RAIRO-Oper. Res. 58 (2024) 3859–3870
https://doi.org/10.1051/ro/2024153

RAIRO Operations Research
www.rairo-ro.org

SPECTRAL SUFFICIENT CONDITIONS FOR GRAPH FACTORS CONTAINING ANY EDGE

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Abstract. A factor of a graph is a spanning subgraph. Spectral sufficient conditions are provided *via* spectral radius and signless Laplacian spectral radius for graphs with (i) a matching of given size (particularly, 1-factor) containing any given edge, and (ii) a star factor with a component isomorphic to stars of order two or three containing any given edge, respectively.

Mathematics Subject Classification. 05C50.

Received April 25, 2024. Accepted July 19, 2024.

1. Introduction

We consider finite, simple, and undirected graphs. A factor of a graph is a spanning subgraph. For integers a and b with $0 \leq a \leq b$, an $[a, b]$-factor is defined as a factor F such that $a \leq d_F(x) \leq b$ for every vertex x, where $d_F(x)$ is the degree of x in F. A 1-factor is a $[1, 1]$-factor. Denote by $K_{1,m}$ the star of order $m+1$ (*i.e.*, a complete bipartite graph with partite sizes 1 and m). For positive integer k, let $S(k) = \{K_{1,1}, \ldots, K_{1,k}\}$. An $S(k)$-factor is factor for which each component is (isomorphic to) one of the stars $K_{1,1}, \ldots, K_{1,k}$. A $\{K_{1,1}\}$-factor is a 1-factor. A star factor is an $S(k)$-factor for some k. The existence of factors with given properties has received special attention; see, *e.g.* [2,5,7,10–12,16,17]. For example, a graph with factor containing any given edge is known as to be factor covered. In recent years, it is of great interest for researchers to find spectral sufficient conditions such that a graph has a given factor; see the survey [3]. Li and Miao [9] gave spectral condition that implies a graph has a factor consisting of vertex disjoint paths on at least two vertices that contains any given edge.

For two vertex disjoint graphs G_1 and G_2, and $G_1 \cup G_2$ denotes the disjoint union of G_1 and G_2, $G_1 \vee G_2$ denotes the join of G_1 and G_2, which is obtained from $G_1 \cup G_2$ by adding all possible edges between any vertex of G_1 and any vertex of G_2. For positive integer k and a graph G, kG denotes the graphs consisting of k vertex disjoint copies of G. Denote by K_n the complete graph of order n.

Given a graph, we denote by $\rho(G)$ the spectral radius of G and $q(G)$ the signless Laplacian spectral radius of G. We are concerned about two types of factors with given properties.

One is 1-factor containing any given edge, for which Little *et al.* [11] gave a characterization. As far as we know, although this concept has been extended to various factors that contain a given edge, there is no spectral sufficient condition for a graph has a 1-factor containing any given edge. Feng *et al.* [4], Suil [14] and Kim *et al.* [8] gave spectral radius conditions that imply a graph on n vertices has a matching of size (at least) $\frac{n-k}{2}$

Keywords. 1-factor, star factor, spectral radius, signless Laplacian spectral radius.

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for $0 \leq k \leq n$. Instead of directly considering 1-factor containing any given edge, we establish spectral radius conditions that imply a graph on n vertices has a matching of size (at least) $\frac{n-k}{2}$ for $0 \leq k \leq n$ containing any given edge. We show the following results.

Theorem 1.1. *Let G be a graph of order $n \geq 5k+6$, where $n \equiv k \pmod 2$. If $\rho(G) \geq \rho(K_2 \vee ((k+1)K_1 \cup K_{n-k-3}))$, then G has a matching of size $\frac{n-k}{2}$ containing any given edge unless $G \cong K_2 \vee ((k+1)K_1 \cup K_{n-k-3})$.*

Theorem 1.2. *Let G be a graph of order $n \geq 5k+7$, where $n \equiv k \pmod 2$. If $q(G) \geq q(K_2 \vee ((k+1)K_1 \cup K_{n-k-3}))$, then G has a matching of size $\frac{n-k}{2}$ containing any given edge unless $G \cong K_2 \vee ((k+1)K_1 \cup K_{n-k-3})$.*

The other is an $S(k)$-factor with a component being $K_{1,1}$ or $K_{1,2}$ containing any given edge, for which Chen, Egawa and Kano [2] gave a characterization. Also, we establish spectral conditions that imply a graph has an $S(k)$-factor with a component being $K_{1,1}$ or $K_{1,2}$ containing any given edge. Our results are as follows.

Theorem 1.3. *Let G be a graph of order $n \geq \frac{3}{2}k+7$ without isolated vertices. If $\rho(G) \geq \rho(K_2 \vee (2K_1 \cup K_{n-4}))$, then G has an $S(k)$-factor in which a component being $K_{1,1}$ or $K_{1,2}$ containing any given edge unless $G \cong K_2 \vee (2K_1 \cup K_{n-4})$.*

Theorem 1.4. *Let G be a graph of order $n \geq 2k+6$ without isolated vertices. If $q(G) \geq q(K_2 \vee (2K_1 \cup K_{n-4}))$, then G has an $S(k)$-factor in which a component being $K_{1,1}$ or $K_{1,2}$ containing any given edge unless $G \cong K_2 \vee (2K_1 \cup K_{n-4})$.*

2. Preliminaries

For a graph G, let $V(G)$ be the vertex set of G and $E(G)$ the edge set of G. For $v \in V(G)$, the neighborhood $N_G(v)$ of v is the set of vertices adjacent to v in G, and the degree of v, denoted by $d_G(v)$, is the number $|N_G(v)|$. For any $S \subseteq V(G)$, let $G[S]$ be the subgraph of G induced by S, and write $G - S = G[V(G) \setminus S]$ if $S \neq V(G)$.

