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DEGREE CONDITIONS FOR PATH-FACTORS IN GRAPHS

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Abstract. A spanning subgraph H of a graph G is called a path-factor if every component of H is a path. Wang and Zhang [*RAIRO:RO* **57** (2023) 2231–2237] conjectured that a connected graph G with $\delta(G) \geq 5$ contains a $\{P_2, P_5\}$-factor if $\delta(G) \geq \frac{3\alpha(G)-1}{4}$, where $\delta(G)$ and $\alpha(G)$ denote the minimum degree and independence number of G, respectively. We show that the conjecture is true except $G \cong X \vee 7K_3$, where X is a spanning subgraph of K_3. Furthermore, we give two degree conditions for the existence of $\{P_2, P_5\}$-factors, one of which is a stronger version of Wang's another conjecture. We also show the degree conditions are best possible.

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1. Introduction

In mathematics and computer science, a graph is a mathematical structure that consists of two main components: vertices (or nodes) and edges. The study of these graphs in various contexts is called graph theory. There are various applications of graph theory in real life such as in computer graphics and networks, biology and many other fields as well. Graph data structures are powerful tools for representing and analyzing complex relationships between objects or entities. They are particularly useful in fields such as social network analysis, recommendation systems and computer networks. In the field of data transmission, graph data structures can be used to analyze and understand the availability and rate between different sites of a network. The existence of special subgraphs, such as paths, is useful to reflect graph data structures. Research on the existence of path factors can help scientists to design and construct networks with high data transmission rates [19]. In this paper, we mainly focus on the conditions for the existence of paths factors, which could present theoretical guidance for data transmission to meet rate or other requirements [19].

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For $v \in V(G)$, we use $d_G(v)$ to denote the degree of v in G. $\delta(G) = \min\{d_G(v)|v \in V(G)\}$ is called the *minimum degree* of G. Let $S \subseteq V(G)$. If any two vertices of S are not adjacent in G, then we call S an *independent set*. The order of the maximum independent set in G is called the *independence number* of G, denoted by $\alpha(G)$. We use $G[S]$ to denote the subgraph of G induced by S, and use $G - S$ to denote the graph obtained from G by deleting the vertices of S and edges with at least one endpoint in S. We often use $d_S(v)$ to denote $d_{G[S]}(v)$ shortly. Let P_k be the path of order k. The two vertices of degree one on P_k are called its *endpoints*. Let $P_{2t+1} = v_1v_2 \dots v_{2t+1}$. Then v_{t+1} is called the *center*

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of P_{2t+1}. For two disjoint graphs G and H, the *union* of G and H, denoted by $G \cup H$, is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. For positive integer k and graph H, we also use kH to denote the union of k graphs isomorphic to H. The *join* of G and H, denoted by $G \vee H$, is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{uv | u \in V(G), v \in V(H)\}$. For positive integer i, we use $c_i(G)$ to denote the number of components of order i in G.

A subgraph H of G is called a *spanning subgraph* of G if $V(H) = V(G)$. For a graph G and a set $\mathcal{H}$ of connected graphs, an $\mathcal{H}$-*factor* of G is a spanning subgraph of G with each component isomorphic to some member in $\mathcal{H}$. A $\{P_2\}$-factor is also called a 1-factor.

Tutte [18] in 1947 found a necessary and sufficient condition for graphs containing 1-factors. After that the path factor problems have received a lot of attention, see for example [3–7, 12–16, 20, 21, 23–26]. There are also many interesting results on other factors, we refer the readers to [2, 17, 22]. Hell and Kirkpatrick [10, 11] proved that if F is a graph with order at least three, then deciding whether a graph G has an F-factor is NP-complete. So, it is interesting and challenging to give a good characterization for a graph to have a P_3-factor or P_5-factor.

For $\{P_2, P_3\}$-factors, Akiyama, Avis and Era [1] in 1980 gave the following classical criterion.

Theorem 1.1 (Akiyama, Avis and Era [1])**.** *Let G be a graph. Then G contains a $\{P_2, P_3\}$-factor if and only if $c_1(G - S) \leq 2|S|$ for any $S \subseteq V(G)$.*

For $\{P_2, P_5\}$-factors, Egawa and Furuya [9] in 2018 gave the following condition.

