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AN EXTENSION OF LOCATING-TOTAL DOMINATION PROBLEM AND ITS COMPLEXITY

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Abstract. An r-dominating set (r-total dominating set) of G is a subset S of $V(G)$ for which $N_r(u) \cap S$ is non-empty for all u not in S (for all u in $V(G)$). An r-locating-dominating set (r-locating-total dominating set) of G is an r-dominating set (r-total dominating set) S of G for which $N_r(u) \cap S$ is different from $N_r(v) \cap S$ for all u and v not in S. This paper presents an extension of the locating-total dominating set of G. Further, we establish a lower bound on r-locating-dominating set and r-locating-total dominating set for k-regular graphs, as well as demonstrate that r-locating-total dominating set is an NP-complete problem. Furthermore, the r-locating-dominating set and r-locating-total dominating set problems are discussed for some standard graphs.

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1. Introduction

Domination in graphs is one of the most studied areas of graph theory due to its many practical applications. These include monitoring devices [38], facility location problems [39], and many others. In addition to the requirement that vertices outside of the dominating sets must have at least one neighbor in the dominating set, there may also be some additional condition(s) on the dominating set. This resulted in the development of new variants of the dominating set. A few of the most interesting variants of dominating sets include total dominating sets, locating-dominating sets, and locating-total dominating sets. In this paper, we refer to the dominant set as DS, the total dominating set as TDS, the locating-dominating set as LDS, and the locating-total dominating set as LTDS. In a similar manner, the domination problem, total domination problem, locating-domination problem, and locating-total domination problem are abbreviated as DP, TDP, LDP, and LTDP, respectively. In addition, the domination number, the total domination number, the locating-domination number, and the locating-total domination number are abbreviated as DN, TDN, LDN, and LTDN, respectively. Berge [4] and Ore [31] introduced the domination problem formally, and it has received considerable attention in graph theory literature in recent decades. In [12], Cockayne *et al.* introduced the concept of the total domination problem by replacing the condition for the vertices that do not belong to S with all the vertices of G. In [38], Slater introduced the concept of the LDP by adding a condition to every pair of vertices that is not in the dominating

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set of G. A similar concept, the LTDP, has been introduced by Haynes *et al.* [19] by replacing the dominating set with the total dominating set in G. Later in [19], the r-LDP was introduced by replacing the dominating set with the r-dominating set in the LDP. Several variations on the domination problem, such as independent domination [6, 16], connected domination [22, 33], power domination [3, 34], double domination [2], restrained roman domination [17, 30], vertex-edge domination [36], semitotal domination [41] etc., have also been defined.

In the following, we provide some of the significant results obtained in the locating-domination and locating-total domination problems. In [9], Charon *et al.* demonstrated that determining locating-dominating number in a graph is NP-complete. Further, some results have been presented for certain classes of graphs, including paths [39], cycles [5], stars and complete multipartite graphs [1], complete graphs [24], bipartite graphs [7, 26], wheels [7], twin-free graphs [14, 15] and hypergraphs [13]. As a result of Canoy and Malacas [8], the locating-domination number has been resolved from the product graphs, as well as bounds for the corona of graphs have been provided. In [32], Rad and Rahbani found new upper bounds on the LDN in trees and identified all trees with equality. In recent years, the concept of locating-total dominating set has been studied by Haynes *et al.* [21], who identified bounds for these parameters within a tree and investigated the ratio between the two parameters. Chellali *et al.* [10] discussed edge critical graphs and Chena *et al.* [11] discussed the bounds on the LTDN of a tree. Foucaud *et al.* [14] proved a general upper bound on the LTDN of a graph in terms of its order and presented conjecture for graphs with no 4-cycles that achieve the bound. In [25], Henning and Rad investigated the LTDN in special families of graphs. In [23], Henning *et al.* characterized the locating-total dominance number of claw-free cubic graphs in one-half of their order and the graphs that achieve this bound. In [29], Miller *et al.* proved that the problem of LTDN of a graph is NP-complete.

Here is how we proceed. The rest of the introduction discusses the r-LDP applications in interconnection networks. In addition, we provide the standard definitions that are required in this paper. Finally, we provide some remarks regarding the LDP. Then, in Section 2, we determine the lower bound for the r-LDS and r-LTDS for k-regular graphs. In Section 3, we show that the r-locating-total dominating set decision problem is NP-complete. In Section 4, we determine the r-LDN and r-LTDN for chain silicate and cyclic silicate graphs with restriction to r. In Section 5, we obtain a lower bound for the r-LDN for corona products, edge corona products; and compute the r-LDN and r-LTDN for necklace graphs with restriction to r.

1.1. Applications of r-locating-domination problem

In parallel computing, each processor is represented by a vertex in graph G, and edges between processors indicate a direct communication link. In this way, the graph formed represents the interconnection networks. The interconnection network ensures fast and reliable communication between the nodes. Depending on the parallel computer architecture in which the network is used, different demands are placed on the network. Consider a situation in which we have limited resources, such as disks, input-output connections, or software modules, and wish to place as many resource units as possible near each processor. Then it would be necessary to construct a minimum r-dominating set for graph G to determine placements. In this parallel architecture, faulty nodes are inevitable. So to safeguard a system it is essential to identify the location of the device designed for monitoring purposes. The monitoring device must be placed in a system in such a way that components not designed for monitoring purposes must be connected to at least one monitoring device. To detect faulty components, we need two critical steps. The first step is to detect faulty components within an r-neighborhood and the second step is to locate the precise location of the faulty components. The process of sensing faulty components among its r-neighbors and locating their exact location is nothing but an r-location-dominating problem.

1.2. The basic prerequisites

Given a graph G and an integer $r \geq 1$, we define the rth *open neighborhood* of u as $N_r(u) = \{x \in V(G) : d(u,x) \leq r\}$ and the rth *closed neighborhood* of u as $N_r[u] = N_r(u) \cup \{u\}$. If $r = 1$, then $N_1(u)$ and $N_1[u]$ is simply referred as $N(u)$ and $N[u]$ respectively. A subset S of $V(G)$ is called an *r-dominating set* (r-DS) of G if every vertex u not in S is such that $N_r(u) \cap S \neq \phi$. The r-domination number, $\gamma_r(G) = \min\{|S| : S$ is an r-DS of $G\}$. Note that 1-DS of G is simply called as a DS of G [37]. The following are the variations of r-dominating sets. An r-dominating set S in a graph G is called an

– *r-total dominating set* (r-TDS) if for all u in $V(G)$, $N_r(u) \cap S \neq \phi$;
– *r-locating-dominating set* (r-LDS) if for all u and v in $V(G) \backslash S$, $N_r(u) \cap S$ is different from $N_r(v) \cap S$;
– *r-locating-total dominating set* (r-LTDS) if S is both an r-TDS and an r-LDS.

The r-total domination number, $\gamma_{t,r}(G) = \min\{|S| : S \text{ is an } r\text{-TDS of } G\}$. Similarly, the r-locating-domination number, $\gamma_r^L(G) = \min\{|S| : S \text{ is an } r\text{-LDS of } G\}$ and the r-locating-total domination number, $\gamma_{t,r}^L(G) = \min\{|S| : S \text{ is an } r\text{-LTDS of } G\}$. Note that the 1-TDS of G is simply referred as the TDS of G. Similarly, the 1-LDS of G is simply referred to as the LDS of G and the 1-LTDS of G is simply referred as the LTDS of G. For our convenience, we call a set $N(u) \cap S$ a locating set of u where $S \subseteq V(G)$ and $u \in V(G) \setminus S$. It has been shown that the above problems defined are all NP-complete problems when $r = 1$ in [9, 20, 29]. The paper focuses on simple, undirected graphs that are connected. Here, $G[S]$ denotes the subgraph of G induced by a subset S of $V(G)$. In the following, the complete graph on n vertices, the radius, and the diameter of graph G are denoted as K_n, $rad(G)$ and $d(G)$ respectively.

1.3. Some remarks

Remark 1.1. Let S be an r-DS of a graph G. If there exist two vertices x and y not in S such that $N_r(x) \cap S = N_r(y) \cap S$ then x and y are called twin vertices. Similarly, if there exist three vertices, namely x, y and z not in S such that $N_r(x) \cap S = N_r(y) \cap S = N_r(z) \cap S$ then x, y and z are called triplet vertices.

Remark 1.2. Let S be an r-DS of a graph G. Two vertices u and v not in S are said to be located by S if $N_r(u) \cap S \neq N_r(v) \cap S$. If S is an r-LDS of G, then S locates every pair of vertices not in S.

Remark 1.3. If there exist two vertices x and y in G such that $N_r(x) = N_r(y)$ or $N_r[x] = N_r[y]$, then any r-LDS of G contains either x or y.

2. Lower bound of r-LDS and r-LTDS for k-regular graphs

In this section, we determine the lower bound for the r-LDS and r-LTDS for k-regular graphs.

