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ON MULTIOBJECTIVE FRACTIONAL PROGRAMS WITH VANISHING CONSTRAINTS

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Abstract. The aim of this article is to combine the study of fractional programming and mathematical programs with vanishing constraints for the first time in literature. This paper deals with multiobjective fractional programs with vanishing constraints (MFPVC) involving continuously differentiable functions. Necessary and sufficient optimality conditions are derived for a feasible point to be an efficient (or local efficient) solution of the (MFPVC). A parametric dual model has been formulated and duality results are established with the primal (MFPVC).

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1. Introduction

A number of decision making problems from real life can be formulated as a fractional programming problem (see, *e.g.* [9]). A vast literature is available for minimax fractional programs [2, 24–26, 28], differentiable multiobjective fractional programs [5, 7, 23, 37], nondifferentiable multiobjective fractional programs with support functions [19, 22, 29, 30, 43], multiobjective fractional programs with nonsmooth generalized convexity [4, 6, 8, 10, 11, 17, 18, 20, 27, 34, 38–42].

On the other hand, mathematical programs with vanishing constraints (MPVC) are the focus of many researchers in the recent past decade. Achtziger and Kanzow [1], Hoheisel and Kanzow [12], Hoheisel and Kanzow [13] initially derived optimality conditions for MPVCs. Abadie and Guignard constraint qualifications for MPVC were studied by Hoheisel and Kanzow [14]. Mishra *et al.* [36] gave duality results for MPVC which were extended by Hu *et al.* [15]. Mishra *et al.* [35] initiated the study of multibjective optimization problems with vanishing constraints (MOPVC) and later some important results were derived by Maurya *et al.* [31], Laha *et al.* [21] etc.

The aim of this article is to study multiobjective fractional programs with vanishing constraints and to derive optimality conditions for the same. The outline of this paper is as follows: in Section 2, we give some preliminary results which will be used in the sequel. In Section 3, we derive necessary and sufficient optimality conditions for a feasible solution of the multiobjective fractional program with vanishing constraints to be an efficient (or

Keywords. Fractional programming, multiobjective optimization, vanishing constraints, generalized convexity, optimality conditions, duality results.

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a local efficient) solution. In Section 4, we formulate a dual model for the multiobjective fractional programs with vanishing constraints and derive some duality results. Section 5 concludes the results of this paper and discuss some future research possibilities.

2. Preliminaries

Mishra *et al.* [35] studied the following multiobjective optimization problem with vanishing constraints (MOPVC):

$$\begin{aligned} \text{(MOPVC)} \quad \min \quad & f(x) := (f_1(x), \dots, f_m(x)) \\ \text{subject to } & H_j(x) \geq 0, \quad \forall j \in P := \{1, \dots, p\}, \\ & G_j(x) H_j(x) \leq 0, \quad \forall j \in P, \\ & \phi_l(x) \leq 0, \quad \forall l \in Q := \{1, \dots, q\}, \\ & \psi_t(x) = 0, \quad \forall t \in R := \{1, \dots, r\}, \end{aligned}$$

where $f_i, G_j, H_j, \phi_l, \psi_t : \mathbb{R}^n \to \mathbb{R},\ i \in M := \{1, 2, \dots, m\}, j \in P, l \in Q, t \in R$ are continuously differentiable functions.

The feasible region of MOPVC is given by

$$\begin{aligned} F := \{x \in \mathbb{R}^n : \quad & H_j(x) \geq 0, \quad G_j(x) H_j(x) \leq 0, \quad \forall j \in P, \\ & \phi_l(x) \leq 0, \quad \forall l \in Q, \quad \psi_t(x) = 0, \quad \forall t \in R\}. \end{aligned}$$

A point $\bar{x} \in F$ is an *efficient solution* of the MOPVC, iff $f(x) - f(\bar{x}) \notin -\mathbb{R}^m_+ \setminus \{0\}$. A point $\bar{x} \in F$ is a *weak efficient solution* of the MOPVC, iff $f(x) - f(\bar{x}) \notin -int\mathbb{R}^m_+$. The set of all efficient and weak efficient solutions of the MOPVC are denoted by F^E_{MOPVC} and F^{WE}_{MOPVC}, respectively.

Define

$$\begin{aligned} & Q^k(\bar{x}) := \{x \in F : f_i(x) \leq f_i(\bar{x}), \quad \forall i \in M, \quad i \neq k\}, \\ & I_+(\bar{x}) := \{j \in P : H_j(\bar{x}) > 0\}, \\ & I_0(\bar{x}) := \{j \in P : H_j(\bar{x}) = 0\}, \\ & I_{0+}(\bar{x}) := \{j \in P : H_j(\bar{x}) = 0, \quad G_j(\bar{x}) > 0\}, \\ & I_{0-}(\bar{x}) := \{j \in P : H_j(\bar{x}) = 0, \quad G_j(\bar{x}) < 0\}, \\ & I_{00}(\bar{x}) := \{j \in P : H_j(\bar{x}) = 0, \quad G_j(\bar{x}) = 0\}, \\ & I_{+0}(\bar{x}) := \{j \in P : H_j(\bar{x}) > 0, \quad G_j(\bar{x}) = 0\}, \\ & I_{+-}(\bar{x}) := \{j \in P : H_j(\bar{x}) > 0, \quad G_j(\bar{x}) < 0\}, \end{aligned}$$

and

$$\begin{aligned} Q^k_{MOPVC}(\bar{x}) := \{x \in \mathbb{R}^n : \quad & f_i(x) \leq f_i(\bar{x}), \quad \forall i \in M, \quad i \neq k, \\ & H_j(x) = 0, \quad G_j(x) \geq 0, \quad \forall j \in I_{0+}(\bar{x}), \\ & H_j(x) \geq 0, \quad G_j(x) \leq 0, \quad \forall j \in I_{0-}(\bar{x}) \cup I_{00}(\bar{x}) \cup I_{+0}(\bar{x}) \cup I_{+-}(\bar{x}), \\ & \phi_l(x) \leq 0, \quad \forall l \in Q, \quad \psi_t(x) = 0, \quad \forall t \in R\} \end{aligned}$$

for any $\bar{x} \in F$.

We say that *GGCQ-MOPVC* ([35], Def. 6.14) is satisfied at $\bar{x} \in F$, iff

$$\bigcap_{k=1}^{m} L\left(Q^k_{MOPVC}(\bar{x}); \bar{x}\right) \subseteq \bigcap_{k=1}^{m} clcoT\left(Q^k(\bar{x}); \bar{x}\right),$$

where the *linearizing cone* to $Q^k_{MOPVC}(\bar{x})$ at $\bar{x} \in F$ is given by

$$\begin{aligned} L\left(Q^k_{MOPVC}(\bar{x});\bar{x}\right) := \{d \in \mathbb{R}^n : & \langle \nabla f_i(\bar{x}), d\rangle \leq 0, \quad \forall i \in M, \quad i \neq k, \\ & \langle \nabla H_j(\bar{x}), d\rangle = 0, \quad \forall j \in I_{0+}(\bar{x}), \\ & \langle \nabla H_j(\bar{x}), d\rangle \geq 0, \quad \forall j \in I_{00}(\bar{x}) \cup I_{0-}(\bar{x}), \\ & \langle \nabla G_j(\bar{x}), d\rangle \leq 0, \quad \forall j \in I_{00}(\bar{x}) \cup I_{+0}(\bar{x}), \\ & \langle \nabla \phi_l(\bar{x}), d\rangle \leq 0, \quad \forall l \in Q, \quad \langle \nabla \psi_t(x), d\rangle = 0, \quad \forall t \in R\} \end{aligned}$$

and the *tangent cone* to $Q^k(\bar{x})$ at $\bar{x} \in clQ^k(\bar{x})$ is given by

$$T(Q^k(\bar{x});\bar{x}) := \left\{ d \in \mathbb{R}^n : \quad \exists \{x^n\} \subseteq Q^k(\bar{x}), \quad \{t_n\} \downarrow 0 \quad \text{such that} \quad x^n \to \bar{x} \quad \text{and} \quad \frac{x^n - \bar{x}}{t_n} \to d \right\}.$$

A KKT type necessary optimality condition for the MOPVC ([35], Thm. 6.4) is given as follows:

Theorem 2.1. *If $\bar{x} \in F^E_{MOPVC}$ such that GGCQ-MOPVC holds at $\bar{x}$, then there exist Lagrange multipliers $\lambda_i \in int\mathbb{R}_+, i \in M,\ \eta^H_j,\ \eta^G_j \in \mathbb{R}, j \in P, \mu_l \in \mathbb{R}, l \in Q, \rho_t \in \mathbb{R}, t \in R$ such that*

$$\sum_{i=1}^{m} \lambda_i \nabla f_i(\bar{x}) - \sum_{j=1}^{p} \eta^H_j \nabla H_j(\bar{x}) + \sum_{j=1}^{p} \eta^G_j \nabla G_j(\bar{x}) + \sum_{l=1}^{q} \mu_l \nabla \phi_l(\bar{x}) + \sum_{t=1}^{r} \rho_t \nabla \psi_t(\bar{x}) = 0$$

and

$$\begin{aligned} &\eta^H_j = 0\,(j \in I_+), \quad \eta^H_j \geq 0\,(j \in I_{00} \cup I_{0-}), \quad \eta^H_j\, free\,(j \in I_{0+}), \quad \eta^H_j H_j(\bar{x}) = 0\,(j \in P), \\ &\eta^G_j = 0\,(j \in I_{0+} \cup I_{0-} \cup I_{+-}), \quad \eta^G_j \geq 0\,(j \in I_{00} \cup I_{+0}), \quad \eta^G_j G_j(\bar{x}) = 0\,(j \in P), \\ &\mu_l \geq 0, \mu_l \phi_l(\bar{x}) = 0, l \in Q, \quad \rho_t \in \mathbb{R}, t \in R. \end{aligned}$$

