called the out-neighbourhood of v_i. The vertices in $N_{v_i}^+(G)$ are called the out-neighbours of v_i. The outdegree of vertex v_i is the number of out-neighbours of this vertex, denoted by $d_G^+(v_i)$, or simply write by d_i^+. The maximum outdegree and second maximum outdegree of G is denoted by Δ_1^+ and Δ_2^+, respectively. An outdegree regular digraph is digraph in which the outdegree of all vertices are equal.

Keywords. Strongly connected digraphs, generalized distance spectral radius, generalized distance energy, bounds.

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Let G be a strongly connected digraph, v_i and v_j be two vertices in G, the distance from vertex v_i to v_j in G, denoted by $d_{v_iv_j}(G)$ or simply by d_{ij}, is the length of the shortest directed path from v_i to v_j. The diameter of G is the maximum distance d_{ij} over all ordered pairs of vertices v_i, v_j. We use $Tr_G(v_i)$ or Tr_i denote the transmission of a vertex v_i, which is defined as $Tr_G(v_i) = Tr_i = \sum\limits_{j=1}^{n} d_{ij}$ $(i = 1, 2, \ldots, n)$. A transmission regular digraph is digraph in which the transmission of all vertices are equal.

For a strongly connected digraph G, the distance matrix $D(G)$ is a real matrix whose (i, j)-entry equal to d_{ij}. The distance signless Laplacian matrix of G is defined as $D^Q(G) = D(G) + Tr(G)$, where $Tr(G) = \mathrm{diag}(Tr_1, Tr_2, \ldots, Tr_n)$ is the diagonal matrix with vertex transmissions of G as diagonal entries.

Recently, Xi *et al.* [16] studied the generalized distance matrix $D_\beta(G)$ of G, which is defined as $D_\beta(G) = \beta Tr(G) + (1-\beta)D(G)$, where $0 \le \beta \le 1$. It is clear that for $\beta = 0$, $D(G) = D_0(G)$, and for $\beta = \frac{1}{2}$, $D^Q(G) = 2D_{\frac{1}{2}}(G)$. From this, it follows that the matrix $D_\beta(G)$ is a generalization of the matrices $D(G)$ and $D^Q(G)$. Therefore, it will be interesting to see the results which already done for the spectral radius of the matrices $D(G)$ and $D^Q(G)$ can be extended to the spectral radius of the matrix $D_\beta(G)$. The eigenvalues of $D_\beta(G)$ are called the generalized distance eigenvalues or D_β-eigenvalues of G, and they are denoted by $\mu_1^\beta(G), \mu_2^\beta(G), \ldots, \mu_n^\beta(G)$. We use $\mu_\beta(G)$ denote the generalized distance spectral radius or D_β-spectral radius of G, which is the spectral radius of $D_\beta(G)$. Since $D_\beta(G)$ is a nonnegative irreducible matrix, we have $\mu_\beta(G)$ is an eigenvalue of $D_\beta(G)$ $(0 \le \beta < 1)$, and there exists a positive unit eigenvector corresponding to $\mu_\beta(G)$, which is called the Perron vector of $D_\beta(G)$.

The investigation on the distance spectral radius, the distance signless Laplacian spectral radius and the generalized distance spectral radius of connected undirected graphs have received a lot of attention of researchers. For some recent results, one may see [1–4, 6, 9, 19] and the references therein. In recent years, researchers have begun to focus on the distance spectral radius and distance signless Laplacian spectral radius of digraphs. In 2013, Lin and Shu [8] determined the extremal digraphs with the minimum distance spectral radius among all strongly connected digraphs with given arc connectivity and dichromatic number, respectively. In 2021, Xi *et al.* [17] completely determine the digraphs has the minimum distance spectral radius among all strongly connected digraphs with given diameter. In 2017, Li *et al.* [7] gave a lower bound on the distance signless Laplacian spectral radius for digraphs with given vertex connectivity, and they also characterized the extremal digraphs. In 2019, Xi *et al.* [15] obtained a sharp lower bound on the distance signless Laplacian spectral radius for digraphs with given arc connectivity, and they also determined the extremal digraphs.

Recently, the research of the matrix $D_\beta(G)$ of strongly connected digraph G has not attracted much attention from the researchers, and there are only a few papers on the spectral radius of $D_\beta(G)$. For example, in [16], the authors determined the digraphs which minimizes the D_β-spectral radius with given dichromatic number, vertex connectivity and arc connectivity, respectively. In [13], the author provided some upper and lower bounds for the D_β-spectral radius of strongly connected digraphs. In this paper, we will first further investigate the generalized distance spectral radius of digraphs.

The energy of a digraph was introduced in 2008 by Peña and Rada [11]. Since then, the study of energy of digraphs has been extended to other matrices assigned to the digraphs. In 2017, Xi and Wang [14] gave the definition of the signless Laplacian energy of a digraph. For recent works on the energy, the Laplacian energy and the signless Laplacian energy of digraphs, we refer to [10, 12, 18]. Motivated by the above works, we will introduce an energy based on the generalized distance matrix $D_\beta(G)$.

The rest of this paper is organized as follow. In Section 2, we give some sharp upper and lower bounds on the generalized distance spectral radius of digraphs and we also characterize the extremal digraphs. In Section 3, we define the generalized distance energy of a strongly connected digraph G as $E_{D_\beta}(G) = \sum\limits_{i=1}^{n} |\mu_i^\beta(G) - \frac{\beta W(G)}{n}|$, where $W(G) = \sum\limits_{i=1}^{n} Tr_i$. We also obtain some lower bounds on the generalized distance energy of strongly connected digraphs.

2. Sharp bounds on the generalized distance spectral radius of strongly connected digraphs

Firstly, we give a useful lemma.