Theorem 1.2 (Egawa and Furuya [9])**.** *Let G be a graph. If G satisfies $c_1(G - X) + \frac{2}{3}c_3(G - X) \leq \frac{4}{3}|X| + \frac{1}{3}$ for any $X \subseteq V(G)$, then G contains a $\{P_2, P_5\}$-factor.*

As shown in [8], the sufficient condition in Theorem 1.2 is not easy to check. So many other parameters have been studied. For example, Dai *et al.* [8] in 2022 considered the minimum degree, toughness and binding number condition and determined some special classes of graphs to have $\{P_2, P_5\}$-factors. Wang and Zhang [20] in 2023 considered the degree condition and posed the following conjectures:

Conjecture 1.3 (Wang and Zhang [20])**.** *Let G be a connected graph with $\delta(G) \geq 5$. If $\delta(G) \geq \frac{3\alpha(G)-1}{4}$, then G contains a $\{P_2, P_5\}$-factor.*

Conjecture 1.4 (Wang and Zhang [20])**.** *Let G be a connected graph of order $n \geq 7$. If G satisfies $\max\{d_G(x), d_G(y)\} \geq \frac{3n-1}{7}$ for any two nonadjacent vertices x, y of G, then G contains a $\{P_2, P_5\}$-factor.*

Let X be a spanning subgraph of K_3, $Y \cong 7K_3$, $G \cong X \vee Y$. We first show that G is a counterexample of Conjecture 1.3. Note that $\alpha(G) = 7, \delta(G) = 5$. So G satisfies the degree condition. Suppose, to the contrary, that G contains a $\{P_2, P_5\}$-factor F. In F, each K_3 in Y must have a vertex adjacent to X and at most two vertices from distinct K_3 in Y could be adjacent to a same vertex in X. This means that there is at least one vertex in Y cannot be in F, a contradiction. So G contains no $\{P_2, P_5\}$-factors. We prove that Conjecture 1.3 holds unless $G \cong X \vee 7K_3$.

Theorem 1.5. *Let G be a connected graph with $\delta(G) \geq 5$ and X be a spanning subgraph of K_3. If $\delta(G) \geq \frac{3\alpha(G)-1}{4}$, then G contains a $\{P_2, P_5\}$-factor unless $G \cong X \vee 7K_3$.*

Furthermore, we give two degree conditions of nonadjacent vertices for the existence of $\{P_2, P_5\}$-factors.

Theorem 1.6. *Let G be a connected graph of order $n \geq 34$. If G satisfies $d_G(u_1) + d_G(u_2) \geq \frac{6n-17}{7}$ for any two nonadjacent vertices u_1 and u_2 of G, then G contains a $\{P_2, P_5\}$-factor.*

Theorem 1.7. *Let G be a connected graph of order $n \geq 9$. If G satisfies $\max\{d_G(w_1), d_G(w_2)\} \geq \frac{3n-4}{7}$ for any two nonadjacent vertices w_1, w_2 of G, then G contains a $\{P_2, P_5\}$-factor.*

Theorem 1.7 is a stronger version of Conjecture 1.4 for $n \geq 9$. The degree conditions in Theorems 1.6 and 1.7 are both best possible.

2. Proof of Theorem 1.5

Now we give the proof of Theorem 1.5. Let G be a connected graph with $\delta(G) \geq 5$ satisfying $\delta(G) \geq \frac{3\alpha(G)-1}{4}$. Suppose G contains no $\{P_2, P_5\}$-factor. Then by Theorem 1.2, there is some $S \subseteq V(G)$ such that

$$c_1(G-S) + \frac{2}{3}c_3(G-S) > \frac{4}{3}|S| + \frac{1}{3}. \tag{1}$$

By (1), we have the following statements.

Claim 2.1. $S \neq \varnothing$.

Proof. Suppose, to the contrary, that $S = \varnothing$. Then by (1), we have $c_1(G) + \frac{2}{3}c_3(G) > \frac{1}{3}$. Since both $c_1(G)$ and $c_3(G)$ are integers, we have $c_1(G) + c_3(G) \geq 1$. Note that G is connected. So $c_1 = 0, c_3 = 1$ or $c_1 = 1, c_3 = 0$. For both cases, we have $|V(G)| \leq 3$, a contradiction. □

By Claim 2.1, we have $|S| \geq 1$. It implies $c_1(G-S) + \frac{2}{3}c_3(G-S) > \frac{5}{3}$.