Let S be an r-DS of G. The *shadow of a vertex* u of G is defined as $S_u = S \cap N_r[u]$ and the *share of a vertex* u in S as $\gamma(u, S) = \sum_{x \in N_r[u]} \frac{1}{|S_x|}$. For simplicity, we refer to $\gamma(u, S)$ as $\gamma(u)$. A similar definition is given in [40], when $r = 1$.

Lemma 2.1. *Let G be a graph with n vertices and S be an r-DS of G. Then $\sum_{u \in S} \gamma(u) = n$.*

The proof of the previous lemma follows from the proof of the lemma given in [40], when $r = 1$. Hence, it is omitted.

Theorem 2.2. *Let G be a k-regular graph with n vertices, $k \geq 2$. Then $\gamma_r^L(G) \geq \left\lceil \frac{(4-2k)n}{6-k[2+(k-1)^r]} \right\rceil$.*

Proof. Let S be an r-LDS of G. We claim that at most two vertices of $N_r[u]$ can have exactly one vertex in its shadow, $u \in S$. Assume that there are three vertices, x, y and z, of $N_r[u]$ with exactly one vertex in its shadow, then $N_r(x) \cap S$, $N_r(y) \cap S$ and $N_r(z) \cap S$ are the same. Hence, the claim. It is evident from the claim that the remaining vertices of $N_r[u]$ must have at least two vertices in their shadows. Hence, $\gamma(u) = \sum_{w \in N_r[u]} \frac{1}{|S_w|} \leq \frac{6-k[2+(k-1)^r]}{4-2k}$, $\forall u \in S$. By Lemma 2.1, $n = \sum_{u \in S} \gamma(u) \leq \frac{6-k[2+(k-1)^r]}{4-2k} |S|$. Therefore, $|S| \geq \left\lceil \frac{(4-2k)n}{6-k[2+(k-1)^r]} \right\rceil$. □

Theorem 2.3. *Let G be a k-regular graph with n vertices, $k \geq 2$. Then $\gamma_{t,r}^L(G) \geq \left\lceil \frac{(4-2k)n}{4-k[1+(k-1)^r]} \right\rceil$.*

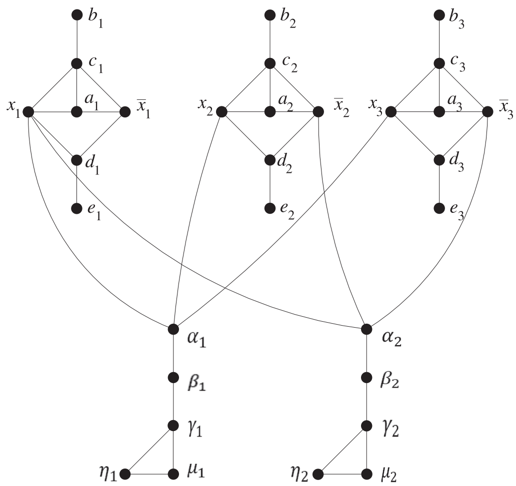

FIGURE 1. Graph of formula $\mathbb{F} = (x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee \overline{x_3})$.

Proof. Let S be an r-LTDS of G. We claim that at most one vertex of $N_r[u]$ can have exactly one vertex in its shadow, $u \in S$. Assume that there are two vertices, x and y, of $N_r[u]$ with exactly one vertex in its shadow, then $N_r(x) \cap S$ is the same as $N_r(y) \cap S$. Hence, the claim. It is evident from the claim that the remaining vertices of $N_r[u]$ must have at least two vertices in their shadows. Hence, $\gamma(u) = \sum_{w \in N_r[u]} \frac{1}{|S_w|} \leq \frac{4-k[1+(k-1)^r]}{4-2k}, \forall u \in S$. By Lemma 2.1, $n = \sum_{u \in S} \gamma(u) \leq \frac{4-k[1+(k-1)^r]}{4-2k} |S|$. Therefore, $|S| \geq \left\lceil \frac{(4-2k)n}{4-k[1+(k-1)^r]} \right\rceil$. □

3. r-LTDS DECISION PROBLEM IS NP-COMPLETE

In this section, we show that the r-LTDS decision problem is NP-complete by reducing the well-known NP-complete problem, namely the 3-SAT problem to the r-LTDS decision problem.

In the following, we give the construction of G' as defined in [29].

Construction of the graph G': first, we consider any instance of 3-SAT, $\mathbb{C} = \{C_1, C_2, \dots, C_m\}$ over the set of variables $X = \{x_1, x_2, \dots, x_n\}$. For each variable x_i of X, we construct a graph $G'_{x_i} = (V_{x_i}, E_{x_i})$ with $V_{x_i} = \{a_i, b_i, c_i, d_i, e_i, x_i, \overline{x}_i\}$ and $E_{x_i} = \{a_i x_i, a_i \overline{x}_i, a_i c_i, b_i c_i, c_i x_i, c_i \overline{x}_i, d_i x_i, d_i \overline{x}_i, d_i e_i\}$, $1 \leq i \leq n$. Next for each clause $C_j = \{u_{j,1}, u_{j,2}, u_{j,3}\}$, we construct the graph $G'_{C_j} = (V_{C_j}, E_{C_j})$, with $V_{C_j} = \{\alpha_j, \beta_j, \gamma_j, \mu_j, \eta_j\}$ and $E_{C_j} = \{\alpha_j \beta_j, \beta_j \gamma_j, \gamma_j \mu_j, \gamma_j \eta_j, \mu_j \eta_j\}$, $1 \leq j \leq m$. Finally, given a formula $F = C_1 \wedge C_2 \wedge \cdots \wedge C_m$, we construct $G' = (V, E)$ with $V = (\bigcup_{i=1}^n V_{x_i}) \cup (\bigcup_{j=1}^m V_{C_j})$ and $E = (\bigcup_{i=1}^n E_{x_i}) \cup (\bigcup_{j=1}^m E_{C_j}) \cup (\bigcup_{j=1}^n \{\alpha_j u_{j,1}, \alpha_j u_{j,2}, \alpha_j u_{j,3}\})$. The graph G' thus constructed has $7n + 5m$ vertices and $9n + 8m$ edges. See Figure 1.

Lemma 3.1. *Let H be a graph with $V(H) = \{k_1, k_2, \dots, k_r, p_1, p_2, \dots, p_{2r+1}, q_1, q_2, \dots, q_{2r+1}\}$ and $E(H) = (\bigcup_{i=1}^{r-1} \{k_i k_{i+1}\}) \cup (\{k_r p_1, k_r q_1\}) \cup (\bigcup_{i=1}^{2r} \{p_i p_{i+1}, q_i q_{i+1}\}) \cup (\bigcup_{i=1}^{2r+1} \{p_i q_i\}) \cup (\bigcup_{i=1}^{2r} \{p_i q_{i+1}, q_i p_{i+1}\})$ (see Fig. 2). Then $\gamma_{t,r}^L(H) = 2r + 1$.*

Proof. Let S be an r-LTDS of H. We claim that $|S| \geq 2r + 1$. By the construction of H, we see that $N_r(p_i) = N_r(q_i)$ for $i = 1, 2, \dots, 2r+1$. Hence, from Remark 1.3, either p_i or q_i belong to S. Therefore, $\gamma_{t,r}^L(H) = |S| \geq 2r + 1$. Now to prove the lower bound is tight, consider the set $S = \{p_1, p_2, \dots, p_{2r+1}\}$. Clearly, S is an r-TDS of H. It remains to prove that S locates every pair of vertices in $V(G) \setminus S$. By the choice of S, we have the following:

- $N_r(k_i) \cap S = \{p_1, p_2, \dots, p_i\}$ where $i = 1, 2, \dots, r$;

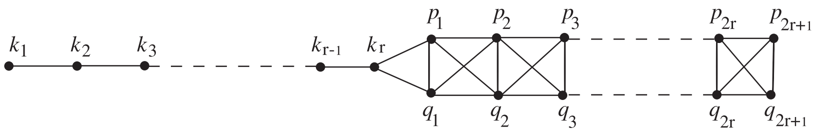

FIGURE 2. Graph H.

– $N_r(q_i) \cap S = \{p_1, p_2, \ldots, p_{r+i}\}$ where $i = 1, 2, \ldots, r+1$;
– $N_r(q_i) \cap S = \{p_{i-r}, p_{i-r+1}, \ldots, p_{2r+1}\}$ where $i = r+2, r+3, \ldots, 2r+1$.

Observe that when $i > j$, there exists a $p_i \in S$ such that $p_i \in N_r(k_i) \cap S$ and $p_i \notin N_r(k_j) \cap S$ where i and j belong to the set $\{1, 2, \ldots, r\}$. Hence, $N_r(k_i) \cap S \neq N_r(k_j) \cap S$, $\forall i, j \in \{1, 2, \ldots, r\}$. In a similar way, we can show that $N_r(u) \cap S \neq N_r(v) \cap S$, $\forall u, v \in V(G) \setminus S$. Hence, S locates every pair of vertices in $V(G) \setminus S$. Therefore, $\gamma^L_{t,r}(H) = |S| \leq 2r + 1$. □

Remark 3.2. The proof of Lemma 3.1 implies that any r-LTDS of H must contain all of its vertices in $\{p_1, p_2, \ldots, p_{2r+1}\} \cup \{q_1, q_2, \ldots, q_{2r+1}\}$.