Lemma 2.1. *([5]) Let $B = (b_{ij})$ be an irreducible nonnegative square matrix of order n, $\rho(B)$ be its spectral radius, and $\psi_i(B)$ be the i-th row sum of B. Then*

$$\min\{\psi_i(B) : 1 \le i \le n\} \le \rho(B) \le \max\{\psi_i(B) : 1 \le i \le n\},$$

any equality holds if and only if $\psi_1(B) = \psi_2(B) = \cdots = \psi_n(B)$.

Now, we obtain a sharp upper bound on the generalized distance spectral radius of strongly connected digraphs.

Theorem 2.2. *Let $G = (V(G), E(G))$ be a strongly connected digraph with $n \ge 2$ vertices, $\{Tr_1, Tr_2, \ldots, Tr_n\}$ be its vertex transmission sequence with $Tr_1 \ge Tr_2 \ge \cdots \ge Tr_n$. Then, for $1 \le i \le n$,*

$$\mu_\beta(G) \le \frac{Tr_1 + \beta Tr_i - (1-\beta) + \sqrt{((1-\beta) + Tr_1 - \beta Tr_i)^2 - 4(1-\beta)(n-i+1)(Tr_1 - Tr_i)}}{2}. \tag{2.1}$$

If $Tr_1 = Tr_i$ or $i = 1$, the equality holds if and only if G is a transmission regular digraph. If $Tr_1 > Tr_i$ and $2 \le i \le n$, the equality holds if and only if G satisfies $Tr_1 = Tr_2 = \cdots = Tr_{i-1} > Tr_i = Tr_{i+1} = \cdots = Tr_n$, and $d_{lj} = 1$ for $1 \le l \le n$ and $i \le j \ne l \le n$.

Proof. If $Tr_1 = Tr_i$ or $i = 1$, then $\frac{Tr_1+\beta Tr_i-(1-\beta)+\sqrt{((1-\beta)+Tr_1-\beta Tr_i)^2-4(1-\beta)(n-i+1)(Tr_1-Tr_i)}}{2} = Tr_1$. Hence, base on Lemma 2.1, we get

$$\mu_\beta(G) \le Tr_1 = \frac{Tr_1 + \beta Tr_i - (1-\beta) + \sqrt{((1-\beta) + Tr_1 - \beta Tr_i)^2 - 4(1-\beta)(n-i+1)(Tr_1 - Tr_i)}}{2},$$

and the equality holds if and only if G is a transmission regular digraph.

Now we suppose $Tr_1 > Tr_i$ and $2 \le i \le n$. Let $V_1 = \{v_1, v_2, \ldots, v_{i-1}\}$ and $V_2 = V(G) \setminus V_1$. Then the matrix $D_\beta(G)$ of the digraph G may be partitioned as

$$D_\beta(G) = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix},$$

where M_{11} is a square matrix of order $i-1$. Let

$$P = \begin{pmatrix} \frac{1}{\alpha} I_{i-1} & 0 \\ 0 & I_{n-i+1} \end{pmatrix},$$

for $0 < \alpha < 1$ (to be determined later) and $D_\beta^* = P^{-1} D_\beta(G) P$, where I_t is the $t \times t$ identity matrix. Then

$$D_\beta^* = \begin{pmatrix} M_{11} & \alpha M_{12} \\ \frac{1}{\alpha} M_{21} & M_{22} \end{pmatrix},$$

is a nonnegative irreducible matrix, and its spectrum is the same as the spectrum of $D_\beta(G)$. In the following we will compute the row sums of D_β^*.

When $1 \le l \le i-1$,

$$\begin{aligned}\psi_l(D_\beta^*) &= \beta Tr_l + (1-\beta)\sum_{j=1}^{i-1} d_{lj} + (1-\beta)\sum_{j=i}^{n} \alpha d_{lj} \\ &= \beta Tr_l + (1-\beta)\sum_{j=1}^{n} d_{lj} + (1-\beta)\sum_{j=i}^{n} (\alpha-1) d_{lj} \\ &= \beta Tr_l + (1-\beta)Tr_l + (1-\beta)(\alpha-1)\sum_{j=i}^{n} d_{lj} \\ &\le Tr_l + (1-\beta)(\alpha-1)(n-i+1) \\ &\le Tr_1 + (1-\beta)(\alpha-1)(n-i+1),\end{aligned}$$

and the equality holds if and only if $Tr_l = Tr_1$ and $d_{lj} = 1$ for $1 \le l \le i-1$ with $i \le j \le n$.

When $i \le l \le n$,

$$\begin{aligned}\psi_l(D_\beta^*) &= \beta Tr_l + \frac{1}{\alpha}(1-\beta)\sum_{j=1}^{i-1} d_{lj} + (1-\beta)\sum_{j=i}^{n} d_{lj} \\ &= \beta Tr_l + \frac{1}{\alpha}(1-\beta)\sum_{j=1}^{n} d_{lj} + (1-\beta)\left(1-\frac{1}{\alpha}\right)\sum_{j=i}^{n} d_{lj} \\ &\le \beta Tr_l + \frac{1}{\alpha}(1-\beta)Tr_l + (1-\beta)\left(1-\frac{1}{\alpha}\right)(n-i) \\ &\le \beta Tr_i + \frac{1}{\alpha}(1-\beta)Tr_i + (1-\beta)\left(1-\frac{1}{\alpha}\right)(n-i),\end{aligned}$$

and the equality holds if and only if $Tr_l = Tr_i$ and $d_{lj} = 1$ for $i \le l \le n$, $i \le j \le n$ and $j \ne l$.

