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COMPONENT FACTORS AND DEGREE SUM CONDITIONS IN GRAPHS

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Abstract. For a set $\mathscr{A}$ of connected graphs, an $\mathscr{A}$-factor is a spanning subgraph of a graph, whose connected components are isomorphic to graphs from the set $\mathscr{A}$. An $\mathscr{A}$-factor is also referred as a component factor. A graph G is called an $(\mathscr{A}, m)$-factor deleted graph if for every $E' \subseteq E(G)$ with $|E'| = m, G - E'$ admits an $\mathscr{A}$-factor. A graph G is called an $(\mathscr{A}, l)$-factor critical graph if for every $V' \subseteq V(G)$ with $|V'| = l, G - V'$ admits an $\mathscr{A}$-factor. Let m, l and k be three positive integers with $k \geq 2$, and write $\mathscr{F} = \{P_2, C_3, P_5, \mathscr{T}(3)\}$ and $\mathscr{H} = \{K_{1,1}, K_{1,2}, \ldots, K_{1,k}, \mathscr{T}(2k+1)\}$, where $\mathscr{T}(3)$ and $\mathscr{T}(2k+1)$ are two special families of trees. Inspired by finding a sufficient condition to check for the existence of path-factors with some special restraints, we focus on the sufficient conditions based on a graphic parameter called *degree sum*:

$$\sigma_k(G) = \min_{X \subseteq V(G)} \Big\{ \sum_{x \in X} d_G(x) : X \text{ is an independent set of } k \text{ vertices} \Big\}.$$

In this article, we verify that: (i) an $(l+2)$-connected graph G of order n is an $(\mathscr{F}, l)$-factor critical graph if $\sigma_3(G) \geq \frac{6n+9l}{5}$; (ii) a $(2m+1)$-connected graph G of order n is an $(\mathscr{F}, m)$-factor deleted graph if $\sigma_{m+2}(G) \geq \frac{6}{5}n$; (iii) an $(l+2)$-connected graph G of order n is an $(\mathscr{H}, l)$-factor critical graph if $\sigma_{2k+1}(G) \geq \frac{6n+(6k+3)l}{2k+3}$; (iv) a $(2m+1)$-connected graph G of order n is an $(\mathscr{H}, m)$-factor deleted graph if $\sigma_{m+2}(G) \geq \frac{6n}{2k+3}$.

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1. Introduction

Graph theory has shown impressive advances in recent years, due to its essential roles providing structural models and indispensable tools in computer science, communication networks and combinatorial optimization problems. Matching theory, or more general factor theory, is one of the fundamental areas in graph theory. It

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studies the structures and properties of matchings and factors, the simplest nontrivial substructures of graphs. Graph factors are the generalizations of matchings, which is also one of the earliest topics to be studied in graph theory. One type of graph factor problems, generally referred as "component factor" problem, which consider a spanning subgraph with components isomorphic to the members of a given list. In this article, We investigate the existence of two types of component factors with given properties in graphs.

Throughout this paper, we discuss only simple connected graphs. We refer to Bondy and Murty [4] for the notation and terminologies not defined here. Let $G = (V(G), E(G))$ be a graph. We denote by $V(G)$ and $E(G)$ the vertex set and the edge set of G, respectively. For $v \in V(G)$, we use $d_G(v)$ and $N_G(v)$ to denote the degree of v and the set of vertices adjacent to v in G, respectively. If $d_G(v) = 0$ for some vertex $v \in V(G)$, then v is said to be an *isolated vertex* in G. The number of isolated vertices of a graph G is denoted by $i(G)$. For any subset $S \subseteq V(G)$, let $G[S]$ denote the subgraph of G induced by S, and $G - S := G[V(G) \setminus S]$ is the resulting graph after deleting the vertices of S from G. For any $M \subseteq E(G)$, we use $G - M$ to denote the subgraph obtained from G by deleting M. Especially, we write $G - x = G - \{x\}$ for $S = \{x\}$ and $G - e = G - \{e\}$ for $M = \{e\}$. We denote by $\kappa(G)$ and $\lambda(G)$ the connectivity and the edge connectivity of G, respectively. We denote by T a tree, and by $Leaf(T)$ the set of leaves in T. An edge of T incident with a leaf is called a pendant edge. Especially, the number of leaves in T is equal to that of pendant edges in T under the case that the order of T is at least 3.

