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A NEWTON DESCENT LOGARITHMIC BARRIER INTERIOR-POINT ALGORITHM FOR MONOTONE LCP

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Abstract. The growing importance of the monotone linear complementarity problems (LCP) lies in the different applications it covers, both in mathematics and in practice. In this paper, based on the optimization techniques, we propose a descent logarithmic barrier interior-point method for solving the (LCP). The idea is to transform the LCP into an equivalent convex quadratic optimization problem, denoted by (CQO). Then, the associated barrier problem to CQO is formulated. The existence and the uniqueness of optimal solution of the barrier problem is showed. For its numerical aspects, the descent direction is computed by using the classical Newton's method. However, to determine the displacement step along this direction, guaranteeing the maintenance of the new iterates inside the domain during the algorithm process and the improvement of the value of the objective function, we apply a new approach using approximation functions known as "minorant and majorant approximating functions". The numerical results obtained are very promising and show the effectiveness of this new strategy.

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1. Introduction

Consider the linear complementarity problem LCP which consist to find a vector $x \in \mathbb{R}^n$ such that

$$x \geq 0,\ Mx + q \geq 0,\ \langle x, Mx + q \rangle = 0 \tag{LCP}$$

where $M \in \mathbb{R}^{n\times n}$ is a given matrix and $q \in \mathbb{R}^n$. Here $x \geq 0$ and $Mx + q \geq 0$ mean that x and $Mx + q$ are nonnegative vectors in $\mathbb{R}^n$ and $\langle x, Mx + q \rangle$ stand for their usual inner product.

For years, the LCP has been one of the most attractive subjects in mathematical programming due to its applications in many areas such as economics, computer science, and physics. For example, equilibrium problems, obstacle problems, and game theory can be transformed into an LCP. Additionally, it includes linear optimization (LO) and convex quadratic optimization (CQO) (see the monograph by Cottle *et al.* [12]).

Among the many solution methods for LCP, the barrier logarithmic interior-point approach deserves a great attention due to its numerical efficiency (see [2, 5, 11, 15, 19, 25]).

Keywords. Linear complementarity problems, interior-point methods, logarithmic barrier approach, Newton method, minorant and majorant functions.

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Generally, computing a displacement step in these methods is based on the classical line search iterative rules such as Armijo-Goldstein, Wolfe and Fibonacci,.... It is shown that when the size of the problem is large, the computational cost in there becomes very high (see *e.g.*, [11, 13]). So recently, to avoid this drawback Crouzeix and Merikhi [13], proposed a novel strategy based on so-called majorant functions for computing a displacement step. Their obtained numerical results *via* this novel strategy shows its efficiency in solving the semidefinite optimization problems (SDO). Subsequently, Leulmi [22], used similar procedure based on minorant functions for computing the displacement step in SDO. Later, this idea was extended successively by Menniche and Benterki [23] and Alzalg [6], for solving LO and second order cone optimization (SOCP) problems.

In this paper, motived by the above works, we extend the technique of minorant and majorant functions for computing an efficient displacement step for solving monotone LCPs. To do so, based on the optimization techniques, we have transformed the LCP problem into an equivalent convex quadratic optimization (CQO) problem, then an unconstrained barrier logarithmic problem is stated. The existence and the uniqueness of an optimal solution of the latter is showed. Further, we prove if the barrier parameter goes to zero, then the limit of the barrier solution tends to a solution of LCP. For its numerical aspects, the descent direction is computed *via* Newton's method. Meanwhile, our contribution in this paper is to exploit the alternative of so-called minorant and majorant functions for determining the displacement step along this direction. The obtained numerical results are promising and show the effectiveness of our new used approach.

The outline of the paper is built as follows. In Section 2, preliminaries are introduced to provide necessary tools in the development of the proposed algorithm. In Section 3, an equivalent optimization problem to monotone LCP and the barrier problem denoted by P_μ associated to it are given. The existence and the uniqueness of an optimal solution of the barrier problem are shown. In addition, as μ tends to zero, the limit of optimal solution of the barrier problem is a solution of the LCP. In Section 4, the numerical aspects is stated. The computation of Newton direction and the prototype algorithm for solving P_μ are given. Four choices of minorant and majorant functions for computing the displacement step are suggested. In Section 5, the numerical implementation of the algorithm on some monotone LCPs is given and compared with the classical Wolfe and Powell line search. A conclusion and future work end the paper in Section 6.

