RAIRO-Oper. Res. 58 (2024) 4861–4888
https://doi.org/10.1051/ro/2024187

RAIRO Operations Research
www.rairo-ro.org

OPTIMAL ORDERING STRATEGY FOR DETERIORATING ITEMS WITH MAXIMUM LIFETIME USING TRADE CREDIT FINANCING UNDER IMPRECISE ENVIRONMENTS

Bappa Mondal[1,2], Chaitali Kar[3,*], Debashis Dutta[1] and Pritha Das[3]

Abstract. In many business scenarios, a retailer is permitted specific credit period to pay back for the products bought earlier. This facility enables retailers to continue their business operations even when they are unable to raise funds or secure a business loan. To boost the market's demand, promotional effort is a very effective business strategy to the retailer for maximizing the profit. On the other hand, a sudden and significant rise in customer demand for an inventory leads to shortages. Moreover, inventory relevant unit cost coefficients become imprecise due to insufficient data, human error etc. Nowadays, neutrosophic set quantifies the impreciseness more realistically. Considering these facts, an imprecise EOQ model for deteriorating items with maximum lifetime is formulated under trade credit facility. In addition, this article allows shortages, which are linearly time-dependent partially backlogged. Here, the unit cost coefficients are expressed as single-valued trapezoidal neutrosophic numbers. Furthermore, particular models are derived under different environments - intuitionistic, fuzzy, and crisp. Step-by-step solution procedures are suggested for all models to obtain optimal solutions. Models are numerically illustrated with real-life data, and some sensitivity analyses are performed. Managerial insights demonstrate that depletion time always depends on demand. Again, the present study suggests to reduce demand by halting the promotional activities during the shortage period and choose products with a larger lifetime.

Mathematics Subject Classification. 90B05, 90B06.

Received October 22, 2022. Accepted September 14, 2024.

1. Introduction

Nowadays, in real-world competitive business, it is necessary to give much attention to different business strategies to increase overall competitiveness. Effective strategies always play a significant role in increasing the marketing scope of any business. Traditionally, in EOQ models, the payment is made just after receiving the items from the suppliers. In recent business transactions, it is often observed that retailer shows much interest in buying more items if the supplier offers some grace credit period to make payment for the supplied items instead of instant payment. These permissible credit limits are frequently changed with the business

Keywords. Trade credit facility, partial backlogging, deterioration, maximum lifetime of products, single-valued trapezoidal neutrosophic number.

[1] National Institute of Technology Warangal, Hanamkonda, Telangana 506004, India.

[2] Asutosh College, University of Calcutta (C.U), Kolkata 700026, West-Bengal, India.

[3] Indian Institute of Engineering Science and Technology, Shibpur, Howrah 711103, West-Bengal, India.

*Corresponding author: chaitalikar12@gmail.com

requirements, current assets, repaying history and goodwill of the retailer. Normally, no interest is charged in such an agreement between supplier and retailer if the incomplete payment is paid within the permitted fixed credit period. Although beyond the permitted period, interest is charged by the supplier for the overdue period on the unpaid balance. Thus, this payment time flexibility is very effective for retailers, especially small companies with limited financing opportunities. In addition, during this allowed period, the retailer sells items, deposits accumulated revenue in an account, and gains interest until the last day of the permitted period. Such payment settlement strategy in business is called trade credit facility. It is an essential tool for financing growth and is very profitable to a retailer. Thus, consideration of trade credit facility in an EOQ model is one of the most realistic fields.

On the other hand, deterioration of physical items, such as fruits, vegetables, etc., is a natural phenomenon. Although modern technology can decrease the deterioration rate of items to some extent, this factor cannot be eradicated. It is undeniable that most items, such as vegetables, fruits, dairy products, cooking items, fish, meat, eggs, etc., that are used in everyday life have an expiration date. These items can be used till the last date of their shelf lives, but once the expiration timeline is crossed, these products must be spoiled/damaged and become useless. A retailer meets some financial loss if such items are not sold within the shelf-life period. The deterioration rate of an item plays an important role in the inventory system. According to Buzby *et al.* [6], in 2010, nearly 31% of food produced in the United States was damaged at the retail and consumer levels. As per FAO[1], about 40% of India's fruits and vegetables perish before reaching consumers. Thus, the analysis of the impact of deterioration in inventory problems is very important.

Retailers always aim to sell the deteriorating items within their expiration dates in order to avoid financial loss and pay back the payment on or before the permitted credit period. Therefore, retailers try to introduce different strategies to boost the demand in a very short time. Promotional effort is an advanced strategy that widely impacts an item's demand in today's business world. This activity includes some sale services such as a replacement facility if any damaged item is found, free delivery service for an allowed period, discount on overprinted selling price, an offer of giving more items at the standard fixed price, 'buy one get one free' offer zone, various attractive coupons with purchased items, etc. Advertising about the brand, new stock and latest offer is also one type of promotional effort on the inventory system. Also, this strategy helps an item be kept in the choice list of the customer and generates a high demand for that item in the market. Sometimes, customers purchase many products rapidly in the market only because of the promotional activities run by retailers. So, introduction of such an effort highly influences the retail industry.

In any inventory system, whenever the demand of an item increases in a very short time, the retailer fails to fulfill the customers' requirements. In such a situation, the customer needs some items, that are unavailable in stock, so shortages arise. During shortages, some customers wait for the stock to be replenished, whereby their demand is subsequently satisfied and it is called backlogged to the retailer. But, at the same time, some customers do not wait for the stock to be replenished, and demand is dropped. This causes a lost sale for the retailer. Shortages play a significant role in the inventory system, especially when the trade credit facility is considered because shortages can affect the benefit from the allowed grace period. In the research field, generally, researchers assume shortages in two ways: fully backlogged (*cf.* Mishra *et al.* [29]; Sundara Rajan and Uthayakumar [55]) or partially backlogged (*cf.* Verma *et al.* [61]; Wu *et al.* [62]). As well as this time-span increases, the chances of accepting backlogging gradually decrease. So, the backlogging rate should be dependent on this waiting time.

There is uncertainty in almost every part of life and society. Uncertainty arises mainly due to insufficient information, lack of exact evidence, incorrect data collection, etc. In real-life situations, sometimes it is hard to find the optimal solutions to many deterministic problems because of unknown data and insufficient information. In order to deal with such impreciseness, several researchers have worked under different imprecise environments like fuzzy, intuitionistic, etc. In 1965, Zadeh [67] introduced fuzzy set, and later many researchers (*cf.* Maiti [27]) applied this concept in different inventory models. After that, Atanassov [4] developed intuitionistic set as a

[1] https://www.ft.com/content/c1f2856e-a518-11e3-8988-00144feab7de

generalization of fuzzy set, and several researchers (*cf.* Man *et al.* [28]; Mondal *et al.* [32]) adopted this concept in their work. Whenever some indeterminate or conflicting information arises in any real-life problem, an intuitionistic set becomes insufficient. At that time, the neutrosophic set is introduced to represent impreciseness more realistically. Smarandache [53] invented the concept of neutrosophic set by considering three independent components: membership, indeterminacy and non-membership. Thus, the neutrosophic set is the most generalized extension of fuzzy set and intuitionistic fuzzy set. Analysis of inventory models under different imprecise environments is very practical. But, the number of such research works is significantly less. Inspired by these facts, we intended to formulate EOQ models under different imprecise (neutrosophic, intuitionistic and fuzzy) environments.

Out of the above-discussed situations, some real-life queries arise:

– What is the effect of different credit periods on a retailer's optimum decision for the partially backlogged inventory model of deteriorating items?
– How do shortages impact a retailer's optimal ordering strategy?
– How do promotional activities affect the customers' demand of an item in inventory?
– What is the significant impact of the maximum lifetime of daily used household products on a retailer's total average cost and optimal order quantity?
– How do the optimal total average costs in an EOQ model for deteriorating items with trade credit facility change under different environments?
– How to develop a cost-minimization inventory model for deteriorating items with maximum lifetime, promotional activity-dependent demand and trade credit facility?

The motivation of this paper is to concentrate on the above-raised queries. This present article is an attempt to develop an inventory problem for deteriorating items with maximum lifetime in market uncertainty under trade credit facility. Also, shortages and partial backlog situations are considered. Promotional effort-dependent demand function is taken, which has been addressed very rarely in the literature. Different cases are derived depending on whether payment is completed before or after the inventory depletion time. The main objective of the present article is to determine the optimal depletion time that minimizes the total average cost for the retailer. The proposed models under neutrosophic environment are converted into equivalent crisp form by applying (α, β, γ)-cuts and weighted arithmetic mean approach. Also, the models are analyzed under different environments like intuitionistic, fuzzy and crisp as particular cases. The solution procedure is described for all models. Moreover, numerical experiments are evaluated to verify and compare the theoretical results. Some sensitivity analyses are described, and managerial decisions are made.

The rest of the present investigation is constructed as follows: A brief literature survey and the research contributions are presented in Section 2. Section 3 describes required symbols and hypotheses for the proposed EOQ model. The mathematical models are formulated under neutrosophic environment in Section 4. In Section 5, some particular cases are derived depending on different environments. Section 6 develops a solution procedure for all models. A numerical application is analyzed, and optimum solutions are presented in Section 7. Section 8 performs a sensitivity analysis against some key parameters, and observations are noted. Managerial insights are suggested, and areas of application to the proposed model are indicated in Section 9. Section 10 presents the conclusions, limitations and future extensions. All required mathematical preliminaries are given in Appendix.

2. Literature review

2.1. Trade credit facility in inventory

The business competition in the retail industry is increasing day by day. In that situation, many retain industries are accepting the trade credit facility. Goyal [13] initiated trade credit policy in an EOQ model. Later, many researchers worked in this direction. Subsequently, Shah and Jani [47] discussed an inventory system that considered varying credit periods for the retailer based on the order quantity. Later, Jani *et al.* [16] demonstrated how to determine the optimal order quantity to maximize profits within a limited time span, considering various payment policies for inventory suppliers. Cárdenas-Barrón *et al.* [7] developed an inventory model with non-linear stock-dependent demand and holding cost with trade credit facility and partial backlogging. A deteriorating inventory model with trade credit and completely backlogged situations is suggested by Tiwari *et al.* [58]. Here, they have considered that the permissible credit period is based on the order quantity. In the recent era, Liao *et al.* [24] investigated an EOQ model for deteriorating items with the trade credit facility. Again, Singh *et al.* [51] presented an inventory model with a three-parameter Weibull deterioration rate where the permissible trade credit facility is adopted. An inventory is formulated and solved under neutrosophic environment considering trade credit for supplier and retailer both by Mohanta *et al.* [31]. Moradi *et al.* [38] investigated an inventory problem with partial trade credit policy. In the recent era, a multi-item inventory model is studied for deteriorating items with trade credit facility by Yadav *et al.* [65].

2.2. Deterioration of items in inventory

Deterioration of an item in the inventory system is a very natural phenomenon. Ghare [12] introduced exponentially decaying items in the inventory system. Time-dependent deterioration rate has been considered by several researchers. An inventory control model for deteriorating products with maximum serviceable lifetime under mixed sales situations is developed by Xie *et al.* [63]. They considered that the rates of demand and deterioration are a function of time. Adak and Mahapatra [2] presented an EOQ model for time and reliability dependent demand and deterioration. An optimal control problem for two-level order linked trade credit with time-dependent deterioration is investigated by Jani *et al.* [15]. Kumar *et al.* [23] investigated a sustainable inventory model for deterioration items under fuzzy environment. Here, preservation technology is used to control the deterioration. A production inventory model with livestock items is invested by Singh and Rana [50]. They considered time dependent deterioration rate of the growing items. Recently, Mahato *et al.* [26] suggested a sustainable inventory model for degrading items with a certain lifetime and dynamic demand.

2.3. Promotional effort dependent demand

Promotional effort is one of the most effective methodologies that help customers to become familiar with the features of the newly arrived items. Demand is always positively related to promotional efforts. In Soni and Chauhan [54] a joint pricing inventory problem is studied by considering the promotional effort. Several supply chain models are investigated with promotional dependent demand by many researchers (*cf.* Ebrahimi *et al.* [10]). Again, Shah *et al.* [48] formulated an EOQ model focusing on optimal strategies, taking into account a trapezoidal demand pattern dependent on price credits. Recently, Nagare *et al.* [41] suggested and solved a problem on discount pricing for deteriorated items subject to promotional effort. The joint effect of selling price and promotional efforts on inventory control is considered by Kumar *et al.* [22]. Paul *et al.* [44] analyzed the impact of promotional activities of a green supply chain. Zhang *et al.* [68] suggested optimal pricing and collection strategies of a manufacturer and retailer where demand depends on product quality and promotional efforts under a fuzzy environment. Their study concluded that the profits of both the manufacturer and retailer will decrease with the increase of the cost factor of promotional efforts.

2.4. Backlogging in inventory system

The inability to satisfy customers' demand from the available stock at an instance is known as backlog. In this situation, some customers choose to wait until their demand is fulfilled and some others go for another source to meet their demand. Backlog is very common situation in retail industry. Abad [1] introduced the idea of customers' impatience in backlogging state and suggested the conversely proportional relationship between backlogging rate and waiting time. Later on, Taleizadeh *et al.* [57] studied an inventory control problem with shortages for fuzzy random quantities and solved it by particle swarm optimization approach. Partial backlogging is introduced in a two warehouse inventory model by Kumar *et al.* [21]. Pakhira *et al.* [43] presented an EOQ model considering partial backlogging and memory effect. They discussed the effect of both the constant and variable backlogging rates. A partially backlogged production inventory model with growing items is developed by Singh and Rana [50]. Again, a smart production inventory model with partial backlogging and constant deterioration is proposed by Yadav *et al.* [66].

