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INTEGER k-MATCHING PRECLUSION OF GRAPHS

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Abstract. As a generalization of matching preclusion number of a graph, we provide the (strong) integer k-matching preclusion number, abbreviated as MP^k number (SMP^k number), which is the minimum number of edges (vertices and edges) whose deletion results in a graph that has neither perfect integer k-matching nor almost perfect integer k-matching. In this paper, we obtain a necessary condition of graphs with an almost-perfect integer k-matching and a relational expression between the matching number and the integer k-matching number of bipartite graphs. And then the MP^k number and the SMP^k number of complete graphs, bipartite graphs and arrangement graphs are obtained, respectively.

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1. Introduction

1.1. Definitions and notations

All graphs considered in this paper are undirected, finite and simple. Let G be a graph. If $|V(G)|$ is odd (even), then G is said to be odd (even). We refer to the book [2] for notations and terminology that are not defined here. For $v \in V(G)$, we denote by $\Gamma(v)$ the set of edges incident with v. For two subsets S, T of $V(G)$, let $E_G(S,T)=\{uv \in E(G)|u \in S, v \in T\}$. A vertex of degree 0 is called an isolated vertex. The number of isolated vertices of G is denoted by $i(G)$. The number of odd components of G is denoted by $c_o(G)$. The number of odd components with at least three vertices is denoted by $odd(G)$. Then $c_o(G) = i(G) + odd(G)$. A complete graph, a path and a cycle on n vertices are denoted by K_n, P_n and C_n, respectively.

A matching of G is a subset of $E(G)$ in which no two edges are adjacent. The matching number of G, denoted by $\mu(G)$, is the maximum size of matchings. For a matching M, a vertex v is M-saturated if $\Gamma(v) \cap M \neq \emptyset$, otherwise v is M-unsaturated. A matching M is perfect if every vertex is M-saturated. A matching M is almost perfect if there exists exactly one M-unsaturated vertex. A fractional matching is a function f: $E(G) \to [0, 1]$ such that $\sum\limits_{e\in\Gamma(v)} f(e) \leq 1$ for any vertex v. Clearly, a fractional matching is a relaxation of matching. Let k be a positive integer. An integer k-matching of a graph G is a function h: $E(G) \to \{0, 1, \ldots, k\}$ such that $\sum_{e\in\Gamma(v)} h(e) \leq k$ for any vertex $v \in V(G)$. Integer k-matching is a kind of generalization of matching. In fact, integer 1-matching is a matching. An edge is said to be 0-edge if $h(e) = 0$. An integer k-matching is perfect if

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$\sum_{e\in\Gamma(v)} h(e) = k$ for each vertex v, that is, $\sum_{e\in E(G)} h(e) = \frac{1}{2}\sum_{v\in V(G)}\sum_{e\in\Gamma(v)} h(e) = \frac{k|V(G)|}{2}$. It is clear that if G has a perfect matching, then G has a perfect integer k-matching which can be constructed by assigning k to every edge in the perfect matching and assigning 0 to other edges. But the opposite is not true, see Figure 1 in [16]. An integer k-matching is almost-perfect [17] if there exists exactly one vertex v' such that $\sum_{e\in\Gamma(v')} h(e) = k-1$ and $\sum_{e\in\Gamma(v)} h(e) = k$ for each other vertex v, that is, $\sum_{e\in E(G)} h(e) = \frac{1}{2}\sum_{v\in V(G)}\sum_{e\in\Gamma(v)} h(e) = \frac{k|V(G)|-1}{2}$.

Brigham *et al.* [3] introduced the concept of matching preclusion, as a measure of robustness in the event of edge failure in interconnection networks. An edge subset F of G is a matching preclusion set (MP set for short) if $G - F$ has neither perfect matching nor almost-perfect matching. The matching preclusion number of G, denoted by $mp(G)$, is the minimum size of MP sets of G. An interconnection network with a large matching preclusion number may be considered as more robust in the event of link failures. Park and Ihm [20] introduced the concept of strong matching preclusion, as a measure of robustness in the event of edge and vertices failure in interconnection networks. A set F of edges and vertices of G is a strong matching preclusion set (SMP set for short) if $G - F$ has neither perfect matching nor almost-perfect matching. The strong matching preclusion number of G, denoted by $smp(G)$, is the minimum size of strong matching preclusion sets of G. The (strong) matching preclusion problems have been studied as interconnection network topologies. We refer the readers to [5, 9, 20] for details.

Liu and Liu [15] introduced the concept of fractional matching preclusion. An edge subset F of G is a fractional matching preclusion set if $G-F$ has no perfect fractional matching. The fractional matching preclusion number of G, denoted by $fmp(G)$, is the minimum size of fractional matching preclusion sets of G. Fractional matching preclusion is a generalization of matching preclusion. The fractional matching preclusion problems of many interconnection networks have been studied. We refer the readers to [18, 19] for details.

As a generalization of matching preclusion number of a graph, Chang, Li and Liu [16] introduce the definitions of (strong) integer k-matching preclusion number. An edge subset F of G is an *integer k-matching preclusion set* (MP^k set for short) if $G - F$ has neither perfect integer k-matching nor almost perfect integer k-matching. The *integer k-matching preclusion number* (MP^k number for short) of G, denoted by $mp^k(G)$, is the minimum size of MP^k sets of G. Then $mp^1(G)=mp(G)$. For any $k \ge 2$, $mp^k(G) \le \delta(G)$ since the set of edges incident with v is an MP^k set for any vertex v. A set F of edges and vertices of G is a *strong integer k-matching preclusion set* (SMP^k set for short) if $G - F$ has neither perfect integer k-matching nor almost perfect integer k-matching. The *strong integer k-matching preclusion number* (SMP^k number for short) of G, denoted by $smp^k(G)$, is the minimum size of SMP^k sets of G. Then $smp^1(G)=smp(G)$. By the definition of $smp^k(G)$, we have that $smp^k(G) \le mp^k(G)$. In [16], we had studied the MP^k number and SMP^k number of twisted cubes and (n, s)-star graphs. More details about MP^k number and SMP^k number can refer [16].

In [16], when k is even, G has a perfect fractional matching if and only if G has a perfect integer k-matching. If G has an almost perfect integer k-matching, say h, then $\sum_{e\in E(G)} h(e) = \frac{k|V(G)|-1}{2}$ is an integer, which means that k is odd and $|V(G)|$ is odd. So every graph has no almost perfect integer k-matching for even k. It follows that $fmp(G) = mp^k(G)$ and $fsmp(G) = smp^k(G)$ for even k. Hence we only consider the case that $k \ge 3$ is odd in this paper.