Some notations used in the paper are as follows. $\mathbb{R}^n$ denotes the space of real n-dimensional vectors and $\mathbb{R}^{n\times n}$ stand for the set of all $n \times n$ real matrices. Given $x \in \mathbb{R}^n$, $X \in \mathbb{R}^{n\times n}$ denotes the diagonal matrix whose elements are the coordinates of x, *i.e.* $X = diag(x)$. Given $x, y \in \mathbb{R}^n$, $\langle x, y\rangle = x^T y = \sum_{i=1}^n x_i y_i$ is their usual inner product. Finally, the Euclidean norm and the vector of ones are denoted by $\|x\| = \sqrt{x^T x}$ and e, respectively.

2. Preliminaries

Throughout the paper, we assume that the following conditions hold for LCP.

- **(PSD)**. The matrix M is positive semi-definite. In this case the LCP is called monotone.
- **(SF)**. $\mathcal{F}^0 = \{x \in \mathbb{R}^n : \ x > 0, \ Mx + q > 0\} \neq \emptyset$. The set of strictly feasible solutions of LCP.

The next lemma states useful inequalities which are crucial in the sequel of the paper.

2.1. Statistical inequalities

Lemma 2.1 ([13], Thm. 5)**.** *Let* $w_1, \dots, w_n$ *s.t.* $w_i > 0$, $\forall i = 1, \dots, n$ *be a sample of size* n*, then*

$$D_1 \leq \sum_{i=1}^{n} \ln(w_i) \leq D_2$$

where

$$D_1 = (n-1)\ln\left(\bar{w} + \frac{\sigma_w}{\sqrt{n-1}}\right) + \ln\left(\bar{w} - \sigma_w\sqrt{n-1}\right),$$

$$D_2 = (n-1)\ln\left(\bar{w} - \frac{\sigma_w}{\sqrt{n-1}}\right) + \ln\left(\bar{w} + \sigma_w\sqrt{n-1}\right),$$

and

$$\bar{w} = \frac{1}{n}\sum_{i=1}^{n} w_i \quad \text{and} \quad \sigma_w = \sqrt{\frac{1}{n}\|w\|^2 - \bar{w}^2} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(w_i - \bar{w})^2},$$

denote the mean value of w and its deviation, respectively.

3. The barrier problem associated to LCP

First, we have transformed LCP into a convex quadratic optimization:

$$m = \inf[g(x) = \langle x, Mx + q\rangle : \; x \geq 0, \; Mx + q \geq 0]. \tag{CQO}$$

We denote by **Sol**(CQO) the set of optimal solutions of this problem and by **Sol**(LCP) the set of solutions of LCP. Clearly, $x \in$ **Sol**(LCP) if and only if $m = 0$ and $x \in$ **Sol**(CQO). It is known that, if **(PSD)** and **(SF)** hold, then $m = 0$, **Sol**(CQO) and **Sol**(LCP) coincide (*e.g*, see, Wright [25], Thm. 8.4).

We apply now the logarithmic barrier approach to CQO to replace the non-negativity conditions $x \geq 0$ and $Mx + q \geq 0$ by additional logarithmic barrier terms to the objective function for $\mu > 0$ (the barrier parameter), we obtain

$$f(x,\mu) = \begin{cases} \langle x, Mx+q\rangle - \mu\sum_{i=1}^{n}\ln x_i\langle Mx+q, e_i\rangle + n\mu\ln\mu, & x \in \mathcal{F}^0, \\ +\infty, & \text{otherwise,} \end{cases}$$

where $\{e_i\}_{i=1}^{n}$ denotes the canonical basis of $\mathbb{R}^n$. Let now introduce the function $m(\mu)$ defined by

$$m(\mu) = \inf_x \left[f_\mu(x) := f(x,\mu) : x \in \mathcal{F}^0\right]. \tag{P_μ}$$

P_μ is the barrier problem associated to CQO. By construction, P_0 is only the problem CQO, and we have $m = m(0) = 0$. So solving LCP is equivalent to solving the perturbed problem P_μ with μ gradually decreases to zero.

Next, we focus on the theoretical study of P_μ *i.e.*, the existence and uniqueness of an optimal solution of P_μ and its convergence to a solution of LCP as μ tends to zero.