A subgraph H of $G = (V(G), E(G))$ is called a spanning subgraph of G if $V(H) = V(G)$ and $E(H) \subseteq E(G)$. For a family of connected graphs $\mathscr{A}$, a spanning subgraph H of a graph G is called an $\mathscr{A}$-factor of G if each component of H is isomorphic to some graph in $\mathscr{A}$. A graph G is called an $(\mathscr{A}, l)$-factor critical graph if $G - V'$ contains an $\mathscr{A}$-factor for any $V' \subseteq V(G)$ with $|V'| = l$. A graph G is called an $(\mathscr{A}, m)$-factor deleted graph if $G - E'$ has an $\mathscr{A}$-factor for any $E' \subseteq E(G)$ with $|E'| = m$. Obviously, an $(\mathscr{A}, 0)$-factor deleted graph is equivalent to a graph having an $\mathscr{A}$-factor.

The first attempt on the study of factors was made by Petersen in 1891, who proved that every graph of even degrees can be decomposed into the union of edge-disjoint 2-factors. This was motivated from the study of an algebraic factorization problem. He also showed that every 2-connected cubic graph has a 1-factor. These two results can be viewed as a forerunner of modern graph factor theory. Since Tutte [22] gave a characterization (*i.e.*, so-called Tutte's 1-factor Theorem) for the existence of perfect matchings in arbitrary graphs in 1947, it has become a cornerstone of factor theory. Till now, this elegant theorem is still one of the most fundamental results in factor theory. Subsequently, Tutte [22] extended the techniques in the proof of 1-factor Theorem to obtain a necessary and sufficient condition for a graph to have an f-factor. For the more comprehensive account of study on matching theory and graph factors, readers can refer to [1, 24] for more information and related references.

As early as 1953, Tutte [23] obtained a necessary and sufficient condition for a graph to have a $\{P_2, C_n : n \geq 3\}$-factor. Egawa, Kano and Yan [11] gave a shorter proof. Klopp and Steffen [19] posed some properties for the existence of $\{K_{1,1}, K_{1,2}, C_m : m \geq 3\}$-factors in graphs. Kano, Lee and Suzuki [16] showed two results for graphs to admit path and cycle factors. Kano, Lu and Yu [17] derived a result for a graph having a $\{K_{1,2}, K_{1,3}, K_5\}$-factor. Amahashi and Kano [3] obtained a criterion for a graph with a $\{K_{1,j} : 1 \leq j \leq k\}$-factor. Kano and Saito [14] used isolated k-toughness to ensure the existence of $\{K_{1,j} : k \leq j \leq 2k\}$-factor. Akiyama, Avis and Era [2] provided a good characterization for a graph admitting a $P_{\geq 2}$-factor in graphs. The following results on component factors of graphs are known.

Theorem 1.1. (Tutte [23]) *A graph G admits a $\{P_2, C_n : n \geq 3\}$-factor if and only if $i(G - S) \leq |S|$ for every $S \subset V(G)$, where $i(G - S)$ denotes the number of isolated vertices in $G - S$.*

Theorem 1.2. (Kano, Lu, Yu [18]) *A graph G admits a $\{K_{1,2}, K_{1,3}, K_5\}$-factor if $i(G - S) \leq \frac{|S|}{2}$ for every $S \subset V(G)$.*

Theorem 1.3. (Amahashi, Kano [3]) *A graph G admits a $\{K_{1,j} : 1 \leq j \leq k\}$-factor if $i(G - S) \leq k|S|$ for every $S \subset V(G)$.*

Theorem 1.4. (Akiyama, Avis, Era. [2]) *A graph G has a $P_{\geq 2}$-factor if and only if $i(G-S) \leq 2|S|$ for all $S \subseteq V(G)$.*