Now let

$$Tr_1 + (1-\beta)(\alpha-1)(n-i+1) = \beta Tr_i + \frac{1}{\alpha}(1-\beta)Tr_i + (1-\beta)\left(1-\frac{1}{\alpha}\right)(n-i),$$

that is

$$(1-\beta)(n-i+1)\alpha^2 - ((1-\beta)(2n-2i+1) - Tr_1 + \beta Tr_i)\alpha + (1-\beta)(n-i) - (1-\beta)Tr_i = 0.$$

Since $Tr_1 > Tr_i \ge n-1 > n-i$,

$$\begin{aligned}&((1-\beta)(2n-2i+1) - Tr_1 + \beta Tr_i)^2 + 4(1-\beta)(n-i+1)((1-\beta)Tr_i - (1-\beta)(n-i)) \\ &= ((1-\beta) + Tr_1 - \beta Tr_i)^2 - 4(1-\beta)(n-i+1)(Tr_1 - Tr_i) \\ &= ((1-\beta) + Tr_1 - \beta Tr_i)^2 - 4(1-\beta)(n-i+1)Tr_1 + 4(1-\beta)(n-i+1)Tr_i \\ &= ((1-\beta) + Tr_1 - \beta Tr_i)^2 - 4(1-\beta)(n-i+1)Tr_1 + 4(1-\beta)(\beta+1-\beta)(n-i+1)Tr_i \\ &= ((1-\beta) + Tr_1 - \beta Tr_i)^2 - 4(1-\beta)(n-i+1)Tr_1 \\ &\quad + 4(1-\beta)\beta(n-i+1)Tr_i + 4(1-\beta)^2(n-i+1)Tr_i \\ &> ((1-\beta) + Tr_1 - \beta Tr_i)^2 - 4(1-\beta)(n-i+1)Tr_1 \\ &\quad + 4(1-\beta)\beta(n-i+1)Tr_i + 4(1-\beta)^2(n-i+1)(n-i) \\ &= ((1-\beta + Tr_1 - \beta Tr_i) - 2(1-\beta)(n-i+1))^2 \ge 0.\end{aligned}$$

Therefore, we have

$$\alpha = \frac{(1-\beta)(2n-2i+1) - Tr_1 + \beta Tr_i + \sqrt{((1-\beta)+Tr_1-\beta Tr_i)^2 - 4(1-\beta)(n-i+1)(Tr_1-Tr_i)}}{2(1-\beta)(n-i+1)}.$$

Moreover,

$$\alpha > \frac{(1-\beta)(2n-2i+1) - Tr_1 + \beta Tr_i + (1-\beta) + Tr_1 - \beta Tr_i - 2(1-\beta)(n-i+1)}{2(1-\beta)(n-i+1)} = 0,$$

$$\alpha < \frac{(1-\beta)(2n-2i+1) - Tr_1 + \beta Tr_i + (1-\beta) + Tr_1 - \beta Tr_i}{2(1-\beta)(n-i+1)} = 1.$$

Hence $0 < \alpha < 1$. Therefore

$$\begin{aligned} &Tr_1 + (1-\beta)(\alpha-1)(n-i+1) \\ &= \frac{Tr_1 + \beta Tr_i - (1-\beta) + \sqrt{((1-\beta)+Tr_1-\beta Tr_i)^2 - 4(1-\beta)(n-i+1)(Tr_1-Tr_i)}}{2}. \end{aligned}$$

Thus

$$\begin{aligned} &\max\{\psi_1(D^*_\beta), \psi_2(D^*_\beta), \ldots, \psi_n(D^*_\beta)\} \\ &\le \frac{Tr_1 + \beta Tr_i - (1-\beta) + \sqrt{((1-\beta)+Tr_1-\beta Tr_i)^2 - 4(1-\beta)(n-i+1)(Tr_1-Tr_i)}}{2}. \end{aligned}$$

Thus, base on Lemma 2.1, we have

$$\begin{aligned} \mu_\beta(G) &\le \max_{1\le l\le n}\{\psi_l(D^*_\beta)\} \\ &\le \frac{Tr_1 + \beta Tr_i - (1-\beta) + \sqrt{((1-\beta)+Tr_1-\beta Tr_i)^2 - 4(1-\beta)(n-i+1)(Tr_1-Tr_i)}}{2}. \end{aligned}$$

Suppose that the equality holds, that is

$$\mu_\beta(G) = \frac{Tr_1 + \beta Tr_i - (1-\beta) + \sqrt{((1-\beta)+Tr_1-\beta Tr_i)^2 - 4(1-\beta)(n-i+1)(Tr_1-Tr_i)}}{2},$$

therefore all the above inequalities must be equalities. Hence,

$$\psi_1(D^*_\beta) = \cdots = \varphi_n(D^*_\beta) = Tr_1 + (1-\beta)(\alpha-1)(n-i+1) = \beta Tr_i + \frac{1}{\alpha}(1-\beta)Tr_i + (1-\beta)\left(1-\frac{1}{\alpha}\right)(n-i).$$

Thus $Tr_1 = Tr_2 = \cdots = Tr_{i-1} > Tr_i = Tr_{i+1} = \cdots = Tr_n$, $d_{lj} = 1$ for $1 \le l \le i-1$ with $i \le j \le n$, and $d_{lj} = 1$ for $i \le l \le n$, $i \le j \ne l \le n$, that is G satisfies $Tr_1 = Tr_2 = \cdots = Tr_{i-1} > Tr_i = Tr_{i+1} = \cdots = Tr_n$, and $d_{lj} = 1$ for $1 \le l \le n$ and $i \le j \ne l \le n$.