The gradient and the Hessian of the objective $f_\mu(x)$ of P_μ are given by

$$\nabla f_\mu(x) = \nabla g(x) - \mu X^{-1}e - \mu\sum_{i=1}^{n}\frac{M^T e_i}{\langle e_i, Mx+q\rangle}, \tag{1}$$

and

$$H := \nabla^2 f_\mu(x) = \nabla^2 g(x) + \mu X^{-2} + \mu\sum_{i=1}^{n}\frac{(M^T e_i)(M^T e_i)^T}{\langle e_i, Mx+q\rangle^2}, \tag{2}$$

where

$$\nabla g(x) = (M + M^T)x + q \;\text{ and }\; \nabla^2 g(x) = (M + M^T), \tag{3}$$

with $X^{-1} := diag(\frac{1}{x})$ and $X^{-2} := diag(\frac{1}{x^2})$.

Lemma 3.1. *Under our hypothesis the objective function $f_\mu(x)$ is strictly convex.*

Proof. We shall prove that the Hessian H in (2), is a positive definite matrix. We prove first that the matrix $(M^Te_i)(M^Te_i)^T$ is symmetric positive semidefinite for all i. We have for all nonzero $y \in \mathbb{R}^n$,

$$\left\langle \left(M^Te_i\right)\left(M^Te_i\right)^T y, y\right\rangle = \left\langle e_ie_i^TMy, My\right\rangle = \left\langle e_i^TMy, e_i^TMy\right\rangle = \left|e_i^TMy\right|^2 \geq 0, \qquad \forall i.$$

This implies that $(M^Te_i)(M^Te_i)^T \in \mathbb{R}^{n\times n}$ is symmetric positive semidefinite for all i. So their sum is also a symmetric positive semidefinite matrix. Next, as M is positive semi-definite and the diagonal matrix μX^{-2} is positive definite for all $x \in \mathcal{F}^0$ and with $\mu > 0$, then the matrix H is positive definite too. This implies that $f_\mu(x)$ is strictly convex. Therefore, if any minimizer of P_μ exists, is unique. This completes the proof. □

Theorem 3.1 ([20], Thm. A.3)**.** *Let the LCP satisfies the conditions **(PSD)** and **(SF)**. Then the problem P_μ has a unique optimal solution $x(\mu)$ for every $\mu > 0$.*

Lemma 3.2. *Let $\mu > 0$ and $x(\mu)$ be an optimal solution of P_μ, then $x^\star = \lim_{\mu\to 0} x(\mu)$ is a solution of LCP.*

Proof. Let $x \in \mathcal{F}^0$ and $\mu > 0$ be given. Since the function $f(x,\mu)$ is convex and differentiable at the point $(x(\mu),\mu)$, we get

$$\begin{aligned} g(x) &= f(x,0) \\ &\geq f(x(\mu),\mu) + \langle x - x(\mu), \nabla_x f(x(\mu),\mu)\rangle + (0-\mu)\nabla_\mu f(x(\mu),\mu). \end{aligned}$$

Now, because $(x(\mu),\mu)$ is an optimal solution of P_μ, therefore $\nabla_x f(x(\mu),\mu) = 0$. Consequently,

$$g(x) \geq f(x(\mu),\mu) - \mu\nabla_\mu f(x(\mu),\mu).$$

Substitution, $\nabla_\mu f(x(\mu),\mu)$ by its value, we obtain

$$\begin{aligned} g(x) &\geq g(x(\mu)) + n\mu\ln\mu - \mu\sum_{i=1}^n \ln(x_i)_\mu - \mu\sum_{i=1}^n \ln\langle e_i, Mx(\mu)+q\rangle \\ &\quad - \mu\left(n + n\ln(\mu) - \mu\sum_{i=1}^n \ln(x_i)_\mu - \mu\sum_{i=1}^n \ln\langle e_i, Mx(\mu)+q\rangle\right) \\ &= g(x(\mu)) - n\mu. \end{aligned}$$

Since $x \in \mathcal{F}^0$ was arbitrary, we then have

$$g(x(\mu)) - n\mu \leq \min_{x\in\mathcal{F}^0} g(x) \leq g(x(\mu)).$$

Letting μ goes to zero and $x^\star := \lim\limits_{\mu\to 0}(x(\mu))$, we get

$$g(x^\star) = g\left(\lim_{\mu\to 0}(x(\mu))\right) = \lim_{\mu\to 0} g(x(\mu)) = \min_{x\in\mathcal{F}^0} g(x).$$

This shows that $x^\star$ is a solution of LCP. This completes the proof. □

From Lemma 3.2, it is clear that solving the original problem $\mathcal{P}$ is equivalent to solving a series of perturbed problems P_μ as μ decreases gradually to zero.