Claim 2.2. $c_1(G-S) = 0$.

Proof. Suppose, to the contrary, that $c_1(G-S) > 0$. Let v be an isolated vertex of $G-S$. Then $d_{G-S}(v) = 0$, which implies $d_G(v) = d_S(v) \leq |S|$. So $\delta(G) \leq |S|$. Note that $\alpha(G) \geq c_1(G-S) + c_3(G-S) \geq c_1(G-S) + \frac{2}{3}c_3(G-S) > \frac{4}{3}|S| + \frac{1}{3} \geq \frac{4}{3}\delta(G) + \frac{1}{3}$, which implies $\delta(G) < \frac{3\alpha(G)-1}{4}$, a contradiction. □

By Claim 2.2 and (1), we have $\frac{2}{3}c_3(G-S) > \frac{4}{3}|S| + \frac{1}{3}$. Then $c_3(G-S) \geq 2|S| + 1 \geq 3$ since $c_3(G-S)$ is an integer. Let C_0 be a component of order three in $G-S$. Choose v from C_0 randomly. Then $d_{G-S}(v) \leq 2$ and $d_G(v) \leq |S| + 2$. Thus $\delta(G) \leq |S| + 2$.

Claim 2.3. $\delta(G) = |S| + 2$.

Proof. Suppose, to the contrary, that $\delta(G) < |S| + 2$. It means $|S| \geq \delta(G) - 1$. Since $\delta(G) \geq 5$, we have

$$\alpha(G) \geq c_3(G-S) \geq 2|S| + 1 \geq 2(\delta(G)-1) + 1 = 2\delta(G) - 1 \geq 9. \tag{2}$$

By (2), we have $\delta(G) \leq \frac{1}{2}\alpha(G) + \frac{1}{2} = \frac{3\alpha(G)-1}{4} + \frac{3}{4} - \frac{1}{4}\alpha(G) < \frac{3\alpha(G)-1}{4}$, a contradiction. □

Note that $c_3(G-S) \geq 3$. Let $c_3(G-S) = t$ and $C_1, \ldots, C_t$ be all components of order three in $G-S$. For $i = 1, \ldots, t$ and $v \in V(C_i)$, $d_G(v) = d_S(v) + d_{G-S}(v) \leq |S| + 2$. By Claim 2.3, we have $d_G(v) = |S| + 2$. So $d_S(v) = |S|$ and $d_{G-S}(v) = 2$. It implies that $C_i \cong K_3$ and $uv \in E(G)$ for any $u \in V(C_i)$ and $v \in S$, where $i = 1, \ldots, t$. Since Claim 2.3 and $\delta(G) \geq 5$, we have

$$\alpha(G) \geq c_3(G-S) \geq 2|S| + 1 = 2(\delta(G)-2) + 1 = 2\delta(G) - 3 \geq 7. \tag{3}$$

By (3), we have

$$\delta(G) \leq \frac{1}{2}\alpha(G) + \frac{3}{2} = \frac{3\alpha(G)-1}{4} + \frac{7}{4} - \frac{1}{4}\alpha(G). \tag{4}$$

If $\alpha(G) \geq 8$, then (4) implies $\delta(G) < \frac{3\alpha(G)-1}{4}$, a contradiction. So $\alpha(G) = 7$. It means (3) is an equality. So $c_3(G-S) = 7, |S| = 3$ and $\delta(G) = 5$.

We now prove $G \cong X \vee 7K_3$, where X is a spanning subgraph of K_3. Since $\alpha(G) = 7 = c_3(G-S)$, there is no other component in $G-S$. So G consists of S and $C_1, \ldots, C_7$. It means $G \cong X \vee 7K_3$, where X is a spanning subgraph of K_3. This completes the proof of Theorem 1.5.

3. Proof of Theorem 1.6

Suppose, to the contrary, that there exists a graph G of order $n \geq 34$ containing no $\{P_2, P_5\}$-factors and satisfying the following condition for any two nonadjacent vertices u_1, u_2:

$$d_G(u_1) + d_G(u_2) \geq \frac{6n-17}{7}. \tag{5}$$

Then by Theorem 1.2, there is some $S \subseteq V(G)$ such that

$$c_1(G-S) + \frac{2}{3}c_3(G-S) > \frac{4}{3}|S| + \frac{1}{3}. \tag{6}$$

By (6), we have the following statements.