The concept of sun was introduced by Kaneko [13]. A graph H is called factor-critical if $H-v$ has a 1-factor for each $v \in V(H)$. Let H be a factor-critical graph and $V(H) = \{v_1, v_2, \ldots, v_n\}$. By adding new vertices $\{u_1, u_2, \ldots, u_n\}$ together with new edges $\{v_i u_i : 1 \leq i \leq n\}$ to H, the resulting graph is called a sun. We also regard K_1 and K_2 as suns. A component of G is called a sun component if it is isomorphic to a sun. Using the concept of sun components, Kaneko [13] presented a good characterization for a graph admitting $P_{\geq 3}$-factors. Kano, Katona and Király [15] gave a simpler proof of Kaneko's result. Kano, Lu and Yu [17] demonstrated that a graph G has a $P_{\geq 3}$-factor if $i(G-S) \leq \frac{2}{3}|S|$ for all $S \subseteq V(G)$. For more sufficient conditions for the graphs admitting $P_{\geq 3}$-factors, we refer to [7–10, 12, 20, 25–28].

Theorem 1.5. (Kano, Lu, Yu [17]) *A graph G has a $P_{\geq 3}$-factor if and only if $i(G-S) \leq \frac{2}{3}|S|$ for all $S \subseteq V(G)$.*

Intuitively, if a graph is dense enough, then it will have a $P_{\geq 3}$-factor. Recall the classic result due to Ore [21]: Let G be a graph of order $n \geq 3$, then G is Hamiltonian if $d_G(u) + d_G(v) \geq n$ for each pair of nonadjacent vertices of G. Our goal is to replace "$d_G(u) + d_G(v) \geq n$ for every pair of nonadjacent vertices of G" by a weaker condition of the same flavor. Here, if G is a graph containing at least k independent vertices, then we use the graphic parameter *degree sum* defined as

$$\sigma_k(G) = \min_{X \subseteq V(G)} \left\{ \sum_{x \in X} d_G(x) : \text{the set } X \text{ is independent and contains } k \text{ vertices} \right\}.$$

Note that when $k = 2$, this corresponds to taking the minimum of $d_G(u) + d_G(v)$ over every pair of nonadjacent vertices of G, part of the crux of the statement of Ore's Theorem [21]. From algorithmic complexity point of view, if k is fixed, the time $O(n^k)$ taken in computing the degree sum parameter $\sigma_k(G)$ is strongly polynomial by simply checking all subsets of size k. On the study of the existence of component factors using the simple Ore-type sufficient condition refer to [5, 6].

In this paper, in terms of degree sum conditions, we investigate the existence of component factors with given properties in graphs, which are shown in Sections 2 and 3.

2. Graph with a $\{P_2, C_3, P_5, \mathscr{T}(3)\}$-factor

Recently, using fractional factors as a tool, Kano, Lu and Yu [18] constructed a new component factor called $\{P_2, C_3, P_5, \mathscr{T}(3)\}$-factor, where $\mathscr{T}(3)$ is defined as follows: for any $\{1,3\}$-tree R (*i.e.*, a tree such that $d_R(x) \in \{1,3\}, \forall x \in V(R)$), a new tree T_R is obtained from R by inserting a new vertex of degree 2 into every edge of R, and by adding a new pendant edge to each leaf of R. Then the tree T_R is a $\{1,2,3\}$-tree admitting $|E(R)| + |Leaf(R)|$ vertices of degree 2 and possesses the same number of leaves as R (*e.g.*, see Fig. 1). The collection of such $\{1,2,3\}$-trees T_R generated from all $\{1,3\}$-trees R is denoted by $\mathscr{T}(3)$.

In this section, we always assume that $\mathscr{F} = \{P_2, C_3, P_5, \mathscr{T}(3)\}$. Kano, Lu and Yu [18] derived a characterization for a graph with $\mathscr{F}$-factors.

Theorem 2.1. (Kano, Lu, Yu [18]) *A graph G admits an $\mathscr{F}$-factor if and only if*

$$i(G-S) \leq \frac{3}{2}|S|$$

for every $S \subset V(G)$.

Using Theorem 2.1, we shall verify the following two results on the graph admitting $\mathscr{F}$-factors.

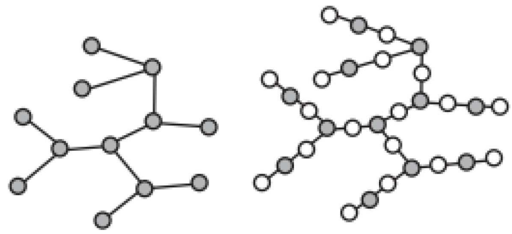

FIGURE 1. A $\{1,3\}$-tree R and the resulting $\{1,2,3\}$-tree T_R obtained from R.