Lemma 3.3. *Let H' be a graph constructed using $r-1$ copies of graph H defined in Lemma 3.1 by adding the edges $\bigcup_{i=1}^{r-2}\{k_1^i k_1^{i+1}\}$, where k_1^i represent the vertex k_1 from the ith copy of H (see Fig. 3). Then $\gamma^L_{t,r}(H') = (r-1)(2r+1)$.*

Proof. Let S be an r-LTDS of H' and H_i be the ith copy of H, $1 \leq i \leq r-1$. By the construction of H', we can observe that any minimal r-LTDS of H_i will not dominate any vertex from H_j, $i \neq j$, since the edges $\bigcup_{i=1}^{r-2}\{k_1^i k_1^{i+1}\}$ connects two copies of H. This implies that we need at least $2r+1$ vertices from each copy of H. Therefore, $|S| \geq (r-1)(2r+1)$. Obviously, $|S| \leq (r-1)(2r+1)$ is straight forward to check. □

Remark 3.4. The proof of Lemma 3.3 implies that one such minimum r-locating-total dominating set of H' consists of minimum r-locating-total dominating sets from each copy of H. Moreover, any minimum r-LTDS of H' must contain all of its vertices in the copies of $\{p_1, p_2, \ldots, p_{2r+1}\} \cup \{q_1, q_2, \ldots, q_{2r+1}\}$.

Theorem 3.5. *3-SAT problem reduces to r-LTDS decision problem.*

Proof. Let $\mathbb{C} = \{C_1, C_2, \ldots, C_m\}$ be any instance of 3-SAT over the set of variables $X = \{x_1, x_2, \ldots, x_n\}$, and we construct a graph G using the graph G' by pasting copies of H' from Lemma 3.3 along each edge of G'. The graph G thus constructed has $(9n+8m)(5r^2-3r-1)-2n-3m$ vertices and $(9n+8m)(11r^2-8r-2)$ edges. Take $k = 3n + 2m + (9n+8m)(r-1)(2r+1)$. We claim that a formula $\mathbb{F}$ can be satisfied if and only if there exists an r-LTDS of size at most k in G.

(i) First, we assume that $\mathbb{F}$ is satisfied, and then we demonstrate the existence of an r-LTDS $S \subseteq V(G)$ of size k. Let $S = X \cup Y \cup Z$, where X contains the minimum r-LTDS of each H' as described in Lemma 3.3; Y contains the vertices c_i, d_i and whichever of x_i and $\overline{x_i}$ that has been set True, $1 \leq i \leq n$; and Z contains the vertices γ_j and μ_j, $1 \leq j \leq m$. Thus, $|S| = 3n + 2m + (9n+8m)(r-1)(2r+1) = k$. Clearly, S is an r-TDS of G. Now we need to show that S is an r-LTDS of G. In G, we observe that the vertices contributing to S from copies of H' as per Lemma 3.3 have no impact on the vertices from G'. In the same way, the vertices contributing to S from G' do not have any impact on the vertices of H'. Thus, the vertices that do not belong to S from each copy of H' are located by the vertices from each copy of H' that belong to S. By the choice of S, the vertices from G' are located as follows by S: $N_r(b_i) \cap S = \{c_i\}$, $N_r(e_i) \cap S = \{d_i\}$, $N_r(\beta_j) \cap S = \{\gamma_j\}$, $N_r(\eta_j) \cap S = \{\gamma_j, \mu_j\}$; $N_r(a_i) \cap S$ and $N_r(\alpha_j) \cap S$ contains the vertex of type x_i or $\overline{x_i}$, because by the assumption that each clause contains at least one true literal. Moreover, $N_r(\alpha_j) \cap S \neq N_r(a_i) \cap S$, $\forall i, j$.

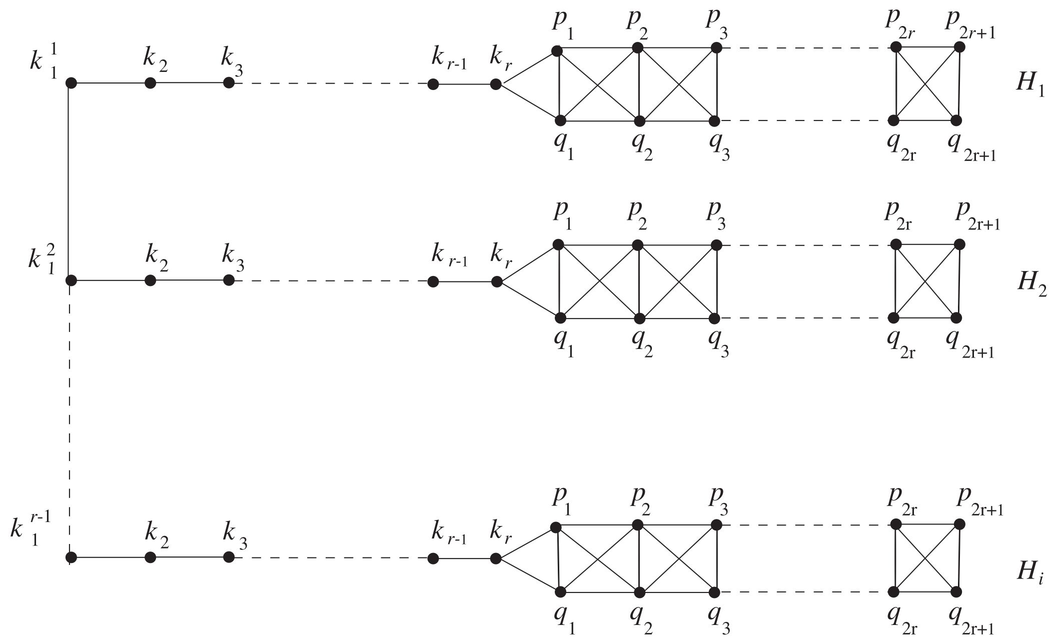

FIGURE 3. Graph H'.

(ii) Now, we assume that there exists an r-LTDS S of size k and then prove that $\mathbb{F}$ is satisfied. According to Lemma 3.3, H' consists of at least $(r-1)(2r+1)$ vertices that belong to copies of the set $\{p_1, p_2, \ldots, p_{2r+1}\} \cup \{q_1, q_2, \ldots, q_{2r+1}\}$. Hence, we need at least $(r-1)(2r+1)(9n+8m)$ vertices from all the copies of H' and these vertices do not impact the vertices that are not part of H' copies. Now observe that $S \cap V_{x_i}$ contains at least three vertices, $\forall i$. Also, if $S \cap V_{x_i}$ contains exactly three vertices, then exactly one of x_i and $\overline{x_i}$ must belong to S. Moreover, it is clear that we need at least two vertices from every clause. Therefore, S contains an r-LTDS with cardinality k. Now a valid truth assignment for X variables is obtained by setting x_i as $True$ if the intersection of S and $\{x_i, \overline{x_i}\}$ is $\{x_i\}$; and x_i as $False$ if the intersection of S and $\{x_i, \overline{x_i}\}$ is $\{\overline{x_i}\}$. Now we discuss about the vertex or vertices that will locate α_j. In V_{C_j}, there is exactly two vertices that belongs to S. It is clear that those two vertices must be among γ_j, μ_j and η_j of V_{C_j}. Hence, if a vertex or vertices locate α_j then it must be from outside V_{C_j}, and it can not be by one of the $(9n+8m)(r-1)(2r+1)$ vertices belonging to the copies of the set $\{p_1, p_2, \ldots, p_{2r+1}\} \cup \{q_1, q_2, \ldots, q_{2r+1}\}$. Now S must contain at least one vertex of type x_i or $\overline{x_i}$ which is at a distance r from α_j so that $N_r(\alpha_j) \cap S$ is not the same as $N_r(v) \cap S$ where $v(\neq \alpha_j)$ belongs to $V(G) \setminus S$. This implies that C_j contains at least one true literal, $\forall j$. In this way, we have a truth assignment that satisfies $\mathbb{F}$.

□

4. CHAIN SILICATE GRAPH AND CYCLIC SILICATE GRAPH

In this section, we discuss the r-LDS and r-LTDS for chain silicate graph and cyclic silicate graph.

Metal oxides or metal carbonates are fused with sand to form silicates. The silicate tetrahedrons (SiO_4) are found in almost all silicates. The corner vertices of the silicate tetrahedrons are actually oxygen atoms, whereas the central vertex represents silicon atom. See Figure 4.

Definition 4.1 ([18]). A *chain silicate graph* of dimension n is obtained by arranging n silicate tetrahedron linearly, denoted as CS_n, $n \geq 3$. See Figure 5a.