1.2. Arrangement graphs

Let n and s be positive integers such that $n \ge 2$ and $1 \le s \le n-1$. The vertex set of arrangement graph $A_{n,s}$ is the set of all permutations on s elements of the set $\{1, 2, \ldots, n\}$. Two vertices corresponding to the permutations $[a_1, a_2, \ldots, a_s]$ and $[b_1, b_2, \ldots, b_s]$ are adjacent if and only if they differ in exactly one position. Then $A_{n,1}$ is the complete graph K_n. We know that $A_{n,n-1}$ is a bipartite graph (see [1]) and $A_{n,n-2}$ is isomorphic to the alternating group graph A_n (see [14]). It can be seen easily that $A_{n,s}$ is $s(n-s)$-regular graph with $\frac{n!}{(n-s)!}$ vertices. For example, $A_{4,2}$ is shown in Figure 1, where every vertex $[i, j]$ of $A_{n,2}$ is denoted by ij.

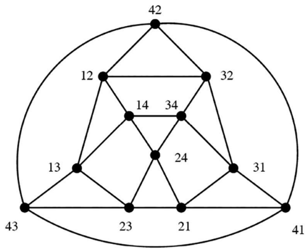

FIGURE 1. Arrangement graph $A_{4,2}$.

The arrangement graphs were introduced in [10] to be a common generalization of star graphs and alternating group graphs, and to provide an even richer class of interconnection networks. Recently, researchers have found that they are excellent candidates as interconnection networks. More results of arrangement graphs can refer to [6,8,13].

Let V_i be the set of vertices representing permutations whose sth element is i and H_i the subgraph of $A_{n,s}$ induced by V_i for each $i \in \{1, \cdots, n\}$. Then H_i can be seen the arrangement graph defined on permutations of $s-1$ elements of $\{1, 2, \ldots, n\} - \{i\}$. Hence H_i is isomorphic to $A_{n-1,s-1}$. Thus H_i has $\frac{(n-1)!}{(n-s)!}$ vertices for each $1 \leq i \leq n$ and $A_{n,s}$ can be decomposed into n subgraphs H_i. The edges whose two end-vertices belong to different H_j are called *cross* edges. Then there are exactly $\frac{(n-2)!}{(n-s-1)!}$ cross edges between H_i and H_j for any $1 \leq i < j \leq n$ which are not adjacent to each other. Each vertex $[a_1, a_2, \ldots, a_{s-1}, i]$ of H_i has exactly $n-s$ neighbors in $V(A_{n,s}) - V(H_i)$, which are $[a_1, a_2, \ldots, a_{s-1}, j]$ $(j \in \{1, 2, \cdots, n\} - \{a_1, a_2, \ldots, a_{s-1}, i\})$ and then they belong to $n-s$ different subgraphs H_j's, called the *outer* neighbors of $[a_1, a_2, \ldots, a_{s-1}, i]$. The subgraph H_i is called adjacent to a vertex u or u is adjacent to H_i if u has an outer neighbor in $V(H_i)$. Then there exist exactly $n-s$ subgraphs among $H_1, \cdots, H_{i-1}, H_{i+1}, \cdots, H_n$ adjacent to v for any vertex v of H_i.

Note that many interconnection networks G have usually an even number of vertices and have the property that $mp(G) = \delta(G)$. So we have that $mp(G) = mp^k(G) = \delta(G)$. Since the arrangement graph $A_{n,s}$ has the property that $mp(A_{n,s}) = \delta(A_{n,s})$ in the case that $2 \leq s \leq n-2$, $mp^k(A_{n,s}) = \delta(A_{n,s})$. So, in this paper, we investigate the SMP^k number of arrangement graph $A_{n,s}$ in the case that $2 \leq s \leq n-2$, and the MP^k numbers and the SMP^k number of $A_{n,1}$ and $A_{n,n-1}$, where $A_{n,1}$ is K_n and $A_{n,n-1}$ is bipartite.

2. MAIN RESULTS

Note that if $G - F$ has a perfect integer k-matching or an almost perfect integer k-matching for any $F \subseteq V(G) \cup E(G)$ such that $|F| = t$, then $G - F_1$ has also a perfect integer k-matching or an almost perfect integer k-matching for any $F_1 \subseteq V(G) \cup E(G)$ such that $|F_1| < t$. So $smp^k(G) \geq t+1$ if and only if $G - F$ has a perfect integer k-matching or an almost perfect integer k-matching for any $F \subseteq V(G) \cup E(G)$ with $|F| = t$.

A Hamiltonian graph G is called k-*fault Hamiltonian* if $G - F$ remains Hamiltonian for every $F \subset V(G) \cup E(G)$ with that $|F| \leq k$. A graph is *Hamiltonian connected* if there is a Hamiltonian path between any two vertices. A Hamiltonian connected graph G is called k-*fault Hamiltonian connected* if $G - F$ remains Hamiltonian connected for every $F \subset V(G) \cup E(G)$ with that $|F| \leq k$. The following lemmas are useful. All the n and s used in the following are integers.

Lemma 2.1 ([12]). *Let $n \geq 4$. Then K_n is $(n-3)$-fault Hamiltonian and $(n-4)$-fault Hamiltonian connected.*

Lemma 2.2 ([3]). *For any even integer $n \geq 2$, $mp(K_n) = n-1$ and for any odd integer $n \geq 11$, $mp(K_n) = 2n-3$.*

Lemma 2.3 ([11]). *Let $n \geq 6$ and $F \subseteq V(K_n) \cup E(K_n)$ such that $|F| \leq n-2$ and $\delta(K_n - F) \geq 2$. Then $K_n - F$ is Hamiltonian.*

Lemma 2.4 ([13]). *Let $F \subseteq V(A_{n,s}) \cup E(A_{n,s})$ with $1 \leq s \leq n-2$. If $|F| \leq s(n-s)-2$, then $A_{n,s} - F$ is Hamiltonian.*

Based on Lemma 2.4, Cheng *et al.* investigated the matching preclusion number and strong matching preclusion number of arrangement graphs $A_{n,s}$ as follows.

Lemma 2.5 ([7]). *For any $2 \leq s \leq n-3$, $mp(A_{n,s}) = s(n-s)$.*

Lemma 2.6 ([6]). *For any $2 \leq s \leq n-2$, $smp(A_{n,s}) = s(n-s)$.*

Since $smp(A_{n,n-2}) \leq mp(A_{n,n-2}) \leq \delta(A_{n,n-2})$, $mp(A_{n,n-2}) = 2(n-2)$ by Lemma 2.6.

2.1. MP^k and SMP^k numbers of K_n

Since $A_{n,1}$ is a complete graph K_n, we study the (strong) integer k-matching preclusion number of complete graphs. Firstly, we need to prove the following lemma.