Theorem 2.2. *Let G be an $(l+2)$-connected graph of order n, where l is a positive integer. Then G is an $(\mathscr{F},l)$-factor critical graph if G satisfies*

$$\max\left\{d_G(x_1), d_G(x_2), d_G(x_3)\right\} \geq \frac{2n+3l}{5}$$

for any independent set $\{x_1,x_2,x_3\}$ of G.

Proof. Let $G' = G - V'$ for any $V' \subseteq V(G)$ with $|V'| = l$. It suffices to verify that G' has an $\mathscr{F}$-factor. On the contrary, we assume that there is no $\mathscr{F}$-factor in G'. By Theorem 2.1, there exists $S \subseteq V(G')$ such that

$$i(G'-S) > \frac{3}{2}|S|. \tag{1}$$

Claim 2.1. $|S| \geq 2$.

Proof. Assume that $|S| \leq 1$. Then the connectivity

$$\kappa(G'-S) = \kappa(G-V'-S) \geq \kappa(G) - |V' \cup S| \geq (l+2)-(l+1) = 1. \tag{2}$$

It follows from (2) that $G'-S$ is connected. So, $i(G'-S)=0$, which contradicts (1). ◊

In terms of (1) and Claim 2.1, we obtain that

$$i(G'-S) = i(G-V'-S) > \frac{3}{2}|S| \geq 3. \tag{3}$$

Let u_1,u_2,u_3 be three distinct isolated vertices in $G'-S$ by (3). It is clear that $\{u_1,u_2,u_3\}$ is independent in G, and $d_{G'-S}(u_i)=0$ for $i=1,2,3$. Hence, we have

$$d_G(u_i) \leq d_{G'}(u_i) + |V'| \leq d_{G'-S}(u_i) + |S| + |V'| \leq |S| + l \tag{4}$$

for $i=1,2,3$. Then in terms of (4) and the degree condition of Theorem 2.2, we obtain

$$|S| + l \geq \max\left\{d_G(u_1), d_G(u_2), d_G(u_3)\right\} \geq \frac{2n+3l}{5},$$

which implies

$$|S| \geq \frac{2}{5}(n-l). \tag{5}$$

Combining (5) with (1), it implies that

$$n \geq |V'| + |S| + i(G'-S) > l + |S| + \frac{3}{2}|S| = l + \frac{5}{2}|S| \geq n,$$

a contradiction. □

Corollary 1. *Let G be an $(l+2)$-connected graph of order n, where l is a positive integer. Then G is an $(\mathscr{F}, l)$-factor critical graph if $\sigma_3(G) \geq \frac{6n+9l}{5}$.*

Theorem 2.3. *Let G be a $(2m+1)$-connected graph of order n, where m is a positive integer. Then G is an $(\mathscr{F}, m)$-factor deleted graph if G satisfies*

$$\max\left\{d_G(x_1), d_G(x_2), \ldots, d_G(x_{m+2})\right\} \geq \frac{2}{5}n$$

for any independent set $\{x_1, x_2, \ldots, x_{m+2}\}$ of G.

Proof. Let $H = G - E'$ for any edge set $E' \subseteq E(G)$ with $|E'| = m$. Then $V(H) = V(G)$ and $E(H) = E(G) \setminus E'$. To prove Theorem 2.3, it suffices to verify that H has a $\mathscr{F}$-factor. On the contrary, we assume that H admits no $\mathscr{F}$-factor. By Theorem 2.1, there exists $S \subseteq V(H)$ such that

$$i(H - S) > \frac{3}{2}|S|. \tag{6}$$

Claim 2.2. $|S| \geq m + 1$.