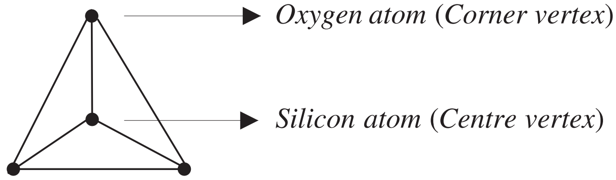

FIGURE 4. Silicate structure.

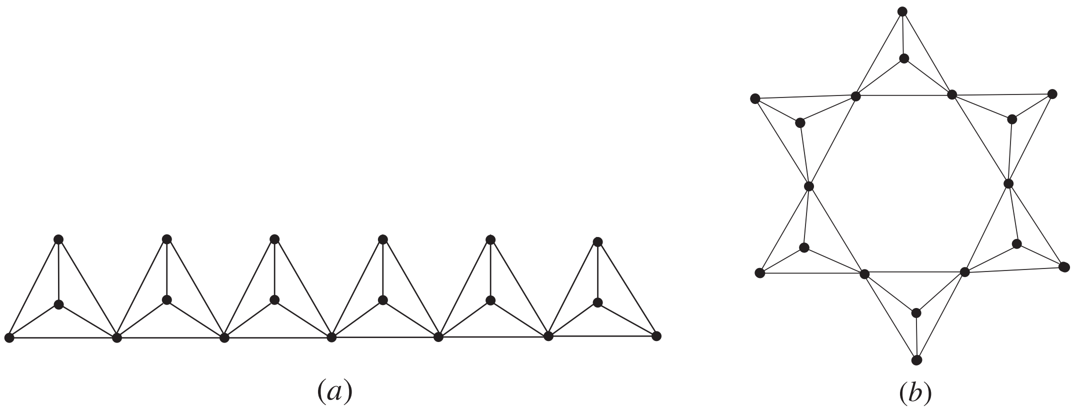

FIGURE 5. (a) CS_6, (b) CC_6.

Definition 4.2 ([18])**.** A *cyclic silicate graph* of dimension n is obtained by connecting n tetrahedrons into a cyclic structure, denoted as CC_n, $n \geq 3$. See Figure 5b.

A silicate tetrahedron is called a terminal tetrahedron if it shares exactly one vertex with another tetrahedron, otherwise, it is called an internal tetrahedron. In CS_n, there are exactly two terminal tetrahedrons, namely the left and right terminal tetrahedrons. The remaining tetrahedrons in CS_n are internal tetrahedrons. See Figure 5a.

Lemma 4.3. *Let G be a chain silicate graph CS_n and S be an r-LDS of G. Then S contains at least one simplicial vertex from each internal tetrahedron and two simplicial vertices from each terminal tetrahedron.*

Proof. Let S be an r-LDS of G. We claim that S contains at least one simplicial vertex from each internal tetrahedron and two simplicial vertices from each terminal tetrahedron. Assume the contrary. Let x and y be two simplicial vertices from an internal tetrahedron such that they belong to $V(G) \setminus S$. Since $N_r(x) = N_r(y)$, by Remark 1.3, either x or y belongs to S, a contradiction. Hence, S contains at least one simplicial vertex from each internal tetrahedron. Similarly, we can show that S contains at least two simplicial vertices from each terminal tetrahedron. Hence, the claim. □

Lemma 4.4. *Let G be a chain silicate graph CS_n and $1 \leq r \leq rad(G)$. Then $\gamma_r^L(G) \geq n + 2r$.*

Proof. Let S be an r-LDS of G. Since G contains $n-2$ internal tetrahedrons and 2 terminal tetrahedrons, by Lemma 4.3, we have $|S| \geq n+2$. We now claim that $|S| \geq n + 2r$.

Case 1: n even.

In G, there are $2(r-1)$ pairs of twin vertices that belong to $V(G) \setminus S$. Thus by Remark 1.1, S contains at least one vertex from each pair of twin vertices. Therefore, $|S| \geq (n+2) + 2(r-1) = n + 2r$.

Case 2: n odd.

The case when $1 \leq r \leq rad(G) - 1$ is similar to Case 1 and hence omitted. When $r = rad(G)$, there are $2(r-2)$ pairs of twin vertices in $V(G) \setminus S$. Thus by Remark 1.1, S contains at least one vertex from each pair of twin vertices. Hence, $|S| \geq (n+2)+2(r-2) = n+2r-2$. We observe that G contains four central vertices, of which two are simplicial and two are non-simplicial. By Lemma 4.3, one of the simplicial vertices belongs to S. Since $r = rad(G)$, the central vertices not in S form a triplet vertices in G. Thus by Remark 1.3, S contains at least two of these vertices. Therefore, $\gamma_r^L(G) = |S| \geq (n + 2r - 2) + 2 = n + 2r$.

□

Theorem 4.5. *Let G be a chain silicate graph CS_n and $1 \leq r \leq rad(G)$. Then $\gamma_r^L(G) = n + 2r$.*

Proof. Let z and z' be simplicial vertices from the left terminal tetrahedron and right terminal tetrahedron of G, respectively. Let $S_1 = \{x : x$ is a non-simplicial vertex such that $d(x, z) \leq r - 1\}$. Let $S_2 = \{y : y$ is a non-simplicial vertex such that $d(y, z') \leq r - 1\}$. We observe that when $r = 1$, $S_1 = S_2 = \emptyset$. Let S_3 be a set that contains a simplicial vertex from each internal tetrahedron and two simplicial vertices from each terminal tetrahedron. Clearly, $|S_1| = |S_2| = r - 1$ and $|S_3| = n + 2$. We claim that $S = S_1 \cup S_2 \cup S_3$ is an r-LDS of G and $|S| \leq n + 2r$. Since S contains at least one vertex from each tetrahedron, S is an r-DS of G. Let $P(x, y)$ be the shortest path from x to y and $G[P(x, y)] = \bigcup_{u \in V(P(x,y))} N[u]$. To prove that S locates every pair of vertices not in S, we proceed as follows: First, we prove that S locates every pair of simplicial vertices not in S. Second, we prove that S locates every pair of non-simplicial vertices not in S. Then, we prove that S locates every pair of vertices consisting of a simplicial vertex and a non-simplicial vertex not in S.

Case 1: x and y be two simplicial vertices not in S.

Clearly there exists $w \in S$ such that $d(x, w) = r$. When $r = 1$, $d(y, w) \geq 2$ since $d(x, w) = r = 1$. Thus, $w \in N(x) \cap S$ and $w \notin N(y) \cap S$. Therefore, $N(x) \cap S \neq N(y) \cap S$. Now, when $2 \leq r \leq rad(G)$, either $d(y, w) = r$ or $d(y, w) > r$ or $d(y, w) < r$.

Case 1.1: $d(y, w) = r$.

In this case, there exists w' in S such that $w' \in N(x)$. This implies that $d(y, w') = d(y, w) + d(w, x) + d(x, w') - 2 = 2r - 1$. Thus, $w' \in N_r(x) \cap S$ and $w' \notin N_r(y) \cap S$. Therefore, $N_r(x) \cap S \neq N_r(y) \cap S$.

Case 1.2: $d(y, w) > r$.

Clearly, $w \notin N_r(y) \cap S$ and $w \in N_r(x) \cap S$ since $d(x, w) = r$. Therefore, $N_r(x) \cap S \neq N_r(y) \cap S$.

Case 1.3: $d(y, w) < r$ and w belongs to a terminal tetrahedron.

Without loss of generality, let w belong to the right terminal tetrahedron. Clearly, $y \in G[P(x, w)]$ and $d(x, y) \geq 2$.

- When $2 \leq r \leq rad(G) - 1$, there exists w' in S such that $d(x, w') = r$ where $w \neq w'$. This implies that $d(y, w') = d(y, x) + d(x, w') - 1 \geq 2 + r - 1 = r + 1$. Thus, $w' \in N_r(x) \cap S$ and $w' \notin N_r(y) \cap S$. Therefore, $N_r(x) \cap S \neq N_r(y) \cap S$. The same argument holds when n is odd and $r = rad(G)$.
- Let n be even and $r = rad(G)$. In this case, the vertex x is a central vertex of G since $d(y, w) < r$ and w belongs to a terminal tetrahedron. Since n is even, there is one more central vertex u of G such that $N_r(x) \cap S = N_r(u) \cap S$. By Remark 1.3, either x or u belongs to S. By our assumption, $x \in V(G) \setminus S$ implying that $u \in S$. Thus, $y \neq u$ since $y \notin S$. Hence, $d(y, x) \geq 3$. Now, by the choice of S, there exists w' in S that belongs to the terminal tetrahedron such that $d(x, w') = r - 1$. This implies that $d(y, w') = d(y, x) + d(x, w') - 1 \geq 3 + (r - 1) - 1 = r + 1$. Thus, $w' \in N_r(x) \cap S$ and $w' \notin N_r(y) \cap S$. Therefore, $N_r(x) \cap S \neq N_r(y) \cap S$. See Figure 6.

Case 1.4: $d(y, w) < r$ and w belongs to an internal tetrahedron.

