RAIRO-Oper. Res. 58 (2024) 5237–5254
https://doi.org/10.1051/ro/2024210 **RAIRO Operations Research**
www.rairo-ro.org

A KIND OF MATCHINGS EXTEND TO HAMILTONIAN CYCLES IN HYPERCUBES

Shujia Wang and Fan Wang*

Abstract. Ruskey and Savage asked the following question: Does every matching in Q_n for $n \geq 2$ extend to a Hamiltonian cycle of Q_n? Kreweras conjectured that every perfect matching of Q_n for $n \geq 2$ can be extended to a Hamiltonian cycle of Q_n. Fink confirmed the conjecture. An edge in Q_n is an edge of direction i if its endpoints differ in the ith position. So all the edges of Q_n can be divided into n directions, *i.e.*, edges of direction $1, \ldots,$ edges of direction n. The set of all edges of direction i of Q_n is denoted by E_i. In this paper, we obtain the following result. For $n \geq 6$, let M be a matching in Q_n with $|M| < 10 \times 2^{n-5}$. If M contains edges in at most 5 directions, then M can be extended to a Hamiltonian cycle of Q_n.

Mathematics Subject Classification. 05C38, 05C45.

Received September 11, 2023. Accepted October 27, 2024.

1. Introduction

The *n-dimensional hypercube*, denoted by Q_n, is a graph whose vertex set consists of all binary strings of length n, *i.e.*, $V(Q_n) = \{u : u = u^1 \cdots u^n \text{ and } u^i \in \{0,1\} \text{ for every } i \in \{1,\ldots,n\}\}$, with two vertices being adjacent whenever the corresponding strings differ in just one position.

The hypercube Q_n is one of the most popular and effective interconnection networks. It is well known that Q_n is Hamiltonian for every $n \geq 2$. This statement dates back to 1872 [8]. Since then, the research on Hamiltonian cycles in hypercubes satisfying certain additional properties has received considerable attention [1–3, 7].

A set of edges in a graph G is called a *matching* if no two edges have an end-vertex in common. A matching of G is *perfect* if it covers all vertices of G and a matching is *maximal* if no matching with larger size contains it. A vertex u is called *covered* by a matching M if $u \in V(M)$. Otherwise, u is called *uncovered* by M.

Ruskey and Savage [10] asked the following question: For $n \geq 2$, does every matching in Q_n extend to a Hamiltonian cycle of Q_n? Kreweras [9] conjectured for $n \geq 2$ that every perfect matching in Q_n extends to a Hamiltonian cycle of Q_n. Fink [5, 6] solved Kreweras' conjecture by proving the following stronger result. Let K_{Q_n} be the complete graph on the vertices of Q_n. Note that Q_n is a spanning subgraph of K_{Q_n}.

Theorem 1.1 ([5, 6]). *For every perfect matching M in K_{Q_n}, $n \geq 2$, there exists a perfect matching F in Q_n such that $M \cup F$ forms a Hamiltonian cycle in K_{Q_n}.*

Keywords. Hypercube, Hamiltonian cycle, matching.

School of Mathematics and Computer Sciences, Nanchang University, Nanchang, Jiangxi 330000, P.R. China.
*Corresponding author: `wangfan@ncu.edu.cn`

Fink [5] pointed out that Ruskey–Savage problem is true for $n = 2, 3, 4$, and Wang [11] provided a complete proof. Also Wang [14] proved that Ruskey–Savage problem is true for $n = 5$.

Theorem 1.2 ([5,11,14])**.** *Every matching in Q_n extends to a Hamiltonian cycle of Q_n for $n \in \{2, 3, 4, 5\}$.*

A forest is *linear*, if each component of it is a path. Dvořák [3] investigated the problem of extending linear forests to Hamiltonian cycles in hypercubes, and obtained the following results [3].

Theorem 1.3 ([3])**.** *For $n \geq 2$, let $F \subseteq E(Q_n)$ such that $|F| \leq 2n - 3$. If F is a linear forest, then there exists a Hamiltonian cycle of Q_n passing through F.*

Theorem 1.4 ([3])**.** *For $n \geq 2$, let $x, y, r, t \in V(Q_n)$ be pairwise distinct vertices such that $p(x) \neq p(r)$ and $p(y) \neq p(t)$. Then (i) there exists a spanning 2-path $P_{xr} + P_{yt}$ in Q_n; (ii) moreover, in the case when $d(x, r) = 1$, path P_{xr} can be chosen such that $P_{xr} = xr$, unless $n = 3$, $d(y, t) = 1$ and $d(xr, yt) = 2$.*

Wang [13] improved Dvořák's result and proved for $n \geq 2$ that every matching of at most $3n - 10$ edges in Q_n extends to a Hamiltonian cycle of Q_n. Dvořák [4] proved for $n \geq 2$ that every matching of at most $\frac{n^2}{16} + \frac{n}{4}$ edges in Q_n extends to a Hamiltonian cycle of Q_n.

An edge in Q_n is an edge of direction i if its endpoints differ in the ith position. So all the edges of Q_n can be divided into n directions, *i.e.*, edges of direction $1, \ldots$, edges of direction n. The set of all edges of direction i of Q_n is denoted by E_i.

In this paper, we obtain the following result. For $n \geq 6$, let M be a matching in Q_n with $|M| < 10 \times 2^{n-5}$. If M contains edges in at most 5 directions, then M can be extended to a Hamiltonian cycle of Q_n.

2. Basic definitions and results

Let H and H' be two subgraphs of G. We use $H + H'$ to denote the graph with the vertex set $V(H) \cup V(H')$ and edge set $E(H) \cup E(H')$. For $F \subseteq E(G)$, let $V(F)$ denote the set of vertices incident with F. We use $H + F$ to denote the graph with the vertex set $V(H) \cup V(F)$ and edge set $E(H) \cup F$. For a set $F \subseteq E(G)$, let $G - F$ denote the resulting graph after removing all edges in F from G. For a set $S \subseteq V(G)$, let $G - S$ denote the graph removing all vertices in S and all the edges incident with S from G. When $S = \{x\}$ and $F = \{e\}$, we simply write $G - S$, $G - F$, $H + F$ and $V(F)$ as $G - x$, $G - e$, $H + e$ and $V(e)$. For a set $E' \subseteq E(G)$, a subgraph H of G *passes through* E' if $E' \subseteq E(H)$.

Let $[n]$ denote the set $\{1, 2, \ldots, n\}$. For any given $j \in [n]$, let $Q_{n-1}^{0,j}$ and $Q_{n-1}^{1,j}$ with the superscript j being omitted when the context is clear, be two $(n-1)$-dimensional subcubes of Q_n induced by all the vertices with the jth positions being 0 or 1, respectively. Clearly, $Q_n - E_j = Q_{n-1}^0 + Q_{n-1}^1$, we say that Q_n is split into two $(n-1)$-dimensional subcubes Q_{n-1}^0 and Q_{n-1}^1 by E_j. For $\delta \in \{0, 1\}$, any vertex $u_\delta \in V(Q_{n-1}^\delta)$ has in $Q_{n-1}^{1-\delta}$ a unique neighbor, denoted by $u_{1-\delta}$. For $M \subseteq E(Q_n)$, let $M_\delta = M \cap E(Q_{(n-1)}^\delta)$.

The *parity* $p(u)$ of a vertex u in Q_n is defined by $p(u) = \sum_{i=1}^n u^i (\text{mod } 2)$. Then there are 2^{n-1} vertices with parity 0 and 2^{n-1} vertices with parity 1 in Q_n. Vertices with parity 0 and 1 are called black vertices and white vertices, respectively. We observe that if vertex u is adjacent to vertex v in Q_n, then $p(u) \neq p(v)$. Consequently, $p(u) \neq p(v)$ if and only if $d(u, v)$ is odd. Hence Q_n is bipartite and vertices of each parity form bipartite sets of Q_n.

A *u, v-path* is a path with endpoints u and v, denoted by $P_{u,v}$ when we specify a particular such path. Let $P = v_1 v_2 \cdots v_i \cdots v_j \cdots v_k$ be a path, we use $P[v_i, v_j]$ to denote the subpath $v_i \cdots v_j$ of P from v_i to v_j. We say that a spanning subgraph of G whose components are k disjoint paths is a *spanning k-path* of G. A spanning 1-path thus is simply a spanning or Hamiltonian path. We use the notation *a spanning* $(\{u, v\}, \{x, y\})$*-path* $P_{u,*} + P_{v,*}$ to denote that there exists a spanning 2-path $P_{u,x} + P_{v,y}$ or $P_{u,y} + P_{v,x}$ in Q_n.

Let u, v be two distinct vertices in Q_n. If $d(u, v) = 1$, then uv is an edge in Q_n. We call such an edge a *short edge*. If $d(u, v) > 1$, then uv is not an edge in Q_n, but an edge in K_{Q_n}. We call such uv a *long edge*. In Q_n, for simplicity, even when $d(u, v) > 1$, we still use the notation uv, and in this case, it refers to a long edge.

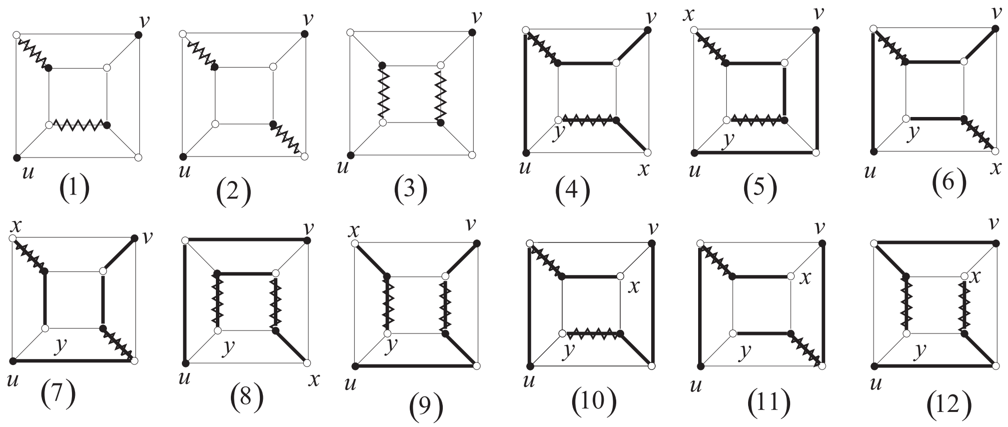

FIGURE 1. Illustration for the proof of Lemma 2.4 with the edges of M curved and the edges of $P_{u,*} + P_{y,*}$ bold.

The following content is devoted to auxiliary results preparing the necessary technique for the constructive proof of the main theorem.

Lemma 2.1 ([12]). *For $n \in \{3, 4\}$, let u, v be two vertices in Q_n with $p(u) \neq p(v)$. If M is a matching in $Q_n - u$, then there exists a Hamiltonian path of Q_n joining u and v passing through M.*

Lemma 2.2 ([12]). *Let u, v be two vertices at distance 2 in Q_3 and let x, y be two distinct vertices in Q_3 such that $d(u, x) = d(v, y) = 1$. If M is a matching in $Q_3 - u - v$, then there exists a spanning 2-path $P_{u,x} + P_{v,y}$ of Q_3 passing through M.*

Lemma 2.3. *Let u, v, x, y be pairwise distinct vertices in Q_3 with $p(u) = p(v) \neq p(x) = p(y)$. If M is a matching in $Q_3 - u - v$, then there exists a spanning $(\{u, v\}, \{x, y\})$-path $P_{u,*} + P_{v,*}$ of Q_3 passing through M.*

Proof. In Q_3, since $p(u) = p(v) \neq p(x) = p(y)$ and u, v, x, y be pairwise distinct vertices, we have $d(u, x) = d(v, y) = 1$ or $d(u, y) = d(v, x) = 1$. By Lemma 2.2 there exists a spanning $(\{u, v\}, \{x, y\})$-path $P_{u,*} + P_{v,*}$ of Q_3 passing through M. □

In the following Lemmas 2.4, 2.7 and 2.8, by the vertex-transitivity of Q_3, we may assume $u = 000$. Then u is a black vertex.