Proof. On the contrary, we fist assume that $S = \emptyset$. By (6), we obtain

$$i(H) = i(H - S) > 0. \tag{7}$$

On the other hand, since G is $(2m+1)$-connected, $H := G - E'$ is a connected graph. So we obtain $i(H) = 0$, which contradicts (7). Hence, $S \neq \emptyset$ and $|S| \geq 1$. This together with (6) implies that $i(H - S) > \frac{3}{2}|S| \geq \frac{3}{2}$. Due to the integrality of $i(H - S)$, we obtain that

$$i(G - E' - S) = i(H - S) \geq 2. \tag{8}$$

If $|S| \leq m$, then

$$\kappa(H - S) = \kappa(G - E' - S) \geq (2m+1) - |E'| - |S| \geq 1,$$

which implies that $H - S$ is connected. So, $i(H - S) = 0$, a contradiction to (8). Then we claim that $|S| \geq m+1$. $\Diamond$

In what follows, we shall consider two cases.

Case 1. S is not a vertex cut set of G.

In this case, we derive $\omega(G - S) = \omega(G) = 1$. By Claim 2.2, we get

$$i(H - S) = i(G - E' - S) \leq \omega(G - E' - S) \leq \omega(G - S) + m = m + 1 \leq \frac{3}{2}|S|,$$

which contradicts (6).

Case 2. S is a vertex cut set of G.

In this case, we possess $\omega(G - S) \geq \omega(G) + 1 = 2$. Combining this with $\kappa(G) \geq 2m + 1$, we derive

$$|S| \geq 2m + 1. \tag{9}$$

In light of (9), we first argue that

$$i(G - S) \geq m + 2. \tag{10}$$

Otherwise,

$$i(H - S) = i(G - E' - S) \leq i(G - S) + 2m \leq (m+1) + 2m \leq |S| + m \leq \frac{3}{2}|S|,$$

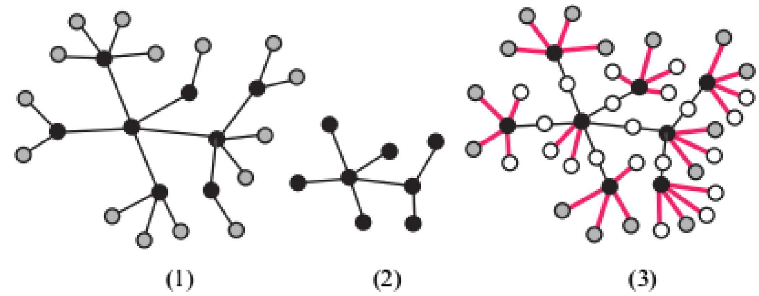

FIGURE 2. (1) A tree R that satisfies (ii) with $k = 4$; (2) The tree $R - Leaf(R)$; (3) The tree T_R obtained from R.

which contradicts (6). Let $\{u_1, u_2, \ldots, u_{m+1}, u_{m+2}\} \subseteq I(G - S)$ by (10), where $I(G - S)$ denotes the set of isolated vertices in $G - S$. Then we obtain an independent subset $\{u_1, u_2, \ldots, u_{m+1}, u_{m+2}\}$ in G. Hence, we have $d_G(u_i) \leq |S|$ for any $1 \leq i \leq m + 2$. Combining with the degree condition of Theorem 2.3, it implies that

$$\frac{2}{5}n \leq \max\left\{d_G(u_i) : 1 \leq i \leq m + 2\right\} \leq |S|. \tag{11}$$

In light of (6) and (11), we deduce

$$n \geq |S| + i(G - S) > |S| + \frac{3}{2}|S| = \frac{5}{2}|S| \geq n,$$

a contradiction. □

Corollary 2. *Let G be a $(2m + 1)$-connected graph of order n, where m is a positive integer. Then G is an $(\mathscr{F}, m)$-factor deleted graph if $\sigma_{m+2}(G) \geq \frac{6}{5}n$.*

3. GRAPH WITH A $\{K_{1,1}, K_{1,2}, \ldots, K_{1,k}, \mathscr{T}(2k + 1)\}$-FACTOR

In this section, we investigate the graph which admits $\{K_{1,1}, K_{1,2}, \ldots, K_{1,k}, \mathscr{T}(2k+1)\}$-factor, where $k \geq 2$ is an integer and $\mathscr{T}(2k+1)$ is a more general class of trees. Let R be a tree that satisfies the following conditions: for each $x \in V(R - Leaf(R))$,

(i) $d_{R-Leaf(R)}(x) \in \{1, 3, \ldots, 2k + 1\}$;

(ii) 2·(the number of leaves adjacent to x in R)+$d_{R-Leaf(R)}(x) \leq 2k + 1$.