Without loss of generality, let the tetrahedron containing w lies to the right of the tetrahedron containing x. Clearly, $y \notin G[P(z, x)]$. Now, we have the following two cases:

- Let $y \in G[P(x, w)]$. By the choice of S, there exists w' in S such that the tetrahedron containing w' lies to the right of the tetrahedron containing w and $d(y, w') \leq r$. Observe that $d(w, w') \geq 2$. Hence,

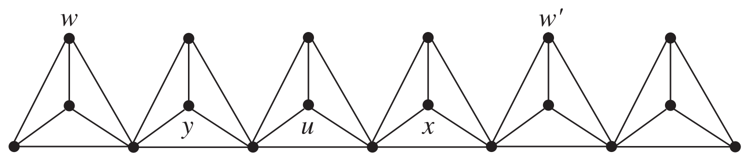

FIGURE 6. Illustrate Case 1.3 when n is even and $r = rad(G)$.

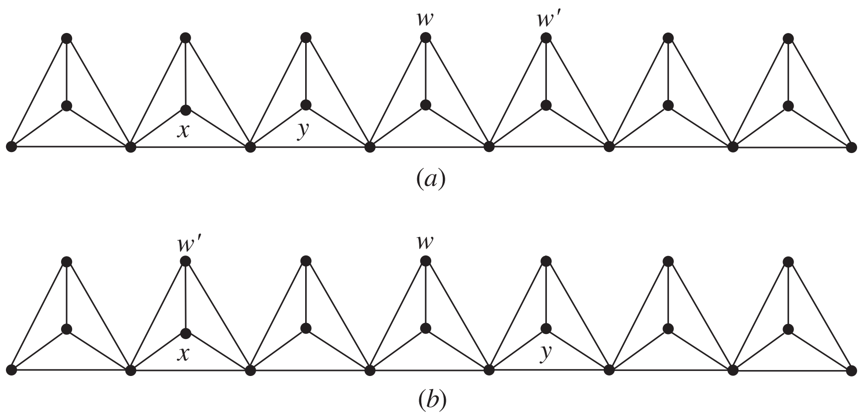

FIGURE 7. Illustrate Case 1.4 (a) when $y \in G[P(x,w)]$ and (b) when $y \notin G[P(x,w)]$.

$d(x,w') = d(x,w) + d(w,w') - 1 \geq r + 2 - 1 = r + 1$. Thus, $w' \notin N_r(x) \cap S$ and $w' \in N_r(y) \cap S$. Therefore, $N_r(x) \cap S \neq N_r(y) \cap S$. See Figure 7a.

- Let $y \notin G[P(z,w)]$. By the choice S, there exists w' in S such that $w' \in N(x) \cap S$. Observe that $d(y,w) \geq 2$. Hence, $d(y,w') = d(y,w) + d(w,x) + d(x,w') - 2 \geq 2 + r + 1 - 2 = r + 1$. Thus, $w' \in N_r(x) \cap S$ and $w' \notin N_r(y) \cap S$. Therefore, $N_r(x) \cap S \neq N_r(y) \cap S$. See Figure 7b.

Therefore, S locates every pair of simplicial vertices not in S.

Case 2: x and y be two non-simplicial vertices not in S.

The proof when the set S locates every pair of non-simplicial vertices not in S is similar to Case 1 and hence omitted.

Case 3: x be a simplicial vertex not in S and y be a non-simplicial vertex not in S.

Clearly, there exists $w \in S$ such that $d(x,w) = r$. When $r = 1$, y is adjacent to two simplicial vertices u and v in S, such that they belong to different tetrahedrons. Suppose $(x,y) \in E(G)$. Since $d(x,w) = 1$, without loss of generality, let $u = w$. This implies that $d(x,v) = 2$. Thus, $v \notin N(x) \cap S$ and $v \in N(y) \cap S$. Therefore, $N(x) \cap S \neq N(y) \cap S$. Suppose $(x,y) \notin E(G)$. Then, $d(x,u)$ and $d(x,v)$ are at least two. Thus, $u \notin N(x) \cap S$, $v \notin N(x) \cap S$ and $\{u,v\} \subseteq N(y) \cap S$. Therefore, $N(x) \cap S \neq N(y) \cap S$. Now, when $2 \leq r \leq rad(G)$, either $d(y,w) = r$ or $d(y,w) > r$ or $d(y,w) < r$.

Case 3.1: $d(y,w) = r$.

Suppose, $(x,y) \notin E(G)$. Then the proof is similar to the Case 1.1 and hence omitted. Suppose, $(x,y) \in E(G)$. If there exists a w' in S such that $d(y,w') = r$ where $w \neq w'$, then $d(x,w') = d(x,y) + d(y,w') = 1 + r$. Therefore, $N_r(x) \cap S \neq N_r(y) \cap S$. If there is no vertex w' in S such that $d(y,w') = r$ where $w \neq w'$, then $d(y,w') \leq r - 1$. This implies that $d(z,y) \leq r - 1$ and hence $y \in S$. This contradicts the fact that $y \notin S$.

Case 3.2: $d(y,w) > r$.

The proof that $N_r(x) \cap S \neq N_r(y) \cap S$ is similar to the Case 1.2 and hence omitted.

Case 3.3: $d(y, w) < r$ and w belongs to a terminal tetrahedron.

Without loss of generality, let w belong to the left terminal tetrahedron. This implies that $d(z, y) \leq r-1$ and hence $y \in S$. This contradicts the fact that $y \notin S$. Therefore, this case does not exist.

Case 3.4: $d(y, w) < r$ and w belongs to an internal tetrahedron.

Without loss of generality, let the tetrahedron containing w lies to the right of the tetrahedron containing x. Clearly, $y \notin G[P(z, x)]$. Now, we have the following two possibilities: $y \in G[P(x, w)]$ and $y \notin G[P(z, w)]$. Proving $N_r(x) \cap S \neq N_r(y) \cap S$ when $y \in G[P(x, w)]$ is similar to the Case 1.4 and hence omitted. Now, let $y \notin G[P(z, w)]$. Suppose $(w, y) \in E(G)$. Since w belongs to a terminal tetrahedron there exists $w' \in N(y) \cap S$ such that the tetrahedron containing w' lies to the right of the tetrahedron containing w. This implies that $d(x, w') = d(x, y) + d(y, w') = r+1$. Thus, $w' \notin N_r(x) \cap S$ and $w' \in N_r(y) \cap S$. Therefore, $N_r(x) \cap S \neq N_r(y) \cap S$. In a similar way, we can prove that $N_r(x) \cap S \neq N_r(y) \cap S$ when $(w, y) \notin E(G)$.

Therefore, S locates every pair of vertices consisting of a simplicial vertex and a non-simplicial vertex not in S.

From the above argument, S is an r-LDS of G and $\gamma_r^L(G) = |S| \leq n+2r$. By Lemma 4.4, $\gamma_r^L(G) \geq n+2r$. Therefore, $\gamma_r^L(G) = n + 2r$. □

In the following, we determine the r-locating-total domination number for the chain silicate graph when (i) $r = 1$ and (ii) $2 \leq r \leq rad(G)$.

Theorem 4.6. *Let G be a chain silicate graph CS_n. Then $\gamma_t^L(G) = n + 2 + \lceil (n-2)/2 \rceil$.*

Proof. Let S be an LTDS of G. By Lemma 4.3, S contains at least one simplicial vertex from each tetrahedron and two simplicial vertices from each terminal tetrahedron. Hence, $|S| \geq n + 2$. Clearly, S is a DS of G, but not a TDS of G. Observe that the simplicial vertices in S that belongs to the internal tetrahedrons are an independent set of vertices in G such that no three of them are adjacent to a vertex of G. This implies that we need at least $\lceil (n-2)/2 \rceil$ more vertices of G in S to make S a TDS of G. Therefore, $|S| \geq n + 2 + \lceil (n-2)/2 \rceil$. Now, we claim that $\gamma_t^L(G) \leq n+2+\lceil (n-2)/2 \rceil$. We begin by labelling the non-simplicial vertices of G starting from left to right as 1 to n. Let S_1 be a set that contains a simplicial vertex from each internal tetrahedron and two simplicial vertices from each terminal tetrahedron. Let S_2 be a set that contains a non-simplicial vertices of G with even labels. Clearly, $|S_1| = n + 2$ and $|S_2| = \lceil (n-2)/2 \rceil$. We claim that $S = S_1 \cup S_2$ is an LTDS of G. Since $S \subseteq S_1$, S is an LDS of G. Now, we need to show that for every x in S, $N(x) \cap S \neq \emptyset$. Let x be a simplicial vertex in S that belongs to a terminal tetrahedron. Then $N(x) \cap S \neq \emptyset$. Let x be a simplicial vertex in S that belongs to an internal tetrahedron. Observe that in G, the simplicial vertices that belong to an internal tetrahedron are adjacent to two non-simplicial vertices. This implies that one of the two non-simplicial vertices belongs to S since one of them has an even label. Thus, for every x in S, $N(x) \cap S \neq \emptyset$. Hence, S is an LTDS of G. Therefore, $\gamma_t^L(G) = |S| \leq n + 2 + \lceil (n-2)/2 \rceil$. □

The following theorem is an easy consequence of Theorem 4.5.