Lemma 2.4. *Let u, v, x, y be pairwise distinct vertices in Q_3 with $p(u) = p(v) \neq p(x) = p(y)$. If M is a matching in $Q_3 - u - v$, then there exists a spanning $(\{u, y\}, \{v, x\})$-path $P_{u,*} + P_{y,*}$ of Q_3 passing through M.*

Proof. Since $p(u) = p(v)$, there is one possibility of (u, v) up to isomorphism (see Fig. 1(1)). It suffices to consider the case that M is a maximal matching in $Q_3 - u - v$. There are three possibilities of (M, u, v) up to isomorphism (see Figs. 1(1)–1(3)).

Case 1. $d(u, x) = d(v, y) = 1$. Since M is a matching in $Q_3 - u - v$ and $p(u) = p(v) \neq p(x) = p(y)$, by Lemma 2.2 there exists a spanning 2-path $P_{u,x} + P_{y,v}$ of Q_3 passing through M.

Case 2. $d(u, x) = 1$ and $d(v, y) = 3$. By examining all possibilities of (M, x, y) up to isomorphism, there exists a spanning 2-path $P_{u,v} + P_{y,x}$ of Q_3 passing through M (see Figs. 1(4)–1(9)).

Case 3. $d(u, x) = 3$ and $d(v, y) = 1$. By Case 2, there exists a spanning 2-path $P_{v,u} + P_{x,y}$ of Q_3 passing through M.

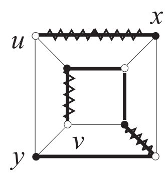

FIGURE 2. Illustration for the proof of Subcase 2.2 in Lemma 2.5 with the edges of M curved and the edges of $P_{u,x} + P_{v,y}$ bold.

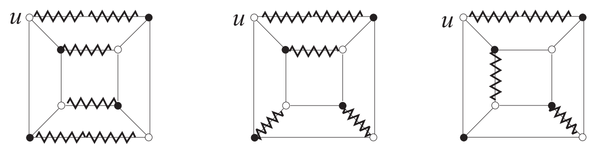

FIGURE 3. All possibilities of (M, u) up to isomorphism in Lemma 2.6.

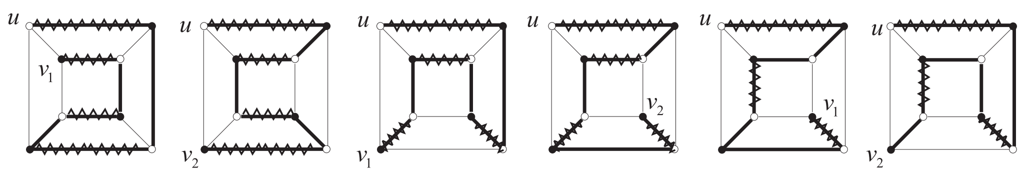

FIGURE 4. Illustration for the proof of Lemma 2.6 with the edges of M curved and the edges of P_{u,v_1} and P_{u,v_2} bold.

Case 4. $d(u, x) = d(v, y) = 3$. By examining all possibilities of (M, x, y) up to isomorphism, there exists a spanning 2-path $P_{u,x} + P_{y,v}$ of Q_3 passing through M (see Figs. 1(10)–1(12)).

□

Lemma 2.5. *Let u, v be two distinct vertices in Q_3 with $p(u) = p(v)$. If M is a matching in Q_3, then there exists a spanning 2-path $P_{u,x}+P_{v,y}$ of Q_3 passing through M, where x,y are two distinct vertices in Q_3 satisfying $p(x) = p(y) \neq p(u)$.*

Proof. It suffices to consider the case that M is a maximal matching in Q_3.

Case 1. M is perfect.

By Theorem 1.1 there exists a Hamiltonian cycle $C : u = v_0, \ldots, v_i = v, v_{i+1}, \ldots, v_8 = u$ of Q_3 containing M. Since M is perfect, the edges of C alternate between M and $C \setminus M$. Since $p(u) = p(v)$, we have $\{uv_1, vv_{i+1}\} \subseteq M$ or $\{v_7u, v_{i-1}v\} \subseteq M$. Without loss of generality, we may assume $\{uv_1, vv_{i+1}\} \subseteq M$. Set $P_{u,x} = v_0, v_1, \ldots, v_{i-1}$ and $P_{v,y} = v_i, v_{i+1}, \ldots, v_7$. Thus the conclusion holds.

Case 2. M is not perfect.

Subcase 2.1. $u \notin V(M)$ (or $v \notin V(M)$).

By Theorem 1.2 there exists a Hamiltonian cycle $C : u = v_0, \ldots, v_i = v, v_{i+1}, \ldots, v_8 = u$ of Q_3 containing M. Since $u \notin V(M)$, we have $\{uv_1, uv_7\} \cap M = \emptyset$. Note that $vv_{i+1} \notin M$ or $v_{i-1}v \notin M$. Without loss of generality, we may assume $v_{i-1}v \notin M$. Set $P_{u,x} = v_0, v_1, \ldots, v_{i-1}$ and $P_{v,y} = v_i, v_{i+1}, \ldots, v_7$. Thus the conclusion holds.

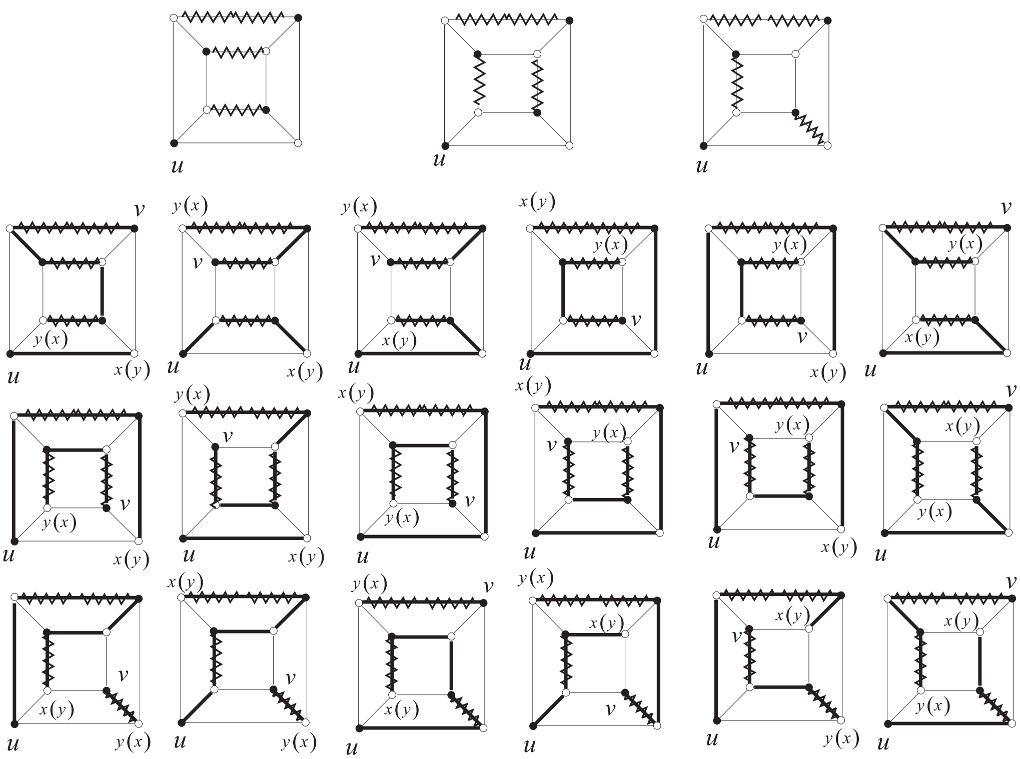

FIGURE 5. Illustration for the proof of Lemma 2.7 with the edges of M curved and the edges of $P_{u,*} + P_{v,*}$ bold.

Subcase 2.2. $\{u, v\} \subseteq V(M)$.

Since M is maximal but not perfect, there is one possibility of M up to isomorphism (see Fig. 2). Since $\{u, v\} \subseteq V(M)$ and $p(u) = p(v)$, there is one possibility of (u, v) up to isomorphism. Thus, the conclusion holds (see Fig. 2).

□

Lemma 2.6. *Let $u \in V(Q_3)$. If M is a matching in Q_3, then there are two Hamiltonian paths of Q_3 joining u with two different vertices and passing both throngh M.*

Proof. If $u \notin V(M)$, then by Lemma 2.1 there are two Hamiltonian paths of Q_3 joining u with two different vertices and passing both throngh M. If $u \in V(M)$, then it suffices to consider the case that M is a maximal matching in Q_3. There are three possibilities of (M, u) up to isomorphism (see Fig. 3). By examining all possibilities of (M, u) up to isomorphism, one can verify that the lemma holds (see Fig. 4). □

Lemma 2.7. *Let u,x,y be three distinct vertices in Q_3 with $p(u) \neq p(x) = p(y)$. If M is a matching in $Q_3 - u$, then there exists a spanning $(\{u, v\}, \{x, y\})$-path $P_{u,*} + P_{v,*}$ of Q_3 passing through M, where v is a vertex in Q_3 satisfying $p(v) = p(u)$.*

Proof. It suffices to consider the case that M is a maximal matching in $Q_3 - u$. There are three non-isomorphic maximal matchings in $Q_3 - u$ (see Fig. 5). Since $p(u) \neq p(x) = p(y)$, we have x, y are two white vertices. By examining all possibilities of (M, u, x, y) up to isomorphism, one can verify that the lemma holds (see Fig. 5). □

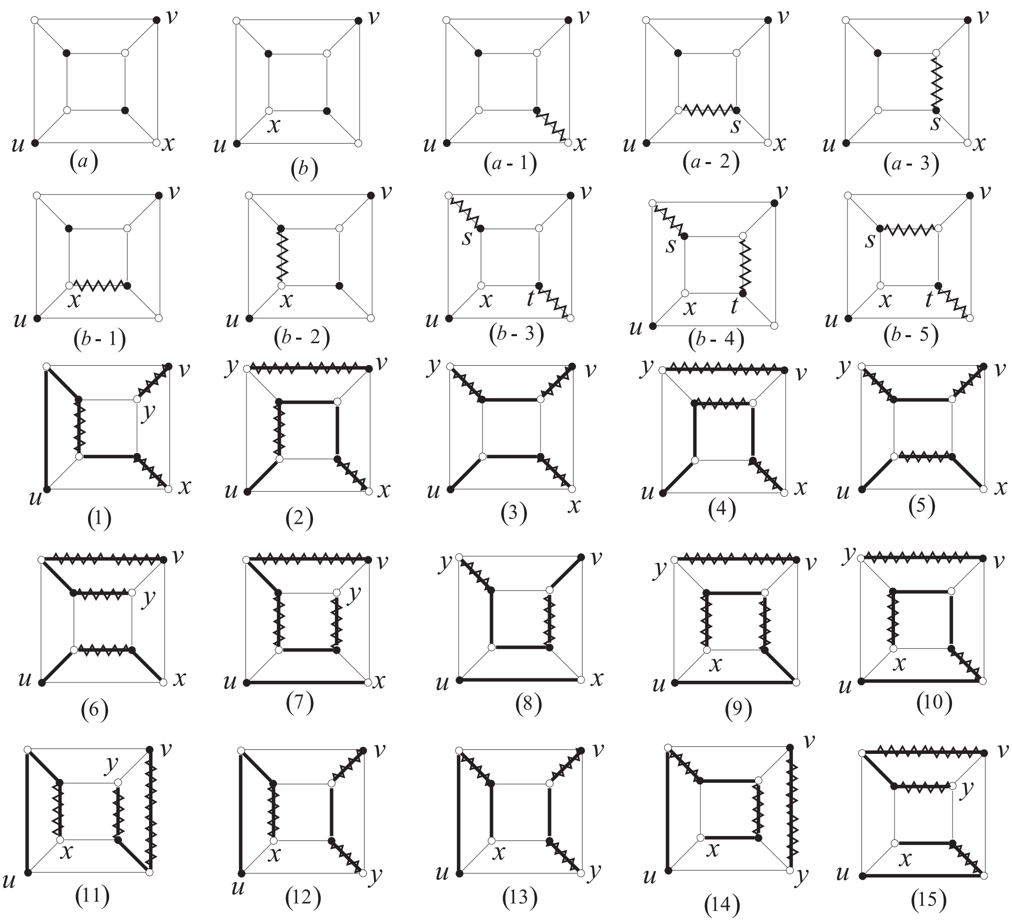

FIGURE 6. Illustration for the proof of Lemma 2.8 with the edges of M curved and the edges of $P_{u,x} + P_{v,y}$ bold.