For such a tree R, we derive a new tree T_R as follows:

(iii) insert a new vertex of degree 2 into each edge of $R - Leaf(R)$;

(iv) for each vertex x of $R - Leaf(R)$ with $d_{R-Leaf(R)}(x) = 2r + 1 < 2k + 1$, add $k - r-$(the number of leaves adjacent to x in R) pendant edges to x.

Then the set of such trees T_R for all trees R satisfying conditions (i) and (ii) is denoted by $\mathscr{T}(2k + 1)$.

In this section, we always assume that $\mathscr{H} = \{K_{1,1}, K_{1,2}, \ldots, K_{1,k}, \mathscr{T}(2k + 1)\}$. Recently, Kano, Lu and Yu [18] derived a characterization for graphs with $\mathscr{H}$-factors.

Theorem 3.1. *Let G be a connected graph, $k \geq 2$ be an integer. Then G admits an $\mathscr{H}$-factors if and only if*

$$i(G - S) \leq \left(k + \frac{1}{2}\right)|S|$$

for every $S \subset V(G)$.

Using Theorem 3.1, we shall verify the following results on the graph admitting $\mathscr{H}$-factors.

Theorem 3.2. *Let G be an $(l+2)$-connected graph of order n, where l is a positive integer. Then G is an $(\mathscr{H}, l)$-factor critical graph if G satisfies*

$$\max\left\{d_G(x_1), d_G(x_2), \ldots, d_G(x_{2k+1})\right\} \geq \frac{2n + (2k+1)l}{2k+3}$$

for any independent set $\{x_1, x_2, \ldots, x_{2k+1}\}$ of G.

Proof. Let $G' = G - V'$ for any $V' \subseteq V(G)$ with $|V'| = l$. It suffices to verify that G' has an $\mathscr{H}$-factor. On the contrary, we assume that there is no $\mathscr{H}$-factor in G'. By Theorem 3.1, there exists $S \subseteq V(G')$ such that

$$i(G' - S) > \left(k + \frac{1}{2}\right)|S|. \tag{12}$$

Claim 3.1. $|S| \geq 2$.

Proof. Assume that $|S| \leq 1$. Then the connectivity

$$\kappa(G' - S) = \kappa(G - V' - S) \geq \kappa(G) - |V' \cup S| \geq (l+2) - (l+1) = 1. \tag{13}$$

It follows from (13) that $G' - S$ is connected. So, $i(G' - S) = 0$, which contradicts (12). ♢

In terms of (12) and Claim 3.1, we obtain that

$$i(G' - S) = i(G - V' - S) > \left(k + \frac{1}{2}\right)|S| \geq 2k + 1. \tag{14}$$

Let $\{u_1, u_2, \ldots, u_{2k}, u_{2k+1}\} \subseteq I(G - S)$ by (14), where $I(G - S)$ denotes the set of isolated vertices in $G - S$. It is clear that $\{u_1, u_2, \ldots, u_{2k}, u_{2k+1}\}$ is an independent subset in G. Hence, we have

$$d_G(u_i) \leq d_{G'}(u_i) + |V'| \leq d_{G'-S}(u_i) + |S| + |V'| \leq |S| + l \tag{15}$$

for $i = 1, 2, \ldots, 2k+1$. Then in terms of (15) and the degree condition of Theorem 3.2, we obtain

$$|S| + l \geq \max\left\{d_G(u_1), d_G(u_2), \ldots, d_G(u_{2k+1})\right\} \geq \frac{2n + (2k+1)l}{2k+3},$$

which implies

$$|S| \geq \frac{2}{2k+3}(n - l). \tag{16}$$

Combining (16) with (12), it implies that

$$n \geq |V'| + |S| + i(G' - S) > l + |S| + \left(k + \frac{1}{2}\right)|S| = l + \frac{2k+3}{2}|S| \geq n,$$

a contradiction. □

Corollary 3. *Let G be an $(l+2)$-connected graph of order n, where l is a positive integer. Then G is an $(\mathscr{H}, l)$-factor critical graph if $\sigma_{2k+1}(G) \geq \frac{6n+(6k+3)l}{2k+3}$.*

Theorem 3.3. *Let G be a $(2m+1)$-connected graph of order n, where m is a positive integer. Then G is an $(\mathscr{H}, m)$-factor deleted graph if G satisfies*

$$\max\left\{d_G(x_1), d_G(x_2), \ldots, d_G(x_{m+2})\right\} \geq \frac{2n}{2k+3}$$

for any independent set $\{x_1, x_2, \ldots, x_{m+2}\}$ of G.