Theorem 4.7. *Let G be a chain silicate graph CS_n and $2 \leq r \leq rad(G)$. Then $\gamma_{t,r}^L(G) = n + 2r$.*

The proof of the following lemma is similar to Lemma 4.3 and hence omitted.

Lemma 4.8. *Let G be a cyclic silicate graph CC_n and S be an r-LDS of G. Then S contains at least one simplicial vertex from each tetrahedron.*

In the following, we first determine the r-locating-domination number for the cyclic silicate graph when (i) $1 \leq r \leq d(G) - 2$ and (ii) $r = d(G) - 1$.

Theorem 4.9. *Let G be a cyclic silicate graph CC_n with $d(G) \geq 3$ and $1 \leq r \leq d(G) - 2$. Then $\gamma_r^L(G) = n$.*

Proof. Let S be an r-LDS of G. By Lemma 4.8, S contains at least one simplicial vertex from each tetrahedron. Hence, $|S| \geq n$. Now, we claim that $\gamma_r^L(G) \leq n$. Let S be a set that contains a simplicial vertex from each tetrahedron. We claim that S is an r-LDS of G. Clearly, S is an r-DS of G. It remains to show that S locates every pair of vertices in $V(G) \setminus S$. Let H be a tetrahedron in G. Let u and v be two simplicial vertices of H; and x and y be two non-simplicial vertices of H. Without loss of generality, let $u \in S$. We claim that S locates every pair of vertices in $V(H) \setminus S$. By the choice of S, there exist vertices x' and y' in S such that $d(x, x') = d(y, y') = r$. Since $r \leq d(G) - 2$, we have the following:

- $d(y, x') = d(y, x) + d(x, x') = 1 + r$ which implies $x' \in N_r(x) \cap S$ and $x' \notin N_r(y) \cap S$;
- $d(x, y') = d(x, y) + d(y, y') = 1 + r$ which implies $y' \notin N_r(x) \cap S$ and $y' \in N_r(y) \cap S$;
- $d(v, x') = d(v, x) + d(x, x') = 1 + r$ and $d(v, y') = d(v, y) + d(y, y') = 1 + r$ which implies that both x' and y' are not in $N_r(v) \cap S$.

Thus, S locates every pair of vertices in $V(H) \setminus S$. This is true for every tetrahedron in G. Hence, S is an r-LDS of G. Therefore, $\gamma_r^L(G) = |S| \leq n$. □

Theorem 4.10. *Let G be a cyclic silicate graph* CC_n*, n is even and $r = d(G) - 1$. Then $\gamma_r^L(G) = 2n - 1$.*

Proof. Let S be an r-LDS of G. We first prove that $|S| \geq 2n - 1$. By Lemma 4.8, S contains at least one simplicial vertex from each tetrahedron. Further, we claim that S contains at least $n-1$ non-simplicial vertices of G. Suppose not, let S contain at most $n-2$ non-simplicial vertices of G. Then there exist two non-simplicial vertices x and y not in S such that $N_r(x) = N_r(y)$ since $r = d(G)-1$. By Remark 1.3, either x or y belongs to S, a contradiction. Hence, S contains at least $n-1$ non-simplicial vertices of G. Therefore, $|S| \geq n+(n-1) = 2n-1$. Now, we claim that $\gamma_r^L(G) \leq 2n-1$. Let S_1 be a set containing a simplicial vertex from each tetrahedron. Let S_2 be a set containing $n-1$ non-simplicial vertices. Clearly, $|S_1| = n$ and $|S_2| = n-1$. We claim that $S = S_1 \cup S_2$ is an r-LDS of G. Observe that $V(G) \setminus S$ contains one non-simplicial vertex, and the remaining are simplicial vertices. Let x and y be two simplicial vertices not in S. Then $d(x, y) \geq 2$. Now, by the choice of S, there exist vertices x' and y' in S such that $d(x, x') = d(y, y') = d(G)$. Hence, $x' \notin N_r(x) \cap S$ and $y' \notin N_r(y) \cap S$. Now, observe that $d(y, y') = d(y, x) + d(x, y') - 1 \geq 2 + d(x, y') - 1$ which implies that $d(x, y') \leq d(y, y') - 1 = d(G) - 1 = r$. Thus, $y' \in N_r(x) \cap S$. Similarly, we can show that $x' \in N_r(y) \cap S$. Therefore, $N_r(x) \cap S \neq N_r(y) \cap S$. Let x be a simplicial vertex not in S and y be a non-simplicial vertex not in S. By the choice of S, $N_r(y) \cap S = S$ and $N_r(x) \cap S \neq S$. Therefore, $N_r(x) \cap S \neq N_r(y) \cap S$. Hence, S is an r-LDS of G. Therefore, $\gamma_r^L(G) = |S| \leq 2n - 1$. □

Theorem 4.11. *Let G be a cyclic silicate graph* CC_n*, n is odd and $r = d(G) - 1$. Then $\gamma_r^L(G) = n$.*

Proof. Let S be an r-LDS of G. By Lemma 4.8, S contains at least one simplicial vertex from each tetrahedron. Hence, $|S| \geq n$. Now, we claim that $\gamma_r^L(G) \leq n$. Let S be a set containing a simplicial vertex from each tetrahedron. We claim that S is an r-LDS of G. Clearly, S is an r-DS of G. To prove that S locates every pair of vertices not in S, we proceed as follows: First, we prove that S locates every pair of simplicial vertices not in S. Next, we prove that S locates every pair of non-simplicial vertices not in S. Then, we prove that S locates every pair of vertices consisting of a simplicial vertex and a non-simplicial vertex not in S.

Case 1: x and y be two simplicial vertices not in S.
Without loss of generality, let the tetrahedron containing x lies to the left of the tetrahedron containing y in the anticlockwise sense. Clearly, $d(x, y) \geq 2$. Suppose $d(x, y) = 2$. By the choice of S, there exists x' in S such that $d(x, x') = r$. This implies that $d(y, x') = d(y, x) + d(x, x') - 1 = r + 1$. Thus, $x' \in N_r(x) \cap S$ and $x' \notin N_r(y) \cap S$. Now, let $d(x, y) \geq 3$. By the choice of S, there exists x' in S such that $d(x, x') = d(G)$. This implies that $d(x, x') = d(x, y) + d(y, x') - 2 \implies d(G) \geq 3 + d(y, x') - 2 \implies d(y, x') \leq d(G) - 1 = r$. Thus, $x' \notin N_r(x) \cap S$ and $x' \in N_r(y) \cap S$. Therefore, S locates every pair of simplicial vertices not in S.

Case 2: x and y be two non-simplicial vertices not in S.

Case 3: x be a simplicial vertex not in S and y be a non-simplicial vertex not in S.

The proof part of Cases 2 and 3 are similar to that of Case 1. Therefore, S is an r-LDS of G and $\gamma_r^L(G) = |S| \leq n$. □

In the following, we determine the r-locating-total domination number for the cyclic silicate graph when (i) $r = 1$, (ii) $2 \leq r \leq d(G) - 2$ and (iii) $r = d(G) - 1$.

Theorem 4.12. *Let G be a cyclic silicate graph CC_n. Then $\gamma_t^L(G) = n + \lceil n/2 \rceil$.*

Proof. Let S be an LTDS of G. By Lemma 4.8, S contains at least one simplicial vertex from each tetrahedron. Hence, $|S| \geq n$. Clearly, S is a DS of G, but not a TDS of G. Observe that these simplicial vertices are an independent set of vertices in G such that no three of them are adjacent to a vertex of G. This implies that we need at least $\lceil n/2 \rceil$ more vertices of G in S to make S a TDS of G. Therefore, $|S| \geq n + \lceil n/2 \rceil$. Now, we claim that $\gamma_t^L(G) \leq n + \lceil n/2 \rceil$. We begin by labelling the non-simplicial vertices of G in a clockwise direction from 1 to n. Let S_1 be a set containing a simplicial vertex from each tetrahedron. Let S_2 be a set containing the non-simplicial vertices with odd labels. Clearly, $|S_1| = n$ and $|S_2| = \lceil n/2 \rceil$. We claim that $S = S_1 \cup S_2$ is an LTDS of G. Since $S_1 \subseteq S$, S is an LDS of G. Now, we need to show that for every x in S, $N(x) \cap S \neq \emptyset$. Let x be a simplicial vertex in S that belongs to a tetrahedron. Observe that in G, a simplicial vertex is adjacent to two non-simplicial vertices u and v. Clearly, either u or v belongs to S since one of them has an odd label. Thus, $N(x) \cap S \neq \emptyset$. Let x be a non-simplicial vertex in S that belongs to a tetrahedron. Observe that in G, a non-simplicial vertex is adjacent to four simplicial vertices. By the choice of S, at least two of them belong to S. Thus, $N(x) \cap S \neq \emptyset$. Hence, S is an LTDS of G. Therefore, $\gamma_t^L(G) = |S| \leq n + \lceil n/2 \rceil$. □

The following theorem is an easy consequence of Theorem 4.9.