For converse, if G satisfies $Tr_1 = Tr_2 = \cdots = Tr_{i-1} > Tr_i = Tr_{i+1} = \cdots = Tr_n$, and $d_{lj} = 1$ for $1 \le l \le n$ and $i \le j \ne l \le n$, then base on the above proof , we get

$$\begin{aligned} &\psi_1(D^*_\beta) = \psi_2(D^*_\beta = \cdots = \psi_n(D^*_\beta) \\ &= \frac{Tr_1 + \beta Tr_i - (1-\beta) + \sqrt{((1-\beta)+Tr_1-\beta Tr_i)^2 - 4(1-\beta)(n-i+1)(Tr_1-Tr_i)}}{2}, \end{aligned}$$

and thus the equality in (2.1) holds. □

In the following, we give an example to illustrate that the digraphs in Theorem 2.2 exist. Let G_1 be a digraph of order n with $V(G_1) = V_1 \cup V_2$, where $V_1 = \{v_1, v_2, \ldots, v_{i-1}\}$ $(3 \le i \le n)$ and $V_2 = V(G_1) \setminus V_1$, such that

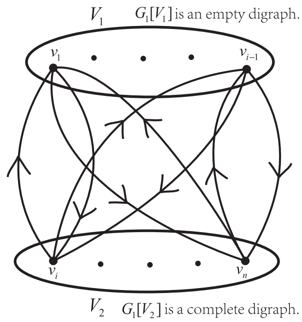

FIGURE 1. The digraph G_1.

$G_1[V_1]$ is an empty digraph, that is any two distinct vertices in V_1 are not adjacent, $G_1[V_2]$ is a complete digraph, and all possible arcs between V_1 and V_2 exist, as shown in Figure 1.

Then $Tr_1 = Tr_2 = \cdots = Tr_{i-1} = 2(i-2) + n - i + 1 = n + i - 3$, $Tr_i = \cdots = Tr_n = n - 1$. Let $y = (y_1, y_2, \ldots, y_n)^T$ be the Perron vector of $D_\beta(G_1)$ with respect to $\mu_\beta(G_1)$, where y_i corresponding to the vertex v_i. Obviously, the entries of y corresponding to vertices in V_1 have the same value, say y_1, the entries of y corresponding to vertices in V_2 have the same value, say y_2. From $D_\beta(G_1)y = \mu_\beta(G_1)y$, we have

$$\mu_\beta(G_1)y_1 = \beta(n+i-3)y_1 + (1-\beta)2(i-2)y_1 + (1-\beta)(n-i+1)y_2,$$

$$\mu_\beta(G_1)y_2 = \beta(n-1)y_2 + (1-\beta)(i-1)y_1 + (1-\beta)(n-i)y_2.$$

Hence

$$\begin{aligned}\mu_\beta(G_1) &= \frac{n+i-4+\beta n + \sqrt{(1-\beta)^2 n - 2(1-\beta)(i-2)n + (5-4\beta)i^2 + (12\beta-16)i - 8\beta + 12}}{2} \\ &= \frac{Tr_1 + \beta Tr_i - (1-\beta) + \sqrt{((1-\beta) + Tr_1 - \beta Tr_i)^2 - 4(1-\beta)(n-i+1)(Tr_1 - Tr_i)}}{2}.\end{aligned}$$

Next we give a sharp lower bound on the generalized distance spectral radius of strongly connected digraphs. The proof of the follow theorem is similar to the the proof of Theorem 2.2, but for completeness, we also give its proof.

Theorem 2.3. *Let $G = (V(G), E(G))$ be a strongly connected digraph with $n \geq 2$ vertices, $\{Tr_1, Tr_2, \ldots, Tr_n\}$ be its vertex transmission sequence with $Tr_1 \geq Tr_2 \geq \cdots \geq Tr_n$. Then, for $1 \leq l \leq n$,*

$$\mu_\beta(G) \geq \frac{Tr_n + \beta Tr_l - (1-\beta) + \sqrt{((1-\beta) + Tr_n - \beta Tr_l)^2 + 4(1-\beta)l(Tr_l - Tr_n)}}{2}. \tag{2.2}$$

If $Tr_l = Tr_n$ or $l = n$, the equality holds if and only if G is a transmission regular digraph. If $Tr_l > Tr_n$ and $1 \leq l \leq n-1$, the equality holds if and only if G satisfies $Tr_1 = Tr_2 = \cdots = Tr_l > Tr_{l+1} = \cdots = Tr_n$, and $d_{ij} = 1$ for $1 \leq i \leq n$ and $1 \leq j \neq i \leq l$.

Proof. If $Tr_l = Tr_n$ or $l = n$, then $\frac{Tr_n + \beta Tr_l - (1-\beta) + \sqrt{((1-\beta)+Tr_n-\beta Tr_l)^2 + 4(1-\beta)l(Tr_l - Tr_n)}}{2} = Tr_n$. Hence, base on Lemma 2.1, we get

$$\mu_\beta(G) \geq Tr_n = \frac{Tr_n + \beta Tr_l - (1-\beta) + \sqrt{((1-\beta) + Tr_n - \beta Tr_l)^2 + 4(1-\beta)l(Tr_l - Tr_n)}}{2},$$

and the equality holds if and only if G is a transmission regular digraph.

Now we suppose $Tr_l > Tr_n$ and $1 \le l \le n-1$. Let

$$U = \begin{pmatrix} \gamma I_l & 0 \\ 0 & I_{n-l} \end{pmatrix},$$

where $\gamma > 1$ (to be determined later) and I_t is the $t \times t$ identity matrix.

Let $B = U^{-1} D_\beta(G) U$. Then B is a nonnegative irreducible matrix, and its spectrum is the same as the spectrum of $D_\beta(G)$. In the following, we will compute the row sums of B.

For $1 \le i \le l$,

$$\begin{aligned} \psi_i(B) &= \beta Tr_i + (1-\beta)\sum_{j=1}^{l} d_{ij} + (1-\beta)\frac{1}{\gamma}\sum_{j=l+1}^{n} d_{ij} \\ &= \beta Tr_i + (1-\beta)\left(1-\frac{1}{\gamma}\right)\sum_{j=1}^{l} d_{ij} + (1-\beta)\frac{1}{\gamma}\sum_{j=1}^{n} d_{ij} \\ &= \beta Tr_i + (1-\beta)\left(1-\frac{1}{\gamma}\right)\sum_{j=1}^{l} d_{ij} + (1-\beta)\frac{1}{\gamma} Tr_i \\ &\ge \beta Tr_l + (1-\beta)\left(1-\frac{1}{\gamma}\right)(l-1) + (1-\beta)\frac{1}{\gamma} Tr_l, \end{aligned}$$

and the equality holds if and only if $Tr_i = Tr_l$ and $d_{ij} = 1$ for $1 \le i \le l$ with $1 \le j \ne i \le l$.