3. Optimality conditions

Consider the following multiobjective fractional programming problem with vanishing constraints (MFPVC) as follows:

$$\min \quad \left(\frac{h}{g}\right)(x) := \left(\left(\frac{h_1}{g_1}\right)(x), \ldots, \left(\frac{h_m}{g_m}\right)(x)\right) \quad \text{subject to} \quad x \in F.$$

A point $\bar{x} \in F$ is an *efficient solution* of the MFPVC, iff $\left(\frac{h}{g}\right)(x) - \left(\frac{h}{g}\right)(\bar{x}) \notin -\mathbb{R}^m_+ \setminus \{0\}$. A point $\bar{x} \in F$ is a *weak efficient solution* of the MFPVC, iff $\left(\frac{h}{g}\right)(x) - \left(\frac{h}{g}\right)(\bar{x}) \notin -int\mathbb{R}^m_+$. The set of all efficient and weak efficient solutions of the MFPVC are denoted by F^E_{MFPVC} and F^{WE}_{MFPVC}, respectively. The set of all local efficient and local weak efficient solutions of the MFPVC are denoted by F^{LE}_{MFPVC} and F^{LWE}_{MFPVC}, respectively. For the sake of convenience, we assume that $g_i(x) > 0, i \in M$ for every $x \in F$.

For any parameter $\alpha := (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m_+$, define a parametric multiobjective optimization problem with vanishing constraints PMOPVC$_\alpha$ as follows:

$$\min\ f(x) := (f_1(x), \ldots, f_m(x)) \ \text{ subject to } \ x \in F,$$

where $f_i(x) := h_i(x) - \alpha_i g_i(x)$ for every $i \in M$.

Define

$$Q^k_\alpha(\bar{x}) := \{x \in F : h_i(x) - \alpha_i g_i(x) \leq h_i(\bar{x}) - \alpha_i g_i(\bar{x}), \quad \forall i \in M, \quad i \neq k\},$$

and

$$Q^k_{PMOPVC_\alpha}(\bar{x}) := \Big\{ x \in \mathbb{R}^n : h_i(x) - \alpha_i g_i(x) \le h_i(\bar{x}) - \alpha_i g_i(\bar{x}), \quad \forall i \in M, \quad i \ne k,$$
$$H_j(x) = 0, \quad G_j(x) \ge 0, \quad \forall j \in I_{0+}(\bar{x}),$$
$$H_j(x) \ge 0, \quad G_j(x) \le 0, \quad \forall j \in I_{0-}(\bar{x}) \cup I_{00}(\bar{x}) \cup I_{+0}(\bar{x}) \cup I_{+-}(\bar{x}),$$
$$\phi_l(x) \le 0, \quad \forall l \in Q, \quad \psi_t(x) = 0, \quad \forall t \in R \Big\}$$

for any $k \in M$, $\alpha := (\alpha_1, \dots, \alpha_m) \in \mathbb{R}^m_+$ and $\bar{x} \in F$.

Definition 3.1. We say that GGCQ-PMOPVC$_\alpha$ is satisfied at $\bar{x} \in F$ for any $\alpha := (\alpha_1, \dots, \alpha_m) \in \mathbb{R}^m_+$, iff

$$\bigcap_{k=1}^{m} L\left(Q^k_{PMOPVC_\alpha}(\bar{x})\,;\bar{x}\right) \subseteq \bigcap_{k=1}^{m} clcoT\left(Q^k_\alpha(\bar{x});\bar{x}\right),$$

where the linearizing cone to $Q^k_{PMOPVC_\alpha}(\bar{x})$ at $\bar{x} \in F$ is given by

$$L\left(Q^k_{PMOPVC_\alpha}(\bar{x});\bar{x}\right) := \Big\{ d \in \mathbb{R}^n : \langle \nabla h_i(\bar{x}), d\rangle \le \alpha_i \langle \nabla g_i(\bar{x}), d\rangle, \quad \forall i \in M, \quad i \ne k,$$
$$\langle \nabla H_j(\bar{x}), d\rangle = 0, \quad \forall j \in I_{0+}(\bar{x}),$$
$$\langle \nabla H_j(\bar{x}), d\rangle \ge 0, \quad \forall j \in I_{00}(\bar{x}) \cup I_{0-}(\bar{x}),$$
$$\langle \nabla G_j(\bar{x}), d\rangle \le 0, \quad \forall j \in I_{00}(\bar{x}) \cup I_{+0}(\bar{x}),$$
$$\langle \nabla \phi_l(\bar{x}), d\rangle \le 0, \quad \forall l \in Q, \quad \langle \nabla \psi_t(x), d\rangle = 0, \quad \forall t \in R \Big\}$$

and the tangent cone to $Q^k_\alpha(\bar{x})$ at $\bar{x} \in clQ^k_\alpha(\bar{x})$ is given by

$$T\left(Q^k_\alpha(\bar{x})\,;\bar{x}\right) := \left\{ d \in \mathbb{R}^n : \exists \{x^n\} \subseteq Q^k_\alpha(\bar{x}), \quad \{t_n\} \downarrow 0 \quad \text{such that} \quad x^n \to \bar{x} \quad \text{and} \quad \frac{x^n - \bar{x}}{t_n} \to d \right\}.$$

The following lemma (see, *e.g.* ([39], Lem. 3.1)) relates MFPVC and PMOPVC$_\alpha$.

Lemma 3.2. *If $\bar{x} \in F^E_{MFPVC}$, then $\bar{x} \in F^E_{PMOPVC_{\bar{\alpha}}}$, where $\bar{\alpha}_i := f_i(\bar{x}) = \left(\frac{h_i}{g_i}\right)(\bar{x})$ for every $i \in M$.*

The necessary optimality condition is given as follows:

Theorem 3.3. *If $\bar{x} \in F^E_{MFPVC}$ such that GGCQ-PMOPVC$_{\bar{\alpha}}$ is satisfied at $\bar{x}$, then there exist Lagrange multipliers $\lambda_i \in int\mathbb{R}_+, i \in M$, η^H_j, $\eta^G_j \in \mathbb{R}, j \in P, \mu_l \in \mathbb{R}, l \in Q, \rho_t \in \mathbb{R}, t \in R$ such that*

$$\begin{cases} \sum_{i=1}^{m} \lambda_i \left[\nabla h_i(\bar{x}) - \frac{h_i(\bar{x})}{g_i(\bar{x})} \nabla g_i(\bar{x})\right] - \sum_{j=1}^{p} \eta^H_j \nabla H_j(\bar{x}) + \sum_{j=1}^{p} \eta^G_j \nabla G_j(\bar{x}) \\ \quad + \sum_{l=1}^{q} \mu_l \nabla \phi_l(\bar{x}) + \sum_{t=1}^{r} \rho_t \nabla \psi_t(\bar{x}) = 0, \\ \eta^H_j = 0, j \in I_+, \quad \eta^H_j \ge 0, j \in I_{00} \cup I_{0-}, \\ \eta^H_j \, free, j \in I_{0+}, \quad \eta^H_j H_j(\bar{x}) = 0, j \in P, \\ \eta^G_j = 0, j \in I_{0+} \cup I_{0-} \cup I_{+-}, \quad \eta^G_j \ge 0, j \in I_{00} \cup I_{+0}, \\ \eta^G_j G_j(\bar{x}) = 0, j \in P, \\ \mu_l \ge 0, \mu_l \phi_l(\bar{x}) = 0, \rho_t \in \mathbb{R}. \end{cases} \tag{3.1}$$

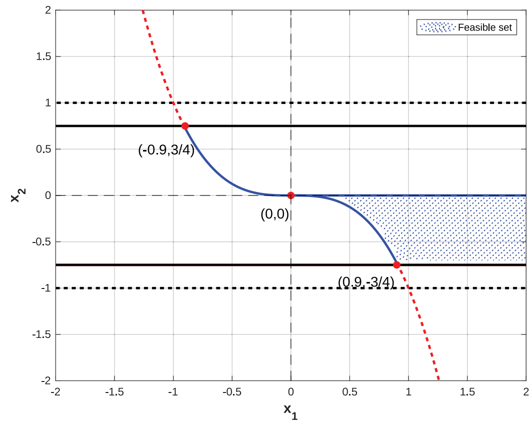

FIGURE 1. Feasible region of the MPVC of Example 3.4.