Theorem 4.13. *Let G be a cyclic silicate graph CC_n and $2 \leq r \leq d(G) - 2$. Then $\gamma_{t,r}^L(G) = n$.*

The following theorem is an easy consequence of Theorems 4.10 and 4.11.

Theorem 4.14. *Let G be a cyclic silicate graph CC_n and $r = d(G) - 1$. Then*

$$\gamma_{t,r}^L(G) = \begin{cases} 2n - 1 & \text{if } n \text{ is even} \\ n & \text{if } n \text{ is odd.} \end{cases}$$

5. Corona products, edge corona products and necklace graphs

In this section, we discuss the r-LDS and r-LTDS for corona products, edge corona products and necklace graphs.

Definition 5.1 ([28])**.** Let G and H be graphs with n and m vertices, respectively. The corona of two graphs G and H is obtained by taking a graph G and n copies of H, and then joining the kth vertex of G to all the vertices of the kth copy of H. It is denoted by $G \circ H$. Here, H_u represents the copy of H in $G \circ H$ corresponding to a vertex u in G.

Definition 5.2 ([28])**.** Let G and H be graphs with n and m vertices, respectively. The edge corona of two graphs G and H is obtained by taking a graph G and n copies of H one-to-one assigned to the edges of G, and $\forall uv \in E(G)$ joining u and v to every vertex of the copy of H associated to uv. It is denoted by $G \diamond H$. Here, H_{uv} represents the copy of H in $G \diamond H$ corresponding to an edge uv in G.

Definition 5.3 ([35])**.** The K-*necklace* is obtained by identifying any one vertex of K_{t_i} with the ith vertex of K_m, $1 \leq i \leq m$, denoted by $KN(K_m; K_{t_1}, K_{t_2}, \ldots, K_{t_m})$. See Figure 8a.

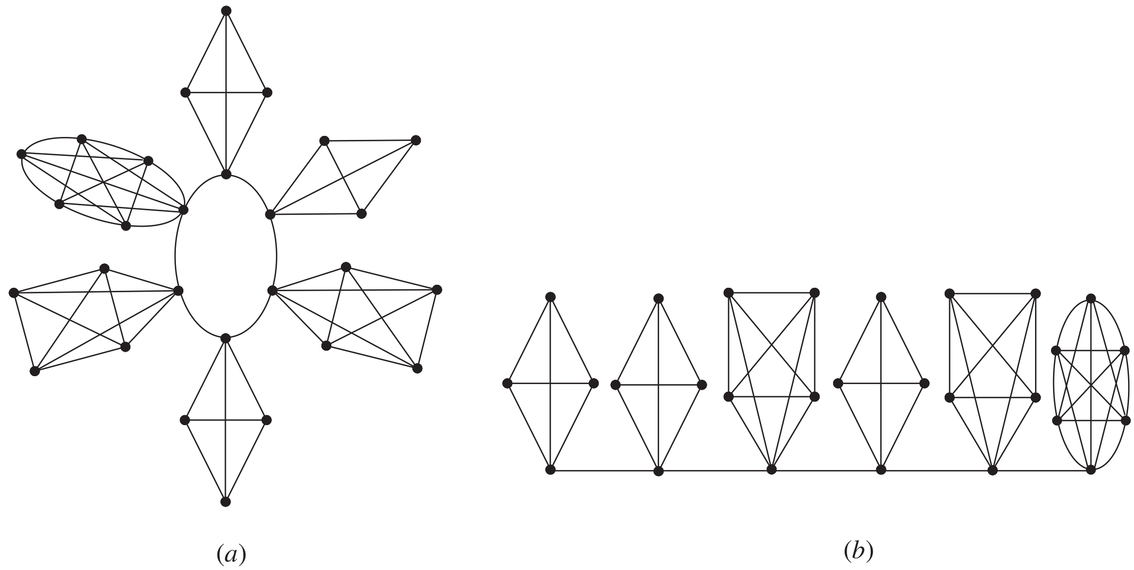

FIGURE 8. (a) K-necklace graph, (b) P-necklace graph.

Definition 5.4 ([35])**.** The *P-necklace* is obtained by identifying a vertex of K_{t_i} with the ith vertex of P_m, $1 \leq i \leq m$, denoted by $PN(P_m; K_{t_1}, K_{t_2}, \ldots, K_{t_m})$. See Figure 8b.

Theorem 5.5. *Let G and H be a graph with n vertices and m vertices respectively, $n, m \geq 2$. Then $\gamma_r^L(G \circ H) \geq n\gamma_r^L(H)$, $1 \leq r \leq 2$.*

Proof. Let S be an r-LDS of $G \circ H$. Let u be a vertex in G and H_u be a copy of H in $G \circ H$ corresponding to u. Observe that $N_r(u) = V(H_u)$. This implies that S contains at least $\gamma_r^L(H)$ vertices from H_u to locate every pair of vertices in H_u. Therefore, $\gamma_r^L(G \circ H) = |S| \geq n\gamma_r^L(H)$. □

The following theorem is an easy consequence of Theorem 5.5.

Theorem 5.6. *Let G and H be a graph with n vertices and m vertices respectively, $n, m \geq 2$. Then $\gamma_r^L(G \diamond H) \geq n\gamma_r^L(H)$, $1 \leq r \leq 2$.*

Theorem 5.7 ([27])**.** *Let G be the K-necklace $KN(K_m; K_{t_1}, K_{t_2}, \ldots, K_{t_m})$, $m \geq 1, t_i \geq 4, 1 \leq i \leq m$ on $n = \sum_{i=1}^{m} t_i$ vertices. Then $\gamma^L(G) = n - 2m + 1$.*

Theorem 5.8. *Let G be the K-necklace $KN(K_m; K_{t_1}, K_{t_2}, \ldots, K_{t_m})$, $m \geq 1, t_i \geq 4, 1 \leq i \leq m$ on $n = \sum_{i=1}^{m} t_i$ vertices. Then $\gamma_2^L(G) = n - m - 1$.*

Proof. Let S be a 2-LDS of G. We claim that $|S| \geq n - m - 1$. We proceed as follows: First, we prove that S contains at least $t_i - 2$ vertices from each K_{t_i}. Then, we prove that S contains at least $m - 1$ vertices from K_m. Let H be a complete graph on p vertices that is identified with a vertex of K_m. We claim that S contains at least $p - 2$ vertices from H. Suppose S contains $p - 3$ vertices from H. Then there exist two vertices x and y not in S from H such that $N_2[x] = N_2[y]$. By Remark 1.3, either x or y belongs to S, a contradiction. Hence, S contains at least $p - 2$ vertices from H. Therefore, we conclude that S contains at least $t_i - 2$ vertices from each K_{t_i}. Hence, $|S| \geq n - 2m$. In a similar way, we can prove that S contains at least $m - 1$ vertices of K_m. Therefore, $|S| \geq (n - 2m) + (m - 1) = n - m - 1$. Now, we claim that $\gamma_2^L(G) \leq n - m - 1$. Let S be a set containing $t_i - 2$ vertices from each K_{t_i} and $m-1$ vertices from K_m. We claim that S is a 2-LDS of G. Clearly, S is a 2-DS of G. To prove that S locates every pair of vertices not in S, we proceed as follows: First, we prove that S locates every pair of vertices not in S from $V(G) \setminus K_m$. Then, we prove that S locates every pair of vertices consisting of a vertex from $V(G) \setminus K_m$ and a vertex not in S from K_m. Let x and y be two vertices not in S from $V(G) \setminus V(K_m)$. Without loss of generality, let $x \in K_{t_i}$ and $y \in K_{t_j}$ where $i \neq j$. Then, $V(K_{t_i}) \setminus \big(V(K_m) \cup \{x\}\big) \subseteq N_2(x) \cap S$ and $V(K_{t_i}) \setminus \big(V(K_m) \cup \{x\}\big) \nsubseteq N_2(y) \cap S$. Thus, $N_2(x) \cap S \neq N_2(y) \cap S$. Now, let x be a vertex not in S from $V(G) \setminus V(K_m)$ and y be a vertex not in S from K_m. Then, $N_2(y) \cap S = S$ and $N_2(x) \cap S \neq S$. Thus, $N_2(y) \cap S \neq N_2(x) \cap S$. Hence, S locates every pair of vertices not in S. Therefore, S is a 2-LDS of G and $\gamma_2^L(G) = |S| \leq n - m - 1$. □

The following theorem is an easy consequence of Theorem 5.8.