2.5. Inventory models under imprecise environments

Several researchers are developing realistic inventory control problems with imprecise parameters in the literature. Kar *et al.* [17, 18]; Mondal *et al.* [33] solved neutrosophic inventory model under uncertain storage capacity constraint. Sarma *et al.* [46] analyzed disaster management problem under uncertain environment by applying trapezoidal neutrosophic numbers in the supply chain. Mondal *et al.* [36] presented an inventory model for seasonal products with delay in payment under neutrosophic environment. In the recent era, Mullai *et al.* [39] investigated a single-valued inventory model where demand is considered as a neutrosophic random variable. Mullai and Surya [40] developed an inventory problem using triangular neutrosophic numbers and solved it by signed distance method. A fuzzy multi-level supplier selection model is developed by Adhami *et al.* [3]. They applied neutrosophic compromise programming approach to solve the problem. Sahoo *et al.* [45] presented an intuitionistic fuzzy EOQ model with backlogging, carbon emission and preservation technology. Here, the unit holding costs are taken as generalized pentagonal intuitionistic fuzzy numbers. A three-stage fuzzy EOQ model is developed considering the effect of carbon emission during transportation and holding of the inventory by Singh *et al.* [52]. Here, the carbon emission costs and transportation costs at each stage are represented by fuzzy numbers. Again, Yadav *et al.* [64] proposed a multi-stage EOQ model with fully backlogged shortages under fuzzy environment. Moreover, a comparison of the present study with earlier investigations is displayed in Table 1.

2.6. Research gaps and research contributions

From the above literature survey, it is observed that some researchers have considered one, two or three situations mentioned above at the same time. But, to the best of our knowledge, till now, none has taken all these factors altogether in an inventory model. The main contributions of the present investigation are as follows:

- An attempt is made to formulate an optimization-based EOQ model with trade credit facility, time-varying partial backlogging rate, and deteriorating items having maximum lifetime.
- As the neutrosophic set represents the impreciseness more realistically, various imprecise coefficients of the proposed models, *i.e.*, inventory relevant unit cost coefficients and rates of interest paid/earned, are taken in the form of generalized single-valued trapezoidal neutrosophic (GSTrN) numbers.
- The application of (α, β, γ)-cut approach of neutrosophic numbers is used to transform neutrosophic parameters into equivalent crisp forms.
- Particular cases are derived under different environments (*i.e.*, intuitionistic fuzzy, fuzzy and crisp) by considering unit cost coefficients and rates of interest paid/earned in the form of generalized trapezoidal intuitionistic fuzzy numbers (GTrIFNs), generalized trapezoidal fuzzy numbers (GTrFNs), and crisp numbers.
- Depending on the trade credit period permitted by the supplier to the retailer, two situations occur in inventory models. Here, an optimization approach is discussed to solve a non-linear unconstrained model with imprecise coefficients for both scenarios.
- Optimal inventory depletion time is obtained that minimizes the defuzzified cost functions and evaluates the corresponding order quantity of all variants.
- The sensitivity analysis is performed for some significant parameters of the proposed model, and afterward, the key managerial acumen is suggested.

3. Problem definition, symbols and hypotheses

In this section, the problem definition is described. The following symbols with their descriptions and hypotheses are considered to formulate the proposed inventory replenishment model.

TABLE 1. Comparison of present work with existing inventory models in various environments.

Authors	Demand rate type	Deterioration rate	Backordering type	Trade credit facility	Environment
Wu *et al.* [62]	Trapezoidal	Time dt.	Partially	NA	Crisp
Soni and Chauhan [54]	Price dt.	Time dt.	Partially	NA	Crisp
Mishra *et al.* [29]	Exponential	Constant	Fully	NA	Fuzzy, Intui.
Mahapatra *et al.* [25]	Prom. effort dt.	Constant	Fully	NA	Fuzzy
Liao *et al.* [24]	Constant	Non-inst.	NA	Fully	Crisp
Singh and Rana [49]	Time dt.	Weibull dist.	NA	Fully	Crisp
Verma *et al.* [61]	Stock dt.	Weibull dist.	Partially	Fully	Crisp
Mondal *et al.* [36]	Time dt.	Weibull dist.	Partially	Fully	Crisp, Neutro.
Present study	Prom. effort dt.	Time dt. with maximum lifetime	Partially	Fully	Neutro., Fuzzy, Intui. and Crisp

Notes. Prom.: Promotional, Intui.: Intuitionistic, Neutro.: Neutrosophic, NA: Not Available, dt.: dependent, dist.: distribution, Non-inst.: Non-instantaneous.

3.1. Problem definition

In order to fulfill the identified lack mentioned in the previous section, this article introduces an imprecise EOQ model to determine optimal ordering strategies for deteriorating items with trade credit financing. This article examines two sub-cases determined by the account settlement time: Case 1, where the credit period is shorter than or equal to the inventory depletion time for settling the account, and Case 2, where the credit period exceeds the inventory depletion time for settling the account. Furthermore, due to insufficient data, human error, and similar causes, impreciseness is incorporated into inventory-relevant unit cost coefficients. In this context, unit cost coefficients and the interest paid and earned are presented as single-valued trapezoidal neutrosophic numbers. Specific models are derived under various environments, like intuitionistic, fuzzy, and crisp. Here, customers' demand is considered as a linearly dependent function of promotional effort. Moreover, considering the significant impact of deterioration on everyday usable perishable products like vegetables, fruits, dairy items, etc., all of which have expiration dates, this analysis incorporates time-dependent deterioration rates. The inventory problem is graphically depicted in Figure 2. Figure 2(a) illustrates the inventory scenario when the settlement time precedes the inventory depletion time, while Figure 2(b) shows the inventory situation when the settlement time surpasses the inventory depletion time for account settlement. Here, the retailer orders the 'Q' quantity from the supplier to fulfill both the demand during the shortage period and the customers' demand. The inventory level $I(t)$ declines due to the combined effects of the demand rate and deterioration. At time t_1, the inventory level reaches zero, and all the demand hereafter (*i.e.*, $t_1 < t \leq T$) is partially backlogged in both cases. Moreover, Figure 1 represents all variants of our proposed EOQ model.

3.2. Symbols:

3.3. Hypotheses:

(1) The total on-hand inventory deteriorates with time dependent deterioration rate $\theta(t)$ and it is expressed as $\theta(t) = \frac{1}{1+L-t}$, where $L > t$ and L is the maximum lifetime of products. When t increases, $\theta(t)$ increases and $\lim_{t\to L}\theta(t) \to 1$ (*cf.* Tai *et al.* [56]).
(2) Shortages are allowed and partially backlogged in the present study. The backlogging rate is supposed to depend on the waiting time for next replenishment. As much as the waiting time $(T-t)$ increases, the backlogging rate becomes smaller. Thus, in the shortage period $(t_1, T]$, the time-varying backlogging rate $\frac{1}{1+\delta(T-t)}$ is considered (*cf.* Mondal *et al.* [35,36]).
(3) The retailer receives a fully permissible delay in payment for the period $[0, \sigma]$, at which they earn interest at the rate I_e. However, if the retailer fails to pay back the payment within the settlement period $[0, \sigma]$, the supplier charges interest at the rate I_p. Also, this study presumes that the rate at which retailer pays

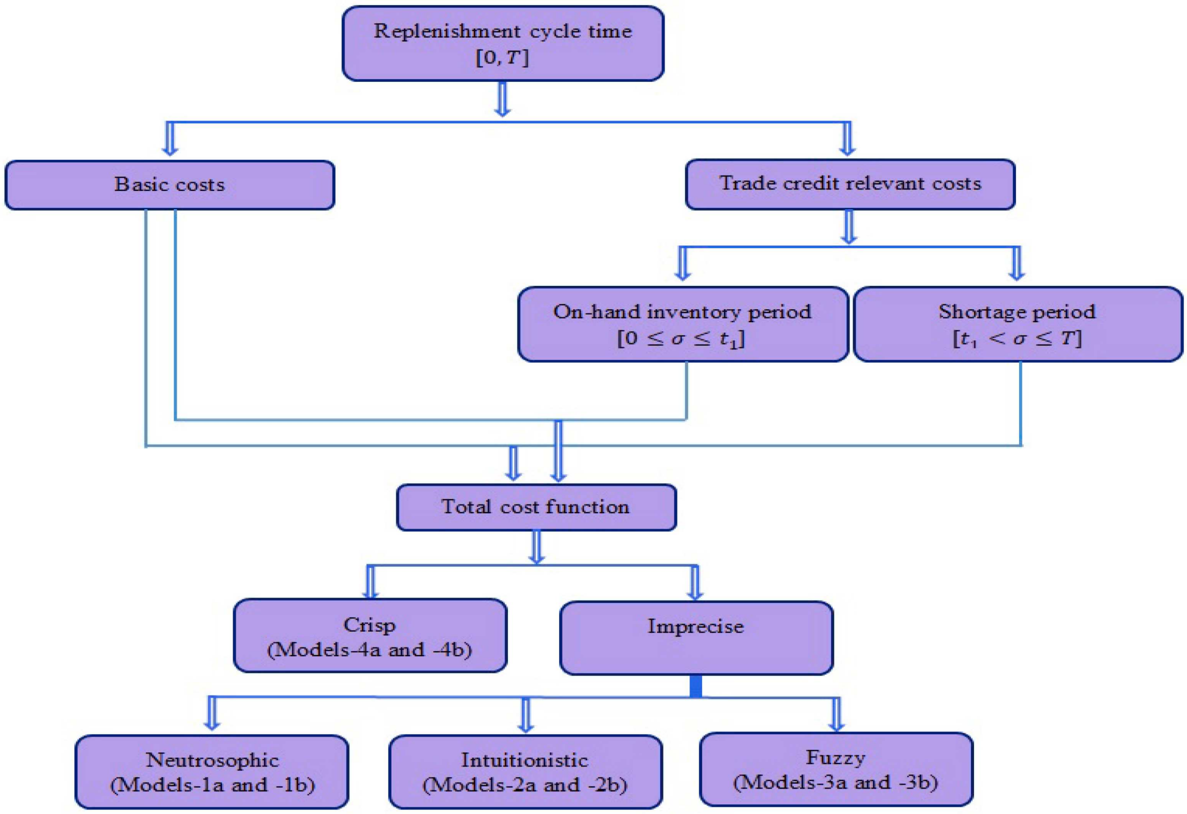


FIGURE 1. Graphical representation of various cases to proposed EOQ model.

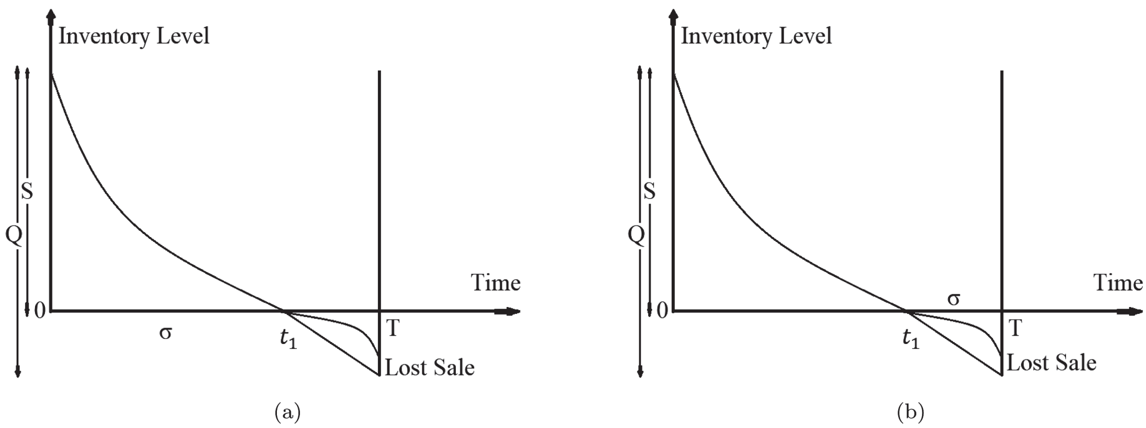


FIGURE 2. Proposed EOQ model across various trade credit intervals over cycle $[0, T]$. (a) Payment before inventory depletion time *i.e.*, $0 \leq \sigma \leq t_1$. (b) Payment after inventory depletion time *i.e.*, $t_1 < \sigma \leq T$.

Inventory related parameters:

Symbol	Explanation
C_1	unit inventory holding cost per unit time (in \$)
C_2	unit backlogging cost per unit time (in \$)
C_3	unit deterioration cost (in \$)
C_4	unit opportunity cost due to lost-sales (in \$)
$a^{'}$	constant part of demand rate ($a^{'} > 0$)
$b^{'}$	promotional effort dependent demand rate coefficient ($b^{'} > 0$)
σ	pre-determined permissible period of delay in settling accounts with supplier (in year)
c	unit purchase cost (in \$)
p	unit selling price (in \$)
η	promotional effort ($\eta \geq 0$)
k	promotional cost for unit promotional effort
δ	fractional parameter of backlogging function ($0 \leq \delta \leq 1$)
L	maximum lifetime of products (in year)
A	replenishment cost per order (in \$)
I_p	interest rate paid by retailer per \$ per unit time
I_e	interest rate earned by retailer per \$ per unit time
S	initial on-hand inventory level (*i.e.*, $I(0) = S$) (units of item)
R	maximum inventory shortage level (units of item)
Q	order quantity per cycle (units of item)
T	replenishment cycle length (in year)
$I(t)$	inventory level at time t over $[0, T]$
$\theta(t)$	time dependent deterioration rate of an item over on-hand inventory period $[0, T]$
TAC	total average cost (in \$)
t_1	inventory depletion time (in year) (decision variable)

interest is higher than the rate at which retailer earns interest, *i.e.*, $I_p > I_e$ (*cf.* Cárdenas-Barrón *et al.* [7]; Mondal *et al.* [35]; Mondal *et al.* [37]).

(4) The demand rate is linearly increasing function of the promotional effort (η) and it is expressed as $D(\eta) = a^{'} + b^{'}\eta$, where $a^{'}$ is the initial demand rate, independent of the sales teams effort η, and $b^{'}$ is a scale parameter of demand change, which varies with sales effort. (*cf.* Ebrahimi *et al.* [10]; Mahapatra *et al.* [25]).