For $l+1 \le i \le n$,

$$\begin{aligned} \psi_i(B) &= \beta Tr_i + (1-\beta)\gamma\sum_{j=1}^{l} d_{ij} + (1-\beta)\sum_{j=l+1}^{n} d_{ij} \\ &= \beta Tr_i + (1-\beta)(\gamma-1)\sum_{j=1}^{l} d_{ij} + (1-\beta)\sum_{j=1}^{n} d_{ij} \\ &\ge \beta Tr_i + (1-\beta)Tr_i + (1-\beta)(\gamma-1)l \\ &\ge Tr_n + (1-\beta)(\gamma-1)l, \end{aligned}$$

and the equality holds if and only if $Tr_i = Tr_n$ and $d_{ij} = 1$ for $l+1 \le i \le n$, $1 \le j \le l$.

Now let

$$\beta Tr_l + (1-\beta)\left(1-\frac{1}{\gamma}\right)(l-1) + (1-\beta)\frac{1}{\gamma} Tr_l = Tr_n + (1-\beta)(\gamma-1)l,$$

that is

$$(1-\beta)l\gamma^2 - ((1-\beta)(2l-1) - Tr_n + \beta Tr_l)\gamma + (1-\beta)(l-1) - (1-\beta)Tr_l = 0\cdot$$

Therefore, we have

$$\gamma = \frac{(1-\beta)(2l-1) - Tr_n + \beta Tr_l + \sqrt{((1-\beta) + Tr_n - \beta Tr_l)^2 + 4(1-\beta)l(Tr_l - Tr_n)}}{2(1-\beta)l}.$$

Obviously, $\gamma > 1$. Therefore

$$\begin{aligned} &Tr_n + (1-\beta)(\gamma-1)l \\ &= \frac{Tr_n + \beta Tr_l - (1-\beta) + \sqrt{((1-\beta) + Tr_n - \beta Tr_l)^2 + 4(1-\beta)l(Tr_l - Tr_n)}}{2}. \end{aligned}$$

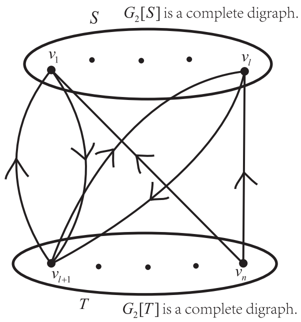

FIGURE 2. The digraph G_2.

Thus

$$\begin{aligned}&\min\{\psi_1(B),\psi_2(B),\ldots,\psi_n(B)\}\\&\geq\frac{Tr_n+\beta Tr_l-(1-\beta)+\sqrt{((1-\beta)+Tr_n-\beta Tr_l)^2+4(1-\beta)l(Tr_l-Tr_n)}}{2}.\end{aligned}$$

Thus, base on Lemma 2.1, we get

$$\begin{aligned}\mu_\beta(G)&\geq\min_{1\leq i\leq n}\{\psi_i(B)\}\\&\geq\frac{Tr_n+\beta Tr_l-(1-\beta)+\sqrt{((1-\beta)+Tr_n-\beta Tr_l)^2+4(1-\beta)l(Tr_l-Tr_n)}}{2}.\end{aligned}$$

Suppose that the equality holds, that is

$$\mu_\beta(G)=\frac{Tr_n+\beta Tr_l-(1-\beta)+\sqrt{((1-\beta)+Tr_n-\beta Tr_l)^2+4(1-\beta)l(Tr_l-Tr_n)}}{2},$$

then all the above inequalities must be equalities. Hence,

$$\psi_1(B)=\cdots=\psi_n(B)=\beta Tr_l+(1-\beta)\left(1-\frac{1}{\gamma}\right)(l-1)+(1-\beta)\frac{1}{\gamma}Tr_l=Tr_n+(1-\beta)(\gamma-1)l.$$

Thus $Tr_1=Tr_2=\cdots=Tr_l>Tr_{l+1}=\cdots=Tr_n$, $d_{ij}=1$ for $1\leq i\leq l$ with $1\leq j\neq i\leq l$, and $d_{ij}=1$ for $l+1\leq i\leq n$, $1\leq j\leq l$, that is G satisfies $Tr_1=Tr_2=\cdots=Tr_l>Tr_{l+1}=\cdots=Tr_n$, and $d_{ij}=1$ for $1\leq i\leq n$ and $1\leq j\neq i\leq l$.

For converse, if G satisfies $Tr_1=Tr_2=\cdots=Tr_l>Tr_{l+1}=\cdots=Tr_n$, and $d_{ij}=1$ for $1\leq i\leq n$ and $1\leq j\neq i\leq l$, then from the proof above, all the inequalities become equalities. Hence, we have the equality in (2.2) holds. □

In the following, we give an example to illustrate that the digraphs in Theorem 2.3 exist. Let G_2 be a digraph of order n with $V(G_2)=S\cup T$, where $S=\{v_1,v_2,\ldots,v_l\}$ $(1\leq l\leq n-2)$ and $T=V(G_2)\setminus S$, such that $G_2[S]$ is a complete digraph, $G_2[T]$ is a complete digraph, and the vertices in S are all out-adjacent to one vertex in T, say v_{l+1}, and for any vertex in T, which is out-adjacent to all vertices in S, as shown in Figure 2.