Lemma 2.7. *Let G be a graph with an almost perfect integer k-matching. Then $odd(G-S)+k\cdot i(G-S) \leq k|S|+1$ for any $S \subseteq V(G)$.*

Proof. Let h be an almost perfect integer k-matching of G and $u \in V(G)$ such that $\sum_{e\in\Gamma(u)} h(e) = k-1$ and $\sum_{e\in\Gamma(v)} h(e) = k$ for any $v \in V(G)-\{u\}$. Let $S \subseteq V(G)$, $Q_1, \cdots, Q_t$ be the odd non-trivial components, $\{v_1, \cdots, v_l\}$ be the isolated vertex set and H be the union of even components of $G-S$, respectively. Let $\overline{S} = V(G) - S$. We distinguish the following cases.

Case 1. There exists a Q_j such that $u \in V(Q_j)$. Then we have that $\sum_{e\in E(\overline{S},S)} h(e) \leq k|S|$, $\sum_{e\in E(v_i,S)} h(e) = k$ for any $i \in \{1,\ldots,l\}$ and $\sum_{e\in E(V(Q_i),S)} h(e) \geq 1$ for any $i \neq j$ since both Q_i and k are odd. Hence

$$odd(G-S)+k\cdot i(G-S) \leq \sum_{i\in\{1,\cdots,t\}-\{j\}} \sum_{e\in E(V(Q_i),S)} h(e) + 1 + \sum_{e\in E(\{v_1,\cdots,v_l\},S)} h(e)$$

$$\leq \sum_{e\in E(\overline{S},S)} h(e) + 1 \leq k|S|+1.$$

Case 2. There exists $j \in \{1,\ldots,l\}$ such that $u = v_j$. Similar to the statement above, we have that

$$odd(G-S)+k\cdot i(G-S) \leq \sum_{i=1}^{t} \sum_{e\in E(V(Q_i),S)} h(e) + \sum_{i\in\{1,\cdots,l\}-\{j\}} \sum_{e\in E(v_i,S)} h(e) + \sum_{e\in E(v_j,S)} h(e) + 1$$

$$\leq \sum_{e\in E(\overline{S},S)} h(e) + 1 \leq k|S|+1.$$

Case 3. $u \in V(H)$. By the similar reasons as above, we have that

$$odd(G-S)+k\cdot i(G-S) \leq \sum_{i=1}^{t} \sum_{e\in E(V(Q_i),S)} h(e) + \sum_{e\in E(\{v_1,\cdots,v_l\},S)} h(e) \leq \sum_{e\in E(\overline{S},S)} h(e) \leq k|S|.$$

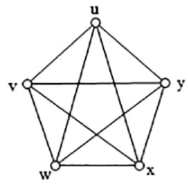

FIGURE 2. K_5.

Case 4. $u \in S$. Then we have that $\sum_{e\in E(\overline{S},S)} h(e) \le k|S| - 1$. Thus

$$odd(G-S) + k\cdot i(G-S) \le \sum_{i=1}^{t}\sum_{e\in E(V(Q_i),S)} h(e) + \sum_{e\in E(\{v_1,\cdots,v_l\},S)} h(e) \le \sum_{e\in E(\overline{S},S)} h(e) \le k|S|-1.$$

□

Theorem 1. *For any complete graph K_n, $mp^k(K_n) = \begin{cases} n-2, & n = 3,5. \\ n-1, & \text{otherwise.}\end{cases}$*

Proof. Since K_3 has an almost perfect integer k-matching, $mp^k(K_3) \ge 1$. But $P_3 = K_3 - e$ has no almost perfect integer k-matching. Thus $mp^k(K_3) \le 1$. Hence $mp^k(K_3) = 1$.

Since $mp^k(K_4) \le \delta(K_4) = 3$ and $mp^k(K_4) \ge mp(K_4) = 3$ by Lemma 2.2, we have that $mp^k(K_4) = 3$.

Clearly, $K_5 - F$ has a Hamiltonian cycle for any $F \subseteq E(K_5)$ with that $|F| = 2$. So $K_5 - F$ has an almost perfect integer k-matching. Thus $mp^k(K_5) \ge 3$. Let $F = \{uv, uw, vw\} \subset E(K_5)$ and $S = \{x, y\} \subset V(K_5)$ (see Fig. 2). Then $odd(K_5 - F - S) + k \cdot i(K_5 - F - S) = 3k > 2k + 1$. So $K_5 - F$ has no almost perfect integer k-matching by Lemma 2.7. Thus $mp^k(K_5) \le 3$. Hence $mp^k(K_5) = 3$.

Let $n \ge 6$. It clear that $mp^k(K_n) \le \delta(K_n) = n - 1$. Let $F \subseteq E(K_n)$ such that $|F| = n - 2$. We prove that F is not MP_k set by the following cases.

Case 1. $\delta(K_n - F) \ge 2$. Then $K_n - F$ is Hamiltonian by Lemma 2.3. Thus $K_n - F$ has a perfect integer k matching or an almost perfect integer k-matching.

Case 2. $\delta(K_n - F) = 1$. Suppose that $d_{K_n-F}(v) = 1$ and $uv \in E(K_n - F)$. It follows that every edge in F is incident with v. Since $K_n - F - \{u, v\} = K_{n-2}$, we can construct a perfect integer k-matching or an almost perfect integer k-matching of $K_n - F$ which assigns k to uv.

Hence $mp_k(K_n) \ge n - 1$. Thus $mp_k(K_n) = n - 1$ for $n \ge 6$. □

Theorem 2. *For any complete graph K_n, $smp^k(K_n) = \begin{cases} n-2, & n = 3,4,5. \\ n-1, & \text{otherwise.}\end{cases}$*

Proof. Since K_3 has an almost perfect integer k-matching, $smp^k(K_3) \ge 1$. And $smp^k(K_3) \le mp^k(K_3) = 1$ by Theorem 1. Thus $smp^k(K_3) = 1$.

Let $v \in V(K_4)$ and $e \in E(K_4 - v)$. Then $K_4 - \{v, e\}$ has neither perfect integer k-matching nor almost perfect integer k-matching. Thus $smp^k(K_4) \le 2$. Since $K_4 - \{u\}$ has an almost perfect integer k-matching for any $u \in V(K_4)$ and $K_4 - \{e\}$ has a perfect integer k-matching for any $e \in E(K_4)$, $smp^k(K_4) \ge 2$. Thus $smp^k(K_4) = 2$.

Clearly, $K_5 - F$ is Hamiltonian for any $F \subset E(K_5) \cup V(K_5)$ with that $|F| = 2$. So $K_5 - F$ has a perfect integer k-matching or an almost perfect integer k-matching. Thus $smp^k(K_5) \ge 3$. And $smp^k(K_5) \le mp^k(K_5) = 3$ by Theorem 1. Hence $smp^k(K_5) = 3$.

Let $n \geq 6$ and $F \subseteq E(K_n) \cup V(K_n)$ such that $|F| = n-2$. Clearly $smp_k(K_n) \leq n-1$. We distinguish the following cases to prove that F is not SMP_k set.

Case 1. $\delta(K_n - F) \geq 2$. Then $K_n - F$ is Hamiltonian by Lemma 2.3. Thus $K_n - F$ has a perfect integer k-matching or an almost perfect integer k-matching.