Lemma 2.8. *Let u, v, x be three distinct vertices in Q_3 with $p(u) = p(v)$ and $d(u, x) = 1$. If M is a matching in $Q_3 - u$ such that $vx \notin M$, then there exists a spanning 2-path $P_{u,x} + P_{v,y}$ of Q_3 passing through M, where y is a vertex in Q_3 satisfying $p(y) \neq p(u)$.*

Proof. Since $p(u) = p(v)$, there is one possibility of (u, v) up to isomorphism (see Fig. 6(a)). Since $d(u, x) = 1$, there are two possibilities of (u, v, x) up to isomorphism (see Figs. 6(a) and 6(b)). It suffices to consider the case that M is a maximal matching in $Q_3 - u - \{vx\}$.

If (u, v, x) is the case (a) in Figure 6, then we consider whether x is covered. When x is covered (see Fig. 6(a-1)), there are four possibilities of (M, u, v, x) up to isomorphism (see Figs. 6(1)–6(4)). When x is uncovered, since M is a maximal matching in $Q_3 - u - \{vx\}$, the third neighbor s of x is covered (see Figs. 6(a-2) and 6(a-3)). Hence there are four possibilities of (M, u, v, x) up to isomorphism (see Figs. 6(5)–6(8)).

If (u, v, x) is the case (b) in Figure 6, then we consider whether x is covered. When x is covered (see Figs. 6(b-1) and 6(b-2)), we find that these two cases are isomorphic, so we only consider the case (b-2). In this case, there are four possibilities of (M, u, v, x) up to isomorphism (see Figs. 6(9)–6(12)). When x is uncovered, since M is a maximal matching in $Q_3 - u - \{vx\}$, the two neighbors s, t of x are both covered (see Figs. 6(b-3)–6(b-5)). Hence there are three possibilities of (M, u, v, x) up to isomorphism (see Figs. 6(13)–6(15)).

By examining all possibilities of (M, u, v, x) up to isomorphism, one can verify that the lemma holds (see Figs. 6(1)–6(15)). $\square$

Lemma 2.9 ([12])**.** *Let u, v be vertices in Q_3 with $p(u) = p(v)$. If M is a matching in $Q_3 - u$, then there exists a spanning 2-path $P_{u,v} + P_{x,y}$ of Q_3 passing through M, where x, y are two distinct vertices in Q_3 satisfying $p(x) = p(y) \neq p(u)$.*

Lemma 2.10. *Let u, v be vertices in Q_4 with $p(u) = p(v)$. If M is a matching in $Q_4 - u$, then there exists a spanning 2-path $P_{u,v} + P_{x,y}$ of Q_4 passing through M, where x, y are two distinct vertices in Q_4 satisfying $p(x) = p(y) \neq p(u)$.*

Proof. It suffices to consider the case that M is a maximal matching in $Q_4 - u$. Since $|M| \leq 7$, there exists $j \in [4]$ such that $|M \cap E_j| \leq 1$. Split Q_4 into subcubes Q_3^0 and Q_3^1 by E_j. By symmetry we may assume $u \in V(Q_3^0)$.

Case 1. $M \cap E_j = \emptyset$.

Subcase 1.1. $v \in V(Q_3^0)$.

Since M_0 is a matching in $Q_3^0 - u$ and $p(u) = p(v)$, by Lemma 2.9 there exists a spanning 2-path $P^0_{u,v} + P^0_{x,s_0}$ of Q_3^0 passing through M_0, where x, s_0 are two distinct vertices in Q_3^0 satisfying $p(x) = p(s_0) \neq p(u)$. By Theorem 1.2 there is a Hamiltonian cycle C_1 of Q_3^1 passing through M_1. Let y be a neighbor of s_1 on C_1 such that $ys_1 \notin M$. Since $p(u) \neq p(x) = p(s_0)$, $p(y) \neq p(s_1)$, we have $p(u) \neq p(x) = p(y), x \neq y$. Let $P_{u,v} = P^0_{u,v}, P_{x,y} = P^0_{x,s_0} + C_1 + s_0s_1 - ys_1$. Hence there exists a spanning 2-path $P_{u,v} + P_{x,y}$ of Q_4 passing through M.

Subcase 1.2. $v \in V(Q_3^1)$.

By Theorem 1.2 there is a Hamiltonian cycle C_1 in Q_3^1 passing through M_1. Let s_1 be a neighbor of v on C_1 such that $vs_1 \notin M$. Since $p(v) \neq p(s_1)$ and $p(s_0) \neq p(s_1)$, we have $p(u) = p(s_0)$. Since $u \notin V(M)$, by Lemma 2.9 there exists a spanning 2-path $P^0_{u,s_0} + P^0_{x,y}$ of Q_3^0 passing through M_0, where x, y are two distinct vertices in Q_3^0 satisfying $p(x) = p(y) \neq p(u)$. Let $P_{u,v} = P^0_{u,s_0} + C_1 + s_0s_1 - vs_1, P_{x,y} = P^0_{x,y}$. Hence there exists a spanning 2-path $P_{u,v} + P_{x,y}$ of Q_4 passing through M.

Case 2. $|M \cap E_j| = 1$. Let $M \cap E_j = \{w_0w_1\}$, where $w_0 \in V(Q_3^0)$.

Subcase 2.1. $v \in V(Q_3^0)$.

Since $u \notin V(M)$ and $p(u) = p(v)$, by Lemma 2.9 there exists a spanning 2-path $P^0_{u,v} + P^0_{x,y}$ of Q_3^0 passing through M_0, where x, y are two distinct vertices in Q_3^0 satisfying $p(x) = p(y) \neq p(u)$. Without loss of generality, we may assume $w_0 \in V(P^0_{x,y})$. Let s_0 be a neighbor of w_0 on $P^0_{x,y}$. Since $p(s_1) \neq p(w_1)$ and $w_1 \notin V(M_1)$, by Lemma 2.1 there exists a Hamiltonian path $P^1_{w_1,s_1}$ of Q_3^1 passing through M_1. Let $P_{u,v} = P^0_{u,v}, P_{x,y} = P^0_{x,y} + P^1_{w_1,s_1} + \{w_0w_1, s_0s_1\} - w_0s_0$. Hence there exists a spanning 2-path $P_{u,v} + P_{x,y}$ of Q_4 passing through M.

Subcase 2.2. $v \in V(Q_3^1)$ and $p(u) = p(w_0)$.

Since $w_0 \notin V(M_0)$, by Lemma 2.9 there exists a spanning 2-path $P^0_{u,w_0} + P^0_{x,y}$ of Q_3^0 passing through M_0, where x, y are two distinct vertices in Q_3^0 satisfying $p(x) = p(y) \neq p(u)$. Since $p(v) \neq p(w_1)$ and $w_1 \notin V(M_1)$, by Lemma 2.1 there exists a Hamiltonian path $P^1_{w_1,v}$ of Q_3^1 passing through M_1. Let $P_{u,v} = P^0_{u,w_0} + P^1_{w_1,v} + w_0w_1, P_{x,y} = P^0_{x,y}$. Hence there exists a spanning 2-path $P_{u,v} + P_{x,y}$ of Q_4 passing through M.

Subcase 2.3. $v \in V(Q_3^1)$ and $p(u) \neq p(w_0)$.

Since $w_1 \notin V(M_1)$ and $p(v) = p(w_1)$, by Lemma 2.9 there exists a spanning 2-path $P^1_{w_1,v} + P^1_{x,y}$ of Q_3^1 passing through M_1, where x, y are two distinct vertices in Q_3^1 satisfying $p(x) = p(y) \neq p(v)$. Since $p(u) \neq p(w_0)$ and $w_0 \notin V(M_0)$, by Lemma 2.1 there exists a Hamiltonian path $P^0_{u.w_0}$ of Q_3^0 passing through M_0. Let $P_{u,v} = P^0_{u,w_0} + P^1_{w_1,v} + w_0w_1, P_{x,y} = P^0_{x,y}$. Hence there exists a spanning 2-path $P_{u,v} + P_{x,y}$ of Q_4 passing through M.

$\square$

Lemma 2.11. *Let u,v,x,y be pairwise distinct vertices in Q_4 with $p(u) = p(v) \neq p(x) = p(y)$. If M is a matching in $Q_4 - u - v$ with $|M| \leq 4$, then there exists a spanning $(\{u,v\},\{x,y\})$-path $P_{u,*} + P_{v,*}$ of Q_4 passing through M.*

Proof. Select $j \in [4]$ such that $|M \cap E_j|$ is as small as possible. Since $|M| \leq 4$, we have $|M \cap E_j| \leq 1$. When $|M \cap E_j| = 0$, without loss of generality, we may assume $j = 4$. When $|M \cap E_j| = 1$, we have $|M \cap E_i| = 1$ for every $i \in [4]$. So there are four possibilities of such j and moreover, we can choose one such that the edge in $M \cap E_j$ is not incident with $\{x,y\}$. Now, since $|M| = 4 > 2$, we have at least two choices of the above j. Without loss of generality, we may assume the two choices of j are positions 3 and 4 (which will be used in Subcase 2.3). Split Q_4 into subcubes Q_3^0 and Q_3^1 by E_4. By the vertex-transitivity of Q_4, we may assume $u = 0000$, then $u \in V(Q_3^0)$.

Case 1. $M \cap E_4 = \emptyset$.

Subcase 1.1. $v \in V(Q_3^1)$.

Since $p(u) = p(v) \neq p(x) = p(y)$ and the vertices u,v are uncovered, the lemma statement is completely symmetric between u and v and between x and y. Hence we only consider the following two cases.

Subcase 1.1.1. $x \in V(Q_3^0), y \in V(Q_3^1)$ (or $x \in V(Q_3^1), y \in V(Q_3^0)$).

Since $p(u) = p(v) \neq p(x) = p(y)$ and M is a matching in $Q_4 - u - v$, by Lemma 2.1 there exists a Hamiltonian path $P^0_{u,x}$ of Q_3^0 and $P^1_{v,y}$ of Q_3^1 passing through M_0 and M_1, respectively. Hence the desired spanning $(\{u,v\},\{x,y\})$-path $P_{u,*} + P_{v,*}$ of Q_4 is formed by edges $E(P^0_{u,x} + P^1_{v,y})$.

Subcase 1.1.2. $\{x,y\} \subseteq V(Q_3^0)$ (or $\{x,y\} \subseteq V(Q_3^1)$).