4. Numerical aspects of P_μ

In this section, we focus on the numerical solution of P_μ, based on the optimality conditions which are necessary and sufficient, x is an optimal of P_μ if:

$$\nabla f_\mu(x) = 0. \tag{4}$$

We solve (4) by using the Newton's method *i.e.*, finding a descent direction $d^{(k)}$ at each iteration satisfying the following system:

$$H^{(k)} d^{(k)} = -\nabla f_\mu(x^{(k)}). \tag{5}$$

For the sake of simplicity, we drop the index μ from $x(\mu)$ and $x(\mu)^{(k)}$, and write x instead of $x(\mu)$ and $x^{(k)}$ instead of $x(\mu)^{(k)}$ and $H^{(k)} \equiv H(x^{(k)})$.

The principal of our algorithm is to generate a sequence $\{x^{(k)}\}$ of interior points, *i.e.*, $x^{(k)}$ remains in $\mathcal{F}^0$ and which converges to an approximate solution of LCP. Therefore at each iteration k, a displacement step $t^{(k)} > 0$ is taken such that

$$x^{(k)} + t^{(k)} d^{(k)} > 0 \text{ and } M\left(x^{(k)} + t^{(k)} d^{(k)}\right) + q > 0. \tag{6}$$

4.1. The prototype algorithm

Now we are ready to describe the Damped-Newton step interior-point method. First, we use an accuracy parameter $\varepsilon > 0$ and a barrier default $\mu > 0$. The algorithm starts by a positive point $x^0 > 0$. Using the obtained search directions d from (5) and the displacement step t from (12) and we take a Damped-Newton step, the algorithm produces a new iterate $x^+ = x + td$. Then, we reduce the barrier parameter μ by a threshold $\rho \in (0, 1)$, that is, $\mu^+ = \rho\mu$. We continue until we obtain an ε-approximate solution of P_μ; that is, the stopping criterion $|\nabla g(x^{(k)}) d^{(k)}| \leq \varepsilon$ is satisfied.

A prototype algorithm for solving P_μ is formally stated in Algorithm 1.

Algorithm 1. Prototype algorithm for solving P_μ.

Require
  An accuracy parameter $\varepsilon > 0$;
  a barrier parameter $\mu^{(0)} > 0$;
  a threshold parameter $0 < \rho < 1$;
Let $x^{(0)} \in \mathcal{F}^0$ and $d^{(0)} \in \mathbb{R}^n$; set $k = 0$;
  While $|\nabla g(x^{(k)}) d^{(k)}| > \varepsilon$ **do**
    Compute the descent direction $d^{(k)}$ from (5);
    Compute the displacement step $t^{(k)}$ from (12);
    Set $x^{(k+1)} = x^{(k)} + t^{(k)} d^{(k)}$;
    Set $\mu^{(k+1)} = \rho\mu^{(k)}$;
    Set $k := k + 1$;
  End While

Next task, is the determination of a displacement step $t^{(k)}$ along the direction $d^{(k)}$ by using the strategy of minorant and majorant functions.

4.2. The determination of the displacement step

Following [13], instead of using the classical line search methods for computing the displacement step $t^{(k)}$ at each iteration k which require minimizing the unidimensional function

$$\min_{t>0} f_\mu(x + td),$$

we approach the function defined by

$$\gamma(t) := \frac{1}{\mu}(f_\mu(x+td) - f_\mu(x)),$$

by simple majorant and minorant approximating functions which give at each iteration k an efficient $t^{(k)}$ which occurs a significant decrease in $\gamma(t)$.

Let us notice that the function $\gamma(t)$ is definite only if $x + td > 0$ and $M(x+td)+q > 0$. Then it is sufficient to find a $t_{\max} > 0$ such that $x + td > 0$ and $M(x+td)+q > 0$ for all $t \in [0, t_{\max})$. So $t_{\max}$ which remains points interior is taken as:

$$t_{\max} = \min_i \left\{ -\frac{x_i}{d_i} : d_i < 0, \, -\frac{\langle e_i, Mx+q\rangle}{(Md)_i} : (Md)_i < 0 \right\}.$$