Then $Tr_1=Tr_2=\cdots=Tr_l=l+2(n-l-1)=2n-l-2$, $Tr_{l+1}=\cdots=Tr_n=n-1$. Let $z=(z_1,z_2,\ldots,z_n)^T$ be the Perron vector of $D_\beta(G_2)$ with respect to $\mu_\beta(G_2)$, where z_i corresponding to the vertex v_i. It is easy to see that the entries of z corresponding to vertices in S have the same value, say z_1, the entries of z corresponding to vertices in T have the same value, say z_2. From $D_\beta(G_2)z=\mu_\beta(G_2)z$, we have

$$\mu_\beta(G_2)z_1=\beta(2n-l-2)z_1+(1-\beta)(l-1)z_1+(1-\beta)z_2+2(1-\beta)(n-l-1)z_2,$$

$$\mu_\beta(G_2)z_2 = \beta(n-1)z_2 + (1-\beta)lz_1 + (1-\beta)(n-l-1)z_2.$$

Hence

$$\begin{aligned}\mu_\beta(G_2) &= \frac{n-\beta-2-\beta l+2\beta n+\sqrt{(1-2\beta)^2n^2-2(2\beta-1)(\beta+\beta l)n+4l(1-\beta)(n-l-1)+(\beta+\beta l)^2}}{2}\\ &= \frac{Tr_n+\beta Tr_l-(1-\beta)+\sqrt{((1-\beta)+Tr_n-\beta Tr_l)^2+4(1-\beta)l(Tr_l-Tr_n)}}{2}.\end{aligned}$$

Theorem 2.4. *Let G be a strongly connected digraph with $n \geq 2$ vertices, Δ_1^+ and Δ_2^+ denote the maximum outdegree and the second maximum outdegree of G, respectively. Then*

$$\mu_\beta(G) \geq \frac{\beta(4n-4-\Delta_1^+-\Delta_2^+)+\sqrt{\beta^2(4n-4-\Delta_1^+-\Delta_2^+)^2+4(1-2\beta)(2n-2-\Delta_1^+)(2n-2-\Delta_2^+)}}{2}, \tag{2.3}$$

with equality if and only if either G is a complete digraph $\overleftrightarrow{K_n}$ or G is an outdegree regular digraph with diameter 2.

Proof. Let $x = (x_1, x_2, \ldots, x_n)^T$ be the Perron vector of $D_\beta(G)$ with respect to $\mu_\beta(G)$, where x_i corresponding to the vertex v_i. Assume that v_k, v_s are two vertices in G such that $x_k = \min\limits_{1\leq i\leq n} x_i$ and $x_s = \min\limits_{1\leq i\leq n, i\neq k} x_i$. Since $D_\beta(G)x = \mu_\beta(G)x$, we obtain

$$\begin{aligned}\mu_\beta(G)x_k &= \beta Tr_k x_k + (1-\beta)\sum_{j=1}^n d_{kj}x_j\\ &\geq \beta(d_k^+ + 2(n-1-d_k^+))x_k + (1-\beta)(d_k^+ + 2(n-1-d_k^+))x_s,\end{aligned}$$

$$\begin{aligned}\mu_\beta(G)x_s &= \beta Tr_s x_s + (1-\beta)\sum_{j=1}^n d_{sj}x_j\\ &\geq \beta(d_s^+ + 2(n-1-d_s^+))x_s + (1-\beta)(d_s^+ + 2(n-1-d_s^+))x_k.\end{aligned}$$

Furthermore,

$$(\mu_\beta(G) - \beta(2n-2-d_k^+))(\mu_\beta(G) - \beta(2n-2-d_s^+)) \geq (1-\beta)^2(2n-2-d_k^+)(2n-2-d_s^+).$$

Hence

$$\begin{aligned}\mu_\beta(G) &\geq \frac{\beta(4n-4-d_k^+-d_s^+)+\sqrt{\beta^2(4n-4-d_k^+-d_s^+)^2+4(1-2\beta)(2n-2-d_k^+)(2n-2-d_s^+)}}{2}\\ &\geq \frac{\beta(4n-4-\Delta_1^+-\Delta_2^+)+\sqrt{\beta^2(4n-4-\Delta_1^+-\Delta_2^+)^2+4(1-2\beta)(2n-2-\Delta_1^+)(2n-2-\Delta_2^+)}}{2}.\end{aligned}$$

The equality holds if and only if all entries of x are equal and the diameter d of G is at most two. For $d = 1$, we get that G is a complete digraph $\overleftrightarrow{K_n}$. For $d = 2$, we get $\mu_\beta(G)x_k = \beta(2n-2-d_k^+)x_k + (1-\beta)(2n-2-d_k^+)x_k = (2n-2-d_k^+)x_k$, and then $\mu_\beta(G) = 2n-2-d_k^+$, which means G is an outdegree regular digraph.

For converse, if G is a complete digraph $\overleftrightarrow{K_n}$, then

$$\begin{aligned}\mu_\beta(G) &= n-1\\ &= \frac{\beta(4n-4-\Delta_1^+-\Delta_2^+)+\sqrt{\beta^2(4n-4-\Delta_1^+-\Delta_2^+)^2+4(1-2\beta)(2n-2-\Delta_1^+)(2n-2-\Delta_2^+)}}{2}.\end{aligned}$$

If G is a Δ^+-regular digraph has diameter 2, thus $Tr_1 = Tr_2 = \cdots = Tr_n = \Delta^+ + 2(n - \Delta^+ - 1) = 2n - 2 - \Delta^+$. Therefore

$$\mu_\beta(G) = 2n - 2 - \Delta^+$$
$$= \frac{\beta(4n - 4 - \Delta_1^+ - \Delta_2^+) + \sqrt{\beta^2(4n - 4 - \Delta_1^+ - \Delta_2^+)^2 + 4(1 - 2\beta)(2n - 2 - \Delta_1^+)(2n - 2 - \Delta_2^+)}}{2}.$$

Therefore, we have the desired result. □

3. Bounds on the generalized distance energy of strongly connected digraphs

In the following, we give the definition of the generalized distance energy of strongly connected digraphs.