The spectral radius of a graph G is the largest eigenvalue of the adjacency matrix of G, which is defined as the symmetric matrix $\mathbf{A}(G) = (a_{uv})_{u,v \in V(G)}$, where $a_{uv} = 1$ if u and v are adjacent, and $a_{uv} = 0$ otherwise. The signless Laplacian spectral radius of a graph G is the largest eigenvalue of its signless Laplacian matrix $\mathbf{Q}(G) = \mathbf{D}(G) + \mathbf{A}(G)$, where $\mathbf{D}(G)$ is the degree diagonal matrix of G.

For a square nonnegative matrix $\mathbf{M}$, let $\lambda(\mathbf{M})$ be its the spectral radius (maximum modulus of the eigenvalues), which is an eigenvalue of $\mathbf{M}$ by the Perron-Frobenius theorem. In particular, we have $\rho(G) = \lambda(\mathbf{A}(G))$ and $q(G) = \lambda(\mathbf{Q}(G))$ for a graph G.

The following lemma, following from the Perron-Frobenius theorem, is well known.

Lemma 2.1. *Let G be a graph and u and v two distinct vertices that are not adjacent in G. Then $\rho(G+uv) \geq \rho(G)$ and $q(G+uv) \geq q(G)$. Both inequalities are strict if $G+uv$ is connected.*

For a graph G, let $V(G) = V_1 \cup \cdots \cup V_s$ be a partition of $V(G)$. For $1 \leq i < j \leq s$, set $\mathbf{B}_{ij}$ denotes the submatrix of $\mathbf{H}(G) \in \{\mathbf{A}(G), \mathbf{Q}(G)\}$ with rows corresponding to vertices in V_i and columns corresponding to vertices in V_j. The matrix $\mathbf{B} = (b_{ij})$, where b_{ij} equals to the average row sums of $\mathbf{B}_{ij}$, is called the quotient matrix of $\mathbf{H}(G)$ with respect to the partition $V_1 \cup \cdots \cup V_s$. Furthermore, if $\mathbf{B}_{ij}$ has constant row sum, then we say that $\mathbf{B}$ is equitable. As an immediate consequence of Lemma 2.3.1 of [1], one gets the following lemma.

Lemma 2.2. *[1] If G is a connected graph and $\mathbf{B}$ is an equitable quotient matrix of $\mathbf{H}(G) \in \{\mathbf{A}(G), \mathbf{Q}(G)\}$, then $\lambda(\mathbf{H}(G))$ is equal to the largest eigenvalue of $\mathbf{B}$.*

Given a connected graph G and $\alpha = 0, 1$, we denote by $\lambda_\alpha(G)$ the spectral radius of the matrix $\alpha\mathbf{D}(G) + \mathbf{A}(G)$. By the Perron-Frobenius theorem, there is a unique unit positive eigenvector corresponding to $\lambda_\alpha(G)$, which is called the Perron vector. It is known that if there is an automorphism ϕ of G such that $\phi(u) = v$, then the entries of the Perron vector at u and v are equal. The following is known in [15] when $\alpha = 0$ and in [6] when $\alpha = 1$, see also [13].

Lemma 2.3. *Let G be a connected graph and u and v be two vertices of G. Let X be the Perron vector of $\alpha\mathbf{D}(G) + \mathbf{A}(G)$ with $x_u \geq x_v$, where $\alpha = \{0, 1\}$. Suppose that $N_G(v) \setminus (N_G(u) \cup \{u\}) \neq \emptyset$. Then for any nonempty $N \subseteq N_G(v) \setminus (N_G(u) \cup \{u\})$,*

$$\lambda_\alpha(G - \{vw : w \in N\} + \{uw : w \in N\}) > \lambda_\alpha(G).$$

For the graph $G = K_s \vee (K_{n_1} \cup \cdots \cup K_{n_t})$ we call the graph K_s appearing first the outer copy of G, and the i-th graph in $K_{n_1} \cup \cdots \cup K_{n_t}$ the i-th inner copy of G, where $i = 1, \ldots, t$.

Lemma 2.4. *For positive integers $s, t, n_1, \ldots n_t$ with $t \geq 2$ and $n_1 \leq \cdots \leq n_t$, let $G = K_s \vee (K_{n_1} \cup \cdots \cup K_{n_t})$ and X be the Perron vector of $\alpha\mathbf{D}(G) + \mathbf{A}(G)$ with $\alpha = \{0, 1\}$, where for $i = 1, \ldots, t$, x_i is the entry of X at any vertex of the i-th inner copy of G. Then $x_i \leq x_{i+1}$ for $i = 1, \ldots, t-1$.*

Proof. Denote by x_0 the entry of X at any vertex of the out copy of G. Then

$$(\lambda_\alpha(G) - \alpha(n_i - 1 + s) + 1 - n_i)x_i = sx_0 = (\lambda_\alpha(G) - \alpha(n_{i+1} - 1 + s) + 1 - n_{i+1})x_{i+1}$$

for $i = 1, \ldots, t-1$. As $n_i \leq n_{i+1}$ and $\lambda_\alpha(G) > \lambda_\alpha(K_{n_{i+1}}) = (\alpha + 1)(n_{i+1} - 1)$, one gets $x_i \leq x_{i+1}$. □

Given a graph G, $o(G)$ the number of odd components of G, and $i(G)$ the number of isolated vertices of G.

3. Graphs with a matching and 1-factor containing any given edge

We need the following lemma.