Case 2. $\delta(K_n - F) = 1$. Then we can assume that $d_{K_n - F}(v) = 1$ and $uv \in E(K_n - F)$. It follows that every edge in F is incident with v and every vertex in F is adjacent to v. Since $K_n - F - \{u, v\}$ is a complete graph, we can construct a perfect integer k-matching or an almost perfect integer k-matching of $K_n - F$ which assigns k to uv.

Hence $smp_k(K_n) \geq n-1$. Thus $smp_k(K_n) = n-1$ for $n \geq 6$. □

2.2. MP^k and SMP^k numbers of bipartite graphs

Note that $A_{n,n-1}$ is a bipartite graph. A connected graph G is said to be an *odd-cycle-tree graph* [16] if every block of G is either an odd cycle or K_2, $\delta(G) \geq 2$ and no two odd cycles have a common vertex.

Lemma 2.8 ([16]). *Let k be an integer, h a maximum integer k-matching of a graph G with the maximum number of 0-edges and H the subgraph of G induced by the set of edges e with that $h(e) \neq 0$. Then every component of H is a single edge or an odd-cycle-tree graph.*

Let G be a bipartite graph and h a maximum integer k-matching of G with the maximum number of 0-edges. Then every component of H is a single edge by Lemma 2.8. So $E(H)$ is a matching. Then $\mu_k(G) \leq k\mu(G)$, where $\mu_k(G)$, named integer k-matching number of G, is the maximum value of $\sum\limits_{e \in E(G)} h(e)$ over all integer k-matching h. On the other hand, for a maximum matching M, we can obtain an integer k-matching by assigning k to each edge in M and 0 to other edges. So $\mu_k(G) \geq k\mu(G)$. Then we have the following corollary.

Corollary 2.9. *If G is a bipartite graph, then $\mu_k(G) = k\mu(G)$.*

By the above corollary, we have the following corollary.

Corollary 2.10. *Let G be a bipartite graph. Then G has a perfect integer k-matching if and only if G has a perfect matching.*

Obviously, if G is a bipartite graph with the odd number of vertices, then $\mu(G) \leq \frac{|V(G)|-1}{2}$. By Corollary 2.9, $\mu_k(G) = k\mu(G) \leq \frac{k|V(G)|-k}{2} < \frac{k|V(G)|-1}{2}$. So we get the following corollary.

Corollary 2.11. *Let G be a bipartite graph with the odd number of vertices. Then G has no almost perfect integer k-matching.*

Theorem 3. *Let G be a bipartite graph. Then $mp^k(G) = smp^k(G) = 0$ if G is odd. Otherwise, $mp^k(G) = mp(G)$ and $smp^k(G) \leq smp(G)$.*

Proof. Let G be odd. By Corollaries 2.10 and 2.11, G contains neither perfect integer k-matching nor almost-perfect integer k-matching. Thus $mp^k(G) = smp^k(G) = 0$. Let G be even. By Corollary 2.10, $mp^k(G) = mp(G)$. Let F be a minimum SMP set of G. Then $|F| = smp(G)$ and $G - F$ contains neither perfect matching nor almost perfect mathing. By Corollaries 2.9 and 2.10, $G - F$ contains no perfect integer k-matching and almost-perfect integer k-matching. So F is a strong integer k-matching preclusion set. Thus $smp^k(G) \leq smp(G)$. □

Note that the equivalence "$smp^k(G) = smp(G)$" is not true. For example, $smp^k(C_4) = 1$ and $smp(C_4) = 2$.

2.3. SMP^k numbers of $A_{n,s}$ $(2 \leq s \leq n-2)$

We investigate the SMP^k number of $A_{n,s}$ in two cases that $3 \leq s \leq n-2$ and $s=2$.

Theorem 4. *Let n and s be integers such that $3 \leq s \leq n-2$. Then $smp^k(A_{n,s}) = s(n-s)$.*

Proof. We have that $smp^k(A_{n,s}) \leq \delta(A_{n,s}) = s(n-s)$. Next we prove that $smp^k(A_{n,s}) \geq s(n-s)$. It suffices to prove that $A_{n,s} - F$ has a perfect integer k-matching or an almost perfect integer k-matching for any $F \subseteq V(A_{n,s}) \cup E(A_{n,s})$ with that $|F| = s(n-s)-1$. Let $F_i = F \cap (V(H_i) \cup E(H_i))$ and $G_i = H_i - F_i$ for any $i \in \{1,\ 2, \cdots, n\}$. Then $F - \cup_{i=1}^{n} F_i$ consists of cross edges in $A_{n,s}$. Without loss of generality, suppose that $|F_1| \geq |F_2| \geq \cdots \geq |F_n|$.

If $A_{n,s} - F$ is even, then $A_{n,s} - F$ has a perfect matching since $smp(A_{n,s}) = s(n-s)$ by Lemma 2.6. It follows that $A_{n,s} - F$ has a perfect integer k-matching. Suppose that $A_{n,s} - F$ is odd. Then the number of odd G_i among $G_1, G_2, \cdots, G_n$ is odd. We distinguish the following cases to prove that $A_{n,s} - F$ has an almost perfect integer k-matching.

Case 1. $|F_1| \leq (s-1)(n-s)-2$. Then $|F_i| \leq |F_1| \leq (s-1)(n-s)-2$ for any $i \geq 2$. By Lemma 2.4, G_i $(i \geq 1)$ is Hamiltonian. Then every even G_i has a perfect integer k-matching. If there exists exactly one odd subgraph G_i among $G_1, G_2, \cdots, G_n$, then G_i has an almost perfect integer k-matching. So $A_{n,s} - F$ has an almost perfect integer k-matching. Suppose that there exists at least three odd subgraphs. For any two odd subgraphs G_i and G_j, $|E_{A_{n,s}-F}(V(G_i), V(G_j))| \geq |E_{A_{n,s}}(V(H_i), V(H_j))| - |F| = \frac{(n-2)!}{(n-s-1)!} - (s(n-s)-1) \geq 1$ according to the property that the cross edges in $E_{A_{n,s}}(V(H_i), V(H_j))$ are independent. Then we can find an even Hamiltonian path of the subgraph induced by $V(G_i) \cup V(G_j)$ in $A_{n,s} - F$. Hence there exists a perfect integer k-matching of the subgraph induced by $V(G_i) \cup V(G_j)$. It follows that $A_{n,s} - F$ has an almost perfect integer k-matching.