Since $p(u) \neq p(x) = p(y)$ and $u \notin V(M)$, by Lemma 2.7 there exists a spanning $(\{u,s_0\},\{x,y\})$-path $P^0_{u,*} + P^0_{s_0,*}$ of Q_3^0 passing through M_0, where s_0 is a vertex in Q_3^0 satisfying $p(s_0) = p(u)$. Since $v \notin V(M)$ and $p(v) \neq p(s_1)$, by Lemma 2.1 there exists a Hamiltonian path P^1_{v,s_1} of Q_3^1 passing through M_1. Hence the desired spanning $(\{u,v\},\{x,y\})$-path $P_{u,*} + P_{v,*}$ of Q_4 is formed by edges $E(P^0_{u,*} + P^0_{s_0,*} + P^1_{v,s_1}) \cup \{s_0s_1\}$.

Subcase 1.2. $v \in V(Q_3^0)$.

In this case, since it is completely symmetric between x and y, we only consider the following three cases.

Subcase 1.2.1. $x \in V(Q_3^0), y \in V(Q_3^1)$ (or $x \in V(Q_3^1), y \in V(Q_3^0)$).

By Lemma 2.6 there exists a Hamiltonian path $P^1_{s_1^i,y}$ of Q_3^1 passing through M_1 for $i \in \{1,2\}$, where s_1^1,s_1^2 are two distinct vertices in Q_3^1 satisfying $p(s_1^1) = p(s_1^2) \neq p(y)$. There exists $i \in \{1,2\}$ such that $s_0^i \neq x$. Without loss of generality, assume $i = 1$. Since M is a matching in $Q_3^0 - u - v$ and $p(u) = p(v) \neq p(x) = p(s_0^1)$, by Lemma 2.3 there exists a spanning $(\{u,v\},\{x,s_0^1\})$-path $P^0_{u,*}+P^0_{v,*}$ of Q_3^0 passing through M_0. Hence the desired spanning $(\{u,v\},\{x,y\})$-path $P_{u,*} + P_{v,*}$ of Q_4 is formed by edges $E(P^0_{u,*} + P^0_{v,*} + P^1_{s_1^1,y}) \cup \{s_0s_1\}$.

Subcase 1.2.2. $\{x,y\} \subseteq V(Q_3^1)$.

Since $p(x) = p(y)$, by Lemma 2.5 there exists a spanning 2-Path $P^1_{x,s_1} + P^1_{y,t_1}$ of Q_3^1 passing through M_1, where s_1,t_1 are two distinct vertices in Q_3^1 satisfying $p(s_1) = p(t_1) \neq p(x)$. Since M is a matching in $Q_3^0 - u - v$ and $p(u) = p(v) \neq p(s_0) = p(t_0)$, by Lemma 2.3 there exists a spanning $(\{u,v\},\{s_0,t_0\})$-path $P^0_{u,*}+P^0_{v,*}$ of Q_3^0 passing through M_0. Hence the desired spanning $(\{u,v\},\{x,y\})$-path $P_{u,*}+P_{v,*}$ of Q_4 is formed by edges $E(P^0_{u,*} + P^0_{v,*} + P^1_{x,s_1} + P^1_{y,t_1}) \cup \{s_0s_1, t_0t_1\}$.

Subcase 1.2.3. $\{x,y\} \subseteq V(Q_3^0)$.

If $|M_1| = 4$, then $|M_0| = 0$. By Theorem 1.2 there is a Hamiltonian cycle C_1 of Q_3^1 passing through M_1. Since M_1 is a perfect matching in Q_3^1 and $|E(C_1) \setminus M_1| - 2 \geq 1$, there exists $s_1t_1 \in E(C_1) \setminus M_1$ such that $\{s_0,t_0\} \cap \{u,v\} = \emptyset$. Since $p(u) = p(v) \neq p(x) = p(y)$, by Lemma 2.3 there exists a spanning $(\{u,v\},\{x,y\})$-path $P^0_{u,*} + P^0_{v,*}$ of Q_3^0 passing through $\{s_0t_0\}$. Hence the desired spanning $(\{u,v\},\{x,y\})$-path $P_{u,*}+P_{v,*}$ of Q_4 is formed by edges $E(P^0_{u,*}+P^0_{v,*}+C_1) \cup \{s_0s_1, t_0t_1\} \setminus \{s_0t_0, s_1t_1\}$. If $|M_1| \leq 3$, then we claim that there exists at most one edge e in $E(Q_3^1) \setminus M_1$ such that $M_1 \cup \{e\}$ is not contained in any Hamiltonian cycle of Q_3^1. When $|M_1| \leq 2$, for any $e \in E(Q_3^1) \setminus M_1$, since $M_1 \cup \{e\}$ is a linear forest of Q_3^1 with $|M_1 \cup \{e\}| \leq 3$, by Theorem 1.3 there is always a Hamiltonian

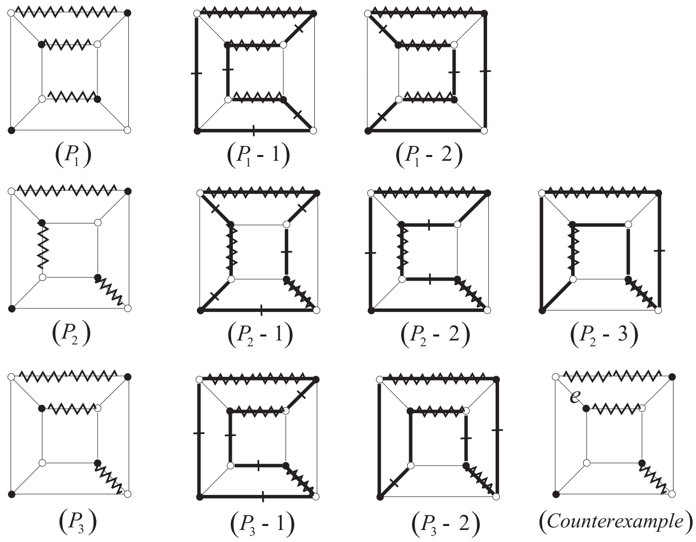

FIGURE 7. Hamiltonian cycles passing through $M_1 \cup \{e\}$ for any $e \in E(Q_3^1) \setminus M_1$ when M_1 is isomorphic to P_1, P_2 or P_3.

cycle of Q_3^1 containing $M_1 \cup \{e\}$. When $|M_1| = 3$, there are three non-isomorphic matchings of size 3 in Q_3, denoted by P_1, P_2 and P_3 (see Fig. 7). By examining all possibilities of (M_1, e), one can verify that the conclusion holds (see Fig. 7). The claim is proved.

Since M_0 is a matching in $Q_3^0 - u - v$, by Lemma 2.3 there exists a spanning $(\{u, v\}, \{x, y\})$-path $P_{u,*}^0 + P_{v,*}^0$ of Q_3^0 passing through M_0. Since $|E(P_{u,*}^0 + P_{v,*}^0)| = 6 > |M_0| + |M_1| + 1$, by the above claim, there exists an edge $s_0t_0 \in E(P_{u,*}^0 + P_{v,*}^0) \setminus M_0$ such that $s_1t_1 \notin M_1$ and $M_1 \cup \{s_1t_1\}$ is contained in some Hamiltonian cycle C_1 of Q_3^1. Hence the desired spanning $(\{u, v\}, \{x, y\})$-path $P_{u,*} + P_{v,*}$ of Q_4 is formed by edges $E(P_{u,*}^0 + P_{v,*}^0 + C_1) \cup \{s_0s_1, t_0t_1\} \setminus \{s_0t_0, s_1t_1\}$.

Case 2. $|M \cap E_4| = 1$. Let $M \cap E_4 = \{w_0w_1\}$. Note that $\{w_0, w_1\} \cap \{x, y\} = \emptyset$.

Subcase 2.1. $v \in V(Q_3^1)$.

In this case, since it is completely symmetric between u and v and between x and y, we only consider the following two cases.

Subcase 2.1.1. $x \in V(Q_3^0), y \in V(Q_3^1)$ (or $x \in V(Q_3^1), y \in V(Q_3^0)$).

Now $p(w_0) = p(u)$ or $p(w_1) = p(v)$. Without loss of generality, assume $p(w_0) = p(u)$. Since $v \notin V(M_1)$, by Lemma 2.1 there exists a Hamiltonian path $P_{v,y}^1$ of Q_3^1 passing through M_1. Let s_1 be a neighbor of w_1 such that $s_0 \neq x$.

If $s_1 \in V(P_{v,y}^1[v, w_1])$, then since M_0 is a matching in $Q_3^0 - u - w_0$, $p(w_0) = p(u) \neq p(x) = p(s_0)$, by Lemma 2.4 there exists a spanning $(\{u, s_0\}, \{x, w_0\})$-path $P_{u,*}^0 + P_{s_0,*}^0$ of Q_3^0 passing through M_0. Hence the desired spanning $(\{u, v\}, \{x, y\})$-path $P_{u,*} + P_{v,*}$ of Q_4 is formed by edges $E(P_{u,*}^0 + P_{s_0,*}^0 + P_{v,y}^1) \cup \{s_0s_1, w_0w_1\} \setminus \{s_1w_1\}$ (see Fig. 8(1)).

If $s_1 \in V(P_{v,y}^1[w_1, y])$, then since M_0 is a matching in $Q_3^0 - u - w_0$ and $p(u) = p(w_0) \neq p(x) = p(s_0)$, by Lemma 2.3 there exists a spanning $(\{u, w_0\}, \{x, s_0\})$-path $P_{u,*}^0 + P_{w_0,*}^0$ of Q_3^0 passing through M_0. Hence the desired spanning $(\{u, v\}, \{x, y\})$-path $P_{u,*} + P_{v,*}$ of Q_4 is formed by edges $E(P_{u,*}^0 + P_{w_0,*}^0 + P_{v,y}^1) \cup \{s_0s_1, w_0w_1\} \setminus \{s_1w_1\}$ (see Fig. 8(2)).

Subcase 2.1.2. $\{x, y\} \subseteq V(Q_3^1)$ (or $\{x, y\} \subseteq V(Q_3^0)$).

If $p(w_0) \neq p(u)$, then $p(v) = p(w_1)$. Since M_1 is a matching in $Q_3^1 - v - w_1$ and $p(v) = p(w_1) \neq p(x) = p(y)$, by Lemma 2.3 there exists a spanning $(\{v, w_1\}, \{x, y\})$-path $P_{v,*}^1 + P_{w_1,*}^1$ of Q_3^1 passing

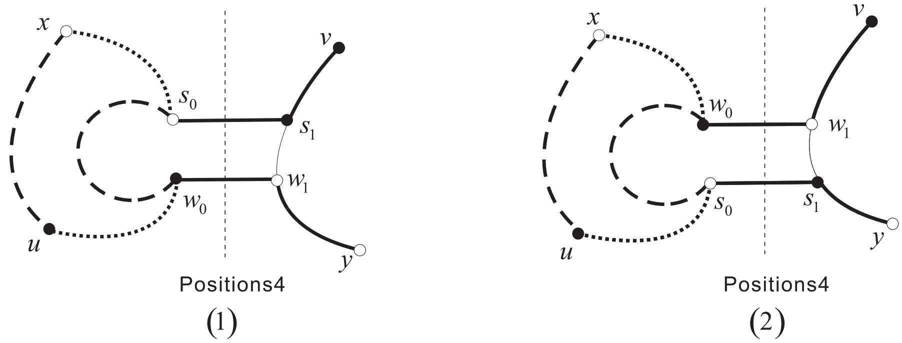

FIGURE 8. Illustration for Subcase 2.1.1.