(5) The promotional effort cost (PEC) is considered as $k\eta^m$, where $k > 0$ and m are constants and values are selected from the best fit of the promotional cost function, which is an increasing function of the promotional effort (*cf.* Soni and Chauhan [54]; Mahapatra *et al.* [25]).

(6) In real-life, the lead time can be constant, variable and stochastic in nature. However, for the sake of simplicity alone, this article considers the lead time is negligible (*cf.* Mondal *et al.* [36]; Chakraborty *et al.* [8]).

(7) Replenishment rate is instantaneously infinite. However, its size is finite and constant (*cf.* Cárdenas-Barrón *et al.* [7]; Chakraborty *et al.* [8]).

4. Formulation of mathematical model

In this section, an order-level inventory model is formulated with a time-dependent deterioration rate and fully permissible delay in payment in various trade credit intervals. The inventory level $I(t)$ at any instant, t, is at its maximum of $Q(= S + R)$ units, of which R units are delivered towards backorders, leaving a balance of S units in the initial inventory at time $t = 0$. The inventory level $I(t)$ declines due to the combined effect of the rate of demand and deterioration. At time t_1, the inventory level is zero and all the demand hereafter (*i.e.*, $t_1 < t \leq T$) is partially backlogged and depends on the waiting time of the customers with backlogging parameter $0 \leq \delta \leq 1$. Total backlogged items' amount is re-established in the next replenishment period. The typical behavior of the proposed inventory system at any time t is depicted in Figure 2. Considering the above situations, in the present article an instantaneous inventory level $I(t)$ at any time t over cycle $[0, T]$ is developed with the help of following boundary value problem (BVP):

$$\frac{dI(t)}{dt} + \frac{1}{1+L-t}I(t) = -D(\eta) \qquad if \quad 0 \le t \le t_1 \tag{4.1}$$

$$\text{and} \quad \frac{dI(t)}{dt} = -\frac{D(\eta)}{1+\delta(T-t)} \qquad if \quad t_1 < t \le T \tag{4.2}$$

with the boundary conditions $I(0) = S,\ I(t_1) = 0$ and $I(T) = -R$.

Solving above BVP (4.1) and (4.2), the time-dependent on-hand inventory level at any time t can be represented as

$$I(t) = \begin{cases} D(\eta)(1+L-t)log\Big(\frac{1+L-t}{1+L-t_1}\Big) & \forall\ 0 \le t \le t_1 \\ \frac{D(\eta)}{\delta}log\Big(1+\delta(T-t)\Big) - R & \forall\ t_1 < t \le T. \end{cases} \tag{4.3}$$

Using the initial condition $I(0) = S$ in the equation[2] (4.3), one can get the initial stock as follows

$$S = D(\eta)(1+L)log\Big(\frac{1+L}{1+L-t_1}\Big)\cdot \tag{4.4}$$

Again, with the help of continuity of $I(t)$ at $t = t_1$ the maximum shortage level is given by

$$R = \frac{D(\eta)}{\delta}log\Big(1+\delta(T-t_1)\Big)\cdot \tag{4.5}$$

In any EOQ model, the order quantity per cycle $[0, T]$ is the sum of initial inventory (S) and total backlogged inventory (R) in period $(t_1, T]$. So, the order quantity per cycle $[0, T]$ becomes

$$Q = D(\eta)\Big\{(1+L)log\Big(\frac{1+L}{1+L-t_1}\Big) + \frac{1}{\delta}log\big(1+\delta(T-t_1)\big)\Big\}. \tag{4.6}$$

Again, after placing the order, managers arrange a suitable place, either owned or rented, to store the products. Even if this storage place is owned, there occurs some expenses for this in the form of insurance, taxes, electricity charge, maintenance, etc. Such cost is termed as holding cost (HC). This is directly proportional to the items' quantity and holding period. Therefore, the total HC per cycle is evaluated as

$$\begin{aligned} HC &= C_1\int_0^{t_1} I(t)dt, \qquad 0 \le t \le t_1 \\ &= \frac{C_1 D(\eta)}{4}\Big\{2(1+L)^2 log\Big(\frac{1+L}{1+L-t_1}\Big) - (2+2L-t_1)t_1\Big\}. \end{aligned} \tag{4.7}$$

In any inventory system, shortages arise whenever available stock becomes inadequate to satisfy the customers' demand. Shortages result in a negative impression of the manager and directly affect the total profit of the business. In the proposed model, shortages occur in the period $(t_1, T]$ (*cf.* Fig. 2), and the retailer confirms backorder to satisfy the demand of the customers waiting for the next replenishment. Therefore, the associated backlogging cost (BC) per cycle can be obtained as follows:

$$\begin{aligned} BC &= -C_2\int_{t1}^{T} I(t)dt, \qquad t_1 < t \le T \\ &= \frac{C_2 D(\eta)}{\delta}\Big\{(T-t_1) - \frac{1}{\delta}log\big(1+\delta(T-t_1)\big)\Big\}. \end{aligned} \tag{4.8}$$

[2]Eq. stands for equation.

In real-life businesses, deteriorating items often get damaged, decayed, vaporized or affected by external factors. The amount of deteriorated items depends on the deterioration rate. The associated deterioration cost (DC) is proportional to the number of total deteriorated items in period $[0, t_1]$. Therefore, the total DC per cycle is calculated as follows:

$$\begin{aligned} DC &= C_3\left(S - \int_0^{t_1} D(\eta)dt\right) \\ &= C_3 D(\eta)\Big\{(1+L)log\Big(\frac{1+L}{1+L-t_1}\Big) - t_1\Big\}. \end{aligned} \tag{4.9}$$

Next, the lost sales are the selling opportunities that are missed out for some reasons like an item is out of stock, suppliers don't carry some particular brands, any other reason comes out, etc. Besides, this cost includes the subsequent loss of intangible goodwill and the cost of sharing information with retailers. Therefore, the time-dependent opportunity cost (OC) due to lost sales at any time t is the difference between total demand $(D(\eta))$ and total time-dependent backorder inventory during $(t_1, T]$. Thus, the total OC per cycle is given by

$$\begin{aligned} OC &= C_4 \int_{t_1}^{T} \left(1 - \frac{1}{1+\delta(T-t_1)}\right) D(\eta)dt \\ &= C_4 D(\eta)\Big\{(T-t_1) - \frac{1}{\delta}log\Big(1+\delta(T-t_1)\Big)\Big\}. \end{aligned} \tag{4.10}$$

A promotional inventory report is a customizable record to avoid fulfillment delays, rush printing fees and unhappy customers. Strategies used by any company that has a product or service inventory to sell the inventory for profits are a general interpretation of monetizing inventory. However, this expression more specifically relates to online publishers and methods used to turn advertising inventory or placement opportunities into advertising revenue. Thus, the total PEC per cycle $[0, T]$ becomes

$$PEC = k\eta^m, k > 0 \text{ and } m \text{ is a positive constant} \tag{4.11}$$

Now, based on the settlement time of the supplier to the retailer, the following cases are derived:

4.1. Case 1:

Payment before total inventory depletion time $(0 \le \sigma \le t_1)$

In this case, the permissible delay payment period $[0, \sigma]$ ends on or before the total inventory reduces to zero completely. A retailer does not need to pay any interest in that period. However, suppose the retailer fails to pay back the total payment within the fixed settlement time. In that case, the supplier incurs some charge at an interest rate I_p on the unpaid balance for the overdue period. Consequently, the total interest paid by retailer per cycle $[0, T]$ is calculated as

$$\begin{aligned} IP_1 &= cI_p \int_{\sigma}^{t_1} I(t)dt, \quad 0 \le t \le t_1 \\ &= \frac{cI_p D(\eta)}{4}\Big\{(1+L-t_1)^2 + (1+L-\sigma)^2\Big(2log\Big(\frac{1+L-\sigma}{1+L-t_1}\Big) - 1\Big)\Big\}. \end{aligned} \tag{4.12}$$

Meanwhile, the retailer sells items and deposits the revenue into some account during the period $[0, \sigma]$ and earns interest at a rate I_e on it. So, the total interest earned by retailer in $[0, T]$ is as follows

$$\begin{aligned} IE_1 &= pI_e \int_0^{\sigma} D(\eta)tdt \\ &= \frac{pI_e}{2} D(\eta)\sigma^2. \end{aligned} \tag{4.13}$$

Hence, the TAC per cycle in this case is evaluated as

$$TAC_a(t_1) = \frac{1}{T}\left[A + HC + BC + DC + OC + PEC + IP_1 - IE_1\right] \tag{4.14}$$

where HC, BC, DC, OC, PEC, IP_1 and IE_1 are respectively given in equations (4.7) to (4.13) along with the fixed replenishment cost A.

4.2. Case 2:

Payment after total inventory depletion time ($t_1 < \sigma \leq T$)

In this scenario, the permissible delay period terminates after the inventory depletion time. So, at the end of settlement period, there is no in-stock item in the inventory system and it is expected that the retailer is able to pay back the total payment to the supplier. Thus, total interest payable in $[0, T]$ is nil *i.e.*, $IP_2 = 0$. In the meantime, retailer has accumulated fund in various ways, like by selling items, by earning interest in positive inventory period $[0, t_1]$ or by investing cash in period $[t_1, \sigma]$ after exhausting inventory at time t_1. So, total interest earned by the retailer in $[0, T]$ is

$$\begin{aligned} IE_2 &= pI_e\left\{\int_0^{t_1} D(\eta)tdt + (\sigma - t_1)\int_0^{t_1} D(\eta)dt\right\} \\ &= pI_eD(\eta)t_1\left(\sigma - \frac{t_1}{2}\right). \end{aligned} \tag{4.15}$$

Analogous to previous case, the TAC per cycle $[0, T]$ in this case is as follows

$$TAC_b(t_1) = \frac{1}{T}\left[A + HC + BC + DC + OC + PEC - IE_2\right] \tag{4.16}$$

where A is the replenishment cost and the expressions for HC, BC, DC, OC, PEC and IE_2 are given in equations (4.7) to (4.11) and (4.15) respectively.

4.3. Model formulation under neutrosophic environment

In the traditional inventory models, it is assumed that all information related to an inventory control system is well-known. But, in today's competitive and dynamic business world, knowing all the necessary information is almost impossible. It is often seen from practical scenarios that in an organization unit cost coefficients like holding cost, shortage cost, deterioration cost, interest rates paid/earned, etc., may not be deterministic all the time due to insufficient data, weak forecasting, human error, etc. In that situation, these parameters become uncertain. Recently, neutrosophic set is introduced to express impreciseness more realistically. Modern researchers use neutrosophic set theory, developed by Smarandache [53], as one of the most effective methods to overcome this uncertainly or vagueness. In the present investigation, various unit cost coefficients such as $\widehat{C}_1, \widehat{C}_2, \widehat{C}_3, \widehat{C}_4$, interest paid $\widehat{I}_p$ and earned $\widehat{I}_e$ are assumed to be in neutrosophic environment and they are represented by GSTrN numbers (*cf.* Definition 3 in Appendix) Let us assume

$$\begin{aligned} \widehat{C}_i &= \left\langle (C_{i1}, C_{i2}, C_{i3}, C_{i4})\,; \mu_{\widehat{C}_i}, \sigma_{\widehat{C}_i}, \nu_{\widehat{C}_i}\right\rangle, \quad i = 1, 2, 3, 4 \\ \widehat{I}_j &= \left\langle (I_{j1}, I_{j2}, I_{j3}, I_{j4})\,; \mu_{\widehat{I}_j}, \sigma_{\widehat{I}_j}, \nu_{\widehat{I}_j}\right\rangle, \quad j = p, e. \end{aligned}$$

Therefore, the two cases of the proposed inventory system are formulated under neutrosophic environment as follows:

4.3.1. Model-1a: Neutrosophic EOQ model for payment before inventory depletion time

In this case the neutrosophic model is represented as follows:

$$\begin{aligned}
&\text{Min } \ \widehat{TAC}_{1a}(t_1)\\
&= \frac{1}{T}\Bigg[A + D(\eta)\Bigg\{\frac{\widehat{C}_1}{4}\Bigg(2(1+L)^2 log\Big(\frac{1+L}{1+L-t_1}\Big) - (2+2L-t_1)t_1\Bigg)\\
&+\frac{(\widehat{C}_2+\widehat{C}_4\delta)}{\delta}\Bigg((T-t_1) - \frac{1}{\delta}log\big(1+\delta(T-t_1)\big)\Bigg) + \widehat{C}_3\Bigg((1+L)log\Big(\frac{1+L}{1+L-t_1}\Big) - t_1\Bigg)\\
&+\frac{c\widehat{I}_p}{4}\Bigg((1+L-t_1)^2 + (1+L-\sigma)^2\Big(2log\Big(\frac{1+L-\sigma}{1+L-t_1}\Big)-1\Big)\Bigg) - \frac{p\widehat{I}_e}{2}\sigma^2\Bigg\} + k\eta^m\Bigg]\\
&\text{subject to } \ t_1 > 0
\end{aligned} \tag{4.17}$$

4.3.2. Model-1b: Neutrosophic EOQ model for payment after inventory depletion time

For this case the neutrosophic model is expressed as given below:

$$\begin{aligned}
&\text{Min } \ \widehat{TAC}_{1b}(t_1)\\
&= \frac{1}{T}\Bigg[A + D(\eta)\Bigg\{\frac{\widehat{C}_1}{4}\Bigg(2(1+L)^2 log\Big(\frac{1+L}{1+L-t_1}\Big) - (2+2L-t_1)t_1\Bigg)\\
&+\frac{(\widehat{C}_2+\widehat{C}_4\delta)}{\delta}\Bigg((T-t_1) - \frac{1}{\delta}log\big(1+\delta(T-t_1)\big)\Bigg) + \widehat{C}_3\Bigg((1+L)log\Big(\frac{1+L}{1+L-t_1}\Big) - t_1\Bigg)\\
&-\Bigg(p\widehat{I}_e t_1(\sigma - \frac{t_1}{2})\Bigg)\Bigg\} + k\eta^m\Bigg]\\
&\text{subject to } \ t_1 > 0
\end{aligned} \tag{4.18}$$

5. Particular cases

In a practical situation, all the data in an inventory system may not be precisely known rather some of them may be imprecise or uncertain. This uncertainty may also be tackled using intuitionistic fuzzy set theory (developed by Atanassov [4]) and fuzzy set theory (introduced by Zadeh [67]). In this section, different models are derived when the unit cost coefficients (C_1, C_2, C_3, C_4), interest paid (I_p) and earned (I_e) are considered under different environments like intuitionistic, fuzzy and crisp. Depending on these assumptions, the following particular models are formulated:

5.1. Model under intuitionistic environment

In the inventory system, when the vagueness or uncertainty is taken to be captured in intuitionistic fuzzy environment; generalized trapezoidal intuitionistic fuzzy numbers can be considered to represent the vagueness of the inventory parameters. Here, all the above said cost coefficients and interest coefficients are considered as GTrI numbers (*cf.* Definition 2 in Appendix) and are represented as given below:

$$\left.\begin{aligned}
\breve{C}_i &= \Big\langle (C_{i1}, C_{i2}, C_{i3}, C_{i4})\,; \mu_{\breve{C}_i}, \sigma_{\breve{C}_i}, \nu_{\breve{C}_i}\Big\rangle, \quad i = 1,2,3,4\\
\breve{I}_j &= \Big\langle (I_{j1}, I_{j2}, I_{j3}, I_{j4})\,; \mu_{\breve{I}_j}, \sigma_{\breve{I}_j}, \nu_{\breve{I}_j}\Big\rangle, \quad j = p, e
\end{aligned}\right\} \tag{5.1}$$

Thus, the two cases of the inventory systems are expressed in the following forms:

5.1.1. Model-2a: Intuitionistic EOQ model for payment before inventory depletion time

In this particular case, the objective function TAC is considered as given in equation (4.14) except the cost coefficients presented in equation (5.1). Thus, the objective function is expressed as $\widetilde{TAC}_{2a}(t_1)$, say.