Case 2. $|F_1| = (s-1)(n-s)-1$. Then $|F_i| \leq |F - F_1| = n-s \leq (s-1)(n-s)-2$. By Lemma 2.4, G_i $(i \geq 2)$ is Hamiltonian and G_1 has a Hamiltonian path, say P.
If G_1 is even, then G_1 has a perfect integer k-matching. For any two subgraphs G_i and G_j $(2 \leq i < j \leq n)$, $|E_{A_{n,s}-F}(V(G_i), V(G_j))| \geq |E_{A_{n,s}}(V(H_i), V(H_j))| - |F - F_1| = \frac{(n-2)!}{(n-s-1)!} - (n-s) \geq 1$. By the similar method in Case 1 as above, we can find an almost perfect integer k-matching of $A_{n,s} - F$.
Suppose that G_1 is odd. Let u and v be two end-vertices of P. Since $|F - F_1| = n-s$ and u and v have $2(n-s)$ outer neighbors in $A_{n,s}$, we can assume that u has an outer neighbor in $A_{n,s} - F$. Since u is adjacent to $n-s$ subgraphs and $|F - F_1| = n-s$, there is a subgraph G_t adjacent to u such that $|F_t| \leq 1$. Suppose that $w \in V(G_t)$ such that $uw \in E(A_{n,s} - F)$. Then the subgraph induced by $V(G_1) \cup \{w\}$ of $A_{n,s} - F$ has a perfect integer k-matching. Let $F'_t = F_t \cup \{w\}$. Then $|F'_t| \leq 2$. By Lemma 2.4, $G'_t = H_t - F'_t$ is Hamiltonian. Similarly, $|E_{A_{n,s}-F}(V(G_i), V(G_j))| \geq 1$ for any pair of subgraphs G_i and G_j among $G_2, \cdots, G'_t, \cdots, G_n$. Then we can obtain an almost perfect integer k-matching of $A_{n,s} - F$ by the similar method in Case 1 as above.

Case 3. $|F_1| = (s-1)(n-s)+\alpha$, where $0 \leq \alpha \leq n-s-1$. We can choose a subset S of F_1 such that $|S| = \alpha+1$ and $F_1 - S$ has even number of vertices. Let $F'_1 = F_1 - S$. Then $|F'_1| = (s-1)(n-s)-1$ and $H_1 - F'_1$ is even. By Lemma 2.6, $H_1 - F'_1$ has a perfect matching, say M. Let $A = S \cap V(H_1) = \{v_1, v_2, \cdots, v_t\}$, $B = S \cap E(H_1) = \{w_1s_1, w_2s_2, \cdots, w_rs_r\}$ and T be the set of all M-unsaturated vertices of G_1. Since $G_1 = H_1 - F_1 = (H_1 - F'_1) - S$, we can assume that $T = \{z_1, \cdots, z_{t'}, w_1, \cdots, w_{r'}, s_1, \cdots, s_{r'}\}$, where $v_1z_1, \cdots, v_{t'}z_{t'} \in M_1$. Then $0 \leq t' \leq t$, $0 \leq r' \leq r$. Thus $|T| = t' + 2r' \leq 2(\alpha+1)$.
Since $|F - F_1| = n-s-1-\alpha \leq n-s-1$ and each vertex has $n-s$ outer neighbors in $A_{n,s}$, each vertex in T has at least one outer neighbor in $A_{n,s} - F$. Let $T' = \{z'_1, \cdots, z'_{t'}, w'_1, \cdots, w'_{r'}, s'_1, \cdots, s'_{r'}\}$, where z'_i, w'_j and s'_j are the outer neighbor of z_i, w_j and s_j in T, respectively. Then $|T'| = |T| \leq 2(\alpha+1)$ and the subgraph induced by $V(G_1) \cup T'$ has a perfect matching. Thus the subgraph induced by $V(G_1) \cup T'$ has a perfect integer k-matching. Furthermore, in $A_{n,s} - F$, there exists at most one vertex in T adjacent only to G_2 since any pair of vertices of G_1 have not common outer neighbors and

$|F - F_1| \leq n - s - 1$. Thus $|T' \cap V(G_2)| \leq 1$. Let $F_i' = F_i \cup (T' \cap V(G_i))$ and $G_i' = H_i - F_i'$ for any $i \geq 2$.

Now we consider the case that $|T| \geq n - s - 1$ and each vertex in T has at least $n - s - 1$ outer neighbors in $A_{n,s} - F$. Then we can find $n - s - 1$ vertices in T, say $\{u_1, \cdots, u_{n-s-1}\}$, such that the outer neighbors $u_1', \cdots, u_{n-s-1}'$ of $u_1, \cdots, u_{n-s-1}$ belong to $n - s - 1$ distinct subgraphs among $G_2, \cdots, G_n$, respectively, and $|(\{u_1', \cdots, u_{n-s-1}'\} \cup T') \cap V(G_2)| \leq 1$. Replace the outer neighbors in T' of $\{u_1, \cdots, u_{n-s-1}\}$ with $\{u_1', \cdots, u_{n-s-1}'\}$, the resulting set is still denoted by T'. Hence there are at least $n - s - 1$ subgraphs G_i $(i \geq 2)$ such that $V(G_i) \cap T' \neq \emptyset$ and $|T' \cap V(G_2)| \leq 1$ and the subgraph induced by $V(G_1) \cup T'$ has also a perfect integer k-matching.

Claim 1. $|F_i'| \leq (s-1)(n-s) - 2$.

If $0 \leq \alpha \leq n - s - 3$ or $|T'| \leq 2\alpha$, then $|F_i'| = |F_i \cup (T' \cap V(G_i))| \leq |F - F_1| + |T'| \leq 2(n-s) - 2 \leq (s-1)(n-s) - 2$. Suppose that $n - s - 2 \leq \alpha \leq n - s - 1$ and $|T'| \geq 2\alpha + 1$. Then $|T| = |T'| \geq n - s - 1$, $|F_2| \leq |F - F_1| = n - s - 1 - \alpha \leq 1$ and $|F_i| = 0$ $(i \geq 3)$. Since $|T' \cap V(G_2)| \leq 1$, $|F_2'| = |F_2 \cup (T' \cap V(G_2))| = |F_2| + |T' \cap V(G_2)| \leq 2 \leq (s-1)(n-s) - 2$. Suppose that $i \geq 3$. Then $F_i' = T' \cap V(G_i)$. If $|T'| \leq 2$, then $|F_i'| = |T' \cap V(G_i)| \leq |T'| = 2 \leq (s-1)(n-s) - 2$. Suppose that $|T'| \geq 3$. Since $|F - F_1| \leq 1$ and each vertex of $A_{n,s}$ has $n - s$ outer neighbors, each vertex in T has at least $n - s - 1$ outer neighbors in $A_{n,s} - F$. In this case, there are at least $n - s - 1$ subgraphs H_i $(i \geq 2)$ such that $V(H_i) \cap T' \neq \emptyset$. Thus $|F_i'| = |T' \cap V(G_i)| \leq |T'| - (n - s - 2) \leq 2(\alpha + 1) - (n - s - 2) \leq 2(n-s) - 2 \leq (s-1)(n-s) - 2$. Thus the claim is true.