Claim 3.1. $S \neq \varnothing$.

The proof of Claim 3.1 is similar to that of Claim 2.1. So we omit it here.

By Claim 3.1, we have $|S| \geq 1$. This implies

$$c_1(G-S) + \frac{2}{3}c_3(G-S) > \frac{5}{3}. \tag{7}$$

Claim 3.2. $c_1(G-S) \geq 2$.

Proof. If $c_1(G-S) = 0$, then by (6), we have $\frac{2}{3}c_3(G-S) > \frac{4}{3}|S| + \frac{1}{3}$. It implies

$$c_3(G-S) \geq 2|S| + 1 \geq 3. \tag{8}$$

Let C_1, C_2 be two components of order three in $G-S$. Choose v_i from C_i for $i = 1, 2$ randomly. Then $d_{G-S}(v_i) \leq 2$ and $d_G(v_i) \leq |S| + 2$ for $i = 1, 2$. Note that v_1, v_2 are not adjacent in G. Then by (5), we have $\frac{6n-17}{7} \leq d_G(v_1) + d_G(v_2) \leq 2|S| + 4$. It implies

$$|S| \geq \frac{6n-17}{14} - 2 = \frac{6n-45}{14}. \tag{9}$$

Note that $n = |S| + \sum_{i \in N_+} c_i(G-S)$. Combining (8) and (9), we have

$$n \geq |S| + 3c_3(G-S) \geq |S| + 3(2|S|+1) = 7|S| + 3 \geq 7 \cdot \frac{6n-45}{14} + 3 = 3n - \frac{39}{2}.$$

It implies $n \leq \frac{39}{4}$, a contradiction.

So we may assume $c_1(G-S) = 1$. By (6), $c_3(G-S) > 2|S| - 1$. This means

$$c_3(G-S) \geq 2|S| \geq 2. \tag{10}$$

Let $C_1 = \{w_1\}$ be the component of order one and C_2 be a component of order three in G. Choose w_2 from C_2 randomly. Then $d_{G-S}(w_1) = 0, d_{G-S}(w_2) \leq 2$. So $d_G(w_1) \leq |S|, d_G(w_2) \leq |S| + 2$. Note that w_1, w_2 are not adjacent in G. Then by (5), we have $\frac{6n-17}{7} \leq d_G(w_1) + d_G(w_2) \leq 2|S| + 2$. It implies

$$|S| \geq \frac{6n-17}{14} - 1 = \frac{6n-31}{14}. \tag{11}$$

Combining (10) and (11), we have

$$n \geq |S| + c_1(G-S) + 3c_3(G-S) \geq |S| + 1 + 3 \cdot 2|S| = 7|S| + 1 \geq 7 \cdot \frac{6n-31}{14} + 1 = 3n - \frac{29}{2}.$$

It implies $n \leq \frac{29}{4}$, a contradiction. ∎

By Claim 3.2, we know that $G-S$ contains at least two components of order one. Let $C_1=\{v_1\}, C_2=\{v_2\}$ be two such components. Then $d_{G-S}(v_i)=0$ and $d_G(v_i)\le |S|$ for $i=1,2$. Note that v_1, v_2 are not adjacent in G. Then by (5), we have $\frac{6n-17}{7}\le d_G(v_1)+d_G(v_2)\le 2|S|$. So

$$|S|\ge \frac{6n-17}{14}. \tag{12}$$

Claim 3.3. $c_3(G-S)=0$.

Proof. Suppose, to the contrary, that $c_3(G-S)\ge 1$. Note that $n\ge |S|+c_1(G-S)+3c_3(G-S)$. Then by (6), we have $n>|S|+(\frac{4}{3}|S|+\frac{1}{3})+\frac{7}{3}c_3(G-S)=\frac{7}{3}|S|+\frac{7}{3}c_3(G-S)+\frac{1}{3}\ge \frac{7}{3}|S|+\frac{8}{3}$. It implies $n\ge \frac{7}{3}|S|+3$. Then by (12), we have $n\ge \frac{7}{3}\cdot\frac{6n-17}{14}+3=n+\frac{1}{6}$, a contradiction. □

Claim 3.4. $c_i(G-S)=0$ for all $i\ge 4$.