Proof. Since $\bar{x} \in F^E_{MFPVC}$ such that GGCQ-PMOPVC$_{\bar{\alpha}}$ is satisfied at $\bar{x}$, by Lemma 3.2 and Theorem 2.1, there exist Lagrange multipliers $\lambda_i \in int\mathbb{R}_+(i \in M)$, η_j^H, $\eta_j^G \in \mathbb{R}(j \in P), \mu_l \in \mathbb{R}, l \in Q, \rho_t \in \mathbb{R}, t \in R$ such that

$$\sum_{i=1}^{m} \lambda_i \left[\nabla h_i(\bar{x}) - \frac{h_i(\bar{x})}{g_i(\bar{x})}\nabla g_i(\bar{x})\right] - \sum_{j=1}^{p} \eta_j^H \nabla H_j(\bar{x}) + \sum_{j=1}^{p} \eta_j^G \nabla G_j(\bar{x})$$
$$+ \sum_{l=1}^{q} \mu_l \nabla\phi_l(\bar{x}) + \sum_{t=1}^{r} \rho_t \nabla\psi_t(\bar{x}) = 0$$

and

$$\eta_j^H = 0\,(j \in I_+)\,, \quad \eta_j^H \geq 0\,(j \in I_{00} \cup I_{0-})\,, \quad \eta_j^H \text{free}(j \in I_{0+}), \quad \eta_j^H H_j(\bar{x}) = 0\,(j \in P)\,,$$
$$\eta_j^G = 0\,(j \in I_{0+} \cup I_{0-} \cup I_{+-})\,, \quad \eta_j^G \geq 0\,(j \in I_{00} \cup I_{+0})\,, \quad \eta_j^G G_j(\bar{x}) = 0\,(j \in P)\,,$$
$$\mu_l \geq 0, \mu_l\phi_l(\bar{x}) = 0, l \in Q, \quad \rho_t \in \mathbb{R}, t \in R.$$

□

The following example illustrates the above result.

Example 3.4. Consider a MFPVC in $\mathbb{R}^2$ as follows:

$$\min \left(\frac{h}{g}\right)(x) := \left(\left(\frac{h_1}{g_1}\right)(x) := \frac{-x_1 - 1}{x_2 + 1}, \left(\frac{h_2}{g_2}\right)(x) := \frac{x_1 - 1}{1 - x_2^2}\right)$$

subject to

$$\phi_1(x) := x_2^2 - \frac{9}{16} \leq 0,$$
$$H_1(x) := x_1^3 + x_2 \geq 0,$$
$$G_1(x)H_1(x) := x_2\left(x_1^3 + x_2\right) \leq 0.$$

The feasible region of the MFPVC is shown in Figure 1. Observe that $\bar{x} := 0_{\mathbb{R}^2}$ is an efficient solution of the MFPVC. Now, $\bar{\alpha}_1 := \frac{h_1(\bar{x})}{g_1(\bar{x})} = -1$ and $\bar{\alpha}_2 := \frac{h_2(\bar{x})}{g_2(\bar{x})} = -1$. Hence, $f_1(x) := h_1(x) - \bar{\alpha}_1 g_1(x) = -x_1 + x_2$ and $f_2(x) := h_2(x) - \bar{\alpha}_2 g_2(x) = x_1 - x_2^2$. This leads us to the associated PMOPVC$_{\bar{\alpha}}$ as follows:

$$\min\left(-x_1 + x_2, x_1 - x_2^2\right) \quad \text{subject to} \quad x \in F.$$

Observe that

$$Q^1_{\bar\alpha}(\bar x) = \left\{x := (x_1, x_2) \in F : x_1 - x_2^2 \leq 0\right\};$$
$$Q^2_{\bar\alpha}(\bar x) = \{x := (x_1, x_2) \in F : -x_1 + x_2 \leq 0\};$$

which gives the associated tangent cones as

$$T\left(Q^1_{\bar\alpha}(\bar x); \bar x\right) = \Big\{d := (d_1, d_2) \in \mathbb{R}^2 : d_1 - d_2^2 \leq 0; \quad d_1^3 + d_2 \geq 0; \quad d_2\left(d_1^3 + d_2\right) \leq 0; \quad \frac{-3}{4} \leq d_2 \leq \frac{3}{4}\Big\};$$
$$T\left(Q^2_{\bar\alpha}(\bar x); \bar x\right) = \Big\{d := (d_1, d_2) \in \mathbb{R}^2 : -d_1 + d_2 \leq 0; \quad d_1^3 + d_2 \geq 0; \quad d_2\left(d_1^3 + d_2\right) \leq 0; \quad \frac{-3}{4} \leq d_2 \leq \frac{3}{4}\Big\};$$

Also,

$$L\left(Q^1_{PMOPVC_{\bar\alpha}}; \bar x\right) = \{0_{\mathbb{R}^2}\} = L\left(Q^2_{PMOPVC_{\bar\alpha}}; \bar x\right).$$

Since

$$L\left(Q^1_{PMOPVC_{\bar\alpha}}; \bar x\right) \cap L\left(Q^2_{PMOPVC_{\bar\alpha}}; \bar x\right) = \{0_{\mathbb{R}^2}\} = clcoT\left(Q^1_{\bar\alpha}(\bar x); \bar x\right) \cap clcoT\left(Q^2_{\bar\alpha}(\bar x); \bar x\right),$$

therefore $GGCQ - PMOPVC_{\bar\alpha}$ is satisfied at $\bar x$. By Theorem 3.3, there exist $\lambda_1, \lambda_2, \mu_1, \eta_1^H, \eta_1^G \in \mathbb{R}$ such that

$$\begin{aligned}
&\lambda_1\left(\nabla h_1(\bar x) - \bar\alpha_1 \nabla g_1(\bar x)\right) + \lambda_2\left(\nabla h_2(\bar x) - \bar\alpha_2 \nabla g_2(\bar x)\right) \\
&\quad + \mu_1 \nabla \phi_1(\bar x) - \eta_1^H \nabla H_1(\bar x) + \eta_1^g \nabla G_1(\bar x) = 0; \\
&\lambda_1, \lambda_2 > 0; \quad \mu_1 \phi_1(\bar x) = 0, \mu_1 \geq 0; \\
&\eta_1^H, \eta_1^G \geq 0,
\end{aligned}$$

which gives

$$\begin{aligned}
&-\lambda_1 + \lambda_2 = 0; \quad \lambda_1 - \eta_1^H + \eta_1^G = 0; \\
&\lambda_1, \lambda_2 > 0; \quad \mu_1 = 0; \quad \eta_1^H, \eta_1^G \geq 0,
\end{aligned}$$

which is solvable for $\lambda_1 = \frac{1}{2} = \lambda_2; \eta_1^H = 1; \eta_1^G = \frac{1}{2}$.

Define the following indices

$$\begin{aligned}
&I_{00}^{+\bullet} := \left\{j \in I_{00} : \eta_j^H > 0\right\}, \ I_{0-}^{+\bullet} := \left\{j \in I_{0-} : \eta_j^H > 0\right\}, \\
&I_{0+}^{+\bullet} := \left\{j \in I_{0+} : \eta_j^H > 0\right\}, \ I_{0+}^{-\bullet} := \{j \in I_{0+} : \eta_j^H < 0\}, \\
&I_{00}^{\bullet +} := \{j \in I_{00} : \eta_j^G > 0\}, \ I_{+0}^{\bullet +} := \left\{j \in I_{+0} : \eta_j^G > 0\right\}, \\
&Q(\bar x) := \{l \in Q : \phi_l(\bar x) = 0\}, \ Q^+ := \{l \in Q(\bar x) : \mu_l > 0\}, \\
&R^+ := \{t \in R : \rho_t > 0\}, \ R^- := \{t \in R : \rho_t < 0\}.
\end{aligned}$$

On the lines of Jeyakumar and Mond [16], we may assume that the PMOPVC$_\alpha$ is V-invex, that is, there exist a vector valued function $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and scalar valued functions $\gamma_i, \beta_j^H, \beta_j^G, \sigma_l, \zeta_t : \mathbb{R}^n \times \mathbb{R}^n \to$

$\mathbb{R}_+ \setminus \{0\}, i \in M, j \in P, l \in Q, t \in R$ such that

$$x, y \in F \implies \begin{cases} \gamma_i(x,y)\left[(h_i(x) - \alpha_i g_i(x)) - (h_i(y) - \alpha_i g_i(y))\right] \\ \geq \langle \nabla h_i(y) - \alpha_i \nabla g_i(y), \eta(x,y)\rangle, \quad i \in M, \\ \beta_j^H(x,y)\left[H_j(x) - H_j(y)\right] \leq \langle \nabla H_j(y), \eta(x,y)\rangle, \quad j \in I_{00}^{+\bullet} \cup I_{0-}^{+\bullet} \cup I_{0+}^{+\bullet}, \\ \beta_j^H(x,y)\left[H_j(x) - H_j(y)\right] \geq \langle \nabla H_j(y), \eta(x,y)\rangle, \quad j \in I_{0+}^{-\bullet}, \\ \beta_j^G(x,y)\left[G_j(x) - G_j(y)\right] \geq \langle \nabla G_j(y), \eta(x,y)\rangle, \quad j \in I_{00}^{\bullet +} \cup I_{+0}^{\bullet +}, \\ \sigma_l(x,y)\left[\phi_l(x) - \phi_l(y)\right] \geq \langle \nabla \phi_l(y), \eta(x,y)\rangle, \quad l \in Q^+, \\ \zeta_t(x,y)\left[\psi_t(x) - \psi_t(y)\right] \geq \langle \nabla \psi_t(y), \eta(x,y)\rangle, \quad t \in R^+, \\ \zeta_t(x,y)\left[\psi_t(x) - \psi_t(y)\right] \leq \langle \nabla \psi_t(y), \eta(x,y)\rangle, \quad t \in R^-. \end{cases}$$