By the claim and Lemma 2.4, G_i' $(i \geq 2)$ is Hamiltonian. For any two subgraphs G_i' and G_j' $(2 \leq i < j \leq n)$, $|E_{A_{n,s}-F}(V(G_i'), V(G_j'))| \geq |E_{A_{n,s}}(V(H_i), V(H_j))| - |(F - F_1) \cup T'| \geq \frac{(n-2)!}{(n-s-1)!} - 2(n-s) \geq 1$. Hence we can obtain an almost perfect integer k-matching of the subgraph induced by $V(G_2') \cup \cdots \cup V(G_n')$ in $A_{n,s} - F$. Thus $A_{n,s} - F$ has an almost perfect integer k-matching. □

Next, we consider the case that $s = 2$. Then a vertex of $A_{n,2}$ is denoted by ij, where $i, j \in \{1, 2, \cdots, n\}$. For convenience, the notations H_i, F_i and G_i in the proof of Theorem 4 are used in the following. In this case, $H_i = K_{n-1}$. According to the definition of $A_{n,s}$, there are exactly $n - 2$ cross edges between H_i and H_j in $A_{n,2}$ which are independent. Each vertex in $A_{n,2}$ have $n - 2$ outer neighbors which belong to $n - 2$ different subgraphs. Moreover, any pair of vertices in $V(H_i)$ are adjacent to $n - 1$ subgraphs in $A_{n,2}$.

Theorem 5. *Let $n \geq 5$ be an integer. Then $smp^k(A_{n,2}) = 2n - 4$.*

Proof. Since $smp^k(A_{n,2}) \leq \delta(A_{n,2}) = 2n - 4$, we only need to prove that $smp^k(A_{n,2}) \geq 2n - 4$. It suffices to prove that $A_{n,2} - F$ has a perfect integer k-matching or an almost perfect integer k-matching for any $F \subseteq V(A_{n,2}) \cup E(A_{n,2})$ with that $|F| = 2n - 5$.

If $A_{n,2} - F$ is even, then $A_{n,2} - F$ has a perfect matching since $smp(A_{n,2}) = 2n - 4$ by Lemma 2.6. It follows that $A_{n,2} - F$ has a perfect integer k-matching. Suppose that $A_{n,2} - F$ is odd. Then the number of odd G_i among $G_1, G_2, \cdots, G_n$ is odd. We distinguish the following cases to prove that $A_{n,2} - F$ has an almost perfect integer k-matching.

Case 1. $|F_1| \leq n - 4$. Then $|F_i| \leq n - 4$ for any $i \geq 2$. By Lemma 2.1, G_i $(i \geq 1)$ is Hamiltonian. If there is exactly one odd subgraph among $G_1, G_2, \cdots, G_n$, then $A_{n,2} - F$ has an almost perfect integer k-matching clearly. Suppose that there exist at least three odd subgraphs among $G_1, G_2, \cdots, G_n$.

Claim 2. There exists at least one pair of subgraphs with a cross edge in $A_{n,2} - F$ among any three subgraphs G_i, G_j, G_k, where $1 \leq i < j < k \leq n$.

For any three subgraphs H_i, H_j, H_k, let

$$S_i = F \cap \{v \in V(H_i) |\ v \text{ is adjacent to both } H_j \text{ and } H_k\},$$

$$S_j = F \cap \{\, v \in V(H_j) |\ v \text{ is adjacent to both } H_i \text{ and } H_k,\ \text{but } v \text{ is not adjacent to } S_i\},$$

$$S_k = F \cap \{v \in V(H_k) |\ v \text{ is adjacent to both } H_i \text{ and } H_j,\ \text{but } v \text{ is adjacent to neither } S_i \text{ nor } S_j\}.$$

Suppose that $S_i = \{l_1 i, l_2 i, \cdots, l_p i\}$, where $\{l_1, l_2, \cdots, l_p\} \subseteq \{1, 2, \cdots, n\} - \{i, j, k\}$. Then

$$S_j \subseteq \{lj | l \in \{1, 2, \cdots, n\} - \{i, j, k\} - \{l_1, l_2, \cdots, l_p\}\}.$$

Let $S_j = \{l_{p+1} j, l_{p+2} j, \cdots, l_{p+p_1} j\}$. Then

$$S_k \subseteq \{lk | l \in \{1, 2, \cdots, n\} - \{i, j, k\} - \{l_1, l_2, \cdots, l_p\} - \{l_{p+1}, l_{p+2}, \cdots, l_{p+p_1}\}\}.$$

Thus $|S_i| + |S_j| + |S_k| \leq n - 3$. Let $m = |S_i| + |S_j| + |S_k|$. Then there are $3(n-2) - 2m$ cross edges among $H_i - S_i,\ H_j - S_j,\ H_k - S_k$. And each vertex in $(F_i - S_i) \cup (F_j - S_j) \cup (F_k - S_k)$ is adjacent to exactly one subgraph among $H_i - S_i,\ H_j - S_j,\ H_k - S_k$. Let $S = S_i \cup S_j \cup S_k$. Then deleting an element in $F - S$ destroys at most one cross edge among $H_i - S_i,\ H_j - S_j,\ H_k - S_k$. Since $|F - S| = 2n - 5 - m$, there are at most $2n - 5 - m$ cross edges among $H_i - S_i,\ H_j - S_j, H_k - S_k$ destroyed after deleting all elements in $F - S$. Since $(3(n-2) - 2m) - (2n - 5 - m) = n - m - 1 \geq n - (n-3) - 1 = 2$, Claim 1 is true.

By Claim 1, we can obtain an almost perfect integer k-matching of $A_{n,2} - F$ since each G_i $(i \geq 1)$ is Hamiltonian.

Case 2. $|F_1| = n - 3$. Then $|F_i| \leq n - 3$ for any $i \geq 2$, $|F - F_1| = n - 2$ and G_1 has a Hamiltonian path, say P. Let x_1 and y_1 be two end-vertices of P.

Firstly, suppose that $|F_2| \leq n - 4$. By Lemma 2.1, $G_i (i \geq 2)$ is Hamiltonian. If G_1 is even, then $A_{n,2} - F$ has an almost perfect integer k-matching since there exists at most a pair of subgraphs among $G_2, G_3, \cdots, G_n$ without cross edges. Suppose that G_1 is odd. Since $|F - F_1| = n - 2$ and x_1 and y_1 are adjacent to $n-1$ subgraphs $H_2, H_3, \cdots, H_n$, without loss of generality, we can assume that $w \in V(H_n)$ such that $x_1 w \in E(A_{n,2} - F)$ and $|F_n| = 0$. Then the subgraph induced by $V(G_1) \cup \{w\}$ has a perfect integer k-matching and $G_n = H_n - \{w\}$ is Hamiltonian by Lemma 2.1. Since $|(F - F_1) \cup \{w\}| = n - 1$ and $|E_{A_{n,2}}(V(H_i), V(H_j))| = n-2$ $(2 \leq i < j \leq n)$, there exists at most a pair of subgraphs in $A_{n,2} - F$ among $G_2, G_3, \cdots, G_n$ without cross edges. Thus we can obtain an almost perfect integer k-matching of $A_{n,2} - F$.