Lemma 3.1. *[11] Let G be a graph on n vertices. Let k be an integer with $0 \leq k \leq n$ and $k \equiv n \pmod 2$. Then G has a matching of size $\frac{n-k}{2}$ containing any given edge if and only if $o(G - S) \leq |S| + k$ for all $S \subset V(G)$ and $o(G - S) = |S| + k$ implies that S is an independent set.*

3.1. Spectral radius

Lemma 3.2. *Let n, s, k be positive integers with $n \geq \max\{5k + 6, 2s + k\}$, $s \geq 2$ and $0 \leq k \leq n$. For fixed n and k, $\rho(K_s \vee ((s + k - 1)K_1 \cup K_{n-2s-k+1}))$ is uniquely maximized when $s = 2$.*

Proof. Let $H_s = K_s \vee ((s+k-1)K_1 \cup K_{n-2s-k+1})$. Partition $V(H_s)$ into $S \cup V_1 \cup V_2$, where $V_1 = V((s+k-1)K_1)$ and $V_2 = V(K_{n-2s-k+1})$. With respect to this partition, the quotient matrix $\mathbf{B}$ of $\mathbf{A}(H_s)$ is equitable, where

$$\mathbf{B} = \begin{pmatrix} s-1 & s+k-1 & n-2s-k+1 \\ s & 0 & 0 \\ s & 0 & n-2s-k \end{pmatrix}.$$

By a simple calculation, the characteristic polynomial of $\mathbf{B}$ is

$$f_s(x) = x^3 - (n - k - s - 1)x^2 - (n + s^2 + (k-2)s - k)x - 2s^3 + (n - 3k + 2)s^2 + ((k-1)n - k^2 + k)s.$$

By Lemma 2.2, $\rho(H_s)$ is equal to the largest root of the equation $f_s(x) = 0$. In particular, for $H_2 = K_2 \vee ((k + 1)K_1 \cup K_{n-k-3})$, $\rho(H_2)$ is equal to the largest root of $f_2(x) = 0$.

Suppose that $s > 2$. We only need to show that $\rho(H_s) < \rho(H_2)$. Note that

$$\frac{f_s(x) - f_2(x)}{s - 2} = h(x) := x^2 - (s + k)x - 2s^2 + (n - 3k - 2)s + (k + 1)n - k^2 - 5k - 4.$$

As a quadratic function of x, $h(x)$ is strictly increasing for $x \geq x_0 := \frac{s+k}{2}$. As $n \geq 5k + 6 \geq \frac{5}{3}k + \frac{8}{3}$ and $n \geq 2s + k$, one gets $x_0 \leq n - k - 2$. Then, for $x \in [n - k - 2, +\infty)$, $h(x)$ is strictly increasing, so

$$h(x) \geq h(n - k - 2)$$

$$\begin{aligned}
&= -2s^2 - 2ks + n^2 - (2k+3)n + k^2 + k \\
&\geq -\frac{(n-k)^2}{2} - k(n-k) + n^2 - (2k+3)n + k^2 + k \\
&= \frac{1}{2}n^2 - (2k+3)n + \frac{3}{2}k^2 + k \\
&\geq \frac{1}{2}(5k+6)^2 - (2k+3)(5k+6) + \frac{3}{2}k^2 + k \\
&= 4k^2 + 4k \\
&\geq 0,
\end{aligned}$$

where the second inequality follows because as a quadratic function of s, $h(n-k-2)$ is strictly decreasing for $2 \leq s \leq \frac{n-k}{2}$, and the third inequality follows because as a quadratic function of n, $\frac{1}{2}n^2 - (2k+3)n - \frac{1}{2}k^2 + k$ is strictly increasing for $n \geq 5k+6 > 2k+3$. Thus, $h(x) > 0$ for $x \in [n-k-2, +\infty)$, as $h(x) = 0$ implies $k = 0$, but then $h(x) \geq h(n-2) > 0$, a contradiction. That is, $f_s(x) > f_2(x)$ for $x \in [n-k-2, +\infty)$. By Lemma 2.1, we have $\rho(H_2) > \rho(K_{n-k-1}) = n-k-2$, so $f_s(x) > f_2(x) \geq 0$ for $x \in [\rho(H_2), +\infty)$. implying that $\rho(H_s) < \rho(H_2)$. □

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Suppose by contradiction that G does not have a matching of size $\frac{n-k}{2}$ containing any given edge. By Lemma 3.1, there exists a vertex subset $S \subset V(G)$ with $|S| = s$ such that $o(G-S) \geq s+k$ and if $G[S]$ is empty, then $o(G-S) > s+k$. Evidently, $n \equiv s + o(G-S) \pmod 2$, *i.e.*, $o(G-S) \equiv n-s \pmod 2$. As $n \equiv k \pmod 2$, one gets $o(G-S) \equiv k-s \equiv s+k \pmod 2$. Thus, if $G[S]$ is empty, then $o(G-S) \geq s+k+2$. Let $t = o(G-S)$, $n_1 \leq \cdots \leq n_{t-1}$ be the orders the odd components of $G-S$ with the first $t-1$ smallest orders, and let $n_t = n - s - n_1 - \cdots - n_{t-1}$. Then G is a spanning subgraph of the graph $G' := G[S] \vee (K_{n_1} \cup \cdots \cup K_{n_t})$. By Lemma 2.1, $\rho(G) \leq \rho(G')$ with equality when $S \neq \emptyset$ if and only if $G \cong G'$. Let $H_s = K_s \vee ((s+k-1)K_1 \cup K_{n-2s-k+1})$.

Case 1. $G[S]$ is not empty.
In this case, $s \geq 2$ and $n \geq s + o(G-S) \geq 2s+k$. So $2 \leq s \leq \frac{n-k}{2}$. By Lemma 2.1 again, $\rho(G) \leq \rho(G^*)$ with equality if and only if $G \cong G^*$, where $G^* := K_s \vee (K_{n_1} \cup \cdots \cup K_{n_t})$. By Lemmas 2.3 and 2.4, we have $\rho(G^*) \leq \rho(H_s)$ with equality if and only if $G^* \cong H_s$. Thus $\rho(G) \leq \rho(H_s)$ with equality if and only if $G \cong H_s$. By the assumption, $\rho(G) \geq \rho(H_2)$. Thus

$$\rho(H_2) \leq \rho(G) \leq \rho(H_s).$$

By Lemma 3.2, $\rho(G) = \rho(H_2)$, so $G \cong H_2$, which is a contradiction.