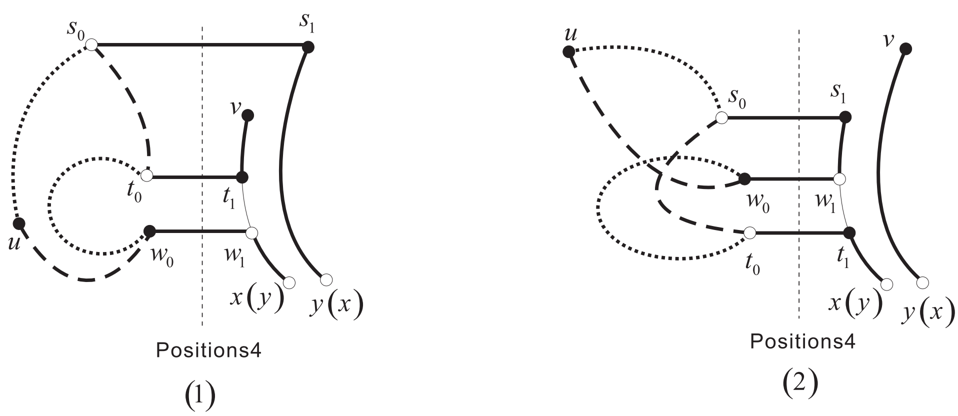

FIGURE 9. Illustration for Subcase 2.1.3.

through M_1. Since M_0 is a matching in $Q_3^0 - u$ and $p(u) \neq p(w_0)$, by Lemma 2.1 there exists a Hamiltonian path P^0_{u,w_0} in Q_3^0 passing through M_0. Hence the desired spanning $(\{u,v\},\{x,y\})$-path $P_{u,*} + P_{v,*}$ of Q_4 is formed by edges $E(P^0_{u,w_0} + P^1_{v,*} + P^1_{w_1,*}) \cup \{w_0w_1\}$.

If $p(w_0) = p(u)$, then since $v \notin V(M_1)$, $p(v) \neq p(x) = p(y)$, by Lemma 2.7 there exists a spanning $(\{v,s_1\},\{x,y\})$-path $P^1_{v,*} + P^1_{s_1,*}$ of Q_3^1 passing through M_1, where s_1 is a vertex in Q_3^1 satisfying $p(s_1) = p(v)$.

If $w_1 \in V(P^1_{v,*})$, let t_1 be a neighbor of w_1 on $P^1_{v,*}[v, w_1]$. Since $p(u) = p(w_0) \neq p(s_0) = p(t_0)$ and M_0 is a matching in $Q_3^0 - u - w_0$, by Lemma 2.4 there exists a spanning $(\{u,t_0\},\{s_0,w_0\})$-path $P^0_{u,*} + P^0_{t_0,*}$ of Q_3^0 passing through M_0. Hence the desired spanning $(\{u,v\},\{x,y\})$-path $P_{u,*} + P_{v,*}$ of Q_4 is formed by edges $E(P^0_{u,*} + P^0_{t_0,*} + P^1_{v,*} + P^1_{s_1,*}) \cup \{w_0w_1, t_0t_1, s_0s_1\} \setminus \{t_1w_1\}$ (see Fig. 9(1)).

If $w_1 \in V(P^1_{s_1,*})$, let t_1 be a neighbor of w_1 on $P^1_{s_1,*}[w_1, *]$. Since $p(u) = p(w_0) \neq p(s_0) = p(t_0)$ and M_0 is a matching in $Q_3^0 - u - w_0$, by Lemma 2.4 there exists a spanning $(\{u,t_0\},\{s_0,w_0\})$-path $P^0_{u,*} + P^0_{t_0,*}$ of Q_3^0 passing through M_0. Hence the desired spanning $(\{u,v\},\{x,y\})$-path $P_{u,*} + P_{v,*}$ of Q_4 is formed by edges $E(P^0_{u,*} + P^0_{t_0,*} + P^1_{v,*} + P^1_{s_1,*}) \cup \{w_0w_1, t_0t_1, s_0s_1\} \setminus \{t_1w_1\}$ (see Fig. 9(2)).

Subcase 2.2. $v \in V(Q_3^0)$ and $p(u) \neq p(w_0)$.

In this case, since it is completely symmetric between x and y, we only consider the following three cases.

Subcase 2.2.1. $x \in V(Q_3^0), y \in V(Q_3^1)$ (or $x \in V(Q_3^1), y \in V(Q_3^0)$).

Since M_1 is a matching in $Q_3^1 - w_1$ and $p(y) \neq p(w_1)$, by Lemma 2.1 there exists a Hamiltonian path $P^1_{w_1,y}$ of Q_3^1 passing through M_1. Since M_0 is a matching in $Q_3^0 - u - v$ and $p(u) = p(v) \neq p(x) = p(w_0)$, by Lemma 2.3 there exists a spanning $(\{u,v\},\{x,w_0\})$-path $P^0_{u,*} + P^0_{v,*}$ of Q_3^0 passing through M_0. Hence the desired spanning $(\{u,v\},\{x,y\})$-path $P_{u,*} + P_{v,*}$ of Q_4 is formed by edges $E(P^0_{u,*} + P^0_{v,*} + P^1_{w_1,y}) \cup \{w_0w_1\}$.

Subcase 2.2.2. $\{x,y\} \subseteq V(Q_3^0)$.

Since M_0 is a matching in $Q_3^0 - u - v$ and $p(u) = p(v) \neq p(x) = p(y)$, by Lemma 2.3 there exists a spanning $(\{u,v\},\{x,y\})$-path $P^0_{u,*} + P^0_{v,*}$ of Q_3^0 passing through M_0. Let s_0 be a neighbor of w_0 on $P^0_{u,*} + P^0_{v,*}$. Since $p(w_1) \neq p(s_1)$ and M_1 is a matching in $Q_3^1 - w_1$, by Lemma 2.1 there exists a Hamiltonian path $P^1_{w_1,s_1}$ in Q_3^1 passing through M_1. Hence the desired spanning $(\{u,v\},\{x,y\})$-path $P_{u,*} + P_{v,*}$ of Q_4 is formed by edges $E(P^0_{u,*} + P^0_{v,*} + P^1_{w_1,s_1}) \cup \{w_0w_1, s_0s_1\} \setminus \{w_0s_0\}$.

Subcase 2.2.3. $\{x,y\} \subseteq V(Q_3^1)$.

Since $p(w_1) \neq p(x) = p(y)$ and M_1 is a matching in $Q_3^1 - w_1$, by Lemma 2.7 there exists a spanning $(\{x,y\},\{w_1,s_1\})$-path $P^1_{x,*} + P^1_{y,*}$ of Q_3^1 passing through M_1, where s_1 is a vertex in Q_3^1 satisfying $p(s_1) = p(w_1)$. Since M_0 is a matching in $Q_3^0 - u - v$, by Lemma 2.3 there exists a spanning $(\{u,v\},\{w_0,s_0\})$-path $P^0_{u,*} + P^0_{v,*}$ of Q_3^0 passing through M_0. Hence the desired spanning $(\{u,v\},\{x,y\})$-path $P_{u,*} + P_{v,*}$ of Q_4 is formed by edges $E(P^0_{u,*} + P^0_{v,*} + P^1_{x,*} + P^1_{y,*}) \cup \{w_0w_1, s_0s_1\}$.

Subcase 2.3. $v \in V(Q_3^0)$ and $p(u) = p(w_0)$.

In the case, we split Q_4 into subcubes $Q_3^{0,3}$ and $Q_3^{1,3}$ by E_3. Note that $|M \cap E_i| = 1$ for every $i \in [4]$. Let $M \cap E_3 = \{w_0'w_1'\}$, where $w_0' \in V(Q_3^{0,3})$. At the beginning of the proof of Lemma 2.11, we showed that the edge in $M \cap E_3$ is not incident with $\{x,y\}$. Hence $\{w_0', w_1'\} \cap \{x,y\} = \emptyset$.

Subcase 2.3.1. $v \in V(Q_3^{1,3})$.

We regard the vertices u, v, x, y, w_0', w_1' in Subcase 2.3.1 as the vertices u, v, x, y, w_0, w_1 in Subcase 2.1, respectively. Since $\{w_0', w_1'\} \cap \{x,y\} = \emptyset$ and $\{w_0, w_1\} \cap \{x,y\} = \emptyset$, the conditions here are the same as in Subcase 2.1. Thus the conclusion holds by Subcase 2.1.

Subcase 2.3.2. $v \in V(Q_3^{0,3})$ and $p(u) = p(v) \neq p(w_0')$.

We regard the vertices u, v, x, y, w_0', w_1' in Subcase 2.3.2 as the vertices u, v, x, y, w_0, w_1 in Subcase 2.2, respectively. Since $\{w_0', w_1'\} \cap \{x,y\} = \emptyset$ and $\{w_0, w_1\} \cap \{x,y\} = \emptyset$, the conditions here are the same as in Subcase 2.2. Thus the conclusion holds by Subcase 2.2.

Subcase 2.3.3. $v \in V(Q_3^{0,3})$ and $p(u) = p(v) = p(w_0')$.

Since $\{u,v\} \subseteq V(Q_3^{0,4}) \cap V(Q_3^{0,3})$, we have $v = 1100$ (see Fig. 10(a)). Since $p(u) = p(v) = p(w_0')$, we have $w_0' = 0101$ or $w_0' = 1001$. Without loss of generality, assume $w_0' = 0101$ (see Fig. 10(a)). So $w_1' = 0111$. Since $p(u) = p(v) = p(w_0)$ and $w_0 \in V(Q_3^{0,4})$, we find $w_0 = 1010$ or $w_0 = 0110$. If $w_0 = 0110$, then $w_1 = 0111 = w_1'$ contradicts that M is a matching. So $w_0 = 1010$, $w_1 = 1011$ (see Fig. 10(a)). Since $M \setminus \{w_0w_1, w_0'w_1'\} \subseteq E_1 \cup E_2$, we have the two edges of $M \setminus \{w_0w_1, w_0'w_1'\}$ only can choose from the four bold edges in Figure 10(a). Since $|M \cap E_1| = |M \cap E_2| = 1$, there are two possibilities of M (see Figs. 10(a-1) and 10(a-2)).

Next, we distinguish two cases to consider based on the position of $\{x,y\}$. If the edge in $M \cap E_1$ or the edge in $M \cap E_2$ is not incident with $\{x,y\}$, without loss of generality, assume $M \cap E_1$ is not incident with $\{x,y\}$, then we split Q_4 by E_1. Now $u \in V(Q_3^{0,1})$ and $v \in V(Q_3^{1,1})$, We regard the vertices u, v, x, y here as the vertices u, v, x, y in Subcase 2.1, respectively. Since the conditions here are the same as in Subcase 2.1, the conclusion holds by Subcase 2.1. If the edge in $M \cap E_1$ and the edge in $M \cap E_2$ are both incident with $\{x,y\}$, then since $p(u) = p(v) \neq p(x) = p(y)$, we have $\{x,y\} = \{0001, 0010\}$ when M is the case (a-1) in Figure 10, and $\{x,y\} = \{1101, 1110\}$ when M is the case (a-2) in Figure 10. Thus, there also exists a spanning $(\{u,v\},\{x,y\})$-path $P_{u,*} + P_{v,*}$ of Q_4 passing through M (see Fig. 11).

□

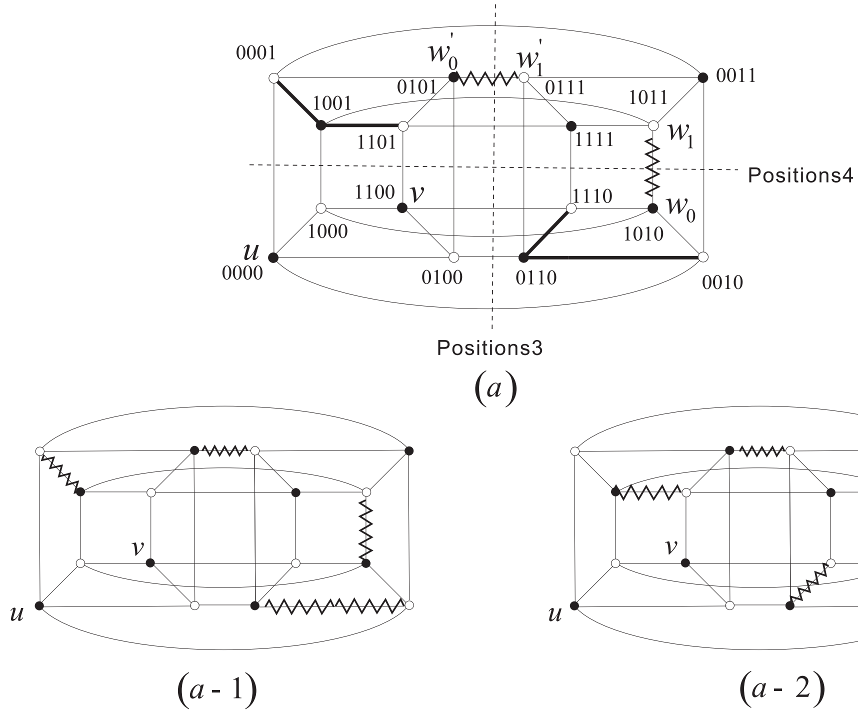

FIGURE 10. The construction of M.