Theorem 5.9. *Let G be the K-necklace $KN(K_m; K_{t_1}, K_{t_2}, \ldots, K_{t_m})$, $m \geq 1, t_i \geq 4, 1 \leq i \leq m$ on $n = \sum_{i=1}^{m} t_i$ vertices. Then $\gamma_{t,2}^L(G) = n - m - 1$.*

Theorem 5.10. *Let G be the P-necklace graph $PN(P_m; K_{t_1}, K_{t_2}, \ldots, K_{t_m})$, $m \geq 3, t_i \geq 4, 1 \leq i \leq m$ on $n = \sum_{i=1}^{m} t_i$ vertices. Then $\gamma_r^L(G) = n - 2m$, $2 \leq r \leq rad(G)$.*

Proof. Let S be an r-LDS of G. The proof of $|S| \geq n - 2m$ is similar to Theorem 5.8 and hence omitted. Therefore, $|S| \geq n - 2m$. Now, we claim that $\gamma_r^L(G) \leq n - 2m$. Let S be a set containing $t_i - 2$ vertices from each K_{t_i}. We claim that S is an r-LDS of G. Clearly, S is an r-DS of G. Let K_{t_1} and K_{t_m} be terminal cliques on t_1 and t_m vertices, respectively. Let K_{t_j} be an internal clique on t_j vertices, $2 \leq j \leq m-1$. Now, the proof that S is an r-LDS of G is similar to the Theorem 4.5 and hence omitted. Therefore, $\gamma_r^L(G) = |S| \leq n - 2m$. □

The following theorem is an easy consequence of previous Theorem 5.10.

Theorem 5.11. *Let G be the P-necklace graph $PN(P_m; K_{t_1}, K_{t_2}, \ldots, K_{t_m})$, $m \geq 3, t_i \geq 4, 1 \leq i \leq m$ on $n = \sum_{i=1}^{m} t_i$ vertices. Then $\gamma_{t,r}^L(G) = n - 2m$, $2 \leq r \leq rad(G)$.*

6. Conclusion

We have extended locating-total domination problem to r-locating-total domination problem, $r \geq 1$ and proved that the problem is NP-complete. We have also discussed the lower bound on $\gamma_r^L(G)$ and $\gamma_{t,r}^L(G)$ when G is a k-regular graph. Furthermore, the exact values of r-locating-domination number and r-locating-total domination number for graphs such as chain silicate graphs, cyclic silicate graphs and necklace graphs have been obtained. It would be an interesting line of research to obtain these parameters for interconnection networks such as circulant networks, butterfly networks, mesh networks and so on.

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References

[1] G.R. Argiroffo, S.M. Bianchi and A.K. Wagler, A polyhedral approach to locating-dominating sets in graphs. *Electron. Notes Dis. Math.* **50** (2015) 89–94.

[2] N.A.A. Aziz, N.J. Rad and H. Kamarulhaili, A note on the double domination number in maximal outerplanar and planar graphs. *RAIRO-Oper. Res.* **56** (2022) 3367–3371.

[3] T.L. Baldwin, L. Mili, M.B. Boisen and R. Adapa, Power system observability with minimal phasor measurement placement. *IEEE Trans. Power Syst.* **8** (1993) 707–715.

[4] C. Berge, Graphs and Hypergraphs. Amsterdam Publications, North Holland (1973).
[5] N. Bertrand, I. Charon, O. Hudry and A. Lobstein, Identifying and locating-dominating codes on chains and cycles. *Eur. J. Combin.* **25** (2004) 969–987.
[6] A. Cabrera-Martinez, A new lower bound for the independent domination number of a tree. *RAIRO-Oper. Res.* **57** (2023) 1951–1956.
[7] J. Caceres, C. Hernando, M. Mora, I.M. Pelayo and M.L. Puertas, Locating-dominating codes: bounds and extremal cardinalities. *Appl. Math. Comput.* **220** (2013) 38–45.
[8] S.R. Canoy, Jr., G.A. Malacas and D. Tarepe, Locating-dominating set in graphs. *Appl. Math. Sci.* **8** (2014) 4381–4388.
[9] I. Charon, O. Hudry and A. Lobstein, Minimizing the size of an identifying or locating-dominating code in a graph is *NP*-hard. *Theor. Comput. Sci.* **290** (2003) 2109–2120.
[10] M. Chellali and N.J. Rad, Locating-total domination critical graphs. *Australas. J. Comb.* **45** (2009) 227–234.
[11] X. Chena and M. Sohn, Bounds on the locating-total domination number of a tree. *Discrete Appl. Math.* **159** (2011) 769–773.
[12] E.J. Cockayne, R.M. Dawes and S.T. Hedetniemi, Total domination in graphs. *Networks* **10** (1980) 211–219.
[13] M. Fazil, I. Javaid, M. Salman and U. Ali, Locating-dominating sets in hypergraphs. *Periodica Math. Hungari.* **72** (2016) 224–234.
[14] F. Foucaud, M.A. Henning and C. Lowenstein, Locating-total dominating sets in twin-free graphs. *Discrete Appl. Math.* **200** (2016) 52–58.
[15] D. Garijo, A. Gonzalez and A. Marquez, The difference between the metric dimension and the determining number of a graph. *Appl. Math. Comput.* **249** (2014) 487–501.
[16] W. Goddard and M.A. Henning, Independent domination, colorings and the fractional idomatic number of a graph. *Appl. Math. Comput.* **382** (2020) 125340.
[17] K. Haghparast, J. Amjadi, M. Chellali and S.M. Sheikholeslami, Restrained {2}-domination in graphs. *RAIRO-Oper. Res.* **57** (2023) 2393–410.
[18] S. Hayat and M. Imran, Computation of topological indices of certain networks. *Appl. Math. Comput.* **240** (2014) 213–228.
[19] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Fundamentals of Domination in Graphs. Marcel Dekker Inc., New York (1998).
[20] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Domination in Graphs Advanced Topic. Boca Raton (1998).
[21] T.W. Haynes, M.A. Henning and J. Howarda, Locating and total dominating sets in trees. *Discrete Appl. Math.* **154** (2006) 1293–1300.
[22] M.A. Henning and P. Kaemawichanurat, Connected domination critical graphs with a block having maximum number of cut vertices. *Appl. Math. Comput.* **406** (2021) 126248.
[23] M.A. Henning and C. Löwenstein, Locating-total domination in claw-free cubic graphs. *Discrete Math.* **312** (2012) 3107–3116.
[24] M.A. Henning and O.R. Oellermann, Metric-locating dominating sets in graphs. *Ars Comb.* **73** (2004) 129–141.
[25] M.A. Henning and N.J. Rad, Locating-total domination in graphs. *Discrete Appl. Math.* **160** (2012) 1986–1993.
[26] C. Hernando, M. Mora and I.M. Pelayo, Locating domination in bipartite graphs and their complements. *Discrete Appl. Math.* **263** (2019) 195–203.
[27] R. Jayagopal, R.S. Rajan and I. Rajasingh, Tight lower bound for locating-total domination number. *Int. J. Pure Appl. Math.* **101** (2015) 661-668.
[28] S. Klavžar and M. Tavakoli, Dominated and dominator colorings over (edge) corona and hierarchical products. *Appl. Math. Comput.* **390** (2021) 125647.
[29] M. Miller, R.S. Rajan, R. Jayagopal, I. Rajasingh and P. Manuel, A note on the locating-total domination in graphs. *Discuss. Math. Graph Theory* **37** (2017) 745–754.
[30] D.A. Mojdeh, I. Masoumi and L. Volkmann, Restrained double Roman domination of a graph. *RAIRO-Oper. Res.* **56** (2022) 2293–2304.
[31] O. Ore, Theory of Graphs. Vol. 38. American Mathematical Society Colloquium Publications, Providence, RI (1962).
[32] N.J. Rad and H. Rahbani, Bounds on the locating-domination number and differentiating-total domination number in trees. *Discuss. Math. Graph Theory* **38** (2018) 455–462.
[33] N.J. Rad and L. Volkmann, Generalization of the total outer-connected domination in graphs. *RAIRO-Oper. Res.* **50** (2016) 233–239.

[34] R.S. Rajan, S. Arulanand, S. Prabhu and I. Rajasingh, 2-power domination number for Knödel graphs and its application in communication networks. *RAIRO-Oper. Res.* **56** (2023) 3157–3168.
[35] I. Rajasingh, B. Rajan and R.S. Rajan, Embedding of hypercubes into necklace, windmill and snake graphs. *Inf. Process. Lett.* **112** (2012) 509–515.
[36] B. Senthilkumar, M. Chellali, H. Naresh Kumar and Y.B. Venkatakrishnan, Graphs with unique minimum vertex-edge dominating sets. *RAIRO-Oper. Res.* **57** (2023) 1785–1795.
[37] P.J. Slater, R-domination in graphs. *Assoc. Comput. Mach.* **23** (1976) 446–450.
[38] P.J. Slater, Domination and location in acyclic graphs. *Networks* **17** (1987) 55–64.
[39] P.J. Slater, Dominating and reference sets in a graph. *J. Math. Phys. Sci.* **22** (1988) 445–455.
[40] P.J. Slater, Fault-tolerant locating-dominating sets. *Discrete Math.* **249** (2002) 179–189.
[41] W. Zhuang, Semitotal domination versus domination and total domination in trees. *RAIRO-Oper. Res.* **58** (2024) 1249–1256.