5.1.2. Model-2b: Intuitionistic EOQ model for payment after inventory depletion time

This case is also formulated in a similar manner where the objective function is represented in equation (4.16) along with the replacement in the crisp cost coefficients by intuitionistic ones as given in equation (5.1). Let us denote this minimization type objective function as $\widetilde{TAC}_{2b}(t_1)$, say.

Therefore, from Sections 5.1.1 and 5.1.2, we have the following two models under intuitionistic environment:

Model-2a:	and	**Model-2b:**
Min $\widetilde{TAC}_{2a}(t_1)$		Min $\widetilde{TAC}_{2b}(t_1)$
subject to $t_1 > 0$		subject to $t_1 > 0$

5.2. Model under fuzzy environment

Sometimes, the uncertainty present in the information of an inventory system can be handled using fuzzy set theory. Here, we have considered generalized trapezoidal fuzzy number to express the uncertain parameters. Thus, in this particular case, all the cost and interest coefficients are considered as GTrF numbers (*cf.* Definition 1 in Appendix). Let us assume,

$$\left.\begin{array}{l}\widetilde{C}_i = \left\langle (C_{i1}, C_{i2}, C_{i3}, C_{i4})\,; \mu_{\widetilde{C}_i}, \sigma_{\tilde{C}_i}, \nu_{\widetilde{C}_i} \right\rangle, \quad i = 1,2,3,4 \\ \widetilde{I}_j = \left\langle (I_{j1}, I_{j2}, I_{j3}, I_{j4})\,; \mu_{\widetilde{I}_j}, \sigma_{\widetilde{I}_j}, \nu_{\widetilde{I}_j} \right\rangle, \quad j = p, e \end{array}\right\} \tag{5.2}$$

Now, the two corresponding cases of the inventory systems are derived in the following forms:

5.2.1. Model-3a: Fuzzy EOQ model for payment before inventory depletion time

The objective function for this particular case is written as $\widetilde{TAC}_{3a}(t_1)$, say, and is developed by replacing the crisp coefficients of equation (4.14) by fuzzy numbers as in equation (5.2). The model is again formulated as minimization type problem.

5.2.2. Model-3b: Fuzzy EOQ model for payment after inventory depletion time

In this case, the objective function is formulated to minimize total cost $\widetilde{TAC}_{3b}(t_1)$, say, which is given in equation (4.16) except that the cost parameters are in fuzzy environment as in equation (5.2).

Thus, from Sections 5.2.1 and 5.2.2, we have the following two models under fuzzy environment:

Model-3a:	and	**Model-3b:**
Min $\widetilde{TAC}_{3a}(t_1)$		Min $\widetilde{TAC}_{3b}(t_1)$
subject to $t_1 > 0$		subject to $t_1 > 0$

5.3. Model under crisp environment

In this case, all the parameters are considered to be known. Therefore, the two cases of the inventory systems are formed using the following expressions:

5.3.1. Model-4a: Crisp EOQ model for payment before inventory depletion time

$$\begin{array}{l}\text{Min} \quad TAC_{4a}(t_1) = TAC_a(t_1) \quad \text{(as given in Eq. (4.14))} \\ \text{subject to} \quad t_1 > 0\end{array}$$

5.3.2. Model-4b: Crisp EOQ model for payment after inventory depletion time

$$\begin{aligned}&\text{Min}\ \ TAC_{4b}(t_1) = TAC_b(t_1)\ \ \text{(as given in Eq. (4.16))}\\&\text{subject to}\ \ t_1 > 0\end{aligned}$$

6. Solution procedure

In this section, we explain the solution procedures for the proposed inventory models. To obtain a solution space, the imprecise models are first converted into an equivalent crisp form. After that, the reduced deterministic problems are solved.

6.1. Solution procedure for Models-1a and -1b

In these two models, some parameters are assumed to be in neutrosophic environment, and these are considered in the form of GSTrN numbers. So, at first, we transform the imprecise parameters into crisp ones. The necessary steps for this transformation are discussed below:

Step 1:
In this step, we calculate the α-cut (for membership), β-cut (for non-membership) and γ-cut (for indeterminacy) of all the neutrosophic unit costs and interest parameters. Using Definition 4 (*cf.* Appendix) we derive the following expressions:

$$\widehat{C}_{i\alpha} = \left[L_{\widehat{C}_i}(\alpha), R_{\widehat{C}_i}(\alpha)\right] = \left[\frac{(\mu_{\widehat{C}_i} - \alpha)C_{i1} + \alpha C_{i2}}{\mu_{\widehat{C}_i}}, \frac{(\mu_{\widehat{C}_i} - \alpha)C_{i4} + \alpha C_{i3}}{\mu_{\widehat{C}_i}}\right] \tag{6.1}$$

$$\widehat{C}_{i\beta} = [L'_{\widehat{C}_i}(\beta), R'_{\widehat{C}_i}(\beta)] = \left[\frac{(1-\beta)C_{i2} + (\beta - \nu_{\widehat{C}_i})C_{i1}}{1-\nu_{\widehat{C}_i}}, \frac{(1-\beta)C_{i3} + (\beta - \nu_{\widehat{C}_i})C_{i4}}{1-\nu_{\widehat{C}_i}}\right] \tag{6.2}$$

$$\widehat{C}_{i\gamma} = [L''_{\widehat{C}_i}(\gamma), R''_{\widehat{C}_i}(\gamma)] = \left[\frac{(1-\gamma)C_{i2} + (\gamma - \sigma_{\widehat{C}_i})C_{i1}}{1-\sigma_{\widehat{C}_i}}, \frac{(1-\gamma)C_{i3} + (\gamma - \sigma_{\widehat{C}_i})C_{i4}}{1-\sigma_{\widehat{C}_i}}\right] \tag{6.3}$$

$$\widehat{I}_{j\alpha} = \left[L_{\widehat{I}_j}(\alpha), R_{\widehat{I}_j}(\alpha)\right] = \left[\frac{(\mu_{\widehat{I}_j} - \alpha)I_{j1} + \alpha I_{j2}}{\mu_{\widehat{I}_j}}, \frac{(\mu_{\widehat{I}_j} - \alpha)I_{j4} + \alpha I_{j3}}{\mu_{\widehat{I}_j}}\right] \tag{6.4}$$

$$\widehat{I}_{j\beta} = [L'_{\widehat{I}_j}(\beta), R'_{\widehat{I}_j}(\beta)] = \left[\frac{(1-\beta)I_{j2} + (\beta - \nu_{\widehat{I}_j})I_{j1}}{1-\nu_{\widehat{I}_j}}, \frac{(1-\beta)I_{j3} + (\beta - \nu_{\widehat{I}_j})I_{j4}}{1-\nu_{\widehat{I}_j}}\right] \tag{6.5}$$

$$\widehat{I}_{j\gamma} = [L''_{\widehat{I}_j}(\gamma), R''_{\widehat{I}_j}(\gamma)] = \left[\frac{(1-\gamma)I_{j2} + (\gamma - \sigma_{\widehat{I}_j})C_{j1}}{1-\sigma_{\widehat{I}_j}}, \frac{(1-\gamma)I_{j3} + (\gamma - \sigma_{\widehat{I}_j})I_{j4}}{1-\sigma_{\widehat{I}_j}}\right]. \tag{6.6}$$

Here, for $i = 1, 2, 3, 4$, above equations (6.1) to (6.3) represent the respective α-cut, β-cut and γ-cut of unit HC, BC, DC and OC. The similar terms for interest paid and earned are given in equations (6.4) to (6.6) for j=p,e respectively.

Step 2:
In this step, $\langle\alpha, \beta, \gamma\rangle$-cuts are calculated for all the neutrosophic parameters with the help of α-cuts, β-cuts and γ-cuts obtained in Step 1. This operation transforms the neutrosophic numbers into equivalent crisp intervals. Thus, using Theorem 2 (*cf.* Appendix), we obtain the following crisp intervals corresponding to the unit costs and interest coefficients:

$$\widehat{C}_i \equiv [L_{\widehat{C}_i}, R_{\widehat{C}_i}]\ (i = 1, 2, 3, 4)\ \ \text{and}\ \ \widehat{I}_j \equiv [L_{\widehat{I}_j}, R_{\widehat{I}_j}]\ (j = p, e)$$

where $L_{\widehat{C}_i} = max\left\{L_{\widehat{C}_i}(\alpha), L^{'}_{\widehat{C}_i}(\beta), L^{''}_{\widehat{C}_i}(\gamma)\right\}, R_{\widehat{C}_i} = min\left\{R_{\widehat{C}_i}(\alpha), R^{'}_{\widehat{C}_i}(\beta), R^{''}_{\widehat{C}_i}(\gamma)\right\},$

$$L_{\widehat{I}_j} = max\left\{L_{\widehat{I}_j}(\alpha), L^{'}_{\widehat{I}_j}(\beta), L^{''}_{\widehat{I}_j}(\gamma)\right\}, \quad R_{\widehat{I}_j} = min\left\{R_{\widehat{I}_j}(\alpha), R^{'}_{\widehat{I}_j}(\beta), R^{''}_{\widehat{I}_j}(\gamma)\right\}.$$

Step 3:
To convert the above-obtained interval numbers into crisp parametric functions, we apply the weighted arithmetic mean approach (*cf.* Theorem 3 in Appendix). Thus, the neutrosophic cost coefficients are converted into the following crisp parametric functions:

$$\widehat{C}_i \equiv L_{\widehat{C}_i}(1-\rho) + R_{\widehat{C}_i}\rho,\ (i=1,2,3,4) \text{ and } \widehat{I}_j \equiv L_{\widehat{I}_j}(1-\rho) + R_{\widehat{I}_j}\rho,\ (j=p,e)\ ,\quad \rho \in [0,1]. \tag{6.7}$$

Therefore, the neutrosophic models are transformed into equivalent crisp models in this step. Let us denote the objective functions of the reduced crisp formulations of the Models-1a and -1b as $TAC^{'}_{1a}(t_1)$ and $TAC^{'}_{1b}(t_1)$, respectively. In these two objective functions, all the terms remain similar as in equations (4.17) and (4.18) except the above obtained parametric functions given in equation (6.7).

Step 4:
To determine the optimal solution for the inventory depletion time t_1 of Models-1a and -1b, set the first order derivatives of the converted objective functions with respect to t_1 (*i.e.*, $\frac{dTAC^{'}_{1a}(t_1)}{dt_1}$ and $\frac{dTAC^{'}_{1b}(t_1)}{dt_1}$) to zero, separately and get the following equations:

$$K_1 + \frac{c(L_{\widehat{I}_p}(1-\rho) + R_{\widehat{I}_p}\rho)(\sigma - t_1)(-2-2L+\sigma+t_1)}{2(1+L-t_1)} = 0 \tag{6.8}$$

and

$$K_1 + p(L_{\widehat{I}_e}(1-\rho) + R_{\widehat{I}_e}\rho)(t_1 - \sigma) = 0 \tag{6.9}$$

respectively where K_1 is an expression given below:

$$K_1 = \Bigg\{(L_{\widehat{C}_4}(1-\rho) + R_{\widehat{C}_4}\rho)\left(-1 + \frac{\delta}{1+\delta(T-t_1)}\right) - \frac{(L_{\widehat{C}_2}(1-\rho) + R_{\widehat{C}_2}\rho)(T-t_1)}{1+\delta(T-t_1)}$$
$$+\frac{(L_{\widehat{C}_3}(1-\rho) + R_{\widehat{C}_3}\rho)t_1}{1+L-t_1} + \frac{(L_{\widehat{C}_1}(1-\rho) + R_{\widehat{C}_1}\rho)(2+2L-t_1)t_1}{2(1+L-t_1)}\Bigg\}.$$

Step 5:
In this step, we evaluate the second order derivatives of the objective functions with respect to t_1 and verify the following conditions:

$$\left(\frac{d^2TAC^{'}_{1a}(t_1)}{dt_1^2}\right) > 0 \text{ and } \left(\frac{d^2TAC^{'}_{1b}(t_1)}{dt_1^2}\right) > 0.$$

For the present problem we have

$$\frac{d^2TAC^{'}_{1a}(t_1)}{dt_1^2} = K_2 + \frac{D(\eta)}{4T}c(L_{\widehat{I}_p}(1-\rho) + R_{\widehat{I}_p}\rho)\left(2 + \frac{2(1+L-\sigma)^2}{(1+L-t_1)^2}\right)$$

and

$$\frac{d^2TAC^{'}_{1b}(t_1)}{dt_1^2} = K_2 + \frac{D(\eta)}{T}p(L_{\widehat{I}_e}(1-\rho) + R_{\widehat{I}_e}\rho)$$

where K_2 is an expression given below:

$$K_2 = \frac{D(\eta)}{4T}\Bigg\{(L_{\widehat{C}_1}(1-\rho) + R_{\widehat{C}_1}\rho)\left(2 + \frac{2(1+L)^2}{(1+L-t_1)^2}\right) + \frac{4(L_{\widehat{C}_3}(1-\rho) + R_{\widehat{C}_3}\rho)(1+L)}{(1+L-t_1)^2}$$
$$+\frac{4((L_{\widehat{C}_2}(1-\rho) + R_{\widehat{C}_2}\rho) + (L_{\widehat{C}_4}(1-\rho) + R_{\widehat{C}_4}\rho)\delta)}{\left(1+\delta(T-t_1)\right)^2}\Bigg\}.$$

Step 6:
Once Step 5 is verified, the optimal decision variable for Model-1a and Model-1b is obtained by solving equations (6.8) and (6.9), given in Step 4, respectively. The corresponding optimal average costs are calculated for both the models by using the value of optimal decision variable.