Example 3.5. Consider the MFPVC of Example 3.4. Observe that

$$\begin{cases} \left[(h_i(x) - \bar{\alpha}_i g_i(x)) - (h_i(\bar{x}) - \bar{\alpha}_i g_i(\bar{x}))\right] \\ - \langle \nabla h_i(\bar{x}) - \bar{\alpha}_i \nabla g_i(\bar{x}), \eta(x,\bar{x})\rangle \quad \geq 0, \quad i \in \{1,2\}, \\ \phi_1(x) - \phi_1(\bar{x}) - \langle \nabla \phi_1(\bar{x}), \eta(x,\bar{x})\rangle \quad \geq 0, \\ H_1(x) - H_1(\bar{x}) - \langle \nabla H_1(\bar{x}), \eta(x,\bar{x})\rangle \leq 0, \\ G_1(x) - G_1(\bar{x}) - \langle \nabla G_1(\bar{x}), \eta(x,\bar{x})\rangle \geq 0, \end{cases} \tag{3.2}$$

which implies that

$$\begin{cases} -x_1 + x_2 + \eta_1(x,\bar{x}) - \eta_2(x,\bar{x}) \geq 0, \\ \eta_1(x,\bar{x}) \leq x_1 - x_2^2, \\ \eta_2(x,\bar{x}) \geq x_1^3 + x_2, \\ \eta_2(x,\bar{x}) \leq x_2, \end{cases} \tag{3.3}$$

where $\eta(x,\bar{x}) := (\eta_1(x,\bar{x}), \eta_2(x,\bar{x}))$. If we choose $\eta_1(x,\bar{x}) = x_1 - x_2^2$ and $x_1^3 + x_2 \leq \eta_2(x,\bar{x}) \leq x_2 - x_2^2$ along with $\gamma_1(x,\bar{x}) = \gamma_2(x,\bar{x}) = \beta_1^H(x,\bar{x}) = \beta_1^G(x,\bar{x}) = \sigma_1(x,\bar{x}) = 1$, then all the conditions of V-invexity will be satisfied.

Now, we are ready to prove a sufficient optimality condition.

Theorem 3.6. *Let $\bar{x} \in F$ for which there exist $\lambda_i \in int\mathbb{R}_+ (i \in M), \eta_j^H, \eta_j^G \in \mathbb{R} (j \in P), \mu_l \in \mathbb{R}, l \in Q, \rho_t \in \mathbb{R}, t \in R$ such that*

$$\sum_{i=1}^{m} \lambda_i \Big[\nabla h_i(\bar{x}) - \frac{h_i(\bar{x})}{g_i(\bar{x})} \nabla g_i(\bar{x})\Big] - \sum_{j=1}^{p} \eta_j^H \nabla H_j(\bar{x}) + \sum_{j=1}^{p} \eta_j^G \nabla G_j(\bar{x}) + \sum_{l=1}^{q} \mu_l \nabla \phi_l(\bar{x}) + \sum_{t=1}^{r} \rho_t \nabla \psi_t(\bar{x}) = 0$$

and

$\eta_j^H = 0\,(j \in I_+), \eta_j^H \geq 0\,(j \in I_{00} \cup I_{0-}), \eta_j^H \mathit{free}\,(j \in I_{0+}), \eta_j^H H_j(\bar{x}) = 0\,(j \in P),$
$\eta_j^G = 0\,(j \in I_{0+} \cup I_{0-} \cup I_{+-}),\ \eta_j^G \geq 0\,(j \in I_{00} \cup I_{+0}), \eta_j^G G_j(\bar{x}) = 0\,(j \in P),$
$\mu_l \geq 0, \mu_l \phi_l(\bar{x}) = 0, \rho_t \in \mathbb{R}.$

Further, assume that there exist a vector valued function $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and scalar valued functions $\gamma_i, \beta_j^H, \beta_j^G, \sigma_l, \zeta_t : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+ \setminus \{0\}, i \in M, j \in P, l \in Q, t \in R$ such that for every $x \in F$, one has

$$\begin{cases} \gamma_i(x,\bar{x})\left[(h_i(x) - \alpha_i g_i(x)) - (h_i(\bar{x}) - \alpha_i g_i(\bar{x}))\right] & \\ \geq \langle \nabla h_i(\bar{x}) - \alpha_i \nabla g_i(\bar{x}), \eta(x,\bar{x})\rangle, & i \in M, \\ \beta_j^H(x,\bar{x})\left[H_j(x) - H_j(\bar{x})\right] \leq \langle \nabla H_j(\bar{x}), \eta(x,\bar{x})\rangle, & j \in I_{00}^{+\bullet} \cup I_{0-}^{+\bullet} \cup I_{0+}^{+\bullet}, \\ \beta_j^H(x,\bar{x})\left[H_j(x) - H_j(\bar{x})\right] \geq \langle \nabla H_j(\bar{x}), \eta(x,\bar{x})\rangle, & j \in I_{0+}^{-\bullet}, \\ \beta_j^G(x,\bar{x})\left[G_j(x) - G_j(\bar{x})\right] \geq \langle \nabla G_j(\bar{x}), \eta(x,\bar{x})\rangle, & j \in I_{00}^{\bullet+} \cup I_{+0}^{\bullet+}, \\ \sigma_l(x,\bar{x})\left[\phi_l(x) - \phi_l(\bar{x})\right] \geq \langle \nabla \phi_l(\bar{x}), \eta(x,\bar{x})\rangle, & l \in Q^+, \\ \zeta_t(x,\bar{x})\left[\psi_t(x) - \psi_t(\bar{x})\right] \geq \langle \nabla \psi_t(\bar{x}), \eta(x,\bar{x})\rangle, & t \in R^+, \\ \zeta_t(x,\bar{x})\left[\psi_t(x) - \psi_t(\bar{x})\right] \leq \langle \nabla \psi_t(\bar{x}), \eta(x,\bar{x})\rangle, & t \in R^-. \end{cases}$$

Then,

(a) *if $I_{0+}^{-\bullet} \cup I_{00}^{\bullet+} \cup I_{+0}^{\bullet+} = \emptyset$, then $\bar{x} \in F_{MFPVC}^{E}$;*
(b) *if $I_{00}^{\bullet+} = \emptyset$, then $\bar{x} \in F_{MFPVC}^{LE}$;*
(c) *if $\bar{x}$ is an interior point with respect to the set $F \cap \{x \in \mathbb{R}^n : H_j(x) = 0, G_j(x) = 0, j \in I_{00}^{\bullet+}\}$, then $\bar{x} \in F_{MFPVC}^{LE}$.*

Proof. (a) Suppose $\bar{x} \notin F_{MFPVC}^{E}$. Then, there exists $y \in F$ such that

$$\frac{h_i(y)}{g_i(y)} \leq \frac{h_i(\bar{x})}{g_i(\bar{x})}, \quad \forall i \in M,$$

with strict inequality for at least one $i \in M$. That is,

$$h_i(y) - \bar{\alpha}_i g_i(y) \leq h_i(\bar{x}) - \bar{\alpha}_i g_i(\bar{x}), \quad \forall i \in M, \tag{3.4}$$

with strict inequality for at least one $i \in M$. Multiplying both sides of (3.4) by $\lambda_i \gamma_i(y,\bar{x}), i \in M$ and adding, one has

$$\sum_{i \in M} \lambda_i \gamma_i(y,\bar{x})\left[(h_i - \bar{\alpha}_i g_i)(y) - (h_i - \bar{\alpha}_i g_i)(\bar{x})\right] < 0. \tag{3.5}$$