Secondly, suppose that $|F_2| = n - 3$. Then $|F_3| \leq 1$ and $|F_i| = 0 (i \geq 4)$. By Lemma 2.1, G_i $(i \geq 3)$ is Hamiltonian and G_2 has a Hamiltonian path, say Q. Let x_2 and y_2 be two end-vertices of Q. Since $|F - F_1 - F_2| = 1$ and $|F_i| = 0$ $(i \geq 4)$, without loss of generality, we can assume that x_1 is adjacent to H_4 and x_2 is adjacent to H_5 in $A_{n,2} - F$. Let $x_1 w_1,\ x_2 w_2 \in E(A_{n,2} - F)$, where $w_1 \in V(H_4)$ and $w_2 \in V(H_5)$. Let M_1 and M_2 be the maximum matching of P and Q such that the M_1-unsaturated and M_2-unsaturated vertices are x_1 and x_2 (if they exist), respectively. Let $X_1 = \{w_1\}$ if G_1 is odd and $X_2 = \{w_2\}$ if G_2 is odd. Otherwise, let $X_1 = \emptyset$ and $X_2 = \emptyset$. Thus the subgraph induced by $V(G_1) \cup X_1$ and $V(G_2) \cup X_2$ have a perfect matching, respectively. By Lemma 2.1, $G_4 = H_4 - X_1$ and $G_5 = H_5 - X_2$ are Hamiltonian. Since $|F - F_1 - F_2| = 1$ and $|\{X_1 \cup X_2\}| \leq 2$, each pair of subgraphs among $G_3, G_4, \cdots, G_n$ has at least one cross edge. Thus we can obtain an almost perfect integer k-matching of $A_{n,2} - F$.

Case 3. $n - 1 \leq |F_1| \leq 2n - 5$. Then $|F - F_1| \leq n - 4$. Let M be a maximum matching of G_1 and X the set of M-unsaturated vertices of G_1. Since $H_1 \cong K_{n-1}$ and $n \geq 5$, $|X| \leq n - 2$. Let $X = \{v_1, v_2, \cdots, v_l\}$, where $1 \leq l \leq n - 2$. Let $G_0 = (X, Y)$ be a bipartite graph with that $Y = \{G_3, G_4, \cdots, G_n\}$ and $v_i G_j \in E(G_0)$ if and only if v_i is adjacent to G_j $(j \geq 3)$ in $A_{n,2} - F$.

Claim 3. G_0 contains a matching that saturates every vertex in X.

By the Hall's theorem, we only need to prove that $|N_{G_0}(S)| \geq |S|$ for any $S \subseteq X$. Since $|F - F_1| \leq n-4$ and each vertex has $n - 2$ outer neighbors in $A_{n,2}$, each vertex in X has at least one outer neighbor in $A_{n,2} - F - G_2$. Thus $|N_{G_0}(S)| \geq 1 = |S|$ for any S with that $|S| = 1$. Since each pair of vertices in $V(A_{n,2})$ are adjacent to $n-1$ different subgraphs H_i and $|F - F_1| \leq n-4$, each pair of vertices in X are adjacent to at least $(n-1)-1-(n-4)$ subgraphs among $G_3, G_4, \cdots, G_n$. So $|N_{G_0}(S)| \geq 2 = |S|$ for any S with that $|S| = 2$. Let $3 \leq |S| \leq n-3$. Since S has at least $|S|-1$ outer neighbors in $V(H_i)$ for each $i \geq 3$, there are at most $\frac{n-4}{|S|-1}$ subgraphs G_i $(i \geq 3)$ not adjacent to S in $A_{n,2} - F$. It follows that $|N_{G_0}(S)| \geq (n-2) - \frac{n-4}{|S|-1} \geq |S|$. Suppose that $|S| = n-2$. Since $|E_{A_{n,2}}(S, V(H_i))| \geq n-3$ for each $i \geq 3$ and $|F - F_1| \leq n-4$, each subgraph $G_i \in Y$ is adjacent to at least one vertex in X of $A_{n,2} - F$. Then $|N_{G_0}(S)| = |Y| = n-2 = |S|$. Hence Claim 2 is true.

By Claim 2, we can assume that $v_1', v_2', \cdots, v_l'$ are the outer neighbor of $v_1, v_2, \cdots, v_l$, respectively, such that $v_1', v_2', \cdots, v_l'$ belong to l distinct subgraphs among $G_3, G_4, \cdots, G_n$. Thus the subgraph induced by $V(G_1) \cup \{v_1', v_2', \cdots, v_l'\}$ has a perfect integer k-matching. Let $F_i' = F_i \cup (V(G_i) \cap \{v_1', v_2', \cdots, v_l'\})$ and $G_i' = H_i - F_i'$, where $i \geq 2$. Then $F_2' = F_2$ and $|F_i'| \leq |F_2| + 1$ for any $i \geq 3$.

If $|F_2| = 0$, then $|F_i| = 0$ for any $i \geq 3$. Thus $|F_i'| \leq 1$ $(i \geq 3)$. By Lemma 2.1, G_i' $(i \geq 2)$ is Hamiltonian. Hence $A_{n,2} - F$ has an almost perfect integer k-matching. Suppose that $|F_2| \geq 1$. Then $|F_i| \leq |F - F_1 - F_2| \leq n-4-1 = n-5$ for any $i \geq 3$. So $|F_i'| \leq n-5+1 = n-4$. Since $|F_2'| = |F_2| \leq |F - F_1| \leq n-4$, $G_i'(i \geq 2)$ is Hamiltonian by Lemma 2.1. Since $|F - F_1| \leq n-4$ and $|E_{A_{n,2}}(V(H_i), V(H_j))| = n-2$, there are at most one pair of subgraphs among $G_2', G_3', \cdots, G_n'$ without cross edges. Thus the subgraph induced by $V(G_2') \cup V(G_3') \cup \cdots \cup V(G_n')$ in $A_{n,2} - F$ has an almost perfect integer k-matching. Hence $A_{n,2} - F$ has an almost perfect integer k-matching.

Case 4. $|F_1| = n-2$. Let M be a maximum matching of G_1 and X the set of M-unsaturated vertices of G_1. Then $|X| \leq n-3$ since $H_1 \cong K_{n-1}$ and $|F_1| = n-2$. Let $X = \{v_1, v_2, \cdots, v_t\}$, where $1 \leq t \leq n-3$. Let $G_0 = (X, Y)$ be a bipartite graph with that $Y = \{G_2, G_3, \cdots, G_n\}$ and $v_iG_j \in E(G_0)$ if and only if v_i is adjacent to G_j in $A_{n,2} - F$.