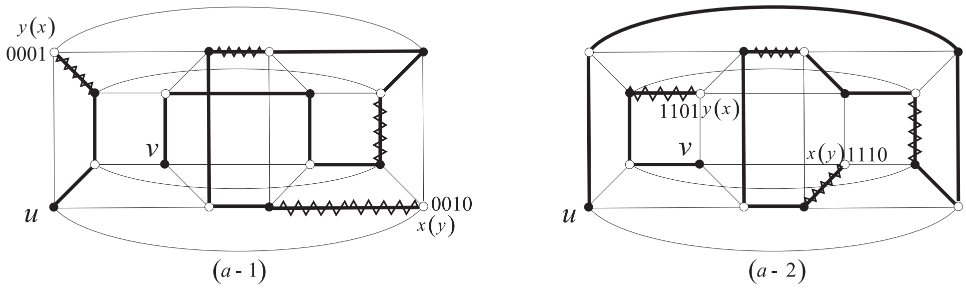

FIGURE 11. A spanning $(\{u, v\}, \{x, y\})$-path $P_{u,*} + P_{v,*}$ of Q_4.

3. Matchings in at most five positions

Theorem 3.1. *Let u, v be two vertices in Q_5 with $p(u) \neq p(v)$. If M is a matching of $Q_5 - u$ with $|M| \leq 9$, then there exists a Hamiltonian path of Q_5 joining u and v passing through M.*

Proof. Since $|M| \leq 9$, there exists $j \in [5]$ such that $|M \cap E_j| \leq 1$. Without loss of generality, assume $j = 5$. Split Q_5 into subcubes Q_4^0 and Q_4^1 by E_5. By symmetry we may assume $u \in V(Q_4^0)$.

Case 1. $M \cap E_5 = \emptyset$.

Subcase 1.1. $v \in V(Q_4^1)$.

By Theorem 1.2 there is a Hamiltonian cycle C_1 of Q_4^1 passing through M_1. Let s_1 be a neighbor of v on C_1 such that $vs_1 \notin M$. Since $p(u) \neq p(v)$ and $p(s_1) \neq p(v)$, we have $p(u) \neq p(s_0)$. Since

$u \notin V(M_0)$, by Lemma 2.1 there exists a Hamiltonian path P^0_{u,s_0} of Q^0_4 passing through M_0. Let $P_{u,v} = P^0_{u,s_0} + C_1 + s_0s_1 - vs_1$. Hence $P_{u,v}$ is a Hamiltonian path of Q_5 joining u and v passing through M.

Subcase 1.2. $v \in V(Q^0_4)$.

If M_1 is not perfect in Q^1_4, there exists $s_1 \in V(Q^1_4) \setminus V(M_1)$ such that $s_0 \neq v$. Since $u \notin V(M_0)$ and $p(u) \neq p(v)$, by Lemma 2.1 there exists a Hamiltonian path $P^0_{u,v}$ of Q^0_4 passing through M_0. Let t_0 be a neighbor of s_0 on $P^0_{u,v}$ such that $t_0s_0 \notin M_0$. Since $s_1 \notin V(M_1)$ and $p(s_1) \neq p(t_1)$, by Lemma 2.1 there exists a Hamiltonian path $P^1_{t_1,s_1}$ of Q^1_4 passing through M_1. Let $P_{u,v} = P^0_{u,v} + P^1_{t_1,s_1} + \{s_0s_1, t_0t_1\} - s_0t_0$. Hence $P_{u,v}$ is a Hamiltonian path of Q_5 joining u and v passing through M.

Otherwise, M_1 is a perfect matching in Q^1_4. Now $|M_1| = 8$ and $|M_0| \leq 1$. By Theorem 1.2 there is a Hamiltonian cycle C_1 of Q^1_4 passing through M_1. Choose an edge $s_1t_1 \in E(C_1) \setminus M_1$ such that $u \notin \{s_0, t_0\}$, $s_0t_0 \notin M_0$ and $M_0 \cup \{s_0t_0\}$ is a matching of Q^0_4. Since $|E(C_1) \setminus M_1| - 2|M_0| - 1 \geq 1$, the above edge s_1t_1 exists. Since $u \notin V(M_0 \cup \{s_0t_0\})$ and $p(u) \neq p(v)$, by Lemma 2.1 there exists a Hamiltonian path $P^0_{u,v}$ of Q^0_4 passing through $M_0 \cup \{s_0t_0\}$. Let $P_{u,v} = P^0_{u,v} + C_1 + \{s_0s_1, t_0t_1\} - \{s_0t_0, s_1t_1\}$. Hence $P_{u,v}$ is a Hamiltonian path of Q_5 joining u and v passing through M.

Case 2. $|M \cap E_5| = 1$. Let $M \cap E_5 = \{w_0w_1\}$.

Subcase 2.1. $v \in V(Q^0_4)$.

By Lemma 2.1 there is a Hamiltonian path $P^0_{u,v}$ of Q^0_4 passing through M_0. Let s_0 be a neighbor of w_0 on $P^0_{u,v}$. Since $w_1 \notin V(M_1)$ and $p(w_1) \neq p(s_1)$, by Lemma 2.1 there exists a Hamiltonian path $P^1_{w_1,s_1}$ of Q^1_4 passing through M_1. Let $P_{u,v} = P^0_{u,v} + P^1_{w_1,s_1} + \{w_0w_1, s_0s_1\} - w_0s_0$. Hence $P_{u,v}$ is a Hamiltonian path of Q_5 joining u and v passing through M.

Subcase 2.2. $v \in V(Q^1_4)$ and $p(u) \neq p(w_0)$.

Since $p(u) \neq p(v)$ and $p(w_0) \neq p(w_1)$, we have $p(w_1) \neq p(v)$. Since $u \notin V(M_0)$ and $w_1 \notin V(M_1)$, by Lemma 2.1 there exist Hamiltonian paths P^0_{u,w_0} of Q^0_4 and $P^1_{w_1,v}$ of Q^1_4 passing through M_0 and M_1, respectively. Let $P_{u,v} = P^0_{u,w_0} + P^1_{w_1,v} + w_0w_1$. Hence $P_{u,v}$ is a Hamiltonian path of Q_5 joining u and v passing through M.

Subcase 2.3. $v \in V(Q^1_4)$ and $p(u) = p(w_0)$. Note that $|M_0| + |M_1| \leq 8$.

Subcase 2.3.1. $|M_0| \leq |M_1|$. Now $|M_0| \leq 4$.

Since $p(w_1) = p(v)$ and M_1 is a matching in $Q^1_4 - w_1$, by Lemma 2.10 there exists a spanning 2-path $P^1_{w_1,v} + P^1_{s_1,t_1}$ of Q^1_4 passing through M_1, where s_1, t_1 are two distinct vertices in Q^1_4 satisfying $p(s_1) = p(t_1) \neq p(w_1)$. Thus $p(u) = p(w_0) \neq p(s_0) = p(t_0)$. Since M_0 is a matching in $Q^0_4 - u - w_0$ with $|M_0| \leq 4$, by Lemma 2.11 there exists a spanning $(\{u, w_0\}, \{s_0, t_0\})$-path $P^0_{u,*} + P^0_{w_0,*}$ of Q^0_4 passing through M_0. Let $P_{u,v} = P^0_{u,*} + P^0_{w_0,*} + P^1_{w_1,v} + P^1_{s_1,t_1} + \{w_0w_1, s_0s_1, t_0t_1\}$. Hence $P_{u,v}$ is a Hamiltonian path of Q_5 joining u and v passing through M.

Subcase 2.3.2. $|M_0| > |M_1|$ and $w_1 = v$. Now $|M_1| \leq 3$.

Choose a neighbor t_0 of w_0 in Q^0_4. Since $u \notin V(M_0)$ and $p(u) = p(w_0) \neq p(t_0)$, by Lemma 2.1 there is a Hamiltonian path P^0_{u,t_0} of Q^0_4 passing through M_0. Let s_0 be the neighbor of w_0 on $P^0_{u,t_0}[u, w_0]$. Since $w_1 \notin V(M_1)$ and w_1s_1, w_1t_1 are two edges in Q^1_4, we find $M_1 \cup \{w_1s_1, w_1t_1\}$ is a linear forest of Q^1_4 with $|M_1 \cup \{w_1s_1, w_1t_1\}| \leq 5$. Thus by Theorem 1.3 there is a Hamiltonian cycle C_1 of Q^1_4 passing through $M_1 \cup \{w_1s_1, w_1t_1\}$. Let $P_{u,v} = P^0_{u,t_0} + C_1 + \{w_0w_1, s_0s_1, t_0t_1\} - \{w_0s_0, w_1s_1, w_1t_1\}$. Hence $P_{u,v}$ is a Hamiltonian path of Q_5 joining u and v passing through M (see Fig. 12).

Subcase 2.3.3. $|M_0| > |M_1|$ and $w_1 \neq v$. Now $|M_1| \leq 3$.

Subcase 2.3.3.1. The four neighbors of w_1 all uncovered by M_1 in Q^1_4.

Choose a vertex t_1 such that $d(t_1, w_1) = 3$ and $t_1 \notin V(M_1)$. Since $|M_1| \leq 3$, the above vertex t_1 exists. Since $u \notin V(M_0)$ and $p(u) = p(w_0) \neq p(t_0)$, by Lemma 2.1 there is a Hamiltonian path P^0_{u,t_0} of Q^0_4 passing through M_0. Let s_0 be the neighbor of w_0 on $P^0_{u,t_0}[w_0, t_0]$. Since the four neighbors of w_1 in Q^1_4 all uncovered by M_1, we have $s_1 \notin V(M_1)$. Since M_1 is a matching of $Q^1_4 - t_1 - s_1$, $p(t_1) = p(s_1) \neq p(w_1) = p(v)$ and $|M_1| \leq 3$, by Lemma 2.11 there exists a spanning $(\{t_1, s_1\}, \{w_1, v\})$-path $P^1_{t_1,*} + P^1_{s_1,*}$ of Q^1_4 passing through M_1. Let $P_{u,v} = P^0_{u,t_0} +$

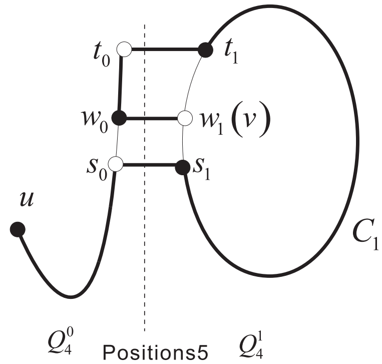

FIGURE 12. Hamiltonian path $P_{u,v}$ in Q_5 passing through M.