Since $\lambda_i > 0 (i \in M), \eta_j^H > 0 (j \in I_{00}^{+\bullet} \cup I_{0-}^{+\bullet} \cup I_{0+}^{+\bullet}), \eta_j^H < 0 (j \in I_{0+}^{-\bullet}), \eta_j^G > 0 (j \in I_{00}^{\bullet+} \cup I_{+0}^{\bullet+}), \mu_l > 0, l \in Q^+, \rho_t > 0, t \in R^+, \rho_t < 0, t \in R^-$, and by V-invexity of PMOPVC$_{\bar{\alpha}}$, one has

$$\begin{cases} \lambda_i \gamma_i(y,\bar{x})\left[(h_i(y) - \alpha_i g_i(y)) - (h_i(\bar{x}) - \alpha_i g_i(\bar{x}))\right] & \\ \geq \langle \lambda_i (\nabla h_i(\bar{x}) - \alpha_i \nabla g_i(\bar{x})), \eta(y,\bar{x})\rangle, & i \in M, \\ -\eta_j^H \beta_j^H(y,\bar{x})\left[H_j(y) - H_j(\bar{x})\right] \geq \left\langle -\eta_j^H \nabla H_j(\bar{x}), \eta(y,\bar{x})\right\rangle, & j \in I_{00}^{+\bullet} \cup I_{0-}^{+\bullet} \cup I_{0+}^{+\bullet}, \\ -\eta_j^H \beta_j^H(y,\bar{x})\left[H_j(y) - H_j(\bar{x})\right] \geq \left\langle -\eta_j^H \nabla H_j(\bar{x}), \eta(y,\bar{x})\right\rangle, & j \in I_{0+}^{-\bullet}, \\ \eta_j^G \beta_j^G(y,\bar{x})\left[G_j(y) - G_j(\bar{x})\right] \geq \left\langle \eta_j^G \nabla G_j(\bar{x}), \eta(y,\bar{x})\right\rangle, & j \in I_{00}^{\bullet+} \cup I_{+0}^{\bullet+}, \\ \mu_l \sigma_l(y,\bar{x})\left[\phi_l(y) - \phi_l(\bar{x})\right] \geq \langle \mu_l \nabla \phi_l(\bar{x}), \eta(y,\bar{x})\rangle, & l \in Q^+, \\ \rho_t \zeta_t(y,\bar{x})\left[\psi_t(y) - \psi_t(\bar{x})\right] \geq \langle \rho_t \nabla \psi_t(\bar{x}), \eta(y,\bar{x})\rangle, & t \in R^+, \\ \rho_t \zeta_t(y,\bar{x})\left[\psi_t(y) - \psi_t(\bar{x})\right] \geq \langle \rho_t \nabla \psi_t(\bar{x}), \eta(y,\bar{x})\rangle, & t \in R^-. \end{cases}$$

Since $y, \bar{x} \in F$, using the definition of indices, one has

$$\begin{cases} \lambda_i \gamma_i(y,\bar{x}) \left[(h_i(y) - \alpha_i g_i(y)) - (h_i(\bar{x}) - \alpha_i g_i(\bar{x})) \right] \\ \geq \langle \lambda_i \left(\nabla h_i(\bar{x}) - \alpha_i \nabla g_i(\bar{x}) \right), \eta(y,\bar{x}) \rangle, & i \in M, \\ 0 \geq \left\langle -\eta_j^H \nabla H_j(\bar{x}), \eta(y,\bar{x}) \right\rangle, & j \in I_{00}^{+\bullet} \cup I_{0-}^{+\bullet} \cup I_{0+}^{+\bullet}, \\ -\eta_j^H \beta_j^H(y,\bar{x}) \left[H_j(y) \right] \geq \left\langle -\eta_j^H \nabla H_j(\bar{x}), \eta(y,\bar{x}) \right\rangle, & j \in I_{0+}^{-\bullet}, \\ \eta_j^G \beta_j^G(y,\bar{x}) \left[G_j(y) \right] \geq \left\langle \eta_j^G \nabla G_j(\bar{x}), \eta(y,\bar{x}) \right\rangle, & j \in I_{00}^{\bullet +} \cup I_{+0}^{\bullet +}, \\ 0 \geq \langle \mu_l \nabla \phi_l(\bar{x}), \eta(y,\bar{x}) \rangle, & l \in Q^+, \\ 0 \geq \langle \rho_t \nabla \psi_t(\bar{x}), \eta(y,\bar{x}) \rangle, & t \in R^+, \\ 0 \geq \langle \rho_t \nabla \psi_t(\bar{x}), \eta(y,\bar{x}) \rangle, & t \in R^-. \end{cases} \tag{3.6}$$

Since $I_{0+}^{-\bullet} \cup I_{00}^{\bullet +} \cup I_{+0}^{\bullet +} = \emptyset$, therefore

$$\begin{cases} \sum_{i \in M} \lambda_i \gamma_i(y,\bar{x}) \left[(h_i(y) - \alpha_i g_i(y)) - (h_i(\bar{x}) - \alpha_i g_i(\bar{x})) \right] \\ \geq \left\langle \sum_{i \in M} \lambda_i \left(\nabla h_i(\bar{x}) - \alpha_i \nabla g_i(\bar{x}) \right) - \sum_{j \in I_{00}^{+\bullet} \cup I_{0-}^{+\bullet} \cup I_{0+}^{+\bullet}} \eta_j^H \nabla H_j(\bar{x}), \eta(y,\bar{x}) \right\rangle \\ + \left\langle \sum_{l \in Q^+} \mu_l \nabla \phi_l(\bar{x}) + \sum_{t \in R^+ \cup R^-} \rho_t \nabla \psi_t(\bar{x}), \eta(y,\bar{x}) \right\rangle \\ = 0 \end{cases}$$

which is a contradiction to (3.5). Hence, $\bar{x} \in F_{MFPVC}^{E}$.

(b) Here $I_{00}^{\bullet +} = \emptyset$. Assume that $\bar{x} \notin F_{MFPVC}^{LE}$. Then, for any $\epsilon > 0$, there exist $y \in B(\bar{x}, \epsilon) \cap F$ such that (3.4) holds. For $j \in I_{+0}^{\bullet +}, H_j(\bar{x}) > 0$ and $G_j(\bar{x}) = 0$, which implies that $H_j(x) > 0$ for any x sufficiently close to $\bar{x}$ and hence $G_j(\bar{x}) \leq 0$ for any x sufficiently close to $\bar{x}$. Now, choose $\epsilon > 0$ in such a way that $y \in B(\bar{x}, \epsilon) \cap F$ is sufficiently close to $\bar{x}$ and hence $G_j(y) \leq 0$.

Similarly, for $j \in I_{0+}^{-\bullet}, G_j(\bar{x}) > 0$ and $H_j(\bar{x}) = 0$, which implies that $G_j(x) > 0$ for any x sufficiently close to $\bar{x}$ and hence $H_j(\bar{x}) \leq 0$ for any x sufficiently close to $\bar{x}$. Now, choose $\epsilon > 0$ in such a way that $y \in B(\bar{x}, \epsilon) \cap F$ is sufficiently close to $\bar{x}$ and hence $H_j(y) \leq 0$. But $H_j(y) \geq 0$, which gives $H_j(y) = 0$. Thus, from (3.6), one has

$$\begin{cases} \lambda_i \gamma_i(y,\bar{x}) \left[(h_i(y) - \alpha_i g_i(y)) - (h_i(\bar{x}) - \alpha_i g_i(\bar{x})) \right] \\ \geq \langle \lambda_i \left(\nabla h_i(\bar{x}) - \alpha_i \nabla g_i(\bar{x}) \right), \eta(y,\bar{x}) \rangle, & i \in M, \\ 0 \geq \left\langle -\eta_j^H \nabla H_j(\bar{x}), \eta(y,\bar{x}) \right\rangle, & j \in I_{00}^{+\bullet} \cup I_{0-}^{+\bullet} \cup I_{0+}^{+\bullet}, \\ 0 \geq \left\langle -\eta_j^H \nabla H_j(\bar{x}), \eta(y,\bar{x}) \right\rangle, & j \in I_{0+}^{-\bullet}, \\ 0 \geq \left\langle \eta_j^G \nabla G_j(\bar{x}), \eta(y,\bar{x}) \right\rangle, & j \in I_{00}^{\bullet +} \cup I_{+0}^{\bullet +}, \\ 0 \geq \langle \mu_l \nabla \phi_l(\bar{x}), \eta(y,\bar{x}) \rangle, & l \in Q^+, \\ 0 \geq \langle \rho_t \nabla \psi_t(\bar{x}), \eta(y,\bar{x}) \rangle, & t \in R^+, \\ 0 \geq \langle \rho_t \nabla \psi_t(\bar{x}), \eta(y,\bar{x}) \rangle, & t \in R^-. \end{cases}$$