Case 2. $G[S]$ is empty.
Suppose first that $s = 0$. Then $t \geq k+2$ and $\rho(G) \leq \rho(G') \leq \rho(K_{n_t}) \leq n-k-2$. Since $\rho(H_2) > \rho(K_{n-k-1}) = n-k-2$, we have $\rho(G) \leq \rho(K_{n_t}) < \rho(H_2)$, which is a contradiction.
Suppose next that $s = 1$. Then $\rho(G) \leq \rho(K_1 \vee (K_{n_1} \cup \cdots \cup K_{n_{k+3}}))$. By Lemmas 2.3 and 2.4, we have

$$\rho(K_1 \vee (K_{n_1} \cup \cdots \cup K_{n_{k+3}})) \leq \rho(K_1 \vee ((k+2)K_1 \cup K_{n-k-3})).$$

By Lemma 2.1, we have

$$\rho(K_1 \vee ((k+2)K_1 \cup K_{n-k-3})) < \rho(H_2).$$

Thus, we have $\rho(G) < \rho(H_2)$, a contradiction.
Now, suppose that $s \geq 2$. By Lemmas 2.3 and 2.4, we have

$$\rho(G') \leq \rho(sK_1 \vee ((s+k+1)K_1 \cup K_{n-2s-k-1}))$$

with equality if and only if $G' \cong sK_1 \vee ((s+k+1)K_1 \cup K_{n-2s-k-1})$. By Lemma 2.1, $\rho(sK_1 \vee ((s+k+1)K_1 \cup K_{n-2s-k-1})) < \rho(H_s)$. It follows that $\rho(G) < \rho(H_s)$. By Lemma 3.2, $\rho(G) < \rho(H_2)$, a contradiction.

□

As an immediate consequence of Theorem 1.1, we have

Corollary 3.3. *Let G be a graph of order $n \geq 6$, where $n \equiv 0 \pmod 2$. If $\rho(G) \geq \rho(K_2 \vee (K_1 \cup K_{n-3}))$, then G has a 1-factor containing any given edge unless $G \cong K_2 \vee (K_1 \cup K_{n-3})$.*

3.2. Signless Laplacian spectral radius

Lemma 3.4. *Let n, s, k be positive integers with $n \geq \max\{5k+7, 2s+k\}$, $s \geq 2$ and $0 \leq k \leq n$. For fixed n and k, $q(K_s \vee ((s+k-1)K_1 \cup K_{n-2s-k+1}))$ is uniquely maximized when $s = 2$.*

Proof. Let $H_s = K_s \vee ((s+k-1)K_1 \cup K_{n-2s-k+1})$. Partition $V(H_s)$ into $S \cup V_1 \cup V_2$, where $V_1 = V((s+k-1)K_1)$ and $V_2 = V(K_{n-2s-k+1})$. With respect to this partition, the quotient matrix

$$\mathbf{B} = \begin{pmatrix} n+s-2 & s+k-1 & n-2s-k+1 \\ s & s & 0 \\ s & 0 & 2n-3s-2k \end{pmatrix}.$$

of $\mathbf{Q}(H_s)$ is equitable. By a simple calculation, the characteristic polynomial of $\mathbf{B}$ is

$$\begin{aligned} f_s(x) = {} & x^3 - (3n-2k-s-2)x^2 - (4s^2 - (n-4k+4)s - 2n^2 + 2(k+2)n - 4k)x \\ & - 2s^3 + (4n-4k-2)s^2 - (2n^2 - 2(2k+1)n + 2k^2 + 2k)s. \end{aligned}$$

By Lemma 2.2, $q(H_s)$ is equal to the largest root of the equation $f_s(x) = 0$. Similarly, let $H_2 = K_2 \vee ((k+1)K_1 \cup K_{n-k-3})$. $q(H_2)$ is equal to the largest root of $f_2(x) = 0$.

Suppose that $s > 2$. It is easy to see that

$$\begin{aligned} \frac{f_s(x) - f_2(x)}{s-2} = h(x) := {} & x^2 + (n-4s-4k-4)x - 2s^2 + (4n-4k-6)s - 2n^2 \\ & + (4k+10)n - 2k^2 - 10k - 12. \end{aligned}$$

Evidently, $h(x)$ is strictly increasing for $x \geq x_0 := \frac{4s-n+4k+4}{2}$. As $n \geq \max\{2k+4, 2s+k\}$, one gets $x_0 \leq 2n-2k-4$. It then follows that $h(x)$ is strictly increasing for $x \in [2n-2k-4, +\infty)$. Thus, for $x \in [2n-2k-4, +\infty)$, one gets

$$\begin{aligned} h(x) & \geq h(2n-2k-4) \\ & = -2s^2 - (4n-4k-10)s + 4n^2 - (14k+18)n + 10k^2 + 30k + 20 \\ & \geq -\frac{(n-k)^2}{2} - \frac{(4n-4k-10)(n-k)}{2} + 4n^2 - (14k+18)n + 10k^2 + 30k + 20 \\ & = \frac{3}{2}n^2 - (9k+13)n + \frac{15}{2}k^2 + 25k + 20 \\ & \geq \frac{3}{2}(5k+7)^2 - (9k+13)(5k+7) + \frac{15}{2}k^2 + 25k + 20 \\ & = 2k + \frac{5}{2} \\ & > 0, \end{aligned}$$

where the second inequality follows because $h(2n-2k-4)$ is strictly decreasing for $2 \le s \le \frac{n-k}{2}$, and the third inequality follows because $\frac{3}{2}n^2 - (9k+13)n + \frac{15}{2}k^2 + 25k + 20$ is strictly increasing for $n \ge 5k+7 > 3k + \frac{13}{3}$. Thus $f_s(x) > f_2(x)$ for $x \in [2n-2k-4, +\infty)$. By Lemma 2.1, we have $q(H_2) > q(K_{n-k-1}) = 2n-2k-4$, so $f_s(x) > f_2(x) \ge 0$ for $x \in [q(H_2), +\infty)$. implying that $q(H_s) < q(H_2)$. □