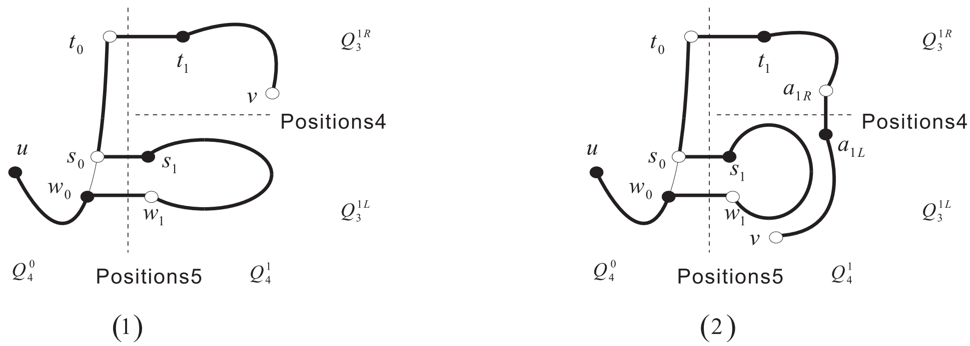

FIGURE 13. Hamiltonian path $P_{u,v}$ in Q_5 passing through M.

$P^1_{t_1,*} + P^1_{s_1,*} + \{w_0w_1, s_0s_1, t_0t_1\} - w_0s_0$. Hence $P_{u,v}$ is a Hamiltonian path of Q_5 joining u and v passing through M.

Subcase 2.3.3.2. There exists a neighbor of w_1 covered by M_1 in Q^1_4.

Since $|M_1| \leq 3$, there exists $j \in [4]$ such that $M_1 \cap E_j = \emptyset$. Without loss of generality, assume $j = 4$. Split Q^1_4 into Q^{1L}_3 and Q^{1R}_3 at position 4. Let $M_{1\delta} = M_1 \cap E(Q^{1\delta}_3)$ for every $\delta \in \{L, R\}$. Note that every vertex $s_{1L} \in V(Q^{1L}_3)$ has in Q^{1R}_3 a unique neighbor, denoted by s_{1R}, and every vertex $t_{1R} \in V(Q^{1R}_3)$ has in Q^{1L}_3 a unique neighbor, denoted by t_{1L}. Without loss of generality, we may assume $w_1 \in V(Q^{1L}_3)$. Choose a vertex t_1 in Q^{1R}_3 such that $d(t_1, w_1) = 3$ and $t_1 \notin V(M_1)$. Since $|M_1| - 1 < 3$, the above vertex t_1 exists. Since $u \notin V(M_0)$ and $p(u) = p(w_0) \neq p(t_0)$, by Lemma 2.1 there is a Hamiltonian path P^0_{u,t_0} of Q^0_4 passing through M_0. Note that w_1 has four neighbors in Q^1_4, where one is in Q^{1R}_3 and the other three are in Q^{1L}_3. So we can choose a neighbor s_0 of w_0 on P^0_{u,t_0} such that $s_1 \in V(Q^{1L}_3)$. Next we will distinguish two cases depending upon $s_1v \in M_1$ or not. When $s_1v \in M_1$, we will further distinguish two cases depending upon $s_0 \in P^0_{u,t_0}[w_0, t_0]$ or $s_0 \in P^0_{u,t_0}[u, w_0]$.

Subcase 2.3.3.2.1. $s_1v \notin M_1$.

If $v \in V(Q^{1R}_3)$, since $w_1 \notin V(M_{1L})$, $p(w_1) \neq p(s_1)$ and $t_1 \notin V(M_{1R})$, $p(t_1) \neq p(v)$, by Lemma 2.1 there is a Hamiltonian path $P^{1L}_{w_1,s_1}$ of Q^{1L}_3 and $P^{1R}_{t_1,v}$ of Q^{1R}_3 passing through M_{1L} and M_{1R}, respectively. Let $P_{u,v} = P^0_{u,t_0} + P^{1L}_{w_1,s_1} + P^{1R}_{t_1,v} + \{w_0w_1, s_0s_1, t_0t_1\} - w_0s_0$. Hence $P_{u,v}$ is a Hamiltonian path of Q_5 joining u and v passing through M (see Fig. 13(1)).

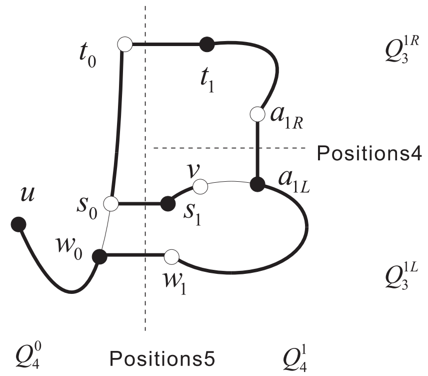

FIGURE 14. Hamiltonian path $P_{u,v}$ in Q_5 passing through M.

Otherwise, $v \in V(Q_3^{1L})$. Since $w_1 \notin V(M_{1L})$ and $p(w_1) = p(v) \neq p(s_1)$, by Lemma 2.8 there exists a spanning 2-path $P^{1L}_{w_1,s_1} + P^{1L}_{v,a_{1L}}$ of Q_3^{1L} passing through M_{1L}, where a_{1L} is a vertex in Q_3^{1L} satisfying $p(v) \neq p(a_{1L})$. Since $t_1 \notin V(M_{1R})$ and $p(t_1) \neq p(a_{1R})$, by Lemma 2.1 there is a Hamiltonian path $P^{1R}_{t_1,a_{1R}}$ of Q_3^{1R} passing through M_{1R}. Let $P_{u,v} = P^0_{u,t_0} + P^{1L}_{w_1,s_1} + P^{1L}_{v,a_{1L}} + P^{1R}_{t_1,a_{1R}} + \{w_0w_1, s_0s_1, t_0t_1, a_{1L}a_{1R}\} - w_0s_0$. Hence $P_{u,v}$ is a Hamiltonian path of Q_5 joining u and v passing through M (see Fig. 13(2)).

Subcase 2.3.3.2.2. $s_1v \in M_1$ and $s_0 \in P^0_{u,t_0}[w_0, t_0]$.

Since $s_1v \in M_1$, $s_1 \in V(Q_3^{1L})$ and $M_1 \cap E_4 = \emptyset$, we have $v \in V(Q_3^{1L})$. Since $w_1 \notin V(M_{1L})$, by Lemma 2.1 there is a Hamiltonian path $P^{1L}_{w_1,s_1}$ of Q_3^{1L} passing through M_{1L}. Let a_{1L} be the neighbor of v on $P^{1L}_{w_1,s_1}[w_1, v]$. Since $s_1v \in M_1$, we have $a_{1L}v \notin M_1$. Since $t_1 \notin V(M_{1R})$ and $p(t_1) \neq p(a_{1R})$, by Lemma 2.1 there is a Hamiltonian path $P^{1R}_{t_1,a_{1R}}$ of Q_3^{1R} passing through M_{1R}. Let $P_{u,v} = P^0_{u,t_0} + P^{1L}_{w_1,s_1} + P^{1R}_{t_1,a_{1R}} + \{w_0w_1, s_0s_1, t_0t_1, a_{1L}a_{1R}\} - \{w_0s_0, a_{1L}v\}$. Hence $P_{u,v}$ is a Hamiltonian path of Q_5 joining u and v passing through M (see Fig. 14).

Subcase 2.3.3.2.3. $s_1v \in M_1$ and $s_0 \in P^0_{u,t_0}[u, w_0]$.

Also, we have $v \in V(Q_3^{1L})$. In this case, due to the special position of s_0, we cannot use s_0. Therefore, we choose the other neighbor r_0 of w_0 that lies on the path $P^0_{u,t_0}[w_0, t_0]$. Now $r_1v \notin M_1$. Next we will distinguish two cases depending upon $r_1 \in V(Q_3^{1L})$ or $r_1 \in V(Q_3^{1R})$.

When $r_1 \in V(Q_3^{1L})$, since $w_1 \notin V(M_{1L})$ and $p(w_1) = p(v) \neq p(r_1)$, by Lemma 2.8 there exists a spanning 2-path $P^{1L}_{w_1,r_1} + P^{1L}_{v,a_{1L}}$ of Q_3^{1L} passing through M_{1L}, where a_{1L} is a vertex in Q_3^{1L} satisfying $p(v) \neq p(a_{1L})$. Since $t_1 \notin V(M_{1R})$ and $p(t_1) \neq p(a_{1R})$, by Lemma 2.1 there is a Hamiltonian path $P^{1R}_{t_1,a_{1R}}$ of Q_3^{1R} passing through M_{1R}. Let $P_{u,v} = P^0_{u,t_0} + P^{1L}_{w_1,r_1} + P^{1L}_{v,a_{1L}} + P^{1R}_{t_1,a_{1R}} + \{w_0w_1, r_0r_1, t_0t_1, a_{1L}a_{1R}\} - w_0r_0$. Hence $P_{u,v}$ is a Hamiltonian path of Q_5 joining u and v passing through M.

When $r_1 \in V(Q_3^{1R})$, since $d(t_1, w_1) = 3$ and $d(r_1, w_1) = 1$, we have $r_1 \neq t_1$. Note that $t_1 \notin V(M_1)$, and $p(t_1) = p(r_1) \neq p(w_1) = p(v)$, and $|M_1| \leq 3$. If $r_1 \notin V(M_1)$, then by Lemma 2.11 there exists a spanning $(\{t_1, r_1\}, \{w_1, v\})$-path $P^1_{t_1,*} + P^1_{r_1,*}$ of Q_4^1 passing through M_1. Let $P_{u,v} = P^0_{u,t_0} + P^1_{t_1,*} + P^1_{r_1,*} + \{w_0w_1, r_0r_1, t_0t_1\} - w_0r_0$. Hence $P_{u,v}$ is a Hamiltonian path of Q_5 joining u and v passing through M.

It remains to consider the case that $r_1 \in V(Q_3^{1R})$ and $r_1 \in V(M_1)$. Note that $t_1, r_1 \in V(Q_3^{1R})$, and $w_1, v \in V(Q_3^{1L})$, and $M_1 \cap E_4 = \emptyset$, and $|M_{1L}| + |M_{1R}| = |M_1| \leq 3$, and $p(w_1) = p(v) \neq p(t_1) = p(r_1)$, and $w_1 \notin V(M_{1L})$, $v \in V(M_{1L})$, $t_1 \notin V(M_{1R})$, $r_1 \in V(M_{1R})$, and the neighbor s_1

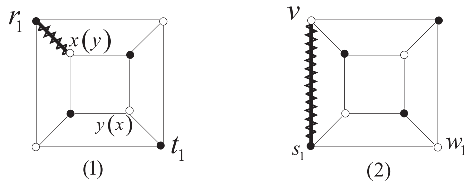

FIGURE 15. (1) The counterexample in Q_3^{1R} when $|M_{1R}| = 1$; (2) The structure of (v, w_1, M_{1L}) in Q_3^{1L} when $|M_{1L}| = 1$.

of v in M_{1L} is adjacent to w_1. First, we claim that there exists a spanning $(\{w_1, v\}, \{t_1, r_1\})$-path $P^1_{w_1,*} + P^1_{v,*}$ of Q^1_4 passing through M_1.

If $|M_{1R}| = 1$, then since $4 - |M_{1L}| \geq 2$, there exist two distinct vertices a_{1L}, b_{1L} such that $p(a_{1L}) = p(b_{1L}) \neq p(w_1) = p(v)$ and $\{a_{1L}, b_{1L}\} \cap V(M_{1L}) = \emptyset$. By Lemma 2.3 there exists a spanning $(\{w_1, v\}, \{a_{1L}, b_{1L}\})$-path $P^{1L}_{w_1,*} + P^{1L}_{v,*}$ of Q^{1L}_3 passing through M_{1L}. Without loss of generality, denote the 2-path by $P^{1L}_{w_1,a_{1L}} + P^{1L}_{v,b_{1L}}$. Now, let us show another selection of $\{a_{1L}, b_{1L}\}$. Since $w_1 \notin V(M_{1L})$ and $p(s_1) \neq p(v) = p(w_1)$, by Lemma 2.1 there is a Hamiltonian path $P^{1L}_{w_1,s_1}$ of Q^{1L}_3 passing through M_{1L}. Note that $s_1 v \in E(P^{1L}_{w_1,s_1})$. Let $b_{1L} = s_1$, and a_{1L} be the neighbor of v on $P^{1L}_{w_1,s_1}[w_1, v]$. So, $a_{1L} v \notin M$. Let $P^{1L}_{w_1,a_{1L}} + P^{1L}_{v,b_{1L}} = P^{1L}_{w_1,s_1} - a_{1L} v$. Under this selection, b_{1L} is covered by M_{1L}, which differs from the previous choice.