Since $I_{00}^{\bullet +} = \emptyset$, one has

$$\begin{cases} \sum_{i \in M} \lambda_i \gamma_i(y,\bar{x}) \left[(h_i(y) - \alpha_i g_i(y)) - (h_i(\bar{x}) - \alpha_i g_i(\bar{x})) \right] \\ \quad \geq \left\langle \sum_{i \in M} \lambda_i \left(\nabla h_i(\bar{x}) - \alpha_i \nabla g_i(\bar{x}) \right) - \sum_{j \in I_{00}^{+\bullet} \cup I_{0-}^{+\bullet} \cup I_{0+}^{+\bullet} \cup I_{0+}^{-\bullet}} \eta_j^H \nabla H_j(\bar{x}) + \sum_{j \in I_{+0}^{\bullet +}} \eta_j^G \nabla G_j(\bar{x}), \eta(y,\bar{x}) \right\rangle, \\ \qquad + \left\langle \sum_{l \in Q^+} \mu_l \nabla \phi_l(\bar{x}) + \sum_{t \in R^+ \cup R^-} \rho_t \nabla \psi_t(\bar{x}), \eta(y,\bar{x}) \right\rangle \\ = 0 \end{cases}$$

which is a contradiction to (3.5). Hence, $\bar{x} \in F^{LE}_{MFPVC}$.
(c) Again, assume that $\bar{x} \notin F^{LE}_{MFPVC}$. Then, for any $\epsilon > 0$, there exist $y \in B(\bar{x}, \epsilon) \cap F$ such that (3.4) holds. Since $\bar{x}$ is an interior point with respect to the set $F \cap \{x \in \mathbb{R}^n : H_j(x) = 0, G_j(x) = 0, j \in I^{\bullet +}_{00}\}$. Hence, for any $x \in F$ sufficiently close to $\bar{x}$, one has

$$H_j(x) = 0, \quad G_j(x) = 0, \quad j \in I^{\bullet +}_{00}.$$

Now, choose $\epsilon > 0$ in such a way that $y \in B(\bar{x}, \epsilon) \cap F$ is sufficiently close to $\bar{x}$ and hence $H_j(y) = 0, G_j(y) = 0, j \in I^{\bullet +}_{00}$. Thus, from (3.6), one has

$$\begin{cases} \lambda_i \gamma_i(y,\bar{x}) \left[(h_i(y) - \alpha_i g_i(y)) - (h_i(\bar{x}) - \alpha_i g_i(\bar{x}))\right] & \\ \geq \left\langle \lambda_i \left(\nabla h_i(\bar{x}) - \alpha_i \nabla g_i(\bar{x})\right), \eta(y,\bar{x}) \right\rangle, & i \in M, \\ 0 \geq \left\langle -\eta^H_j \nabla H_j(\bar{x}), \eta(y,\bar{x}) \right\rangle, & j \in I^{+\bullet}_{00} \cup I^{+\bullet}_{0-} \cup I^{+\bullet}_{0+}, \\ 0 \geq \left\langle -\eta^H_j \nabla H_j(\bar{x}), \eta(y,\bar{x}) \right\rangle, & j \in I^{-\bullet}_{0+}, \\ 0 \geq \left\langle \eta^G_j \nabla G_j(\bar{x}), \eta(y,\bar{x}) \right\rangle, & j \in I^{\bullet +}_{00} \cup I^{\bullet +}_{+0}, \\ 0 \geq \left\langle \mu_l \nabla \phi_l(\bar{x}), \eta(y,\bar{x}) \right\rangle, & l \in Q^+, \\ 0 \geq \left\langle \rho_t \nabla \psi_t(\bar{x}), \eta(y,\bar{x}) \right\rangle, & t \in R^+, \\ 0 \geq \left\langle \rho_t \nabla \psi_t(\bar{x}), \eta(y,\bar{x}) \right\rangle, & t \in R^-, \end{cases}$$

which gives

$$\begin{cases} \sum_{i \in M} \lambda_i \gamma_i(y,\bar{x}) \left[(h_i(y) - \alpha_i g_i(y)) - (h_i(\bar{x}) - \alpha_i g_i(\bar{x}))\right] \\ \quad \geq \left\langle \sum_{i \in M} \lambda_i \left(\nabla h_i(\bar{x}) - \alpha_i \nabla g_i(\bar{x})\right) - \sum_{j \in I^{+\bullet}_{00} \cup I^{+\bullet}_{0-} \cup I^{+\bullet}_{0+} \cup I^{-\bullet}_{0+}} \eta^H_j \nabla H_j(\bar{x}) + \sum_{j \in I^{\bullet +}_{+0} \cup I^{\bullet +}_{00}} \eta^G_j \nabla G_j(\bar{x}), \eta(y,\bar{x}) \right\rangle \\ \quad\quad + \left\langle \sum_{l \in Q^+} \mu_l \nabla \phi_l(\bar{x}) + \sum_{t \in R^+ \cup R^-} \rho_t \nabla \psi_t(\bar{x}), \eta(y,\bar{x}) \right\rangle \\ = 0 \end{cases}$$

which is a contradiction to (3.5). Hence, $\bar{x} \in F^{LE}_{MFPVC}$. This completes the proof. □

4. Duality

We consider the parametric vector dual model for the MFPVC, denoted by (MPD-VC(x)), depending upon $x \in F$, as follows:

$$\max \ \alpha := (\alpha_1, \dots, \alpha_m)$$

subject to

$$\sum_{i=1}^{m} \lambda_i \Big[\nabla h_i(y) - \alpha_i \nabla g_i(y)\Big] - \sum_{j=1}^{p} \eta^H_j \nabla H_j(y) + \sum_{j=1}^{p} \eta^G_j \nabla G_j(y) + \sum_{l=1}^{q} \mu_l \nabla \phi_l(y) + \sum_{t=1}^{r} \rho_t \nabla \psi_t(y) = 0, \tag{4.1}$$

$$\lambda \geq 0, \ h_i(y) - \alpha_i g_i(y) \geqq 0, \ i = 1, \dots, m, \tag{4.2}$$

$$-\eta^H_j H_j(y) \geq 0, \ \forall j \in P, \eta^H_j \geq 0, \ \forall j \in I_+(x), \eta^H_j \in \mathbb{R}, \ \forall j \in I_0(x), \tag{4.3}$$

$$\begin{aligned} &\eta^G_j G_j(y) \geq 0, \ \forall j \in P, \ \eta^G_j \geq 0, \ \forall j \in I_{0-}(x) \cup I_{+-}(x), \\ &\quad \eta^G_j \leq 0, \ \forall j \in I_{0+}(x), \eta^G_j \in \mathbb{R}, \forall j \in I_{+0}(x) \cup I_{00}(x), \end{aligned} \tag{4.4}$$

$$\mu_l \phi_l(y) \geq 0, \quad \mu_l \geq 0, \quad \forall l \in Q, \tag{4.5}$$

$$\rho_t \psi_t(y) \geq 0, \quad \forall t \in R. \tag{4.6}$$

The set of all feasible solutions of the (MPD-VC(x)) is given by $F_D(x) \subseteq \mathbb{R}^{n+2m+2p+q+r}$ and the projection of $F_D(x)$ on $\mathbb{R}^n$ is given by

$$pr_{\mathbb{R}^n} F_D(x) := \left\{ y \in \mathbb{R}^n : (y, \alpha, \lambda, \eta^H, \eta^G, \mu, \rho) \in F_D(x) \right\}.$$

Also, let us denote $pr_{\mathbb{R}^n} F_D(x) = Y_D$.

The following theorem establish weak duality result between (MFPVC) and (MPD-VC(x)).

Theorem 4.1 (Weak duality)**.** *Let $x \in F$ and let $(y, \alpha, \lambda, \eta^H, \eta^G, \mu, \rho) \in \mathbb{R}^{n+2m+2p+q+r}$ be a feasible solution to the problem (MPD-VC(x)). Further, suppose that there exist a vector valued function $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and scalar valued functions $\gamma_i, \beta_j^H, \beta_j^G, \sigma_l, \zeta_t : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+ \setminus \{0\}, i \in M, j \in P, l \in Q, t \in R$ such that*

$$x, y \in F \cup Y_D \implies \begin{cases} \gamma_i(x,y)\left[(h_i(x) - \alpha_i g_i(x)) - (h_i(y) - \alpha_i g_i(y))\right] & \\ \geq \langle \nabla h_i(y) - \alpha_i \nabla g_i(y), \eta(x,y) \rangle, & i \in M, \\ \beta_j^H(x,y)\left[H_j(x) - H_j(y)\right] \leq \langle \nabla H_j(y), \eta(x,y) \rangle, & j \in I_{00}^{+\bullet} \cup I_{0-}^{+\bullet} \cup I_{0+}^{+\bullet}, \\ \beta_j^H(x,y)\left[H_j(x) - H_j(y)\right] \geq \langle \nabla H_j(y), \eta(x,y) \rangle, & j \in I_{0+}^{-\bullet}, \\ \beta_j^G(x,y)\left[G_j(x) - G_j(y)\right] \geq \langle \nabla G_j(y), \eta(x,y) \rangle, & j \in I_{00}^{\bullet+} \cup I_{+0}^{\bullet+}, \\ \sigma_l(x,\bar{x})\left[\phi_l(x) - \phi_l(\bar{x})\right] \geq \langle \nabla \phi_l(\bar{x}), \eta(x,\bar{x}) \rangle, & l \in Q^+, \\ \zeta_t(x,\bar{x})\left[\psi_t(x) - \psi_t(\bar{x})\right] \geq \langle \nabla \psi_t(\bar{x}), \eta(x,\bar{x}) \rangle, & t \in R^+, \\ \zeta_t(x,\bar{x})\left[\psi_t(x) - \psi_t(\bar{x})\right] \leq \langle \nabla \psi_t(\bar{x}), \eta(x,\bar{x}) \rangle, & t \in R^-, \end{cases}$$

holds. Then, $\Phi(x) := \frac{h(x)}{g(x)} \not< \alpha$.