Definition 3.1. Let G be a strongly connected digraph with n vertices and $W(G) = \sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}$. Then the generalized distance energy of strongly connected digraph G is defined as

$$E_{D_\beta}(G) = \sum_{i=1}^{n} |\mu_i^\beta(G) - \frac{\beta W(G)}{n}|,$$

where $\mu_1^\beta(G), \mu_2^\beta(G), \ldots, \mu_n^\beta(G)$ are the eigenvalues of the generalized distance matrix $D_\beta(G)$ of G.

Let $\varepsilon_i = \mu_i^\beta(G) - \frac{\beta W(G)}{n}$. We have the following observations.

Lemma 3.2. *Let $G = (V(G), E(G))$ be a strongly connected digraph on n vertices with transmission sequence $\{Tr_1, Tr_2, \ldots, Tr_n\}$, and $W(G) = \sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}$. Then*

(1). $\sum_{i=1}^{n} \mu_i^\beta(G) = \sum_{i=1}^{n} \beta Tr_i = \beta W(G)$,

(2). $\sum_{i=1}^{n} (\mu_i^\beta(G))^2 = \sum_{i=1}^{n} \beta^2 Tr_i^2 + (1-\beta)^2 \sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}d_{ji}$,

(3). $\sum_{i=1}^{n} \varepsilon_i^2 = (1-\beta)^2 \sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}d_{ji} + \sum_{i=1}^{n} \left(\beta Tr_i - \frac{\beta W(G)}{n}\right)^2$.

Proof.(1). We have $\sum_{i=1}^{n} \mu_i^\beta(G) = trac(D_\beta(G)) = \sum_{i=1}^{n} \beta Tr_i = \beta W(G)$.

(2). Since $D_\beta^2(G) = \beta^2 Tr^2(G) + \beta(1-\beta)Tr(G)D(G)) + \beta(1-\beta)D(G)Tr(G) + (1-\beta)^2 D^2(G)$,

$$\sum_{i=1}^{n} (\mu_i^\beta(G))^2 = trac(D_\beta^2(G))$$
$$= \beta^2 \sum_{i=1}^{n} Tr_i^2 + (1-\beta)^2 trac(D^2(G))$$
$$= \beta^2 \sum_{i=1}^{n} Tr_i^2 + (1-\beta)^2 \sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}d_{ji}.$$

(3). Since $\sum_{i=1}^{n}\varepsilon_i=\sum_{i=1}^{n}(\mu_i^{\beta}(G)-\frac{\beta W(G)}{n})=0$,

$$\begin{aligned}\sum_{i=1}^{n}\varepsilon_i^2&=\sum_{i=1}^{n}(\mu_i^{\beta}(G)-\frac{\beta W(G)}{n})^2\\&=\sum_{i=1}^{n}(\mu_i^{\beta}(G))^2+\sum_{i=1}^{n}\left(\frac{\beta W(G)}{n}\right)^2-2\frac{\beta W(G)}{n}\sum_{i=1}^{n}\mu_i^{\beta}(G)\\&=\beta^2\sum_{i=1}^{n}Tr_i^2+(1-\beta)^2\sum_{i=1}^{n}\sum_{j=1}^{n}d_{ij}d_{ji}+\sum_{i=1}^{n}\left(\frac{\beta W(G)}{n}\right)^2-2\frac{\beta W(G)}{n}\sum_{i=1}^{n}\beta Tr_i\\&=(1-\beta)^2\sum_{i=1}^{n}\sum_{j=1}^{n}d_{ij}d_{ji}+\sum_{i=1}^{n}(\beta Tr_i-\frac{\beta W(G)}{n})^2.\end{aligned}$$

□

Let $H(G)=(1-\beta)^2\sum_{i=1}^{n}\sum_{j=1}^{n}d_{ij}d_{ji}$, $M(G)=(1-\beta)^2\sum_{i=1}^{n}\sum_{j=1}^{n}d_{ij}d_{ji}+\sum_{i=1}^{n}\left(\beta Tr_i-\frac{\beta W(G)}{n}\right)^2$. Since $\frac{W(G)}{n}$ is the average transmission, we have $M(G)=H(G)>0$ if and only if G is a transmission regular digraph, and $M(G)>H(G)>0$ otherwise.

Theorem 3.3. *Let $G=(V(G),E(G))$ be a strongly connected digraph on n vertices, and $\mu_1^{\beta}(G),\mu_2^{\beta}(G),\ldots,\mu_n^{\beta}(G)$ be the eigenvalues of the generalized distance matrix $D_\beta(G)$ of G. Then*

$$E_{D_\beta}(G)\le\sqrt{n\left((1-\beta)^2\sum_{i=1}^{n}\sum_{j=1}^{n}d_{ij}^2+\sum_{i=1}^{n}\left(\beta Tr_i-\frac{\beta W(G)}{n}\right)^2\right)}.$$

Proof. By Lemma 3.2, $\sum_{i=1}^{n}\mu_i^{\beta}(G)=\sum_{i=1}^{n}\beta Tr_i$. Hence $Re(\sum_{i=1}^{n}\mu_i^{\beta}(G))=\sum_{i=1}^{n}\beta Tr_i$, where $Re(z)$ is the real part of the complex number z.