Proof. Otherwise, there is an integer $j\ge 4$ such that $c_j(G-S)\ge 1$. Then $n\ge |S|+c_1(G-S)+jc_j(G-S)\ge |S|+c_1(G-S)+4$. Combining Claim 3.3 and (6), we have $n>\frac{7}{3}|S|+\frac{13}{3}$. By (12), we have $n>\frac{7}{3}\cdot\frac{6n-17}{14}+\frac{13}{3}=n+\frac{3}{2}$, a contradiction. □

Claim 3.5. $c_2(G-S)=0$.

Proof. By Claims 3.3 and 3.4, we have $c_i(G-S)=0$ for all $i\ge 3$. If $c_2(G-S)\ge 2$, then we can get a contradiction similar to the proof of Claim 3.4.

So we may assume $c_2(G-S)=1$. Then $n=|S|+c_1(G-S)+2c_2(G-S)=|S|+c_1(G-S)+2$. Combining Claim 3.3 and (6), we have $n>|S|+(\frac{4}{3}|S|+\frac{1}{3})+2=\frac{7}{3}|S|+\frac{7}{3}$. This implies $n\ge\frac{7}{3}|S|+\frac{8}{3}$. It follows that $|S|\le\frac{3n-8}{7}$. Then by (12), we have $\frac{6n-17}{7}\le 2|S|\le\frac{6n-16}{7}$.

If $n\equiv t\ (mod\ 7)$ for some $t\in\{0,1,2,3,4,6\}$, it is easy to check that there is no even number between $\frac{6n-17}{7}$ and $\frac{6n-16}{7}$. So we pay our attention to the case $n\equiv 5\ (mod\ 7)$. Let $n=7k+5$ for some positive integer k. $n\ge 34$ implies $k\ge 5$. The only even number between $\frac{6n-17}{7}$ and $\frac{6n-16}{7}$ is $6k+2$. So $2|S|=6k+2$, $|S|=3k+1$. Then $|V(G)\backslash S|=4k+4$. Note that $c_2(G-S)=1$. So $G-S\cong K_2\cup(4k+2)K_1$. Let $i,j\in\{1,\dots,4k+2\}$, uv be the only edge in $G-S$ and $v_1,\dots,v_{4k+2}$ be the isolated vertices in $G-S\cup\{u,v\}$. Then $d_G(v_i)=d_S(v_i)\le|S|$. By (5), for any two nonadjacent vertices v_i,v_j in $G-S\cup\{u,v\}$, we have $2|S|\ge d_G(v_i)+d_G(v_j)\ge\frac{6n-17}{7}=6k+2=2|S|$. So $d_G(v_i)+d_G(v_j)=2|S|$. This means $d_G(v_i)=|S|$, which implies $v_iw\in E(G)$ for all $i\in\{1,\dots,4k+2\}$ and any $w\in V(S)$.

We now prove that $G-\{u,v\}$ contains a $\{P_2,P_5\}$-factor with xP_5 and yP_2 satisfying all the endpoints and centers of P_5 in $G-S$ and exact one endpoint of P_2 in $G-S$. We consider the following equations:

$$\begin{cases}3x+y=4k+2,\\ 2x+y=3k+1.\end{cases}$$

Note that it has non-negative integer solutions:

$$\begin{cases}x=k+1,\\ y=k-1.\end{cases}$$

Thus $G-\{u,v\}$ contains a $\{P_2,P_5\}$-factor consisting of $(k+1)$ P_5 and $(k-1)$ P_2. It implies G contains a $\{P_2,P_5\}$-factor consisting of $(k+1)$ P_5 and k P_2, a contradiction. □

Claim 3.6. $\frac{6n-17}{7}\le 2|S|\le\frac{6n-4}{7}$.

Proof. By Claims 3.1–3.5, we have $c_i(G-S)=0$ for all $i\geq 2$. This means that $G-S$ consists of $n-|S|$ isolated vertices. Then by (6), we have

$$n=|S|+c_1(G-S)=|S|+c_1(G-S)+\frac{2}{3}c_3(G-S)>|S|+\frac{4}{3}|S|+\frac{1}{3}=\frac{7}{3}|S|+\frac{1}{3}. \tag{13}$$

It implies $n\geq \frac{7}{3}|S|+\frac{2}{3}$, which means $|S|\leq \frac{3n-2}{7}$. Then by (12), we have $\frac{6n-17}{7}\leq 2|S|\leq \frac{6n-4}{7}$. □

We now prove G contains a $\{P_2,P_5\}$-factor to get a contradiction.