Proof. Let x and $(y, \alpha, \lambda, \eta^H, \eta^G, \mu, \rho)$ be feasible solutions to the problems (MFPVC) and (MPD-VC(x)), respectively. This implies that

$$\begin{cases} -\eta_j^H H_j(x) \leq 0 \leq -\eta_j^H H_j(y), & \forall j \in I_+(x), \\ -\eta_j^H H_j(x) = 0 \leq -\eta_j^H H_j(y), & \forall j \in I_0(x), \\ \eta_j^G G_j(x) \leq 0 \leq \eta_j^G G_j(y), & \forall j \in I_{+-}(x) \cup I_{0-}(x) \cup I_{0+}(x), \\ \eta_j^G G_j(x) = 0 \leq \eta_j^G G_j(y), & \forall j \in I_{+0}(x) \cup I_{00}(x), \\ \mu_l \phi_l(x) \leq 0 \leq \mu_l \phi_l(y), & \forall l \in Q, \\ \rho_t \psi_t(x) = 0 \leq \rho_t \psi_t(y), & \forall t \in R. \end{cases} \tag{4.7}$$

Now, proceeding by contradiction, we suppose contrary to the result of the theorem, that

$$\Phi(x) = \frac{h(x)}{g(x)} < \alpha,$$

that is,

$$\frac{h_i(x)}{g_i(x)} < \alpha_i, \ \forall i \in M.$$

From $g_i(x) > 0, i \in M$, it follows that $h_i(x) - \alpha_i g_i(x) < 0, i \in M$. So, by condition (4.2) of (MPD-VC(x)), one has

$$(h_i(x) - \alpha_i g_i(x)) - (h_i(y) - \alpha_i g_i(y)) < 0, \forall i \in M. \tag{4.8}$$

Since $\lambda_i > 0 (i \in M), \eta_j^H > 0 (j \in I_{00}^{+\bullet} \cup I_{0-}^{+\bullet} \cup I_{0+}^{+\bullet}),\ \eta_j^H < 0 (j \in I_{0+}^{-\bullet}), \eta_j^G > 0 (j \in I_{00}^{\bullet+} \cup I_{+0}^{\bullet+}), \mu_l > 0, l \in Q^+, \rho_t > 0, t \in R^+, \rho_t < 0, t \in R^-$, therefore, the hypothesis of the theorem implies that

$$\begin{cases} \lambda_i \gamma_i(x,y)\left[(h_i(x) - \alpha_i g_i(x)) - (h_i(y) - \alpha_i g_i(y))\right] & \\ \geq \lambda_i \left\langle \nabla h_i(y) - \alpha_i \nabla g_i(y), \eta(x,y) \right\rangle, & i \in M, \\ \eta_j^H \beta_j^H(x,y)\left[H_j(x) - H_j(y)\right] \leq \eta_j^H \left\langle \nabla H_j(y), \eta(x,y) \right\rangle), & j \in I_{00}^{+\bullet} \cup I_{0-}^{+\bullet} \cup I_{0+}^{+\bullet}, \\ \eta_j^H \beta_j^H(x,y)\left[H_j(x) - H_j(y)\right] \leq \eta_j^H \left\langle \nabla H_j(y), \eta(x,y) \right\rangle), & j \in I_{0+}^{-\bullet}, \\ \eta_j^H \beta_j^G(x,y)\left[G_j(x) - G_j(y)\right] \geq \eta_j^H \left\langle \nabla G_j(y), \eta(x,y) \right\rangle), & j \in I_{00}^{\bullet+} \cup I_{+0}^{\bullet+}, \\ \mu_l \sigma_l(x,\bar{x})\left[\phi_l(x) - \phi_l(\bar{x})\right] \geq \left\langle \mu_l \nabla \phi_l(\bar{x}), \eta(x,\bar{x}) \right\rangle), & l \in Q^+, \\ \rho_t \zeta_t(x,\bar{x})\left[\psi_t(x) - \psi_t(\bar{x})\right] \geq \left\langle \rho_t \nabla \psi_t(\bar{x}), \eta(x,\bar{x}) \right\rangle), & t \in R^+, \\ \rho_t \zeta_t(x,\bar{x})\left[\psi_t(x) - \psi_t(\bar{x})\right] \geq \left\langle \rho_t \nabla \psi_t(\bar{x}), \eta(x,\bar{x}) \right\rangle), & t \in R^-. \end{cases} \tag{4.9}$$

Using the definition of indices and adding both sides of inequalities (4.9), and then using inequalities (4.7) and (4.8), one has

$$\sum\nolimits_{i=1}^{m} \lambda_i \Big[\nabla h_i(y) - \alpha_i \nabla g_i(y)\Big] - \sum\nolimits_{j=1}^{p} \eta_j^H \nabla H_j(y) + \sum_{j=1}^{p} \eta_j^G \nabla G_j(y) + \sum\nolimits_{l=1}^{q} \mu_l \nabla \phi_l(y) + \sum\nolimits_{t=1}^{r} \rho_t \nabla \psi_t(y) < 0,$$

which is contradiction to (4.1). This completes the proof. □

Similarly, we can prove following weak duality result under the stronger hypothesis.

Theorem 4.2 (Weak duality). *Let $x \in F$ and $(y, \alpha, \lambda, \eta^H, \eta^G, \mu, \rho) \in \mathbb{R}^{n+2m+2p+q+r}$ be a feasible solution to the problem (MPD-VC(x)). Further, suppose that there exist a vector valued function $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and scalar valued functions $\gamma_i, \beta_j^H, \beta_j^G : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+ \setminus \{0\}, i \in M, j \in P$ such that*

$$x, y \in F \cup Y_D \implies \begin{cases} \gamma_i(x,y)\left[(h_i(x) - \alpha_i g_i(x)) - (h_i(y) - \alpha_i g_i(y))\right] & \\ > \left\langle \nabla h_i(y) - \alpha_i \nabla g_i(y), \eta(x,y) \right\rangle, & i \in M, \\ \beta_j^H(x,y)\left[H_j(x) - H_j(y)\right] \leq \left\langle \nabla H_j(y), \eta(x,y) \right\rangle, & j \in I_{00}^{+\bullet} \cup I_{0-}^{+\bullet} \cup I_{0+}^{+\bullet}, \\ \beta_j^H(x,y)\left[H_j(x) - H_j(y)\right] \geq \left\langle \nabla H_j(y), \eta(x,y) \right\rangle, & j \in I_{0+}^{-\bullet}, \\ \beta_j^G(x,y)\left[G_j(x) - G_j(y)\right] \geq \left\langle \nabla G_j(y), \eta(x,y) \right\rangle, & j \in I_{00}^{\bullet+} \cup I_{+0}^{\bullet+} \\ \sigma_l(x,y)\left[\phi_l(x) - \phi_l(y)\right] \geq \left\langle \nabla \phi_l(y), \eta(x,y) \right\rangle, & l \in Q^+, \\ \zeta_t(x,y)\left[\psi_t(x) - \psi_t(y)\right] \geq \left\langle \nabla \psi_t(y), \eta(x,y) \right\rangle, & t \in R^+, \\ \zeta_t(x,y)\left[\psi_t(x) - \psi_t(y)\right] \leq \left\langle \nabla \psi_t(y), \eta(x,y) \right\rangle, & t \in R^-. \end{cases}$$

holds. Then, $\Phi(x) \nleq \alpha$.

The following theorem establish strong duality result between (MFPVC) and (MPD-VC(x)).

Theorem 4.3 (Strong duality). *Let $\bar{x}$ be a weak efficient solution of the (MFPVC) and constraint qualification (GGCQ-PMOPVC$_{\bar{\alpha}}$) is satisfied at $\bar{x}$ with $\bar{\alpha} \in \mathbb{R}_+^m$. Then, there exist $\bar{\lambda} \in \mathbb{R}_+^m,\ \bar{\eta}^H, \bar{\eta}^G \in \mathbb{R}^p, \bar{\mu} \in \mathbb{R}^q, \bar{\rho} \in \mathbb{R}^r$ such that $(\bar{x}, \bar{\alpha}, \bar{\lambda}, \bar{\eta}^H, \bar{\eta}^G, \bar{\mu}, \bar{\rho})$ is a feasible solution of the dual problem (MPD-VC($\bar{x}$)) and $\frac{h(\bar{x})}{g(\bar{x})} = \bar{\alpha}$.*

Moreover, if all the hypotheses of the weak duality Theorem 4.1 are satisfied, then $(\bar{x}, \bar{\alpha}, \bar{\lambda}, \bar{\eta}^H, \bar{\eta}^G, \bar{\mu}, \bar{\rho})$ is a weak efficient solution of the dual problem (MPD-VC($\bar{x}$)).

Proof. Since $\bar{x}$ is a weak efficient solution of the (MFPVC) and constraint qualification (GGCQ-PMOPVC$_{\bar{\alpha}}$) is satisfied at $\bar{x}$. Therefore, by KKT type necessary conditions, there exists $\bar{\lambda}_i \in int\mathbb{R}_+ (i \in M),\ \bar{\eta}_j^H,\ \bar{\eta}_j^G \in \mathbb{R} (j \in P), \bar{\mu}_l \geq 0, l \in Q, \bar{\rho}_t \in \mathbb{R}, t \in R$ such that conditions of (3.1) are satisfied, which implies that $(\bar{x}, \bar{\alpha}, \bar{\lambda}, \bar{\eta}^H, \bar{\eta}^G, \bar{\mu}, \bar{\rho})$ is a feasible solution of the (MPD-VC($\bar{x}$)). Also, it is obvious that the corresponding objective function value is equal to $\frac{h(\bar{x})}{g(\bar{x})}$.