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. Suppose by contradiction that G does not have a matching of size $\frac{n-k}{2}$ containing any given edge. By Lemma 3.1, there exists a vertex subset $S \subset V(G)$ with $|S| = s$ such that either $o(G-S) \ge s+k$ or $G[S]$ is empty and $o(G-S) > s+k$. Evidently, $n \equiv s + o(G-S) \pmod 2$, *i.e.*, $o(G-S) \equiv n - s \pmod 2$. As $n \equiv k \pmod 2$, one gets $o(G-S) \equiv k - s \equiv s + k \pmod 2$. Thus, if $G[S]$ is empty, then $o(G-S) \ge s+k+2$. Let $t = o(G-S)$, $n_1 \le \cdots \le n_{t-1}$ be the orders the odd components of $G-S$ with the first $t-1$ smallest orders, and let $n_t = n - s - n_1 - \cdots - n_{t-1}$. Then G is a spanning subgraph of the graph $G' := G[S] \vee (K_{n_1} \cup \cdots \cup K_{n_t})$. By Lemma 2.1, $q(G) \le q(G')$ with equality when $S \ne \emptyset$ if and only if $G \cong G'$. Let $H_s = K_s \vee ((s+k-1)K_1 \cup K_{n-2s-k+1})$.

Case 1. $G[S]$ is not empty.
In this case, $s \ge 2$ and $n \ge s + o(G-S) \ge 2s+k$. So $2 \le s \le \frac{n-k}{2}$. By Lemma 2.1 again, $q(G) \le q(G^*)$ with equality if and only if $G \cong G^*$, where $G^* := K_s \vee (K_{n_1} \cup \cdots \cup K_{n_t})$. By Lemmas 2.3 and 2.4, we have $q(G^*) \le q(H_s)$ with equality if and only if $G^* \cong H_s$. Thus $q(G) \le q(H_s)$ with equality if and only if $G \cong H_s$. By the assumption, $q(G) \ge q(H_2)$. Thus $q(H_2) \le q(G) \le q(H_s)$ By Lemma 3.4, $q(G) = q(H_2)$, so $G \cong H_2$, which is a contradiction.

Case 2. $G[S]$ is empty.
Suppose first that $s = 0$. Then $t \ge k+2$ and $q(G) \le q(G') \le q(K_{n_t}) \le 2n-2k-4$. Since $q(H_2) > q(K_{n-k-1}) = 2n-2k-4$, we have $q(G) \le q(K_{n_t}) < q(H_2)$, which is a contradiction.
Suppose next that $s = 1$. Then $q(G) \le q(K_1 \vee (K_{n_1} \cup \cdots \cup K_{n_{k+3}}))$. By Lemmas 2.3 and 2.4, we have

$$q(K_1 \vee (K_{n_1} \cup \cdots \cup K_{n_{k+3}})) \le q(K_1 \vee ((k+2)K_1 \cup K_{n-k-3})).$$

By Lemma 2.1, we have

$$q(K_1 \vee ((k+2)K_1 \cup K_{n-k-3})) < q(H_2).$$

Thus, we have $q(G) < q(H_2)$, a contradiction.
Finally, suppose that $s \ge 2$. By Lemmas 2.3 and 2.4, we have

$$q(G') \le q(sK_1 \vee ((s+k+1)K_1 \cup K_{n-2s-k-1}))$$

with equality if and only if $G' \cong sK_1 \vee ((s+k+1)K_1 \cup K_{n-2s-k-1})$. By Lemma 2.1, $q(sK_1 \vee ((s+k+1)K_1 \cup K_{n-2s-k-1})) < q(H_s)$. It thus follows that $q(G) < q(H_s)$. By Lemma 3.4, $q(G) < q(H_2)$, a contradiction.

□

By Theorem 1.2, we have

Corollary 3.5. *Let G be a graph of order $n \ge 7$, where $n \equiv 0 \pmod 2$. If $q(G) \ge q(K_2 \vee (K_1 \cup K_{n-3}))$, then G has a 1-factor containing any given edge unless $G \cong K_2 \vee (K_1 \cup K_{n-3})$.*

4. Star factor with a component $K_{1,1}$ or $K_{1,2}$ containing any given edge

We need the following lemma.

Lemma 4.1. *[2] Let G be a graph and let k be an integer, $k \geq 2$. Then G has an $S(k)$-factor with a component being $K_{1,1}$ or $K_{1,2}$ containing any given edge if and only if, for any proper subset S of $V(G)$,*

$$i(G-S) \leq \begin{cases} k|S| & \text{if } G[S] \text{ is empty,} \\ k|S|-2k+1 & \text{otherwise.} \end{cases}$$

4.1. Spectral radius

Lemma 4.2. *Let n, s, k be positive integers with $n \geq \max\{\frac{3}{2}k+5, (k+1)s+1\}$, $s \geq 1$ and $k \geq 2$. For fixed n and k, $\rho(sK_1 \vee ((ks+1)K_1 \cup K_{n-(k+1)s-1}))$ is uniquely maximized when $s=1$.*

Proof. Let $G_s = sK_1 \vee ((ks+1)K_1 \cup K_{n-(k+1)s-1})$. Partition $V(G_s)$ into $S \cup V_1 \cup V_2$, where $V_1 = V((ks+1)K_1)$ and $V_2 = V(K_{n-(k+1)s-1})$. With respect to this partition, the quotient matrix

$$\mathbf{B} = \begin{pmatrix} 0 & ks+1 & n-(k+1)s-1 \\ s & 0 & 0 \\ s & 0 & n-(k+1)s-2 \end{pmatrix}.$$

of $\mathbf{A}(G_s)$ is equitable. By a simple calculation, the characteristic polynomial of $\mathbf{B}$ is

$$f_s(x) = x^3 - (n-s-ks-2)x^2 - (ns-s^2)x - (k^2+k)s^3 + (kn-3k-1)s^2 + (n-2)s.$$

By Lemma 2.2, $\rho(G_s)$ is equal to the largest root of the equation $f_s(x)=0$. In particular, $\rho(G_1)$ is equal to the largest root of $f_1(x)=0$.