For $t\in\{1,2\}$ and $n\equiv t$ $(mod\ 7)$, let $n=7k+t$ for some positive integer k. Since $n\geq 34$, we have $k\geq 5$. It is easy to check that the only even number between $\frac{6n-17}{7}$ and $\frac{6n-4}{7}$ is $6k$. So $2|S|=6k$, $|S|=3k$. Then $|V(G)\backslash S|=4k+t$. This means that $G-S$ consists of $4k+t$ isolated vertices. Let $v_1,\dots,v_{4k+t}$ be the $4k+t$ isolated vertices of $G-S$. Then $d_G(v_i)=d_S(v_i)\leq |S|$ for $i\in\{1,\dots,4k+t\}$. By (5), we have $2|S|\geq d_G(v_i)+d_G(v_j)\geq \frac{6n-17}{7}=6k+\frac{6t-17}{7}=2|S|-\frac{17-6t}{7}$ for all distinct $i,j\in\{1,\dots,4k+t\}$. So $d_G(v_i)+d_G(v_j)=2|S|$ for $t=2$, $d_G(v_i)+d_G(v_j)=2|S|$ or $2|S|-1$ for $t=1$.

If $d_G(v_i)+d_G(v_j)=2|S|$ for all distinct $i,j\in\{1,\dots,4k+t\}$, then $d_G(v_i)=|S|$ for all $i\in\{1,\dots,4k+t\}$. This means $v_iw\in E(G)$ for all $i=1,\dots,4k+t$ and any $w\in V(S)$. We now prove that G contains a $\{P_2,P_5\}$-factor with all the endpoints and centers of P_5 in $G-S$ and exact one endpoint of P_2 in $G-S$. We consider the following equations:

$$\begin{cases}3x+y=4k+t,\\ 2x+y=3k.\end{cases}$$

Note that it has non-negative integer solutions:

$$\begin{cases}x=k+t,\\ y=k-2t.\end{cases}$$

Thus G contains a $\{P_2,P_5\}$-factor consisting of $(k+t)$ P_5 and $(k-2t)$ P_2, a contradiction.

If there are $i,j\in\{1,\dots,4k+t\}$ such that $d_G(v_i)+d_G(v_j)=2|S|-1$, then we have $t=1$ and $\{d_G(v_i),d_G(v_j)\}$=$\{|S|,|S|-1\}$. Suppose, without loss of generality, that $d_G(v_i)=|S|-1$. Then for all $\ell\in\{1,\dots,4k+1\}\setminus\{i\}$, since $d_G(v_i)+d_G(v_\ell)\geq 2|S|-1$, we have $d_G(v_\ell)=|S|$. Let $w\in V(S)$ such that $v_iw\in E(G)$. We now prove that $G-\{v_i,w\}$ contains a $\{P_2,P_5\}$-factor with with all the endpoints and centers of P_5 in $G-S$ and exact one endpoint of P_2 in $G-S$. We consider the following equations:

$$\begin{cases}3x+y=4k,\\ 2x+y=3k-1.\end{cases}$$

Note that it has non-negative integer solutions:

$$\begin{cases}x=k+1,\\ y=k-3.\end{cases}$$

Thus $G-\{v_i,w\}$ contains a $\{P_2,P_5\}$-factor consisting of $(k+1)$ P_5 and $(k-3)$ P_2. It implies G contains a $\{P_2,P_5\}$-factor consisting of $(k+1)$ P_5 and $(k-2)$ P_2, a contradiction.

For $t\in\{3,4,5\}$ and $n\equiv t$ $(mod\ 7)$, let $n=7k+t$ for some positive integer k. Since $n\geq 34$, we have $k\geq 5$. It is easy to check that the only even number between $\frac{6n-17}{7}$ and $\frac{6n-4}{7}$ is $6k+2$. So $2|S|=6k+2$, $|S|=3k+1$. Then $|V(G)\setminus S|=4k+t-1$. This means that $G-S$ consists of $4k+t-1$ isolated vertices. Let $v_1,\dots,v_{4k+t-1}$ be the $4k+t-1$ isolated vertices of $G-S$. Then $d_G(v_i)=d_S(v_i)\leq |S|$ for $i=1,\dots,4k+t-1$. By (5), we have $2|S|\geq d_G(v_i)+d_G(v_j)\geq \frac{6n-17}{7}=6k+\frac{6t-17}{7}=2|S|-\frac{31-6t}{7}$ for all distinct $i,j\in\{1,\dots,4k+t-1\}$. So $d_G(v_i)+d_G(v_j)=2|S|$ for $t=5$, and $d_G(v_i)+d_G(v_j)=2|S|$ or $2|S|-1$ for $t=3,4$.