Theorem 4.1. *The function $\gamma(t)$ is well-defined on $t \in [0, t_{\max})$, and can be written as follows:*

$$\gamma(t) = \left(\frac{t^2-2t}{2}\right)a + tb - \sum_{i=1}^n \ln(1+tz_i) - \sum_{i=1}^n \ln(1+ts_i),$$

where

$$a = \frac{1}{\mu}\langle \nabla^2 g(x)d, d\rangle, \; b = \sum_{i=1}^n z_i + \sum_{i=1}^n s_i - \|z\|^2 - \|s\|^2, z_i = \frac{d_i}{x_i}$$

and

$$s_i = \frac{\langle e_i, Md\rangle}{\langle e_i, Mx+q,\rangle}, \qquad \forall i.$$

Proof. Let $x \in \mathcal{F}^0$, then we have

$$\begin{aligned} \gamma(t) &= \frac{1}{\mu}(f_\mu(x+td) - f_\mu(x)) \\ &= \frac{t}{\mu}\left[d^T(M+M^T)x + d^Tq + td^TMd - \mu\sum_{i=1}^n \ln\left(1+t\frac{d_i}{x_i}\right) - \mu\sum_{i=1}^n \ln\left(1+t\frac{\langle e_i, Md\rangle}{\langle e_i, Mx+q\rangle}\right)\right]. \end{aligned}$$

Next, according to (3), we obtain

$$\gamma(t) = \frac{t}{\mu}\left[d^T\nabla g(x) + t\frac{\langle \nabla^2 g(x)d, d\rangle}{2}\right] - \sum_{i=1}^n \ln\left(1+t\frac{d_i}{x_i}\right) - \sum_{i=1}^n \ln\left(1+t\frac{\langle e_i, Md\rangle}{\langle e_i, Mx+q\rangle}\right).$$

Now, by (1), we have

$$\begin{aligned} \langle \nabla f_\mu(x), d\rangle = d^T\nabla f_\mu(x) &= d^T\nabla g(x) - \mu d^TX^{-1}e - \mu d^T\sum_{i=1}^n \frac{M^Te_i}{\langle e_i, Mx+q\rangle} \\ &= d^T\nabla g(x) - \mu d^TX^{-1}e - \mu\sum_{i=1}^n \frac{\langle e_i, Md\rangle}{\langle e_i, Mx+q\rangle} \end{aligned}$$

from which we deduce that

$$d^T\nabla g(x) = d^T\nabla f_\mu(x) + \mu d^TX^{-1}e + \mu\sum_{i=1}^n \frac{\langle e_i, Md\rangle}{\langle e_i, Mx+q\rangle}.$$

Due to the fact that the direction d satisfies $\nabla^2 f_\mu(x)d = -\nabla f_\mu(x)$, then we have

$$d^T\nabla^2 f_\mu(x)d = -d^T\nabla f_\mu(x),$$

and

$$d^T\nabla g(x) = -d^T\nabla^2 f_\mu(x)d + \mu d^T X^{-1}e + \mu\sum_{i=1}^{n}\frac{\langle e_i, Md\rangle}{\langle e_i, Mx+q\rangle}. \tag{7}$$

From (2), we have

$$\begin{aligned} d^T\nabla^2 f_\mu(x)d &= \langle\nabla^2 g(x)d, d\rangle + \mu d^T X^{-2}d + \mu\sum_{i=1}^{n}\frac{d^T(M^Te_i)(M^Te_i)^Td}{\langle e_i, Mx+q\rangle^2} \\ &= \langle\nabla^2 g(x)d, d\rangle + \mu d^T X^{-2}d + \mu\sum_{i=1}^{n}\frac{\langle e_i, Md\rangle\times\langle e_i, Md\rangle}{\langle e_i, Mx+q\rangle^2} \\ &= \langle\nabla^2 g(x)d, d\rangle + \mu d^T X^{-2}d + \mu\sum_{i=1}^{n}\frac{\langle e_i, Md\rangle^2}{\langle e_i, Mx+q\rangle^2}. \end{aligned}$$

Letting $z_i = \frac{d_i}{x_i}$ and $s_i = \frac{\langle e_i, Md\rangle}{\langle e_i, Mx+q\rangle}$ then

$$d^T\nabla^2 f_\mu(x)d = \langle\nabla^2 g(x)d, d\rangle + \mu\|z\|^2 + \mu\|s\|^2. \tag{8}$$

Substituting (8) into (7), we get

$$\begin{aligned} d^T\nabla g(x) &= -\big(\langle\nabla^2 g(x)d, d\rangle + \mu\|z\|^2 + \mu\|s\|^2\big) + \mu d^T X^{-1}e + \mu\sum_{i=1}^{n}\frac{\langle e_i, Md\rangle}{\langle e_i, Mx+q\rangle} \\ &= -\langle\nabla^2 g(x)d, d\rangle - \mu\|z\|^2 - \mu\|s\|^2 + \mu\sum_{i=1}^{n} z_i + \mu\sum_{i=1}^{n} s_i. \end{aligned}$$