6.2. Solution procedure for Models-2a and -2b

The required step-by-step solution procedure for conversion of intuitionistic models into crisp models and evaluation of optimum solution are discussed below:

Step 1:
This step evaluates the α-cut for membership and β-cut for non-membership degrees of intuitionistic unit costs and interest parameters. The expressions are the same as given in equations (6.1) to (6.2) and (6.4) to (6.5). Let us denote them as $\breve{C}_{i\alpha}$, $\breve{C}_{i\beta}$, $\breve{I}_{j\alpha}$ and $\breve{I}_{j\beta}$, $(i = 1, 2, 3, 4, \quad j = p, e)$ respectively.

Step 2:
Now, we calculate the $\langle\alpha, \beta\rangle$-cuts for the intuitionistic cost coefficients by using Theorem 1 (*cf.* Appendix), which transforms the parameters into the following crisp intervals:

$$\breve{C}_i \equiv [L_{\breve{C}_i}, R_{\breve{C}_i}] \ \ (i = 1, 2, 3, 4) \quad \text{and} \quad \breve{I}_j \equiv [L_{\breve{I}_j}, R_{\breve{I}_j}] \ \ (j = p, e)$$

where $$L_{\breve{C}_i} = max\left\{L_{\breve{C}_i}(\alpha), L'_{\breve{C}_i}(\beta)\right\},\ R_{\breve{C}_i} = min\left\{R_{\breve{C}_i}(\alpha), R'_{\breve{C}_i}(\beta)\right\},$$

$$L_{\breve{I}_j} = max\left\{L_{\breve{I}_j}(\alpha), L'_{\breve{I}_j}(\beta)\right\}, \qquad R_{\breve{I}_j} = min\left\{R_{\breve{I}_j}(\alpha), R'_{\breve{I}_j}(\beta)\right\}.$$

Step 3:
Here, Step 3 to Step 6 of Section 6.1 are applied to convert the models into crisp parametric problems and to find the optimum solutions.

6.3. Solution procedure for Models-3a and -3b

The necessary steps to solve fuzzy inventory models are discussed below:

Step 1:
This step calculates the α-cut of all the fuzzy unit costs and interest coefficients. Let us denote them as $\widetilde{C}_{i\alpha}$ $(i = 1, 2, 3, 4)$ and $\widetilde{I}_{j\alpha}$ $(j = p, e)$. Now these expressions for the fuzzy parameters are calculated as similar to the expression given in equations (6.1) and (6.4), respectively.

Step 2:
The obtained crisp intervals are converted into parametric functions by using Theorem 3 (*cf.* Appendix), and the corresponding functions become as follows:

$$\widetilde{C}_i \equiv L_{\widetilde{C}_i}(\alpha)(1-\rho) + R_{\widetilde{C}_i}(\alpha)\rho, \quad (i = 1, 2, 3, 4) \quad \text{and} \quad \widetilde{I}_j \equiv L_{\widetilde{I}_j}(\alpha)(1-\rho) + R_{\widetilde{I}_j}(\alpha)\rho, \quad (j = p, e)\ , \quad \rho \in [0, 1].$$

Step 3:
Now, the fuzzy problems are reduced to their equivalent crisp formulations. Therefore, Step 4 to Step 6 of Section 6.1 are followed to find the optimum solutions.

6.4. Solution procedure for Models-4a and -4b

The crisp models (Models-4a and -4b) are solved by following Step 4 to Step 6 of Section 6.1 as only crisp parameters are present in these models.

TABLE 2. Values of various parameters of the proposed model.

Parameters	Values	Parameters	Values
a	100 units/year	δ	0.56
b	0.6 units/year	η	2.5
c	\$8/unit item	m	2
p	\$10/unit item	k	12
L	0.8 year	σ	0.7/0.4 year
A	\$500/order	T	1 year

TABLE 3. Values of various cost coefficients of the proposed models in different environments.

Environments	Parameters' values	
Neutrosophic	$\widehat{C}_1 = \langle(0.7, 0.8, 0.9, 1.0); 0.6, 0.4, 0.2\rangle$,	$\widehat{C}_4 = \langle(1, 5, 8, 11); 0.7, 0.3, 0.4\rangle$,
	$\widehat{C}_2 = \langle(3, 5, 7, 9); 0.8, 0.2, 0.3\rangle$,	$\widehat{I}_e = \langle(0.02, 0.04, 0.08, 0.10); 0.5, 0.2, 0.3\rangle$
	$\widehat{C}_3 = \langle(2, 6, 10, 14); 0.4, 0.5, 0.1\rangle$,	$\widehat{I}_p = \langle(0.05, 0.10, 0.15, 0.20); 0.7, 0.4, 0.3\rangle$.
Intuitionistic	$\breve{C}_1 = \langle(0.7, 0.8, 0.9, 1.0); 0.6, 0.2\rangle$,	$\breve{C}_4 = \langle(1, 5, 8, 11); 0.7, 0.4\rangle$,
	$\breve{C}_2 = \langle(3, 5, 7, 9); 0.8, 0.3\rangle$,	$\breve{I}_e = \langle(0.02, 0.04, 0.08, 0.10); 0.5, 0.3\rangle$,
	$\breve{C}_3 = \langle(2, 6, 10, 14); 0.4, 0.1\rangle$,	$\breve{I}_p = \langle(0.05, 0.10, 0.15, 0.20); 0.7, 0.3\rangle$.
Fuzzy	$\widetilde{C}_1 = \langle(0.7, 0.8, 0.9, 1.0); 0.6\rangle$,	$\widetilde{C}_4 = \langle(1, 5, 8, 11); 0.7, 0.4\rangle$,
	$\widetilde{C}_2 = \langle(3, 5, 7, 9); 0.8\rangle$,	$\widetilde{I}_e = \langle(0.02, 0.04, 0.08, 0.10); 0.5\rangle$,
	$\widetilde{C}_3 = \langle(2, 6, 10, 14); 0.4\rangle$,	$\widetilde{I}_p = \langle(0.05, 0.10, 0.15, 0.20); 0.7\rangle$.
Crisp	$C_1 = 0.9, C_2 = 5, C_3 = 8, C_4 = 6,$	$I_e = 0.08, I_p = 0.1$

7. Numerical application

Input data:
In order to investigate the theoretical results and acquire management insights into the present study, authors have considered numerical data from the articles of Mondal *et al.* [36] and Chakraborty *et al.* [8] along with some additional data to adapt the model. All these essential inputs are presented in Table 2. Furthermore, the inventory and trade credit related unit costs for different environments are adorned in Table 3. Moreover, the values of the defuzzification parameters are considered as $\alpha = 0.5$, $\beta = 0.4$ and $\gamma = 0.45$.

Optimal solutions
Assuming the above presented input data sets, all the proposed models are solved by suggested solution procedures. The optimum inventory depletion time (t_1^*), minimum cost per cycle ($TAC^*(t_1^*)$) and the corresponding order quantity (Q^*) for neutrosophic models (Models-1a and -1b) are given in Table 4. The similar optimum results for intuitionistic (Models-2a and -2b), fuzzy (Models-3a and -3b) and crisp (Models-4a and -4b) models are represented in Tables 5, 6 and 7 respectively. Moreover, the graphical representation of optimal TAC and Q *versus* weighted arithmetic mean parameter (ρ) in different imprecise environments are drawn in Figures 3 and 4 for the various trade credit intervals.

8. Sensitivity analysis

In the present section, a sensitivity analysis is performed to observe the behaviour of optimal TAC, Q and t_1 with respect to the changes in key parameters $A, L, \sigma, a, c, p, \delta$ and m. Optimal results are obtained by changing these important parameters' values by -50%, -20%, 20% and 50%, one parameter at a time while leaving the

TABLE 4. Optimal solution of neutrosophic models for different values of arithmetic mean parameter ρ.

	Model-1a ($0 \le \sigma \le t_1^*$)			Model-1b ($t_1^* < \sigma \le T$)		
ρ	t_1^* (Year)	$TAC^*(t_1^*)$ (\$)	Q^* (Unit items)	t_1^* (Year)	$TAC^*(t_1^*)$ (\$)	Q^* (Unit items)
0.0	0.5833	533.74	109.57	0.5976	525.61	110.55
0.2	0.6059	515.40	111.12	0.6211	505.22	112.20
0.4	0.6249	496.35	112.47	0.6408	484.06	113.62
0.5	0.6333	486.60	113.07	0.6494	473.26	114.25
0.6	0.6411	476.74	113.64	0.6575	462.32	114.85
0.8	0.6551	456.67	114.67	0.6719	440.10	115.94
1.0	0.6673	436.24	115.59	0.6845	417.49	116.91

TABLE 5. Optimal solution of intuitionistic models for different values of arithmetic mean parameter ρ.

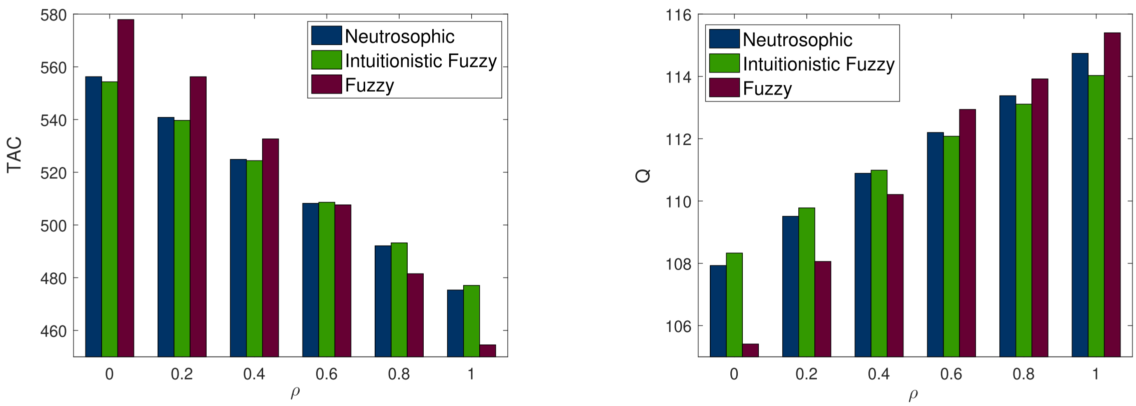
	Model-2a ($0 \le \sigma \le t_1^*$)			Model-2b ($t_1^* < \sigma \le T$)		
ρ	t_1^* (Year)	$TAC^*(t_1^*)$ (\$)	Q^* (Unit items)	t_1^* (Year)	$TAC^*(t_1^*)$ (\$)	Q^* (Unit items)
0.0	0.5854	533.24	109.71	0.5985	525.38	110.61
0.2	0.6071	515.06	111.21	0.6216	505.16	112.23
0.4	0.6253	496.22	112.49	0.6409	484.21	113.63
0.5	0.6331	486.62	113.05	0.6493	473.51	114.23
0.6	0.6407	476.87	113.61	0.6573	462.68	114.84
0.8	0.6540	457.12	114.59	0.6715	440.70	115.91
1.0	0.6655	437.03	115.46	0.6838	418.35	116.86

TABLE 6. Optimal solution of fuzzy models for different values of arithmetic mean parameter ρ.

	Model-3a ($0 \le \sigma \le t_1^*$)			Model-3b ($t_1^* < \sigma \le T$)		
ρ	t_1^* (Year)	$TAC^*(t_1^*)$ (\$)	Q^* (Unit items)	t_1^* (Year)	$TAC^*(t_1^*)$ (\$)	Q^* (Unit items)
0.0	0.5459	560.43	107.09	0.5586	553.33	107.92
0.2	0.5853	533.95	109.70	0.5996	524.59	110.68
0.4	0.6168	505.51	111.89	0.6324	493.75	113.01
0.5	0.6304	490.72	112.86	0.6465	477.71	114.04
0.6	0.6427	475.61	113.76	0.6594	461.34	114.99
0.8	0.6643	444.60	115.37	0.6820	427.73	116.71
1.0	0.6827	412.71	116.77	0.7012	393.17	118.22

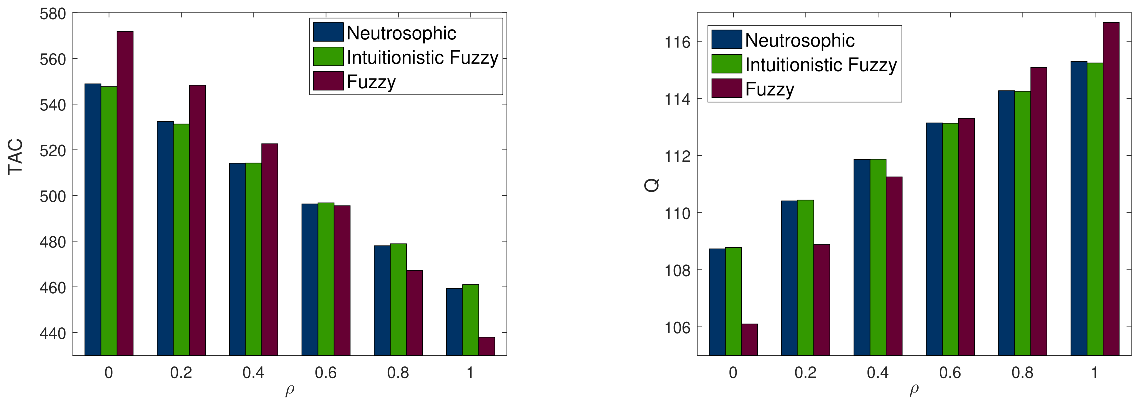
TABLE 7. Optimal solution for crisp models.