If $d_G(v_i)+d_G(v_j) = 2|S|$ for all distinct $i, j \in \{1, \ldots, 4k+t-1\}$, then $d_G(v_i) = |S|$ for all $i \in \{1, \ldots, 4k+t-1\}$. This means $v_i w \in E(G)$ for all $i \in \{1, \ldots, 4k+t-1\}$ and any $w \in V(S)$. We now prove that G contains a $\{P_2, P_5\}$-factor with all the endpoints and centers of P_5 in $G - S$ and exact one endpoint of P_2 in $G - S$. We consider the following equations:

$$\begin{cases} 3x + y = 4k + t - 1, \\ 2x + y = 3k + 1. \end{cases}$$

Note that it has non-negative integer solutions:

$$\begin{cases} x = k + t - 2, \\ y = k - 2t + 5. \end{cases}$$

Thus G contains a $\{P_2, P_5\}$-factor consisting of $(k + t - 2)$ P_5 and $(k - 2t + 5)$ P_2, a contradiction.

If there are $i, j \in \{1, \ldots, 4k + t - 1\}$ such that $d_G(v_i) + d_G(v_j) = 2|S| - 1$, then we have $t = 3$ or 4 and $\{d_G(v_i), d_G(v_j)\} = \{|S|, |S| - 1\}$. Suppose, without loss of generality, that $d_G(v_i) = |S| - 1$. Then for all $\ell \in \{1, \ldots, 4k + t - 1\} \setminus \{i\}$, since $d_G(v_i) + d_G(v_\ell) \geq 2|S| - 1$, we have $d_G(v_\ell) = |S|$. Let $w \in V(S)$ such that $v_i w \in E(G)$. We now prove that $G - \{v_i, w\}$ contains a $\{P_2, P_5\}$-factor with all the endpoints and centers of P_5 in $G - S$ and exact one endpoint of P_2 in $G - S$. We consider the following equations:

$$\begin{cases} 3x + y = 4k + t - 2, \\ 2x + y = 3k. \end{cases}$$

Note that it has non-negative integer solutions:

$$\begin{cases} x = k + t - 2, \\ y = k - 2t + 4. \end{cases}$$

Thus $G - \{v_i, w\}$ contains a $\{P_2, P_5\}$-factor consisting of $(k + t - 2)$ P_5 and $(k - 2t + 4)$ P_2. It implies G contains a $\{P_2, P_5\}$-factor consisting of $(k + t - 2)$ P_5 and $(k - 2t + 5)$ P_2, a contradiction.

For $n \equiv 6$ $(mod\ 7)$ or $n \equiv 0$ $(mod\ 7)$, we can get a $\{P_2, P_5\}$-factor of G similarly, so we omit the details here. This completes the proof of Theorem 1.6.

4. Proof of Theorem 1.7

Suppose, to the contrary, that there exists a graph G of order $n \geq 9$ containing no $\{P_2, P_5\}$-factors and satisfying the following condition for any two nonadjacent vertices w_1, w_2:

$$\max\{d_G(w_1), d_G(w_2)\} \geq \frac{3n - 4}{7}. \tag{14}$$

Then by Theorem 1.2, there is some $S \subseteq V(G)$ such that

$$c_1(G - S) + \frac{2}{3} c_3(G - S) > \frac{4}{3}|S| + \frac{1}{3}. \tag{15}$$

We can prove $|S| \geq \frac{3n-4}{7}$ and $c_i(G - S) = 0$ for all $i \geq 2$ similar to the proof of Claims 3.1–3.5. So we omit the details here. This means that $G - S$ consists of $n - |S|$ isolated vertices. Then by (15), we have

$$n = |S| + c_1(G - S) = |S| + c_1(G - S) + \frac{2}{3} c_3(G - S) > |S| + \frac{4}{3}|S| + \frac{1}{3} = \frac{7}{3}|S| + \frac{1}{3}. \tag{16}$$

If $|S| \geq \frac{3n-1}{7}$, then we can get a contradiction from (16). So we may assume $\frac{3n-4}{7} \leq |S| \leq \frac{3n-2}{7}$.