Moreover, let $(y, \alpha, \lambda, \eta^H, \eta^G, \mu, \rho)$ be any feasible solution of the (MPD-VC($\bar{x}$)). Then, by weak duality Theorem 4.1, the inequality $\bar{\alpha} \nleq \alpha$ holds true. This implies that $(\bar{x}, \bar{\alpha}, \bar{\lambda}, \bar{\eta}^H, \bar{\eta}^G, \bar{\mu}, \bar{\rho})$ is a weak efficient solution of the (MPD-VC($\bar{x}$)). This completes the proof. □

Similarly, we have the following strong duality theorem.

Theorem 4.4 (Strong duality). *Let $\bar{x}$ be an efficient solution of the (MFPVC) and constraint qualification (GGCQ-PMOPVC$_{\bar{\alpha}}$) is satisfied at $\bar{x}$ with $\bar{\alpha} \in \mathbb{R}^m_+$. Then, there exist $\bar{\lambda} \in \mathbb{R}^m_+$, $\bar{\eta}^H, \bar{\eta}^G \in \mathbb{R}^p, \bar{\mu} \in \mathbb{R}^q_+, \bar{\rho} \in \mathbb{R}^r$, such that $(\bar{x}, \bar{\alpha}, \bar{\lambda}, \bar{\eta}^H, \bar{\eta}^G, \bar{\mu}, \bar{\rho})$ is a feasible solution of the dual problem (MPD-VC($\bar{x}$)) and $\frac{h(\bar{x})}{g(\bar{x})} = \bar{\alpha}$.*

Moreover, if all the hypotheses of the weak duality Theorem 4.2 are satisfied, then $(\bar{x}, \bar{\alpha}, \bar{\lambda}, \bar{\eta}^H, \bar{\eta}^G, \bar{\mu}, \bar{\rho})$ is an efficient solution of the dual problem (MPD-VC($\bar{x}$)).

We have the following converse duality theorem between (MFPVC) and (MPD-VC(x)).

Theorem 4.5 (Converse duality). *Let $(\bar{y}, \bar{\alpha}, \bar{\lambda}, \bar{\eta}^H, \bar{\eta}^G, \bar{\mu}, \bar{\rho})$ be a feasible solution of the MPD-VC(x) for every $x \in F$ and $\bar{y} \in F$. Further, suppose that there exists a vector-valued function $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and scalar valued functions $\gamma_i, \beta^H_j, \beta^G_j, \sigma_l, \zeta_t : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+ \setminus \{0\}, i \in M, j \in P, l \in Q, t \in R$ such that*

$$x, \bar{y} \in F \cup Y_D \implies \begin{cases} \gamma_i(x,\bar{y})\left[(h_i(x) - \bar{\alpha}_i g_i(x)) - (h_i(\bar{y}) - \bar{\alpha}_i g_i(\bar{y}))\right] & \\ \qquad \geq (or >) \left\langle \nabla h_i(\bar{y}) - \bar{\alpha}_i \nabla g_i(\bar{y}), \eta(x,\bar{y}) \right\rangle, & i \in M, \\ \beta^H_j(x,\bar{y})\left[H_j(x) - H_j(\bar{y})\right] \leq \left\langle \nabla H_j(\bar{y}), \eta(x,\bar{y}) \right\rangle, & j \in I^{+\bullet}_{00} \cup I^{+\bullet}_{0-} \cup I^{+\bullet}_{0+}, \\ \beta^H_j(x,\bar{y})\left[H_j(x) - H_j(\bar{y})\right] \geq \left\langle \nabla H_j(\bar{y}), \eta(x,\bar{y}) \right\rangle, & j \in I^{-\bullet}_{0+}, \\ \beta^G_j(x,\bar{y})\left[G_j(x) - G_j(\bar{y})\right] \geq \left\langle \nabla G_j(\bar{y}), \eta(x,\bar{y}) \right\rangle, & j \in I^{\bullet+}_{00} \cup I^{\bullet+}_{+0} \\ \sigma_l(x,\bar{y})\left[\phi_l(x) - \phi_l(\bar{y})\right] \geq \left\langle \nabla \phi_l(\bar{y}), \eta(x,\bar{y}) \right\rangle, & l \in Q^+, \\ \zeta_t(x,\bar{y})\left[\psi_t(x) - \psi_t(\bar{y})\right] \geq \left\langle \nabla \psi_t(\bar{y}), \eta(x,\bar{y}) \right\rangle, & t \in R^+, \\ \zeta_t(x,\bar{y})\left[\psi_t(x) - \psi_t(\bar{y})\right] \leq \left\langle \nabla \psi_t(\bar{y}), \eta(x,\bar{y}) \right\rangle, & t \in R^- \end{cases}$$

holds, then $\bar{y}$ is a weak efficient solution (or an efficient solution) of (MFPVC).

Proof. The proof of this theorem follows directly from weak duality Theorem 4.1 (or Thm. 4.2). □

Also, we have the following strict converse duality theorem between (MFPVC) and (MPD-VC(x)).

Theorem 4.6 (Strict Converse Duality). *Let $\bar{x}$ and $(\bar{y}, \bar{\alpha}, \bar{\lambda}, \bar{\eta}^H, \bar{\eta}^G, \bar{\mu}, \bar{\rho})$ be efficient solutions of the (MFPVC) and (MPD-VC(x)), respectively, with $\bar{\alpha}_i = \frac{h_i(\bar{x})}{g_i(\bar{x})}, i \in M$. Further, assume that there exist a vector valued function $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and scalar valued functions $\gamma_i, \beta^H_j, \beta^G_j, \sigma_l, \zeta_t : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+ \setminus \{0\}, i \in M, j \in P, l \in Q, t \in R$ such that*

$$x, \bar{y} \in F \cup Y_D \implies \begin{cases} \gamma_i(x,\bar{y})\left[(h_i(x) - \bar{\alpha}_i g_i(x)) - (h_i(\bar{y}) - \bar{\alpha}_i g_i(\bar{y}))\right] & \\ > \left\langle \nabla h_i(\bar{y}) - \bar{\alpha}_i \nabla g_i(\bar{y}), \eta(x,\bar{y}) \right\rangle, & i \in M, \\ \beta^H_j(x,\bar{y})\left[H_j(x) - H_j(\bar{y})\right] \leq \left\langle \nabla H_j(\bar{y}), \eta(x,\bar{y}) \right\rangle, & j \in I^{+\bullet}_{00} \cup I^{+\bullet}_{0-} \cup I^{+\bullet}_{0+}, \\ \beta^H_j(x,\bar{y})\left[H_j(x) - H_j(\bar{y})\right] \geq \left\langle \nabla H_j(\bar{y}), \eta(x,\bar{y}) \right\rangle, & j \in I^{-\bullet}_{0+}, \\ \beta^G_j(x,\bar{y})\left[G_j(x) - G_j(\bar{y})\right] \geq \left\langle \nabla G_j(\bar{y}), \eta(x,\bar{y}) \right\rangle, & j \in I^{\bullet+}_{00} \cup I^{\bullet+}_{+0}, \\ \sigma_l(x,\bar{y})\left[\phi_l(x) - \phi_l(\bar{y})\right] \geq \left\langle \nabla \phi_l(\bar{y}), \eta(x,\bar{y}) \right\rangle, & l \in Q^+, \\ \zeta_t(x,\bar{y})\left[\psi_t(x) - \psi_t(\bar{y})\right] \geq \left\langle \nabla \psi_t(\bar{y}), \eta(x,\bar{y}) \right\rangle, & t \in R^+, \\ \zeta_t(x,\bar{y})\left[\psi_t(x) - \psi_t(\bar{y})\right] \leq \left\langle \nabla \psi_t(\bar{y}), \eta(x,\bar{y}) \right\rangle, & t \in R^-, \end{cases}$$

hold. Then, $\bar{x} = \bar{y}$, i.e. *$\bar{y}$ is an efficient solution of the (MFPVC).*

5. Conclusions

In this paper, we have studied a multiobjective fractional programming problem with vanishing constraints involving continuously differentiable functions for the first time in literature. Using a parametric approach, we have derived a necessary optimality condition for a feasible solution to be an efficient solution of the multiobjective fractional programming problem with vanishing constraints. We have also proved that V-invexity assumptions of the multiobjective fractional programming problem with vanishing constraints are sufficient conditions for a Karush–Kuhn–Tucker point to be either an efficient solution or a local efficient solution. Duality results are derived between the primal MFPVC and a parametric dual model. Further, under different suitable assumptions other dual models can be investigated and saddle point results can be developed for the problem under consideration. Moreover, approximate solution concepts (see, *e.g.* [3, 32, 33]) can be explored for the MFPVC under consideration. In addition, multiobjective fractional programs can be investigated involving semi-infinite constraints [44, 45] and in a different setting of Hadamard manifolds [46, 47].

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