Suppose that $s>1$. It is easy to see that

$$\begin{aligned} \frac{f_s(x)-f_1(x)}{s-1} &= h(x) := (k+1)x^2 - (n-s-1)x - (k^2+k)s^2 + (kn-k^2-4k-1)s \\ &\quad + (k+1)n - k^2 - 4k - 3. \end{aligned}$$

Note that $h(x)$ is strictly increasing for $x \geq x_0 := \frac{n-s-1}{2(k+1)}$. As $n \geq \frac{2k^2+6k+2}{2k+1}$ and $s \geq 1$, one gets $x_0 \leq n-k-2$. It then follows that $h(x)$ is strictly increasing for $x \in [n-k-2, +\infty)$. Thus, one may check that, for $x \in [n-k-2, +\infty)$,

$$\begin{aligned} h(x) &\geq h(n-k-2) \\ &= -(k^2+k)s^2 + ((k+1)n-k^2-5k-3)s + kn^2 - (2k^2+4k)n + k^3 + 4k^2 + 3k - 1 \\ &\geq -(k^2+k) + ((k+1)n-k^2-5k-3) + kn^2 - (2k^2+4k)n + k^3 + 4k^2 + 3k - 1 \\ &= kn^2 - (2k^2+3k-1)n + k^3 + 2k^2 - 3k - 4 \\ &\geq k\left(\frac{3}{2}k+5\right)^2 - (2k^2+3k-1)\left(\frac{3}{2}k+5\right) + k^3 + 2k^2 - 3k - 4 \\ &= \frac{1}{4}k^3 + \frac{5}{2}k^2 + \frac{17}{2}k + 1 \\ &> 0. \end{aligned}$$

Thus $f_s(x) > f_1(x)$ for $x \in [n-k-2, +\infty)$. By Lemma 2.1, we have $\rho(G_1) > \rho(K_{n-k-1}) = n-k-2$, so $f_s(x) > f_1(x) \geq 0$ for $x \in [\rho(H_1), +\infty)$ implying that $\rho(G_s) < \rho(G_1)$. □

Lemma 4.3. *Let n, s, k be positive integers with $n \geq \max\{k+7, (k+1)s-2k+2\}$, $s \geq 2$ and $k \geq 2$. For fixed n and k, $\rho(K_s \vee ((ks-2k+2)K_1 \cup K_{n-(k+1)s+2k-2}))$ is uniquely maximized when $s = 2$.*

Proof. Let $G_s = K_s \vee ((ks-2k+2)K_1 \cup K_{n-(k+1)s+2k-2})$. Partition $V(G_s)$ into $S \cup V_1 \cup V_2$, where $V_1 = V((ks-2k+2)K_1)$ and $V_2 = V(K_{n-(k+1)s+2k-2})$. With respect to this partition, the quotient matrix of $\mathbf{Q}(G_s)$ is

$$\mathbf{B} = \begin{pmatrix} s-1 & ks-2k+2 & n-(k+1)s+2k-2 \\ s & 0 & 0 \\ s & 0 & n-(k+1)s+2k-3 \end{pmatrix},$$

which is equitable. By a simple calculation, the characteristic polynomial of $\mathbf{B}$ is

$$\begin{aligned} f_s(x) = x^3 &- (n+2k-ks-4)x^2 - (n+ks^2-(3k-2)s+2k-3)x - (k^2+k)s^3 \\ &+ (kn+4k^2-3k-2)s^2 - (2(k-1)n+4k^2-10k+6)s. \end{aligned}$$

By Lemma 2.2, $\rho(G_s)$ is equal to the largest root of the equation $f_s(x) = 0$. Particularly, $\rho(G_2)$ is equal to the largest root of $f_2(x) = 0$.

Suppose that $s > 2$. It is easy to see that

$$\frac{f_s(x) - f_2(x)}{s-2} = h(x) := kx^2 - (ks-k+2)x - (k^2+k)s^2 + (kn+2k^2-5k-2)s + 2n - 10.$$

Note that $h(x)$ is is strictly increasing for $x \geq x_0 := \frac{ks-k+2}{2k}$. As $n \geq \max\{\frac{7k^2+5k+2}{2k^2+k}, (k+1)s+1\}$, one gets $x_0 \leq n-3$. Thus, for $x \in [n-3, +\infty)$, we have

$$\begin{aligned} h(x) &\geq h(n-3) \\ &= -(k^2+k)s^2 + (2k^2-2k-2)s + kn^2 - 5kn + 6k - 4 \\ &\geq -(k^2+k)\left(\frac{n+2k-2}{k+1}\right)^2 + (2k^2-2k-2)\frac{n+2k-2}{k+1} + kn^2 - 5kn + 6k - 4 \\ &= \frac{k^2}{k+1}n^2 - \frac{7k^2+3k+2}{k+1}n + \frac{6k^2-2k}{k+1} \\ &\geq \frac{k^2}{k+1}(k+7)^2 - \frac{7k^2+3k+2}{k+1}(k+7) + \frac{6k^2-2k}{k+1} \\ &= \frac{1}{k+1}(k^4+7k^3+3k^2-25k-14) \\ &> 0. \end{aligned}$$

Thus $f_s(x) > f_2(x)$ for $x \in [n-3, +\infty)$. By Lemma 2.1, we have $\rho(G_2) > \rho(K_{n-2}) = n-3$, so $f_s(x) > f_2(x) \geq 0$ for $x \in [\rho(G_2), +\infty)$, implying that $\rho(H_s^*) \leq \rho(G_2)$. □

Now we are ready to prove Theorem 1.3.