For $t \in \{0, 2, 4, 5\}$ and $n \equiv t \ (mod\ 7)$, there is no integer between $\frac{3n-4}{7}$ and $\frac{3n-2}{7}$. We now consider $n = 7k+1$ for some positive integer k. Since $n \geq 9$, we have $k \geq 2$. The only integer between $\frac{3n-4}{7}$ and $\frac{3n-2}{7}$ is $3k$. So $|S| = 3k$. Then $|V(G) \setminus S| = 4k + 1$. This means that $G - S$ consists of $4k + 1$ isolated vertices. Let $V(G - S) = \{v_1, \ldots, v_{4k+1}\}$. For $v_i, v_j \in V(G - S)$, we have $\max\{d_G(v_i), d_G(v_j)\} \geq \frac{3n-4}{7}$ by (14). Note that $n = 7k + 1$ and $d_G(v_i), d_G(v_j)$ are integers. So we have $\max\{d_G(v_i), d_G(v_j)\} \geq \frac{3n-3}{7} = |S|$ for any $v_i, v_j \in V(G - S)$. This implies that at least $4k$ vertices of $G - S$, say $\{v_1, \ldots, v_{4k}\}$, are adjacent to all the vertices of S. Since G is connected, v_{4k+1} is adjacent to at least one vertex of S. Let $v_{4k+1}u \in E(G)$ and $P = v_{4k+1}uxyz$ be a path of order five with endpoints and center in $G - S$. We now prove that $G - V(P)$ contains a $\{P_2, P_5\}$-factor with all the endpoints and centers of P_5 in $G - S$ and exact one endpoint of each P_2 in $G - S$. We consider the following equations:

$$\begin{cases} 3x + y = 4k + 1 - 3, \\ 2x + y = 3k - 2. \end{cases}$$

Note that it has non-negative integer solutions:

$$\begin{cases} x = k, \\ y = k - 2. \end{cases}$$

So $G - V(P)$ contains a $\{P_2, P_5\}$-factor consisting of k P_5 and $(k - 2)$ P_2. Thus G contains a $\{P_2, P_5\}$-factor consisting of $(k + 1)$ P_5 and $(k - 2)$ P_2, a contradiction.

For the case $n = 7k + 3$ or $n = 7k + 6$, we can get a $\{P_2, P_5\}$-factor of G similarly. So we omit the details here. This completes the proof of Theorem 1.7. □

5. Concluding remarks

Remark 5.1. Now we explain that the degree sum condition in Theorem 1.6 is best possible. Let $G \cong K_{23,15}$ with bipartitions A, B of order $23, 15$, respectively. Then $n = 38$ and for any two nonadjacent vertices x, y of G, $d_G(x) + d_G(y) \geq 30 = \frac{6n-18}{7}$. Suppose, to the contrary, that G contains a $\{P_2, P_5\}$-factor. Let x, y, z denote the number of P_2, the number of P_5 with endpoints in A, the number of P_5 with endpoints in B, respectively. Then we have that

$$\begin{cases} x + 3y + 2z = 23, \\ x + 2y + 3z = 15. \end{cases} \tag{17}$$

Since (17) has no non-negative integer solution, G contains no $\{P_2, P_5\}$-factor, a contradiction.

Remark 5.2. Now we explain that the maximum degree condition in Theorem 1.7 is best possible. Let $H \cong K_{11,7}$ with bipartitions A, B of order $11, 7$, respectively. Then $n = 18$ and for any two nonadjacent vertices x, y of H, $\max\{d_H(x), d_H(y)\} \geq 7 = \frac{3n-5}{7}$. Similar to Remark 5.1, it is easy to check H contains no $\{P_2, P_5\}$-factors.

In this paper, we give three degree conditions for the existence of $\{P_2, P_5\}$-factors according to two conjectures of Wang and Zhang [20]. It is interesting and challenging to further consider the existence of long-path factors.

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Conflicts of Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data Availability Statement

No data was used for the research described in the article.

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