Model-4a ($0 \le \sigma \le t_1^*$)			Model-4b ($t_1^* < \sigma \le T$)		
t_1^* (Year)	$TAC^*(t_1^*)$ (\$)	Q^* (Unit items)	t_1^* (Year)	$TAC^*(t_1^*)$ (\$)	Q^* (Unit items)
0.6024	497.33	110.88	0.6172	482.38	111.92



(a) (b)

FIGURE 3. Comparative analysis of TAC^* and Q^* *versus* different ρ in the interval $0 \leq \sigma \leq t_1$.



(a) (b)

FIGURE 4. Comparative analysis of TAC^* and Q^* *versus* different ρ in the interval $t_1 < \sigma \leq T$.

remaining parameters fixed. The results of this sensitivity analysis are summarised in Table 8. Moreover, the effect on total TAC and Q subject to the changes in parameters L, σ and δ across various trade credit intervals are graphically presented in Figures 5 to 7 respectively, leading to the following observations.

– **Observation 1.** It is noticeable from Table 8 along with Figure 5 when the value of maximum lifetime L of the products increases, the value of TAC^* decreases and that of Q^* increases, both of which occur moderately, and in that case the value of t_1^* is observed to increase significantly. In addition, similar changes are monitored against the change in σ in various trade credit intervals. But, the optimal t_1^*, Q^* and TAC^* are less sensitive with respect to the changes in σ, which are graphically adorned by Figure 6.
– **Observation 2.** Next, the authors present the effect of changes of non-negative backordering parameter δ through Table 8 and Figure 7. The value of TAC^* increases tremendously, and the value of Q^* decreases moderately, and in that case, the value of t_1^* increases significantly.
– **Observation 3.** At any time within the cycle, TAC^* is highly sensitive concerning changes in values of A and m. However, t_1^* and Q^* are completely insensitive for any change in values of these parameters. On the

Table 8. Sensitivity analysis of the system parameters across various trade credit intervals.

Parameter	% change	$0 \le \sigma \le t_1^*$ % change of t_1^*	$0 \le \sigma \le t_1^*$ % change of TAC^*	$0 \le \sigma \le t_1^*$ % change of Q^*	$t_1^* < \sigma \le T$ % change of t_1^*	$t_1^* < \sigma \le T$ % change of TAC^*	$t_1^* < \sigma \le T$ % change of Q^*
A	50	–	48.3962	–	–	49.4784	–
	20	–	19.3585	–	–	19.7914	–
	−20	–	−19.3585	–	–	−19.7914	–
	−50	–	−48.3962	–	–	−49.4784	–
L	50	10.1454	−4.9073	4.3942	10.4966	−5.3563	4.7164
	20	4.24314	−2.0547	1.7849	4.3812	−2.2383	1.9079
	−20	−4.5055	2.1873	−1.8176	−4.6373	2.3757	−1.9316
	−50	−11.7845	5.7373	−4.5971	−12.0955	6.2156	−4.8629
σ	50	1.7521	−1.4724	0.7282	1.6759	−2.4636	0.7200
	20	0.7421	−0.6289	0.3069	0.6738	−0.9805	0.2881
	−20	−0.7967	0.6917	−0.3271	−0.6784	0.9739	−0.2881
	−50	−2.0936	1.8620	−0.8543	−1.7049	2.4223	−0.7204
a	50	$1.96 * 10^{-14}$	−5.5720	$2.54 * 10^{-14}$	$3.84 * 10^{-14}$	−6.7815	–
	20	$1.96 * 10^{-14}$	−2.2288	$2.54 * 10^{-14}$	–	−2.7126	–
	−20	$1.96 * 10^{-14}$	2.2288	$2.54 * 10^{-14}$	–	2.7126	–
	−50	$1.96 * 10^{-14}$	5.5720	$2.54 * 10^{-14}$	$5.76 * 10^{-14}$	6.7815	–
c	50	−0.7186	0.1385	−0.2951	–	–	–
	20	−0.2915	0.0562	−0.1199	–	–	–
	−20	0.2971	−0.0574	0.1226	–	–	–
	−50	0.7538	−0.1458	0.3118	–	–	–
p	50	–	−0.4716	–	0.2908	−1.4356	0.1241
	20	–	−0.1886	–	0.1174	−0.5741	0.0501
	−20	–	0.1886	–	−0.1189	0.5739	−0.0506
	−50	–	0.4716	–	−0.2999	1.4343	−0.1276
δ	50	−10.5548	38.2810	−4.1405	−10.4393	39.5322	−4.2297
	20	−4.3897	17.8841	−1.7718	−4.3384	18.4582	−1.8098
	−20	4.5893	−23.1295	1.9338	4.5300	−23.8479	1.9743
	−50	11.7700	−75.2066	5.1397	11.6040	−77.4621	5.2442
m	50	–	21.7783	–	–	22.2653	–
	20	–	6.4275	–	–	6.5712	–
	−20	–	−4.4551	–	–	−4.5548	–
	−50	–	−8.7113	–	–	−8.9061	–

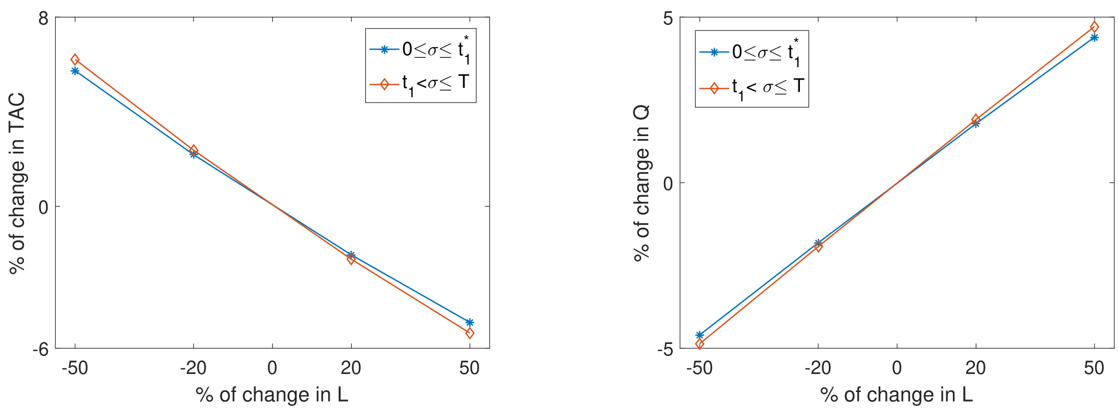


Figure 5. Effect of changes of parameter L on TAC and Q.

other hand, t_1^* and Q^* are both low sensitivity with respect to changes in values of parameter a, while TAC^* is moderately sensitive. Also, trade credit parameters such as c and p are less significant in all circumstances.

– **Observation 4.** This analysis finds that with each increase in the convex set parameter ρ, the TAC^* decreases, while Q^* and t_1^* increase. These findings do not depend on the trade credit period intervals. Moreover, the above circumstances remain consistent in the case of all imprecise environments (like neutrosophic, intuitionistic, and fuzzy). These findings are embellished by Tables 4 to 6.

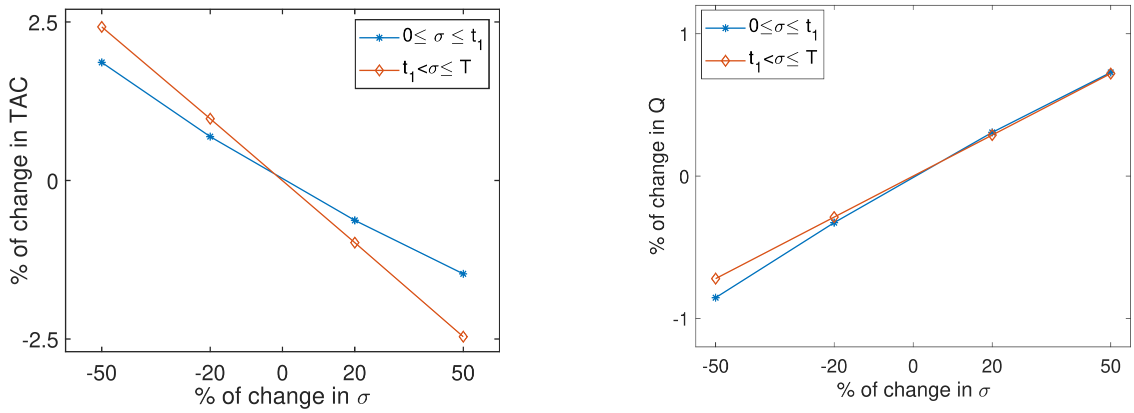


FIGURE 6. Effect of changes of parameter σ on TAC and Q.

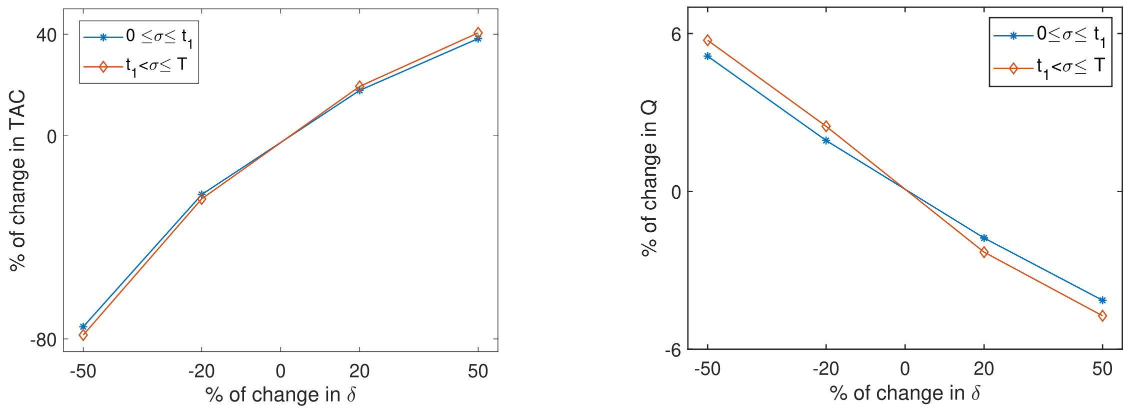


FIGURE 7. Effect of changes of parameter δ on TAC and Q.

– **Observation 5.** The authors have well noticed in Table 8 that the decision variable (t_1^*) and dependent variables ($i.e.$, Q^*, TAC^*) subject to the changes in each parameter have a greater impact when the fixed trade credit period σ belongs to the shortage period ($i.e.$, $t_1^* < \sigma \leq T$) than the on-hand inventory period ($i.e.$, $0 \leq \sigma \leq t_1^*$).
– **Observation 6.** It is observed that in pessimistic scenario ($i.e.$, $0 \leq \rho < 0.5$) for the case of intuitionistic model, the retailer is able to buy more items at lower cost, whereas, in the optimistic scenario ($i.e.$, $0.5 < \rho \leq 1$), the fuzzy model is providing such results. However, at the moderate value of the weighted mean parameter ($i.e.$, $\rho = 0.5$), the neutrosophic model gives minimum TAC with maximum Q than other environments. These findings are depicted in Tables 4 to 7 along with Figures 2 and 3 and above mentioned discussions are independent of trade credit policy.

9. MANAGERIAL INSIGHTS

The following findings are obtained from the current investigation's sensitivity analysis about management guidelines for decision-makers in relevant industries.

- **Insight 1.** It's important to note that Mondal *et al.* [35] demonstrated that inventory depletion time does not depend on demand in the on-hand inventory period but depends only on the shortage period. However, from observation 3, inventory depletion time depends on the demand function. Also, it can be stated from this observation that the TAC is decreasing as the demand increases, but the order quantity and shortage period remain more or less the same. In order to increase demand, the present author suggests increasing promotional activities during the on-hand inventory period. Since the demand function is directly proportional to the promotional effect in this model (*cf.* hypothesis 4). It's also important to avoid over-promotions, as they drive up costs, especially during the shortage. If the demand increases, the shortage period will increase, negatively impacting a business, so promotional activities should plan to halt during the shortage period.
- **Insight 2.** Retailers always try to run their businesses smoothly not only to maintain their popularity in today's competitive market but also to keep their regular customers satisfied. In addition, inventory management systems are not always profitable. If the inventory system is not beneficial, it should not be promoted, and product quality should be monitored. We know product quality and business profitability are intertwined in the long-term business. This is because companies that focus on product quality enhance their profit potential since high-quality goods can be sold at higher prices. Additionally, when companies focus on improving their product quality, it makes regular customers happier, leading to increased sales and profits. Satisfied customers also tell others about the products, which helps the company sell more and become more popular, resulting in improved sales and a larger market share.
- **Insight 3.** Observation 1 of previous Section 8 shows that the maximum lifetime of products has the most significant impact on both retailers total average cost per unit time and optimal order quantity. The deterioration rate is an increasing function and is proportional to the holding duration (*cf.* hypothesis 1). Naturally, buyers always choose to purchase products with longer expiration dates to keep the products fresh for a longer period of time after purchase. As a result, products with a longer lifetime attract more customers and contribute to revenue growth to a greater extent than products with a shorter lifetime. On the other hand, with the reduction of the shortage period, the retailer will continue his business for a long time, and the retailer-customer relationship will be well maintained. The management is therefore recommended to select products with a larger lifetime.
- **Insight 4.** In the analysis of the proposed EOQ model, observation 1 of previous Section 8 reveals that with an equal settlement time from supplier to retailer, costs during shortage periods are notably lower than during on-hand periods, while the order quantity remains relatively the same. As a result, retailers should accept a supplier's offer with a longer permissible delay period.
- **Insight 5.** Observation 2 of the previous Section 8 presents that the time-dependent backlogging rate function decreases as the non-negative backlogging parameter δ increases. As a result, the shortage period extends, and the lost-sale value increases. For the aforementioned to happen, the retailer must spend a lot of money to buy a small amount of inventory. In such a circumstance, the DM will move the next replenishment period a few times ahead of schedule in order to prevent clients from switching to another and maintaining market reputation.