Proof of Theorem 1.3. Suppose by contradiction that G does not have an $S(k)$-factor containing any given edge. By Lemma 4.1, there exists a vertex subset $S \subset V(G)$ with $|S| = s$ such that

$$i(G-S) \geq \begin{cases} ks+1 & \text{if } G[S] \text{ is empty,} \\ ks-2k+2 & \text{otherwise.} \end{cases}$$

As G has no isolated vertices, $s \geq 1$. By Lemma 2.1, $\rho(G) \leq \rho(G_s)$ with equality if and only if $G \cong G_s$, where

$$G_s := \begin{cases} sK_1 \vee ((ks+1)K_1 \cup K_{n-(k+1)s-1}) & \text{if } G[S] \text{ is empty,} \\ K_s \vee ((ks-2k+2)K_1 \cup K_{n-(k+1)s+2k-2}) & \text{otherwise.} \end{cases}$$

Suppose first that $G[S]$ is empty. It is evident that $s \geq 1$. From $n \geq s + i(G - S)$, we have $n \geq (k+1)s + 1$. So $2 \leq s \leq \frac{n-1}{k+1}$. By Lemmas 4.2 and 2.1, $\rho(G) \leq \rho(G_s) \leq \rho(G_1) < \rho(G_2)$, contracting the assumption.

Suppose next $G[S]$ is not empty. Then $s \geq 2$. From $n \geq s + i(G - S)$, we have $n \geq (k+1)s - 2k + 2$. Thus $2 \leq s \leq \frac{n+2k-2}{k+1}$. By Lemma 4.3, $\rho(G) \leq \rho(G_s) \leq \rho(G_2)$, By the assumption, $\rho(G) \geq \rho(G_2)$. Thus $\rho(G) = \rho(G_2)$, implying that $G \cong G_2$, a contradiction. □

4.2. Signless Laplacian spectral radius

Lemma 4.4. *Let n, s, k be positive integers with $n \geq \max\{2k+4, (k+1)s+1\}$, $s \geq 1$ and $k \geq 2$. For fixed n and k, $q(sK_1 \vee ((ks+1)K_1 \cup K_{n-(k+1)s-1}))$ is uniquely maximized when $s = 1$.*

Proof. Let $H'_s = sK_1 \vee ((ks+1)K_1 \cup K_{n-(k+1)s-1})$. Partition $V(H'_s)$ into $S \cup V_1 \cup V_2$, where $V_1 = V((ks+1)K_1)$ and $V_2 = V(K_{n-(k+1)s-1})$.

It is easy to see that this partition is an equitable partition. The corresponding quotient matrix is

$$\mathbf{B} = \begin{pmatrix} n-s & ks+1 & n-(k+1)s-1 \\ s & s & 0 \\ s & 0 & 2n-(2k+1)s-4 \end{pmatrix}.$$

By a simple calculation, the characteristic polynomial of $\mathbf{B}$ is

$$\begin{aligned} f_s(x) = {} & x^3 - (3n-(2k+1)s-4)x^2 + (2n^2-4n-(2k+1)ns)x - (2k^2+4k+2)s^3 \\ & + (4(k+1)n-6k-6)s^2 - (2n^2-6n+4)s. \end{aligned}$$

By Lemma 2.2, $q(H'_s)$ is equal to the largest root of the equation $f_s(x) = 0$, and $q(H'_1)$ is equal to the largest root of $f_1(x) = 0$.

Suppose that $s > 1$. It is easy to see that

$$\begin{aligned} \frac{f_s(x) - f_1(x)}{s-1} = h(x) := {} & (2k+1)x^2 - (2k+1)nx - (2k^2+4k+2)s^2 \\ & + (4kn+4n-2k^2-10k-8)s - 2n^2 + (4k+10)n - 2k^2 - 10k - 12. \end{aligned}$$

Note that $h(x)$ is strictly increasing for $x \geq x_0 := \frac{n}{2}$. As $n \geq \frac{4}{3}k + \frac{8}{3}$, one gets $x_0 \leq 2n - 2k - 4$. It then follows that $h(x)$ is strictly increasing for $x \in [2n-2k-4, +\infty)$. Thus

$$\begin{aligned} h(x) &\geq h(2n-2k-4) \\ &= -(2k^2+4k+2)s^2 + (4(k+1)n-2k^2-10k-8)s + 4kn^2 - (12k^2+26k+2)n \\ &\quad + 8k^3 + 34k^2 + 38k + 4 \\ &\geq -(2k^2+4k+2) + (4(k+1)n-2k^2-10k-8) + 4kn^2 - (12k^2+26k+2)n \\ &\quad + 8k^3 + 34k^2 + 38k + 4 \\ &= 4kn^2 - (12k^2+22k-2)n + 8k^3 + 30k^2 + 24k - 6 \\ &\geq 4k(2k+4)^2 - (12k^2+22k-2)(2k+4) + 8k^3 + 30k^2 + 24k - 6 \\ &= 2k^2 + 4k + 2 \\ &> 0. \end{aligned}$$

Thus $f_s(x) > f_1(x)$ for $x \in [2n-2k-4, +\infty)$. By Lemma 2.1, we have $q(H_1) > q(K_{n-k-1}) = 2n-2k-4$, so $f_s(x) > f_1(x) \geq 0$ for $x \in [q(H_1), +\infty)$, implying that $q(H'_s) \leq q(H_1)$. □

Lemma 4.5. *Let n, s, k be positive integers with $n \geq \max\{2k+6, (k+1)s-2k+2\}$, $s \geq 2$ and $k \geq 2$. For fixed n and k, $q(K_s \vee ((ks-2k+2)K_1 \cup K_{n-(k+1)s+2k-2}))$ is uniquely maximized when $s = 2$.*

Proof. Let $H_s^* = K_s \vee ((ks-2k+2)K_1 \cup K_{n-(k+1)s+2k-2})$. Partition $V(H_s^*)$ into $S \cup V_1 \cup V_2$, where $V_1 = V((ks-2k+2)K_1)$ and $V_2 = V(K_{n-(k+1)s+2k-2})$.