By Schur's unitary triangularization theorem [5], there is a unitary matrix P such that $P^*D_\beta(G)P=M=(m_{ij})_{n\times n}$, where $M=(m_{ij})_{n\times n}$ is an upper triangular matrix with diagonal entries $m_{ii}=\mu_i^{\beta}(G)$, $i=1,2,\ldots,n$. Therefore

$$M^*M=P^*D_\beta^*(G)PP^*D_\beta(G)P=P^*D_\beta^*(G)D_\beta(G)P.$$

Therefore, we have $tr(M^*M)=tr(D_\beta^*(G)D_\beta(G))$, and

$$\begin{aligned}\sum_{i=1}^{n}|\mu_i^{\beta}(G)|^2=\sum_{i=1}^{n}|m_{ii}|^2\le\sum_{i,j=1}^{n}|m_{ij}|^2=tr(M^*M)=tr(D_\beta^*(G)D_\beta(G))\\=\beta^2\sum_{i=1}^{n}Tr_i^2+(1-\beta)^2\sum_{i=1}^{n}\sum_{j=1}^{n}d_{ij}^2.\end{aligned}$$

Applying the Cauchy-Schwarz inequality to the vector $(|\varepsilon_1|,|\varepsilon_2|,\ldots,|\varepsilon_n|)$ and $(1,1,\ldots,1)$, we obtain

$$E_{D_\beta}(G)=\sum_{i=1}^{n}|\mu_i^{\beta}(G)-\frac{\beta W(G)}{n}|=\sum_{i=1}^{n}|\varepsilon_i|\le\sqrt{n\sum_{i}^{n}|\varepsilon_i|^2}.$$

However,

$$\begin{aligned}\sum_{i=1}^{n}|\varepsilon_i|^2 &= \sum_{i=1}^{n}|\mu_i^{\beta}(G)-\frac{\beta W(G)}{n}|^2 = \sum_{i=1}^{n}(\mu_i^{\beta}(G)-\frac{\beta W(G)}{n})(\overline{\mu_i^{\beta}(G)}-\frac{\beta W(G)}{n})\\ &= \sum_{i=1}^{n}|\mu_i^{\beta}(G)|^2 - 2\frac{\beta W(G)}{n}\sum_{i=1}^{n}Re(\mu_i^{\beta}(G)) + \sum_{i=1}^{n}\left(\frac{\beta W(G)}{n}\right)^2\\ &\le \beta^2\sum_{i=1}^{n}Tr_i^2 + (1-\beta)^2\sum_{i=1}^{n}\sum_{j=1}^{n}d_{ij}^2 - 2\frac{\beta W(G)}{n}\sum_{i=1}^{n}\beta Tr_i + \sum_{i=1}^{n}\left(\frac{\beta W(G)}{n}\right)^2\\ &= (1-\beta)^2\sum_{i=1}^{n}\sum_{j=1}^{n}d_{ij}^2 + \sum_{i=1}^{n}(\beta Tr_i - \frac{\beta W(G)}{n})^2.\end{aligned}$$

Thus

$$E_{D_\beta}(G) \le \sqrt{n\left((1-\beta)^2\sum_{i=1}^{n}\sum_{j=1}^{n}d_{ij}^2 + \sum_{i=1}^{n}(\beta Tr_i - \frac{\beta W(G)}{n})^2\right)}.$$

□

Theorem 3.4. *Let G be a strongly connected digraph on n vertices. Then*

$$E_{D_\beta}(G) \ge \sqrt{2\left((1-\beta)^2\sum_{i=1}^{n}\sum_{j=1}^{n}d_{ij}d_{ji} + \sum_{i=1}^{n}\left(\beta Tr_i - \frac{\beta W(G)}{n}\right)^2\right)}.$$

Proof. By the definition of the generalized distance energy of strongly connected digraph G, we obtain

$$(E_{D_\beta}(G))^2 = (\sum_{i=1}^{n}|\varepsilon_i|)^2 = \sum_{i=1}^{n}|\varepsilon_i|^2 + 2\sum_{1\le i<j\le n}|\varepsilon_i||\varepsilon_j|.$$

Furthermore, by Lemma 3.2,

$$\sum_{i=1}^{n}\varepsilon_i^2 = (1-\beta)^2\sum_{i=1}^{n}\sum_{j=1}^{n}d_{ij}d_{ji} + \sum_{i=1}^{n}\left(\beta Tr_i - \frac{\beta W(G)}{n}\right)^2.$$

Hence

$$(1-\beta)^2\sum_{i=1}^{n}\sum_{j=1}^{n}d_{ij}d_{ji} + \sum_{i=1}^{n}\left(\beta Tr_i - \frac{\beta W(G)}{n}\right)^2 = |\sum_{i=1}^{n}\varepsilon_i^2| \le \sum_{i=1}^{n}|\varepsilon_i|^2.$$

Moreover, $\sum_{i=1}^{n}\varepsilon_i = 0$. Then

$$0 = (\sum_{i=1}^{n}\varepsilon_i)^2 = \sum_{i=1}^{n}\varepsilon_i^2 + 2\sum_{1\le i<j\le n}\varepsilon_i\varepsilon_j,$$

that is

$$2\sum_{1\le i<j\le n}\varepsilon_i\varepsilon_j = -\sum_{i=1}^{n}\varepsilon_i^2 = -\left((1-\beta)^2\sum_{i=1}^{n}\sum_{j=1}^{n}d_{ij}d_{ji} + \sum_{i=1}^{n}\left(\beta Tr_i - \frac{\beta W(G)}{n}\right)^2\right).$$

Therefore,

$$(1-\beta)^2\sum_{i=1}^{n}\sum_{j=1}^{n}d_{ij}d_{ji}+\sum_{i=1}^{n}\left(\beta Tr_i-\frac{\beta W(G)}{n}\right)^2=|2\sum_{1\le i<j\le n}\varepsilon_i\varepsilon_j|\le 2\sum_{1\le i<j\le n}|\varepsilon_i||\varepsilon_j|.$$

Thus

$$(E_{D_\beta}(G))^2\ge 2\left((1-\beta)^2\sum_{i=1}^{n}\sum_{j=1}^{n}d_{ij}d_{ji}+\sum_{i=1}^{n}\left(\beta Tr_i-\frac{\beta W(G)}{n}\right)^2\right).$$

So we have the desired result. □

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