Consequently,

$$\begin{aligned} \gamma(t) &= \frac{t}{\mu}\left[d^T\nabla g(x) + t\frac{\langle\nabla^2 g(x)d, d\rangle}{2}\right] - \sum_{i=1}^{n}\ln(1+tz_i) - \sum_{i=1}^{n}\ln(1+ts_i) \\ &= \left(\frac{t^2-2t}{2}\right)a + tb - \sum_{i=1}^{n}\ln(1+tz_i) - \sum_{i=1}^{n}\ln(1+ts_i), \qquad \forall t\in[0, t_{\max}), \end{aligned}$$

where

$$a = \frac{1}{\mu}\langle\nabla^2 g(x)d, d\rangle,\ \ b = \sum_{i=1}^{n} z_i + \sum_{i=1}^{n} s_i - \|z\|^2 - \|s\|^2.$$

This completes the proof. □

The first and the second derivatives of γ are given by:

$$\begin{aligned} \gamma'(t) &= (t-1)a + b - \sum_{i=1}^{n}\frac{z_i}{1+tz_i} - \sum_{i=1}^{n}\frac{s_i}{1+ts_i}, \\ \gamma''(t) &= a + \sum_{i=1}^{n}\frac{z_i^2}{(1+tz_i)^2} + \sum_{i=1}^{n}\frac{s_i^2}{(1+ts_i)^2} > 0, \qquad \forall t\in[0, t_{\max}). \end{aligned}$$

We remark that the function γ satisfies the following properties:

$$\gamma(0) = 0,\ -\gamma'(0) = \gamma''(0) = (a + \|z\|^2 + \|s\|^2) > 0.$$

It is clear that γ is strictly convex and differentiable on its domain of definition and so it reaches a unique minimum $\hat{t} > 0$ when $\gamma'(t) = 0$. It is reported that finding a solution of it, is a hard task. So it is worth to find upper and lower approximating functions easy to handle than γ.

In this paper, we construct such approximating functions of γ by using the strategy of minorant and majorant functions of γ. Such minorant and majorant functions $\hat{\gamma}(t)$ are constructed to be close enough to γ and satisfying the following properties

$$\hat{\gamma}(0) = 0,\ -\hat{\gamma}'(0) = \hat{\gamma}''(0) = (a + \|z\|^2 + \|s\|^2) > 0. \tag{9}$$

4.3. Minorant and majorant functions

4.3.1. The first minorant and majorant function

We may define the first minorant function γ_1 by

$$\gamma_1(t) = \left(\frac{t^2 - 2t}{2}\right)a + tb - (n-1)\ln(1 + t\delta) - \ln(1 + t\lambda) - (n-1)\ln(1 + t\kappa) - \ln(1 + t\nu),$$

on $[0, \tilde{t}_1)$ where

$$\tilde{t}_1 = \min\left\{t_{\max},\ -\frac{1}{\delta} : \delta < 0,\ -\frac{1}{\lambda} : \lambda < 0,\ -\frac{1}{\kappa} : \kappa < 0,\ -\frac{1}{\nu} : \nu < 0,\right\},$$

where

$$\delta = \bar{z} - \frac{\sigma_z}{\sqrt{n-1}},\ \lambda = \bar{z} + \sigma_z\sqrt{n-1},\ \kappa = \bar{s} - \frac{\sigma_s}{\sqrt{n-1}},\ \nu = \bar{s} + \sigma_s\sqrt{n-1}, \tag{10}$$

and

$$\bar{z} = \frac{1}{n}\sum_{i=1}^{n} z_i,\ \sigma_z = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(z_i - \bar{z})^2},\ \bar{s} = \frac{1}{n}\sum_{i=1}^{n} s_i,\ \sigma_s = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(s_i - \bar{s})^2}.$$