9.1. Areas of application to proposed model

The strategy outlined in this paper offers potential benefits to average inventory managers operating in supermarkets like Carrefour and Wal-Mart. Moreover, it holds broad applicability across various industries, including businesses dealing with perishable products (*e.g.*, vegetables, fruits, pharmacies), electronic components (*e.g.*, capacitors, transistors, silicon-controlled rectifiers), the e-commerce sector (*e.g.*, apparel manufacturing, agriculture, textiles, and food courts), as well as household items (*e.g.*, refrigerators, air conditioners, mixers).

10. Conclusions

In the present investigation, an unconstrained EOQ model for deteriorating items under trade credit facility with consideration of imprecise coefficients is formulated. The deterioration rate depends on the maximum lifetimes of the items. Furthermore, partial backlogging is allowed. In the case of neutrosophic model, various unit cost coefficients and trade credit related costs are considered as GSTrN numbers. In order to convert the model into its equivalent crisp form, (α, β, γ)- cuts are calculated, and weighted arithmetic mean approach is applied. Furthermore, the model is solved under intuitionistic, fuzzy and crisp environments as particular cases. Step-by-step solution procedures are suggested for all models. Some sensitivity analyses are made against various key parameters, and the observations are explained. Through managerial insights, it is demonstrated that inventory depletion time always depends on demand. Also, it is suggested to halt the expenses on promotional activities to reduce demand in the shortage period. For the present model, at the moderate value of weighted arithmetic mean parameter ($\rho = 0.5$), neutrosophic environment gives minimum TAC with maximum Q compared with other environments- intuitionistic, fuzzy and crisp. Moreover, the present study recommends monitoring products' quality when business is not beneficial to the retailer.

Limitations and future extensions:
The following limitations are identified, and future extensions are suggested:

- The present study is a single-objective problem. But, it may be developed as a multi-objective problem (*cf.* Torabzadeh *et al.* [59]; Zhou *et al.* [69]; Garai *et al.* [11]).
- Preservation technology can be introduced against wastage for deteriorated items (*cf.* Soni and Chauhan [54]; Sahoo *et al.* [45]).
- The present problem is an unconstrained model. In this model, one can introduce restrictions on capital, storage, etc (*cf.* Kar *et al.* [17]; Mondal *et al.*[33]).
- Here, the demand depends on promotional effort. In the future, demand can be considered as ramp type, power, probabilistic, etc (*cf.* Cárdenas-Barrón *et al.* [7]; Mondal *et al.* [35]; Utama *et al.* [60]).
- Present article can be formulated with a finite replenishment rate (*cf.* Mahapatra *et al.* [25]; Pakhira *et al.* [42]).
- The model can be investigated under different environments like random (*cf.* Barron [5]), fuzzy-rough (*cf.* Mohammed *et al.* [30]), fuzzy-random (*cf.* Kumar *et al.* [20]), type-2 fuzzy (*cf.* Zhou *et al.* [69]), etc.

Appendix A.

Definition 1. (Hemalatha and Annadurai [14])
A generalized trapezoidal fuzzy number (GTrF-number) $\widetilde{a}$ is a fuzzy set of real line R (set of real number) having the form $\widetilde{a} = \langle (a_1, a_2, a_3, a_4); w_{\widetilde{a}} \rangle$ where $w_{\widetilde{a}} \in [0, 1]$ be any real numbers and $a_1, a_2, a_3, a_4 \in R,\ a_1 \leq a_2 \leq a_3 \leq a_4$ and the membership function $\mu_{\widetilde{a}}(x)$ is defined as follows:

$$\mu_{\widetilde{a}}(x) = \begin{cases} \frac{x-a_1}{a_2-a_1} w_{\widetilde{a}} & if \quad a_1 \leq x < a_2 \\ w_{\widetilde{a}} & if \quad a_2 \leq x \leq a_3 \\ \frac{a_4-x}{a_4-a_3} w_{\widetilde{a}} & if \quad b_3 < x \leq b_4 \\ 0 & otherwise \end{cases}$$

Definition 2. (Kheiri and Cao [19]; Mondal *et al.* [34])
A generalized trapezoidal intuitionistic number (GTrI-number) $\breve{a}$ is defined as a subset of R (set of real number) and it is denoted as

$$\breve{a} = \langle (a_1, a_2, a_3, a_4); w_{\breve{a}}, y_{\breve{a}} \rangle$$

where $w_{\breve{a}}, y_{\breve{a}} \in [0,1]$ be any real numbers and $a_1, a_2, a_3, a_4 \in R$, $a_1 \le a_2 \le a_3 \le a_4$ are the values of the trapezoidal number. Here, membership function $\mu_{\breve{a}}(x)$ and non-membership function $\nu_{\breve{a}}(x)$ are defined as follows:

$$\mu_{\breve{a}}(x) = \begin{cases} \frac{x-a_1}{a_2-a_1} w_{\breve{a}} & if \quad a_1 \le x < a_2 \\ w_{\breve{a}} & if \quad a_2 \le x \le a_3 \\ \frac{a_4-x}{a_4-a_3} w_{\breve{a}} & if \quad b_3 < x \le b_4 \\ 0 & otherwise \end{cases} \text{ and } \nu_{\breve{a}}(x) = \begin{cases} \frac{a_2-x+y_{\breve{a}}(x-a_1)}{a_2-a_1} & if \quad a_1 \le x < a_2 \\ y_{\breve{a}} & if \quad a_2 \le x \le a_3 \\ \frac{x-a_3+y_{\breve{a}}(a_4-x)}{a_4-a_3} & if \quad a_3 < x \le a_4 \\ 0 & otherwise \end{cases}$$

respectively.

Definition 3. (Deli and Şubaş [9])
A generalized single-valued trapezoidal neutrosophic number (GSTrN-number) $\widehat{a}$ is a special neutrosophic set on R (set of real number) having the form

$$\widehat{a} = \langle (a_1, a_2, a_3, a_4); w_{\widehat{a}}, u_{\widehat{a}}, y_{\widehat{a}} \rangle$$

where $w_{\widehat{a}}, u_{\widehat{a}}, y_{\widehat{a}} \in [0,1]$ be any real numbers and $a_1, a_2, a_3, a_4 \in R$, $a_1 \le a_2 \le a_3 \le a_4$ are the values of the trapezoidal number. Here, membership function $\mu_{\widehat{a}}(x)$, indeterminacy function $\sigma_{\widehat{a}}(x)$ and non-membership function $\nu_{\widehat{a}}(x)$ are defined as follows:

$$\mu_{\widehat{b}}(x) = \begin{cases} \frac{x-a_1}{a_2-a_1} w_{\widehat{a}} & if \quad a_1 \le x < a_2 \\ w_{\widehat{a}} & if \quad a_2 \le x \le a_3 \\ \frac{a_4-x}{a_4-a_3} w_{\widehat{a}} & if \quad b_3 < x \le b_4 \\ 0 & otherwise \end{cases}, \quad \sigma_{\widehat{a}}(x) = \begin{cases} \frac{a_2-x+u_{\widehat{a}}(x-a_1)}{a_2-a_1} & if \quad a_1 \le x < a_2 \\ u_{\widehat{a}} & if \quad a_2 \le x \le a_3 \\ \frac{x-a_3+u_{\widehat{a}}(a_4-x)}{a_4-a_3} & if \quad a_3 < x \le a_4 \\ 0 & otherwise \end{cases}$$

and

$$\nu_{\widehat{a}}(x) = \begin{cases} \frac{a_2-x+y_{\widehat{a}}(x-a_1)}{a_2-a_1} & if \quad a_1 \le x < a_2 \\ y_{\widehat{a}} & if \quad a_2 \le x \le a_3 \\ \frac{x-a_3+y_{\widehat{a}}(a_4-x)}{a_4-a_3} & if \quad a_3 < x \le a_4 \\ 0 & otherwise \end{cases}$$

respectively.

Definition 4. (Deli and Şubaş [9])
Let $\widehat{a} = \langle (a_1, a_2, a_3, a_4); w_{\widehat{a}}, u_{\widehat{a}}, y_{\widehat{a}} \rangle$ be an arbitrary GSTrN-number. Then
1. α-cut of $\widehat{a}$ for membership is defined as

$$\widehat{a}_\alpha = [L_{\widehat{a}}(\alpha), R_{\widehat{a}}(\alpha)] = \left[\frac{(w_{\widehat{a}}-\alpha)a_1+\alpha a_2}{w_{\widehat{a}}}, \frac{(w_{\widehat{a}}-\alpha)a_4+\alpha a_3}{w_{\widehat{a}}}\right]$$

where $\alpha \in [0, w_{\widehat{a}}]$.
2. β-cut of $\widehat{a}$ for non-membership is defined as

$$\widehat{a}_\beta = [L'_{\widehat{a}}(\beta), R'_{\widehat{a}}(\beta)] = \left[\frac{(1-\beta)a_2+(\beta-y_{\widehat{a}})a_1}{1-y_{\widehat{a}}}, \frac{(1-\beta)a_3+(\beta-y_{\widehat{a}})a_4}{1-y_{\widehat{a}}}\right]$$

where $\beta \in [y_{\widehat{a}}, 1]$.
3. γ-cut of $\widehat{a}$ for indeterminacy is defined as

$$\widehat{a}_\gamma = [L''_{\widehat{a}}(\gamma), R''_{\widehat{a}}(\gamma)] = \left[\frac{(1-\gamma)a_2+(\gamma-u_{\widehat{a}})a_1}{1-u_{\widehat{a}}}, \frac{(1-\gamma)a_3+(\gamma-u_{\widehat{a}})a_4}{1-u_{\widehat{a}}}\right]$$

where $\gamma \in [u_{\widehat{a}}, 1]$.

Theorem 1. *Let* $\breve{a} = \langle (a_1, a_2, a_3, a_4); w_{\breve{a}}, y_{\breve{a}} \rangle$ *be an arbitrary GTrI-number. Then,* $\breve{a}_{\langle \alpha, \beta \rangle} = \breve{a}_\alpha \cap \breve{a}_\beta$ *is hold for any* $0 < \alpha < w_{\breve{a}}$, *and* $y_{\breve{a}} < \beta < 1$ *where* $0 \leq \alpha + \beta \leq 1$.

Proof. See for instance, Kheiri and Cao [19]. □

Theorem 2. *Let* $\widehat{a} = \langle (a_1, a_2, a_3, a_4); w_{\widehat{a}}, u_{\widehat{a}}, y_{\widehat{a}} \rangle$ *be an arbitrary GSTrN-number. Then,* $\widehat{a}_{\langle \alpha, \beta, \gamma \rangle} = \widehat{a}_\alpha \cap \widehat{a}_\beta \cap \widehat{a}_\gamma$ *is hold for any* $0 < \alpha < w_{\widehat{a}}$, $y_{\widehat{a}} < \beta < 1$ *and* $u_{\widehat{a}} < \gamma < 1$ *where* $0 \leq \alpha + \beta + \gamma \leq 3$.

Proof. See for instance, Deli and Şubaş [9] □

Theorem 3. *Let* $A = [a, b], a, b > 0$ *be a closed interval with weights* $w_1(> 0), w_2(> 0)$. *Then the interval can be represented by a function using the weighted arithmetic mean*

$$WAM_A(\rho) = \frac{w_1 a + w_2 b}{w_1 + w_2} = a(1 - \rho) + b\rho$$

where $\rho = \frac{w_2}{w_1 + w_2}$, $\rho \in [0, 1]$.

Proof. See for instance, Mondal *et al.* [36]. □

Acknowledgements

The first author Bappa Mondal acknowledges the National Institute of Technology Warangal, India, for their financial assistance for this research work.

References

[1] P.L. Abad, Optimal pricing and lot-sizing under conditions of perishability and partial backordering. *Manag. Sci.* **42** (1996) 1093–1104.
[2] S. Adak and G.S. Mahapatra, Effect of reliability on multi-item inventory systemMonday, September 30, 2024 at 9:27 pm with shortages and partial backlog incorporating time dependent demand and deterioration. *Ann. Oper. Res.* (2020) 1–21.
[3] A.Y. Adhami, A. Melethil and F. Ahmad, Neutrosophic programming approach to multilevel decision-making model for supplier selection problem in a fuzzy situation. *RAIRO:RO* **57** (2023) 1307–1328.
[4] K.T. Atanassov, Intuitionistic Fuzzy Sets. Physica, Heidelberg (1999) 1–137.
[5] Y. Barron, A replenishment inventory model with a stock-dependent demand and age–stock-dependent cost functions in a random environment. *Asia-Pac. J. Oper. Res.* **39** (2022) 2150035.
[6] J.C. Buzby, H. Farah-Wells and J. Hyman, The estimated amount, value, and calories of postharvest food losses at the retail and consumer levels in the United States. *USDA-ERS Econ. Inf. Bull.* **121** (2014).
[7] L.E. Cárdenas-Barrón, A.A. Shaikh, S. Tiwari and G. Treviño-Garza, An eoq inventory model with nonlinear stock dependent holding cost, nonlinear stock dependent demand and trade credit. *Comput. Ind. Eng.* **139** (2020) 105557.
[8] D. Chakraborty, D.K. Jana and T.K. Roy, Two-warehouse partial backlogging inventory model with ramp type demand rate, three-parameter weibull distribution deterioration under inflation and permissible delay in payments. *Comput. Ind. Eng.* **123** (2018) 157–179.
[9] I. Deli and Y. Şubaş, A ranking method of single valued neutrosophic numbers and its applications to multi-attribute decision making problems. *Int. J. Mach. Learn. Cybern.* **8** (2017) 1309–1322.
[10] S. Ebrahimi, S.M. Hosseini-Motlagh and M. Nematollahi, Proposing a delay in payment contract for coordinating a two-echelon periodic review supply chain with stochastic promotional effort dependent demand. *Int. J. Mach. Learn. Cybern.* **10** (2019) 1037–1050.
[11] A. Garai, B. Mondal and T.K. Roy, Optimisation of multi-objective commercial bank balance sheet management model: a parametric t-set approach. *Int. J. Math. Oper. Res.* **15** (2019) 395–416.
[12] P.M. Ghare, A model for an exponentially decaying inventory. *J. Ind. Eng.* **14** (1963) 238–243.
[13] S.K. Goyal, Economic order quantity under conditions of permissible delay in payments. *J. Oper. Res. Soc.* (1985) 335–338.