Proof. Let $G' = G - E'$ for any edge set $E' \subseteq E(G)$ with $|E'| = m$. Then $V(G') = V(G)$ and $E(G') = E(G) \setminus E'$. To prove Theorem 3.3, it suffices to verify that G' has an $\mathscr{H}$-factor. On the contrary, we assume that G' admits no $\mathscr{H}$-factor. By Theorem 3.1, there exists $S \subseteq V(G')$ such that

$$i(G' - S) > \left(k + \frac{1}{2}\right)|S|. \tag{17}$$

Claim 3.2. $|S| \geq m + 1$.

Proof. On the contrary, we assume that $|S| \leq m$, then

$$\kappa(G' - S) = \kappa(G - E' - S) \geq (2m + 1) - |E'| - |S| \geq 1. \tag{18}$$

On the other hand, by (17), we obtain $i(G' - S) > (k + \frac{1}{2})|S| \geq 0$. Due to the integrality of $i(G' - S)$, we obtain that

$$i(G - E' - S) = i(G' - S) \geq 1,$$

which contradicts (18). So, we claim that $|S| \geq m + 1$. ♢

In what follows, we shall consider two cases.

Case 1. S is not a vertex cut set of G.

In this case, we derive $\omega(G - S) = \omega(G) = 1$. By Claim 3.2, we get

$$i(G' - S) = i(G - E' - S) \leq \omega(G - E' - S) \leq \omega(G - S) + m = m + 1 \leq \left(k + \frac{1}{2}\right)|S|,$$

which contradicts (17).

Case 2. S is a vertex cut set of G.

In this case, we possess $\omega(G - S) \geq \omega(G) + 1 = 2$. Combining this with $\kappa(G) \geq 2m + 1$, we derive

$$|S| \geq 2m + 1. \tag{19}$$

In light of (19), we first argue that

$$i(G - S) \geq m + 2. \tag{20}$$

Otherwise,

$$i(G' - S) = i(G - E' - S) \leq i(G - S) + 2m \leq (m + 1) + 2m \leq |S| + m \leq \left(k + \frac{1}{2}\right)|S|,$$

which contradicts (17). Let $\{u_1, u_2, \ldots, u_{m+1}, u_{m+2}\} \subseteq I(G - S)$ by (20), where $I(G - S)$ denotes the set of isolated vertices in $G - S$. Then we obtain an independent subset $\{u_1, u_2, \ldots, u_{m+1}, u_{m+2}\}$ in G. Hence, we have $d_G(u_i) \leq |S|$ for any $1 \leq i \leq m + 2$. Combining with the degree condition of Theorem 3.3, it implies that

$$\frac{2n}{2k + 3} \leq \max\left\{d_G(u_i) : 1 \leq i \leq m + 2\right\} \leq |S|. \tag{21}$$

In light of (17) and (21), we deduce

$$n \geq |S| + i(G - S) > |S| + \left(k + \frac{1}{2}\right)|S| = \frac{2k + 3}{2}|S| \geq n,$$

a contradiction. □

Corollary 4. *Let G be a $(2m + 1)$-connected graph of order n, where m is a positive integer. Then G is an $(\mathscr{H}, m)$-factor deleted graph if $\sigma_{m+2}(G) \geq \frac{6n}{2k+3}$.*

4. Conclusions

In this paper, we establish the relationships between sum degree conditions and component factors of graphs, and derive some sum degree conditions for graphs to be $\{P_2, C_3, P_5, \mathscr{T}(3)\}$-factor critical/deleted graphs, and $\{K_{1,1}, K_{1,2}, \ldots, K_{1,k}, \mathscr{T}(2k+1)\}$-factor critical/deleted graphs, where $k \geq 2$.