Before ending the proof, we point out a small conclusion. In Q^{1R}_3, for any two distinct vertices x, y satisfying $p(x) = p(y) \neq p(r_1) = p(t_1)$, if (r_1, t_1, x, y, M_{1R}) is not the case in Figure 15(1), then there exists a spanning $(\{r_1, t_1\}, \{x, y\})$-path $P^{1R}_{r_1,*} + P^{1R}_{t_1,*}$ of Q^{1R}_3 passing through M_{1R}. Let us prove this conclusion. If $\{x, y\} \cap V(M_{1R}) = \emptyset$, then by Lemma 2.3 there exists a spanning $(\{r_1, t_1\}, \{x, y\})$-path $P^{1R}_{r_1,*} + P^{1R}_{t_1,*}$ of Q^{1R}_3 passing through M_{1R}. If $\{x, y\} \cap V(M_{1R}) \neq \emptyset$, then $M_{1R} = \{r_1 x\}$ (or $M_{1R} = \{r_1 y\}$). By Theorem 1.4, there exists a spanning 2-path $r_1 x + P^{1R}_{t_1,y}$ of Q^{1R}_3 except the case in Figure 15(1). Thus, the above conclusion holds.

Since there are two selections of $\{a_{1L}, b_{1L}\}$ in Q^{1L}_3, we can choose one such that $(r_1, t_1, a_{1R}, b_{1R}, M_{1R})$ is not the case in Figure 15(1). By the above conclusion there exists a spanning $(\{r_1, t_1\}, \{a_{1R}, b_{1R}\})$-path $P^{1R}_{r_1,*} + P^{1R}_{t_1,*}$ of Q^{1R}_3 passing through M_{1R}. Let $P^1_{w_1,*} + P^1_{v,*} = P^{1L}_{w_1,a_{1L}} + P^{1L}_{v,b_{1L}} + P^{1R}_{r_1,*} + P^{1R}_{t_1,*} + \{a_{1L} a_{1R}, b_{1L} b_{1R}\}$ (see Fig. 16(1)).

If $|M_{1R}| = 2$, then now $|M_{1L}| = 1$ and $M_{1L} = \{s_1 v\}$. Choose two distinct vertices a_{1R}, b_{1R} such that $p(a_{1R}) = p(b_{1R}) \neq p(r_1) = p(t_1)$ and $\{a_{1R}, b_{1R}\} \cap V(M_{1R}) = \emptyset$. By Lemma 2.3 there exists a spanning $(\{r_1, t_1\}, \{a_{1R}, b_{1R}\})$-path $P^{1R}_{r_1,*} + P^{1R}_{t_1,*}$ of Q^{1R}_3 passing through M_{1R}. In Q^{1L}_3, now $p(w_1) = p(v)$, and $M_{1L} = \{s_1 v\}$, and s_1 is adjacent to w_1 (see Fig. 15(2)). Thus, $(v, w_1, a_{1L}, b_{1L}, M_{1L})$ is not isomorphic to the case in Figure 15(1). By the above conclusion there exists a spanning $(\{v, w_1\}, \{a_{1L}, b_{1L}\})$-path $P^{1L}_{v,*} + P^{1L}_{w_1,*}$ of Q^{1L}_3 passing through M_{1L}. Let $P^1_{w_1,*} + P^1_{v,*} = P^{1L}_{v,*} + P^{1L}_{w_1,*} + P^{1R}_{r_1,*} + P^{1R}_{t_1,*} + \{a_{1L} a_{1R}, b_{1L} b_{1R}\}$. The claim is proved.

Let $P_{u,v} = P^0_{u,t_0} + P^1_{w_1,*} + P^1_{v,*} + \{w_0 w_1, r_0 r_1, t_0 t_1\} - w_0 r_0$. Hence, $P_{u,v}$ is a Hamiltonian path of Q_5 joining u and v passing through M (see Fig. 16(2)).

□

Theorem 3.2. *For $n \geq 6$, let M be a matching in Q_n with $|M| < 10 \times 2^{n-5}$. If M contains edges in at most 5 directions, then there exists a Hamiltonian cycle of Q_n passing through M.*

Proof. Since M contains edges in at most 5 directions, without loss of generality we may assume M contains edges only in directions $\{1, 2, \ldots, 5\}$. Then $M \subseteq E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5$.

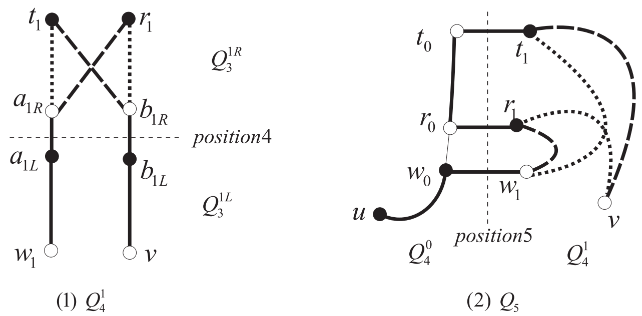

FIGURE 16. (1) Spanning $(\{w_1, v\}, \{t_1, r_1\})$-path $P^1_{w_1,*} + P^1_{v,*}$ of Q_4^1; (2) Hamiltonian path $P_{u,v}$ in Q_5 passing through M.

Let Q_{n-5} be a $(n-5)$-dimensional hypercube. When $n = 6$, let $V(Q_{n-5}) = \{x_0, x_1\}$. When $n \geq 7$, choose a Hamiltonian cycle $C = x_0, x_1, \ldots, x_{2^{n-5}-1}, x_0$ of Q_{n-5}. Note that for every $k \in \{0, 1, \ldots, 2^{n-5}-1\}$, x_k is a binary string of length $(n-5)$.

For every $k \in \{0, 1, \ldots, 2^{n-5}-1\}$, let $Q_5^{x_k}$ be the 5-dimensional subcube of Q_n induced by the vertex set $\{y \in V(Q_n) : y^i = x_k^{i-5}$ for every $i \in \{6, \ldots, n\}\}$. Then $Q_n - E_6 - \cdots - E_n = Q_5^{x_0} + Q_5^{x_1} + \cdots + Q_5^{x_{2^{n-5}-1}}$ and $\bigcup_{k=0}^{2^{n-5}-1} E(Q_5^{x_k}) = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5$. Hence $M \subseteq \bigcup_{k=0}^{2^{n-5}-1} E(Q_5^{x_k})$. Let $M_k = M \cap E(Q_5^{x_k})$ for every $k \geq 0$. Then $M = \bigcup_{k=0}^{2^{n-5}-1} M_k$.

Since $|M| < 10 \times 2^{n-5}$, there exists some M_i such that $|M_i| \leq 9$. Without loss of generality, assume $|M_0| \leq 9$. First apply Theorem 1.2 to obtain a Hamiltonian cycle C_k of $Q_5^{x_k}$ passing through M_k for every $k \in \{1, \ldots, 2^{n-5}-1\}$.

For every $k \in \{0, 1, \ldots, 2^{n-5}-1\}$, since x_k is adjacent to x_{k+1} in Q_{n-5}, every vertex $y \in V(Q_5^{x_k})$ has of $Q_5^{x_{k+1}}$ a unique neighbor $y^1y^2y^3y^4y^5x^0_{k+1} \cdots x^{n-5}_{k+1}$, with subscripts taken modulo 2^{n-5}. Let $u_0 \in V(Q_5^{x_0}) \setminus V(M_0)$ and v_1 be the neighbor of u_0 in $Q_5^{x_1}$. Then $p(u_0) \neq p(v_1)$. From $k = 1$ to $2^{n-5}-1$, let u_k be a neighbor of v_k on C_k such that $u_k v_k \notin M$ and let v_{k+1} be the neighbor of u_k in $Q_5^{x_{k+1}}$, where the subscripts modulo 2^{n-5}. Then $p(u_k) \neq p(v_k)$ and $p(u_k) \neq p(v_{k+1})$ for every $k \in \{1, \ldots, 2^{n-5}-1\}$. Hence $p(u_0) \neq p(v_0)$. Since M_0 is a matching of $Q_5^{x_0} - u_0$, by Theorem 3.1 there exists a Hamiltonian path P_{u_0,v_0} of $Q_5^{x_0}$ passing through M_0. Then the desired Hamiltonian cycle of Q_n is formed by edges of $E(P_{u_0,v_0}) \cup (\bigcup_{k=1}^{2^{n-5}-1}(E(C_k) \cup \{u_{k-1}v_k\} \setminus \{u_k v_k\})) \cup \{u_{2^{n-5}-1}v_0\}$. $\square$

Acknowledgements

The authors would like to express their gratitude to the anonymous referees whose helpful comments and suggestions have led to a substantial improvement of the paper.

Funding

This work is supported by NSFC (grant no. 12061047), and supported by Jiangxi Provincial Natural Science Foundation (nos. 20212BAB201027 and 20192BAB211002).

References

[1] R. Caha and V. Koubek, Spanning multi-paths in hypercubes. *Discrete Math.* **307** (2007) 2053–2066.
[2] D. Dimitrov, T. Dvořák, P. Gregor and R. Škrekovski, Gray codes avoiding matchings. *Discrete Math. Theor. Comput. Sci.* **11** (2009) 123–148.
[3] T. Dvořák, Hamiltonian cycles with prescribed edges in hypercubes. *SIAM J. Discrete Math.* **19** (2005) 135–144.

[4] T. Dvořák and J. Fink, Gray codes extending quadratic matchings. *J. Graph Theory* **90** (2019) 123–136.
[5] J. Fink, Perfect matchings extend to Hamilton cycles in hypercubes. *J. Comb. Theory Ser. B* **97** (2007) 1074–1076.
[6] J. Fink, Matching graphs of hypercubes and complete bipartite graphs. *Eur. J. Comb.* **30** (2009) 1624–1629.
[7] P. Gregor, T. Novotný and R. Škrekovski, Extending perfect matchings to gray codes with prescribed ends. *Electron. J. Comb.* **25** (2018) P2.56.
[8] L. Gros, Théorie du Baguenodier. Aimé Vingtrinier, Lyon (1872).
[9] G. Kreweras, Matchings and Hamiltonian cycle on hypercubes. *Bull. Inst. Comb. Appl.* **16** (1996) 87–91.
[10] F. Ruskey and C. Savage, Hamilton cycles that extend transposition matchings in Cayley graphs of S_n. *SIAM J. Discrete Math.* **6** (1993) 152–166.
[11] F. Wang and H. Zhang, Prescribed matchings extend to Hamiltonian cycles in hypercubes with faulty edges. *Discrete Math.* **321** (2014) 35–44.
[12] F. Wang and H. Zhang, Two types of matchings extend to Hamiltonian cycles in hypercubes. *ARS Comb.* **118** (2015) 269–283.
[13] F. Wang and H. Zhang, Small matchings extend to Hamiltonian cycles in hypercubes. *Graphs Comb.* **32** (2016) 363–376.
[14] F. Wang and W. Zhao, Matchings extend to Hamiltonian cycles in 5-cube. *Discuss. Math. Graph Theory* **38** (2018) 217–231.