[14] S. Hemalatha and K. Annadurai, A fuzzy eoq inventory model with advance payment and various fuzzy numbers. *Materials Today: Proceedings* (2020).

[15] M.Y. Jani, N.H. Shah and U. Chaudhari, Inventory control policies for time-dependent deteriorating item with variable demand and two-level order linked trade credit. In: *Optimization and Inventory Management.* Springer (2020) 55–67.

[16] M.Y. Jani, M.R. Betheja, U. Chaudhari and B. Sarkar, Effect of future price increase for products with expiry dates and price-sensitive demand under different payment policies. *Mathematics* **11** (2023) 263.

[17] C. Kar, B. Mondal and T.K. Roy, An inventory model under space constraint in neutrosophic environment: a neutrosophic geometric programming approach. *Neutrosophic Sets Syst. Int. Book Ser. Inf. Sci. Eng.* **21** (2018) 201893.

[18] C. Kar, T.K. Roy and M. Maiti, EOQ model with price, marketing, service and green dependent neutrosophic demand under uncertain resource constraint: A geometric programming approach. *Neutrosophic Sets Syst.* **51** (2022) 797–823.

[19] Z. Kheiri and B.Y. Cao, Posynomial geometric programming with intuitionistic fuzzy coefficients. In: *Fuzzy Systems & Operations Research and Management.* Springer International Publishing (2016) 15–30.

[20] M. Kumar, R.S. Kumar and A.K. Saha, Continuous review inventory system for intuitionistic fuzzy random demand under service level constraint. *Sādhanā* **47** (2022) 103.

[21] N. Kumar, S. Dahiya and S. Kumar, Two warehouse inventory model for deteriorating items with fixed shelf-life stock-dependent demand and partial backlogging. *J. Math. Comput. Sci.* **12** (2022).

[22] S. Kumar, S.R. Singh, S. Agarwal and D. Yadav, Joint effect of selling price and promotional efforts on retailer's inventory control policy with trade credit, time-dependent holding cost, and partial backlogging under inflation. *RAIRO:RO* **57** (2023) 1491–1522.

[23] S. Kumar, S. Sami, S. Agarwal and D. Yadav, Sustainable fuzzy inventory model for deteriorating item with partial backordering along with social and environmental responsibility under the effect of learning. *Alex. Eng. J.* **69** (2023) 221–241.

[24] J. Liao, K.N. Huang, K.J. Chung, S.D. Lin, S.T. Chuang and H.M. Srivastava, Optimal ordering policy in an economic order quantity (eoq) model for non-instantaneous deteriorating items with defective quality and permissible delay in payments. *Rev. Real Acad. Cienc. Exactas Fís. Nat. Ser. A: Mat.* **114** (2020) 1–26.

[25] A.S. Mahapatra, B. Sarkar, M.S. Mahapatra, H.N. Soni and S.K. Mazumder, Development of a fuzzy economic order quantity model of deteriorating items with promotional effort and learning in fuzziness with a finite time horizon. *Inventions* **4** (2019) 36.

[26] F. Mahato, M. Choudhury and G.C. Mahata, Inventory models for deteriorating items with fixed lifetime, partial backordering and carbon emissions policies. *J. Manag. Anal.* **10** (2023) 129–190.

[27] A.K. Maiti, Multi-item fuzzy inventory model for deteriorating items in multi-outlet under single management. *J. Manag. Anal.* **7** (2020) 44–68.

[28] Y. Man, J. Ren, R. Lin and Y. Liu, A multicriteria intuitionistic fuzzy group decision-making method for sustainability ranking of biofuel production pathways. In: *Biofuels for a More Sustainable Future.* Elsevier (2020) 357–376.

[29] U. Mishra, R.H. Waliv and H.P. Umap, Optimizing of multi-objective inventory model by different fuzzy techniques. *Int. J. Appl. Comput. Math.* **5** (2019) 1–15.

[30] R.T. Mohammed, A.H. Alamoodi, O.S. Albahri, A.A. Zaidan, H. AlSattar, U. Aickelin, A.S. Albahri, B.B. Zaidan, A.R. Ismail and R.Q. Malik, A decision modeling approach for smart e-tourism data management applications based on spherical fuzzy rough environment. *Appl. Soft Comput.* **143** (2023) 110297.

[31] K. Mohanta, A.K. Jha, A. Dey and A. Pal, An application of neutrosophic logic on an inventory model with two-level partial trade credit policy for time-dependent perishable products. *Soft Comput.* **27** (2023) 4795–4822.

[32] B. Mondal, A. Garai and T.K. Roy, Optimization of eoq model with space constraint: An intuitionistic fuzzy geometric programming approach. *Notes Inst Fuzzy Sets* **24** (2018) 172–189.

[33] B. Mondal, C. Kar, A. Garai and T.K. Roy, Optimization of EOQ Model with Limited Storage Capacity by Neutrosophic Geometric Programming. Infinite Study (2018).

[34] B. Mondal, S. Biswas, A. Garai and T.K. Roy, Posynomial geometric programming in eoq model with interval neutrosophic number. In: *International Conference on Information Technology and Applied Mathematics.* Springer (2019) 434–449.

[35] B. Mondal, A. Garai and T.K. Roy, Optimization of generalized order-level inventory system under fully permissible delay in payment. *RAIRO:RO* **55** (2021) S195–S224.

[36] B. Mondal, A. Garai, A. Mukhopadhyay and S.K. Majumder, Inventory policies for seasonal items with logistic-growth demand rate under fully permissible delay in payment: a neutrosophic optimization approach. *Soft Comput.* **25** (2021) 3725–3750.

[37] B. Mondal, A. Garai and T.K. Roy, Operational strategies of fuzzy inventory models for costly metallic items with conditional trade-credit policy linked to purchasing cost. *Eur. J. Ind. Eng.* **17** (2023) 696–739.

[38] S. Moradi, M.R. Gholamian and A. Sepehri, An inventory model for imperfect quality items considering learning effects and partial trade credit policy. *Opsearch* **60** (2023) 276–325.

[39] M. Mullai, R. Sangeetha, G. Madhan, R. Jeyabalan and S. Broumi, A single valued neutrosophic inventory model with neutrosophic random variable. *Int. J. Neutrosophic Sci.* **1** (2020) 52.

[40] M. Mullai and R. Surya, Neutrosophic inventory backorder problem using triangular neutrosophic numbers. *Neutrosophic Sets Syst.* **31** (2020) 148–155.

[41] M. Nagare, P. Dutta and P. Suryawanshi, Optimal procurement and discount pricing for single-period non-instantaneous deteriorating products with promotional efforts. *Oper. Res.* **20** (2020) 89–117.

[42] N. Pakhira, M.K. Maiti and M. Maiti, Two-level supply chain of a seasonal deteriorating item with time, price, and promotional cost dependent demand under finite time horizon. *Am. J. Math. Manag. Sci.* **36** (2017) 292–315.

[43] R. Pakhira, U. Ghosh, H. Garg and V.N. Mishra, An inventory model for partial backlogging items with memory effect. *Soft Comput.* **27** (2023) 1–18.

[44] A. Paul, T. Garai and B.C. Giri, Sustainable supply chain coordination with greening and promotional effort dependent demand. *Int. J. Procure. Manag.* **16** (2023) 196–233.

[45] S. Sahoo, M. Acharya and S. Patnaik, Sustainable intuitionistic fuzzy inventory models with preservation technology investment and shortages. *Int. J. Reason.-Based Intell. Syst.* **14** (2022) 8–18.

[46] D. Sarma, A. Das, U.K. Bera and I.M. Hezam, Redistribution for cost minimization in disaster management under uncertainty with trapezoidal neutrosophic number. *Comput. Ind.* **109** (2019) 226–238.

[47] N. Shah and M. Jani, Economic order quantity model for non-instantaneously deteriorating items under order-size-dependent trade credit for price-sensitive quadratic demand. *AMSE J.* **37** (2016) 1–19.

[48] N. Shah, M. Jani and U. Chaudhari, Retailer's optimal policies for price-credit dependent trapezoidal demand under two-level trade credit. *Int. J. Oper. Quant. Manag.* **23** (2017) 115–130.

[49] S.R. Singh and K. Rana, Effect of inflation and variable holding cost on life time inventory model with multi variable demand and lost sales. *Int. J. Recent Technol. Eng.* **8** (2020) 5513–5519.

[50] S.R. Singh and K. Rana, A sustainable production inventory model for growing items with trade credit policy under partial backlogging. *Int. J. Adv. Oper. Manag.* **15** (2023) 64–81.

[51] T. Singh, M.M. Muduly, N. Asmita, C. Mallick and H. Pattanayak, A note on an economic order quantity model with time-dependent demand, three-parameter weibull distribution deterioration and permissible delay in payment. *J. Stat. Manag. Syst.* **23** (2020) 643–662.

[52] R. Singh, A. Kumar and D. Yadav, Effect of carbon emission on multi-stage economic order quantity model with backlogging under fuzzy environment. In: *AIP Conference Proceedings*. AIP Publishing, **2253** (2020) 020007.

[53] F. Smarandache, Neutrosophic set-a generalization of the intuitionistic fuzzy set. *Int. J. Pure Appl. Math.* **24** (2005) 287.

[54] H.N. Soni and A.D. Chauhan, Joint pricing, inventory, and preservation decisions for deteriorating items with stochastic demand and promotional efforts. *J. Ind. Eng. Int.* **14** (2018) 831–843.

[55] R. Sundara Rajan and R. Uthayakumar, Analysis and optimization of an eoq inventory model with promotional efforts and back ordering under delay in payments. *J. Manag. Anal.* **4** (2017) 159–181.

[56] A.H. Tai, Y. Xie, W. He and W.K. Ching, Joint inspection and inventory control for deteriorating items with random maximum lifetime. *Int. J. Prod. Econ.* **207** (2019) 144–162.

[57] A.A. Taleizadeh, S.T.A. Niaki and R.G. Meibodi, Replenish-up-to multi-chance-constraint inventory control system under fuzzy random lost-sale and backordered quantities. *Knowl.-Based Syst.* **53** (2013) 147–156.

[58] S. Tiwari, L.E. Cárdenas-Barrón, A.A. Shaikh and M. Goh, Retailer's optimal ordering policy for deteriorating items under order-size dependent trade credit and complete backlogging. *Comput. Ind. Eng.* **139** (2020) 105559.

[59] S.A. Torabzadeh, E. Nejati, A. Aghsami and M. Rabbani, A dynamic multi-objective green supply chain network design for perishable products in uncertain environments, the coffee industry case study. *Int. J. Manag. Sci. Eng. Manag.* **17** (2022) 220–237.

[60] D.M. Utama, I. Santoso, Y. Hendrawan and W.A. Dania, Sustainable Production-inventory model with multi-material, quality degradation, and probabilistic demand: From bibliometric analysis to a robust model. *Indones. J. Sci. Technol.* **8** (2023) 171–196.

[61] S.K. Verma, M. Rizwanullah and C. Singh, Optimal ordering policy for deteriorating items with stock dependent demand, partial backlogging and trade credit period. *Int. J. Logist. Syst. Manag.* **40** (2021) 95–120.

[62] J. Wu, K. Skouri, J.T. Teng and Y. Hu, Two inventory systems with trapezoidal-type demand rate and time-dependent deterioration and backlogging. *Expert Syst. Appl.* **46** (2016) 367–379.

[63] Y. Xie, A.H. Tai, W.K. Ching, Y.H. Kuo and N. Song, Joint inspection and inventory control for deteriorating items with time-dependent demand and deteriorating rate. *Ann. Oper. Res.* **300** (2021) 225–265.

[64] D. Yadav, A. Kumar and R. Singh, A fuzzy economic order quantity model for multiple stage supply chain with fully backlogged shortages derived without derivatives under the effect of human learning. *Int. J. Oper. Res.* (2022).

[65] D. Yadav, S.R. Singh and M. Sarin, Multi-item EOQ model for deteriorating items having multivariate dependent demand with variable holding cost and trade credit. *Int. J. Oper. Res.* **47** (2023) 202–244.

[66] D. Yadav, U. Chand, R. Goel and B. Sarkar, Smart production system with random imperfect process, partial backordering, and deterioration in an inflationary environment. *Mathematics* **11** (2023) 440.

[67] L.A. Zadeh, Fuzzy sets. *Inf. control* **8** (1965) 338–353.

[68] H. Zhang, Z. Wang, X. Hong and Q. Zhong, Fuzzy closed-loop supply chain models with quality and marketing effort-dependent demand. *Expert Syst. Appl.* **207** (2022) 118081.

[69] J. Zhou, S. Wu and J. Gao, Two-stage fuzzy interactive multi-objective approach under interval type-2 fuzzy environment with application to the remanufacture of old clothes. *Symmetry* **14** (2022) 1785.