It is easy to see that this partition is an equitable partition. The corresponding quotient matrix is

$$\mathbf{B} = \begin{pmatrix} n+s-2 & ks-2k+2 & n-(k+1)s+2k-2 \\ s & s & 0 \\ s & 0 & 2n-(2k+1)s+4k-6 \end{pmatrix}.$$

By a simple calculation, the characteristic polynomial of $\mathbf{B}$ is

$$\begin{aligned} f_s(x) = x^3 &- (3n-(2k-1)s+4k-8)x^2 + (2n^2+(4k-10)n-4ks^2 \\ &- ((2k-3)n-12k+12)s-8k+12)x - 2k^2s^3 + (4kn+8k^2-14k)s^2 \\ &- (2n^2+(8k-14)n+8k^2-28k+24)s. \end{aligned}$$

By Lemma 2.2, $q(H_s^*)$ is equal to the largest root of the equation $f_s(x)=0$, and $q(H_2^*)$ is equal to the largest root of $f_2(x)=0$.

Suppose that $s>2$. It is easy to see that

$$\begin{aligned} \frac{f_s(x)-f_2(x)}{s-2} = h(x) &:= (2k-1)x^2 - ((2k-3)n+4ks-4k+12)x - 2k^2s^2 \\ &+ (4kn+4k^2-14k)s - 2n^2 + 14n - 24. \end{aligned}$$

As $n \geq \max\{\frac{28k^2+12k}{6k^2+k-1}, (k+1)s-2k+2\}$, one gets $x_0 := \frac{(2k-3)n+4ks-4k+12}{2(2k-1)} \leq 2n-6$. So $h(x)$ is strictly increasing for $x \in [2n-6, +\infty)$. Thus

$$\begin{aligned} h(x) &\geq h(2n-6) \\ &= -2k^2s^2 - (4kn-4k^2-10k)s + 4kn^2 - (28k+4)n + 48k + 12 \\ &\geq -2k^2\left(\frac{n+2k-2}{k+1}\right)^2 - (4kn-4k^2-10k)\frac{n+2k-2}{k+1} + 4kn^2 - (28k+4)n + 48k + 12 \\ &= \frac{1}{(k+1)^2}\left((4k^3+2k^2)n^2 - (40k^3+38k^2+18k+4)n + 84k^3 + 92k^2 + 52k + 12\right) \\ &\geq \frac{1}{(k+1)^2}((4k^3+2k^2)(2k+6)^2 - (40k^3+38k^2+18k+4)(2k+6) \\ &\quad + 84k^3 + 92k^2 + 52k + 12) \\ &= \frac{16k^5+24k^4-40k^3-100k^2-64k-12}{(k+1)^2} \\ &= 4(4k^3-2k^2-10k-3) \\ &> 0. \end{aligned}$$

Thus $f_s(x) > f_2(x)$ for $x \in [2n-6, +\infty)$. By Lemma 2.1, we have $q(H_2^*) > q(K_{n-2}) = 2n-6$, so $f_s(x) > f_2(x) \geq 0$ for $x \in [q(H_2^*), +\infty)$, implying that $q(H_s^*) \leq q(H_2^*)$. ☐

Now we are ready to prove Theorem 1.4.

Proof of Theorem 1.4. Suppose by contradiction that G does not have an $S(k)$-factor containing any given edge. By Lemma 4.1, there exists a vertex subset $S \subset V(G)$ with $|S| = s$ such that

$$i(G-S) \geq \begin{cases} ks+1 & \text{if } G[S] \text{ is empty,} \\ ks-2k+2 & \text{otherwise.} \end{cases}$$

As G has no isolated vertices, $s \geq 1$. By Lemma 2.1, $q(G) \leq q(G_s)$ with equality if and only if $G \cong G_s$, where

$$G_s := \begin{cases} sK_1 \vee ((ks+1)K_1 \cup K_{n-(k+1)s-1}) & \text{if } G[S] \text{ is empty,} \\ K_s \vee ((ks-2k+2)K_1 \cup K_{n-(k+1)s+2k-2}) & \text{otherwise.} \end{cases}$$

Suppose first that $G[S]$ is empty. It is evident that $s \geq 1$. From $n \geq s + i(G-S)$, we have $n \geq (k+1)s+1$. So $2 \leq s \leq \frac{n-1}{k+1}$. By Lemmas 4.4 and 2.1, $q(G) \leq q(G_s) \leq q(G_1) < q(G_2)$, contracting the assumption.

Suppose next $G[S]$ is not empty. Then $s \geq 2$. From $n \geq s + i(G-S)$, we have $n \geq (k+1)s - 2k + 2$. Thus $2 \leq s \leq \frac{n+2k-2}{k+1}$. By Lemma 4.5, $q(G) \leq q(G_s) \leq q(G_2)$, By the assumption, $q(G) \geq q(G_2)$. Thus $q(G) = q(G_2)$, implying that $G \cong G_2$, a contradiction. □

Acknowledgements

The authors thank the referees for helpful comments.

Funding

This work was supported by National Natural Science Foundation of China (No. 12071158).

Conflict of Interest

The authors declare that they have no conflicts of interest.

Data availability statement

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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