Now, we define the first majorant function γ_2 by

$$\gamma_2(t) = \left(\frac{t^2 - 2t}{2}\right)a + tb - (n-1)\ln(1 + t\beta) - \ln(1 + t\eta) - (n-1)\ln(1 + t\tau) - \ln(1 + t\omega),$$

on $t \in [0, \tilde{t}_2)$ where

$$\tilde{t}_2 = \min\left\{t_{\max},\ -\frac{1}{\beta} : \beta < 0,\ -\frac{1}{\eta} : \eta < 0,\ -\frac{1}{\tau} : \tau < 0,\ -\frac{1}{\omega} : \omega < 0\right\},$$

where

$$\beta = \bar{z} + \frac{\sigma_z}{\sqrt{n-1}},\ \ \eta = \bar{z} - \sigma_z\sqrt{n-1},\ \ \tau = \bar{s} + \frac{\sigma_s}{\sqrt{n-1}},\ \ \omega = \bar{s} - \sigma_s\sqrt{n-1}, \tag{11}$$

and

$$\bar{z} = \frac{1}{n}\sum_{i=1}^{n} z_i,\ \ \sigma_z = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(z_i - \bar{z})^2},\ \ \bar{s} = \frac{1}{n}\sum_{i=1}^{n} s_i,\ \ \sigma_s = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(s_i - \bar{s})^2}.$$

Next lemma shows that γ_1 is a minorant function of γ and γ_2 is a majorant function for γ, respectively.

Lemma 4.1. *Let $t \in [0, \hat{t}_1)$, then $\gamma_1(t) \le \gamma(t) \le \gamma_2(t)$ for all $t \in [0, \hat{t}_1)$, where $\hat{t}_1 = \min\{\tilde{t}_1, \tilde{t}_2\}$.*

Proof. From the right hand of the inequality in Lemma 2.1, we have

$$-D_2 \leq -\sum_{i=1}^{n} \ln w_i.$$

Letting first, $w_i = 1 + tz_i$, so $\bar{w} = 1 + t\bar{z}$, $\sigma_w = t\sigma_z$, these implies on one hand that

$$-(n-1)\ln(1+t\delta) - \ln(1+t\lambda) \leq -\sum_{i=1}^{n} \ln(1+tz_i).$$

On the other hand if we set $w_i = 1 + ts_i$, so $\bar{w} = 1 + t\bar{s}$, $\sigma_w = t\sigma_s$, these implies that

$$-(n-1)\ln(1+t\kappa) - \ln(1+t\nu) \leq -\sum_{i=1}^{n} \ln(1+ts_i),$$

where δ, λ, κ and ν are defined in (10). By summation of these two inequalities, we get,

$$-(n-1)\ln(1+t\delta) - \ln(1+t\lambda) - (n-1)\ln(1+t\kappa) - \ln(1+t\nu) \leq -\sum_{i=1}^{n} \ln(1+tz_i) - \sum_{i=1}^{n} \ln(1+ts_i).$$

Now, adding the term $\left[\left(\frac{t^2-2t}{2}\right)a + tb\right]$ for both sides to the previous inequality, we obtain

$$\gamma_1(t) = \left(\frac{t^2-2t}{2}\right)a + tb - (n-1)\ln(1+t\delta) - \ln(1+t\lambda) - (n-1)\ln(1+t\kappa) - \ln(1+t\nu) \leq \gamma(t).$$

This shows that $\gamma_1(t) \leq \gamma(t)$ for all $t \in [0, \tilde{t}_1)$. For the proof of the first majorant function we have omitted it since it is similar to the one used for the first minorant function, where we use the inequality $-\sum_{i=1}^{n} \ln w_i \leq -D_1$ in Lemma 2.1. Then, it is easy to deduce that $\gamma(t) \leq \gamma_2(t)$ for all $t \in [0, \tilde{t}_2)$. In addition, we have

$$\begin{aligned}
\gamma_1'(t) &= (t-1)a + b - \frac{(n-1)\delta}{1+t\delta} - \frac{\lambda}{1+t\lambda} - \frac{(n-1)\kappa}{1+t\kappa} - \frac{\nu}{1+t\nu},\\
\gamma_1''(t) &= a + b + \frac{(n-1)\delta^2}{(1+t\delta)^2} + \frac{\lambda^2}{(1+t\lambda)^2} + \frac{(n-1)\kappa^2}{(1+t\kappa)^2} + \frac{\nu^2}{(1+t\nu)^2} > 0,\\
\gamma_2'(t) &= (t-1)a + b - \frac{(n-1)\beta}{1+t\beta} - \frac{\eta}{1+t\eta} - \frac{(n-1)\tau}{1+t\tau} - \frac{\omega}{1+t\omega},
\end{aligned}$$

and

$$\gamma_2''(t) = a + \frac{(n-1)\beta^2}{(1+t\beta)^2} + \frac{\eta^2}{(1+t\eta)^2} + \frac{(n-1)\tau^2}{(1+t\tau)^2} + \frac{\omega^2}{(1+t\omega)^2} > 0,$$