Claim 4. G_0 contains a matching that saturates every vertex in X.

Similar to the method of the proof of Claim 2, we can prove that Claim 3 is true.

By Claim 3, we can assume that $v_1', v_2', \cdots, v_t'$ are the outer neighbor of $v_1, v_2, \cdots, v_t$, respectively, such that $v_1', v_2', \cdots, v_t'$ belong to t distinct subgraphs among $G_2, G_3, \cdots, G_n$. Thus the subgraph induced by $V(G_1) \cup \{v_1', v_2', \cdots, v_t'\}$ has a perfect integer k-matching. Let $F_i' = F_i \cup (V(G_i) \cap \{v_1', v_2', \cdots, v_t'\})$ and $G_i' = H_i - F_i'$, where $i \geq 2$. Then $F_i' = F_i$ or $F_i' = F_i \cup \{v_j'\}$. Note that $|F_2| \leq |F - F_1| = n-3$.

Subcase 4.1. $|F_2| \leq n-5$. Then $|F_i| \leq n-5$ $(i \geq 3)$. Thus $|F_i'| \leq n-4$ for any $i \geq 2$. By Lemma 2.1, $G_i'(i \geq 2)$ is Hamiltonian. Since $|F - F_1| = n-3$, there exist at most two pair of subgraphs among $G_2', G_3', \cdots, G_n'$ without cross edges. Hence the subgraph induced by $V(G_2') \cup \cdots \cup V(G_n')$ in $A_{n,2} - F$ has an almost perfect integer k-matching. Thus $A_{n,2} - F$ has an almost perfect integer k-matching.

Subcase 4.2. $|F_2| = n-3$. Then $F = F_1 \cup F_2$. Thus $|F_2'| \leq n-2$ and $|F_i'| \leq 1$ $(i \geq 3)$. By Lemma 2.1, G_i' $(i \geq 3)$ is Hamiltonian and G_2' contains a spanning subgraph which consist of two paths, say P and Q, respectively. Let x_1 and y_1 are the two end-vertices of P and x_2 and y_2 are the two end-vertices of Q. Since $t \leq n-3$, there is a subgraph among $H_3, H_4, \cdots, H_n$, say H_3 such that $V(H_3) \cap \{v_1', v_2', \cdots, v_t'\} = \emptyset$. Since $F - F_1 - F_2 = \emptyset$, $G_3' = H_3$. Since $\{x_1, y_1, x_2, y_2\} \subseteq V(H_2)$, there is at most one in $\{x_1, y_1, x_2, y_2\}$ not adjacent to H_3. Without loss of generality, we can assume that $w_1, w_2 \in V(H_3)$ are the outer neighbor of x_1 and x_2, respectively. Let M be a maximum matching of G_2' and X the set of M-unsaturated vertices of G_2' such that $X \subseteq \{x_1, x_2\}$ (it is possible that $X = \emptyset$). Let X' be the outer neighbor of X such that $X' \subseteq \{w_1, w_2\}$. Then the subgraph induced by $V(G_2') \cup X'$ has a perfect matching and $H_3 - X' = K_2$ (in this case, $n = 5$) or $H_3 - X'$ is Hamiltonian by Lemma 2.1. Clearly, each pair of subgraphs among $G_4', G_5', \cdots, G_n'$ has cross edges. Hence we can obtain an almost perfect integer k-matching of $A_{n,2} - F$.

Subcase 4.3. $|F_2| = n-4$. Then $|F_3| \leq 1$ and $|F_i| = 0$ $(i \geq 4)$. Suppose that $|F_2'| = n-3$. Then we can assume that $v_1' \in V(G_2)$. By Lemma 2.1, G_2' has a Hamiltonian path, say P. Let x and y be two end-vertices of P. If P is even, then $A_{n,2}-F$ has an almost perfect matching. Suppose that P is odd. Since x and y are adjacent to $n-2$ subgraphs $H_3, H_4, \cdots, H_n$ and $|\{v'_2, v'_3, \cdots, v'_t\}| \leq n-4$ and $|F-F_1-F_2| = 1$, there is a subgraph, say G'_i, adjacent to x or y in $A_{n,2}-F-\{v'_2, v'_3, \cdots, v'_t\}$ such that $|F_i| = 0$. Suppose that $w \in V(G'_i)$ and w is an outer neighbor of x or y in $A_{n,2} - F - \{v'_2, v'_3, \cdots, v'_t\}$. Then the subgraph induced by $V(G'_2) \cup \{w\}$ has a perfect matching. By Lemma 2.1, $G'_i - w = K_2$ (in this case, $n = 5$ and $|F'_i| = 1$) or $G'_i - w$ is Hamiltonian and $G'_j(j \geq 4, j \neq i)$ is Hamiltonian. If $i = 3$, then $G'_3 - w = K_2$ or $G'_3 - w$ is Hamiltonian and $G'_j(j \geq 4)$ is Hamiltonian. Thus we can obtain an almost perfect integer k-matching of $A_{n,2}-F$.

Now we consider the case that $i \neq 3$. For G'_3, we have that $|F'_3| \leq |F_3| + 1 \leq 2$. If $|F'_3| = 1$ or $n \geq 6$, then G'_3 is Hamiltonian by Lemma 2.1. Suppose that $|F'_3| = 2$ and $n = 5$. Then $G'_3 = K_2$ (in this case $F'_3 \subset V(H_3)$) or $G'_3 = K_3 - e$ (in this case, F'_3 consist of one edge and one vertex). When G'_3 is Hamiltonian or $G'_3 = K_2$, we can obtain an almost perfect integer k-matching of $A_{n,2} - F$ since $G'_j(j \geq 4, j \neq i)$ is Hamiltonian and there are at most two pair of subgraphs among $G'_3, \cdots, G'_i - w, \cdots, G'_n$ without cross edges. Suppose that $G'_3 = K_3 - e$. Then $n = 5$, $t = 2$, $|F'_3| = 2$ and $F = F_1 \cup F_2 \cup F_3$. Thus we can assume that $v'_2 \in V(G_3)$. Let u and v be two end-vertices of G'_3. W.L.O.G., we can assume that $i = 4$. Then u or v has an outer neighbor $z \in V(H_5)$. Thus the subgraph induced by $V(G'_3) \cup \{z\}$ has a perfect matching and $H_5 - z$ is Hamiltonian by Lemma 2.1. Hence $A_{n,2} - F$ has an almost perfect integer k-matching.

Finally, we consider the case that $|F'_2| = |F_2| = n - 4$. Then every G'_j $(j \neq 1, 3)$ is Hamiltonian. By the same discussion as above, $G'_3 = K_2$ or $G'_3 = K_3 - e$ or G'_3 is Hamiltonian. And then we obtain an almost perfect integer k-matching of $A_{n,2} - F$. □

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