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References

[1] J. Akiyama and M. Kano, Factors and Factorizations of Graphs. Springer, Berlin, Lecture Notes in Mathematics, 2031 (2011) 1–347.
[2] J. Akiyama, D. Avis and H. Era, On a $\{1, 2\}$-factor of a graph. *TRU Math.* **16** (1980) 97–102.
[3] A. Amahashi and M. Kano, On factors with given components. *Discrete Math.* **42** (1982) 1–6.
[4] J.A. Bondy and U.S.R. Murty, Graph Theory. Springer (2008).
[5] G. Dai, Degree sum and restricted $\{P_2, P_5\}$-factor in graphs. *Proc. Romanian Acad. Ser. A* **24** (2023) 105–111.
[6] G. Dai, Degree sum conditions for path-factor uniform graphs. *Indian J. Pure Appl. Math.* (2023) DOI: https://doi.org/10.1007/s13226-023-00446-7
[7] G. Dai, The existence of path-factor covered graphs. *Discuss. Math. Graph Theory* **43** (2023) 5–16.
[8] G. Dai and Z. Hu, P_3-factors in the square of a tree. *Graphs Comb.* **36** (2020) 1913–1925.
[9] G. Dai, Z. Zhang, Y. Hang and X. Zhang, Some degree conditions for $P_{\geq k}$-factor covered graphs. *RAIRO:RO* **55** (2021) 2907–2913.
[10] G. Dai, Y. Hang, X. Zhang, Z. Zhang and W. Wang, Sufficient conditions for graphs with $\{P_2, P_5\}$-factor. *RAIRO:RO* **56** (2022) 2895–2901.
[11] Y. Egawa, M. Kano and Z. Yan, Star-cycle factors of graphs. *Discuss. Math. Graph Theory* **34** (2014) 193–198.
[12] W. Gao and W. Wang, Remarks on component factors. *J. Oper. Res. Soc. China* **11** (2023) 657–666.
[13] A. Kaneko, A necessary and sufficient condition for the existence of a path factor every component of which is a path of length at least two. *J. Combin. Theory Ser. B.* **88** (2003) 195–218.
[14] M. Kano and A. Saito, Star-factors with large components. *Discrete Math.* **312** (2012) 2005–2008.
[15] M. Kano, G.Y. Katona and Z.Z. Király, Packing paths of length at least two. *Discrete Math.* **283** (2004) 129–135.
[16] M. Kano, C. Lee and K. Suzuki, Path and cycle factors of cubic bipartite graphs. *Discuss. Math. Graph Theory* **28** (2008) 551–556.
[17] M. Kano, H. Lu and Q. Yu, Component factors with large components in graphs. *Appl. Math. Lett.* **23** (2010) 385–389.
[18] M. Kano, H. Lu and Q. Yu, Fractional factors, component factors and isolated vertex conditions in graphs. *Electron. J. Comb.* **26** (2019) DOI: https://doi.org/10.37236/8498
[19] A. Klopp and E. Steffen, Fractional matchings, component-factors and edge-chromatic critical graphs. *Graphs Comb.* **37** (2021) 559–580.
[20] H. Liu, Binding number for path-factor uniform graphs. *Proc. Romanian Acad. ser. A* **23** (2022) 25–32.
[21] O. Ore, Notes on Hamilton circuits. *Am. Math. Mon.* **67** (1960) 55.
[22] W.T. Tutte, The factors of graphs. *Can. J. Math.* **4** (1952) 314–328.
[23] W.T. Tutte, The 1-factors of oriented graphs. *Proc. Am. Math. Soc.* **4** (1953) 922–931.
[24] Q. Yu and G. Liu, Graph Factors and Matching Extensions. Higher Education Press, Beijing (2009).
[25] P. Zhang and S. Zhou, Characterizations for $P_{\geq 2}$-factor and $P_{\geq 3}$-factor covered graphs. *Discrete Math.* **309** (2009) 2067–2076.
[26] S. Zhou, Some results about component factors in graphs. *RAIRO:RO* **53** (2019) 723–730.
[27] S. Zhou, J. Wu and T. Zhang, The existence of $P_{\geq 3}$-factor covered graphs. *Discuss. Math. Graph Theory* **37** (2017) 1055–1065.

[28] S. Zhou, Z. Sun and H. Liu, Sun toughness and $P_{\geq 3}$-factors in graphs. *Contrib. Discrete Math.* **14** (2019) 167–174.