Further, the properties in (9) are satisfied by γ_1 and γ_2. Also it is clear that $\min_t \gamma_1(t) \leq \min_t \gamma(t) \leq \min_t \gamma_2(t)$ for $t \in [0, \hat{t}_1)$. This completes the proof. □

4.3.2. The second minorant and majorant function

We may define the second minorant function γ_3 by

$$\gamma_3(t) = \left(\frac{t^2-2t}{2}\right)a + t\big(\|z\| + \|s\| - \|z\|^2 - \|s\|^2\big) + \ln(1 - t\|z\|) + \ln(1 - t\|s\|),$$

on $[0, \tilde{t}_3)$, and the second majorant function γ_4 by

$$\gamma_4(t) = \left(\frac{t^2-2t}{2}\right)a - t\big(\|z\| + \|s\| + \|z\|^2 + \|s\|^2\big) - \ln(1 - t\|z\|) - \ln(1 - t\|s\|),$$

on $[0, \tilde{t}_3)$ where

$$\tilde{t}_3 = \max\{t > 0 : 1 - t\|z\| > 0,\ 1 - t\|s\| > 0\}.$$

Next, to show that γ_3 is a minorant function of γ and γ_4 is a majorant function of γ, respectively, for all $t \in [0, \tilde{t}_3)$, we need to prove the following technical lemma.

Lemma 4.2. *For all $t \in \mathbb{R}$ and $z, s \in \mathbb{R}^n$ we have*

$$t\left(\sum_{i=1}^{n} z_i - \|z\| + \sum_{i=1}^{n} s_i - \|s\|\right) - \sum_{i=1}^{n} \ln(1 + tz_i) - \ln(1 - t\|z\|) - \sum_{i=1}^{n} \ln(1 + ts_i) - \ln(1 - t\|s\|) \geq 0,$$

and

$$t\left(\sum_{i=1}^{n} z_i + \|z\| + \sum_{i=1}^{n} s_i + \|s\|\right) - \sum_{i=1}^{n} \ln(1 + tz_i) + \ln(1 - t\|z\|) - \sum_{i=1}^{n} \ln(1 + ts_i) + \ln(1 - t\|s\|) \leq 0.$$

Proof. Let

$$h_1(t) = t\left(\sum_{i=1}^{n} z_i - \|z\| + \sum_{i=1}^{n} s_i - \|s\|\right) - \sum_{i=1}^{n} \ln(1 + tz_i) - \ln(1 - t\|z\|) - \sum_{i=1}^{n} \ln(1 + ts_i) - \ln(1 - t\|s\|),$$

and

$$h_2(t) = t\left(\sum_{i=1}^{n} z_i + \|z\| + \sum_{i=1}^{n} s_i + \|s\|\right) - \sum_{i=1}^{n} \ln(1 + tz_i) + \ln(1 - t\|z\|) - \sum_{i=1}^{n} \ln(1 + ts_i) + \ln(1 - t\|s\|),$$

then

$$\begin{aligned}
h_1'(t) &= \sum_{i=1}^{n} z_i - \|z\| + \sum_{i=1}^{n} s_i - \|s\| - \sum_{i=1}^{n} \frac{z_i}{1 + tz_i} + \frac{\|z\|}{1 - t\|z\|} - \sum_{i=1}^{n} \frac{s_i}{1 + ts_i} + \frac{\|s\|}{1 - t\|s\|} \\
&= \sum_{i=1}^{n} z_i\left(1 - \frac{1}{1 + tz_i}\right) + \|z\|\left(\frac{1}{1 - t\|z\|} - 1\right) + \sum_{i=1}^{n} s_i\left(1 - \frac{1}{1 + ts_i}\right) + \|s\|\left(\frac{1}{1 - t\|s\|} - 1\right) \\
&= \sum_{i=1}^{n} z_i\left(\frac{tz_i}{1 + tz_i}\right) + \|z\|\left(\frac{t\|z\|}{1 - t\|z\|}\right) + \sum_{i=1}^{n} s_i\left(\frac{ts_i}{1 + ts_i}\right) + \|s\|\left(\frac{t\|s\|}{1 - t\|s\|}\right) \\
&= \sum_{i=1}^{n} \frac{tz_i^2}{1 + tz_i} + \frac{t\|z\|^2}{1 - t\|z\|} + \sum_{i=1}^{n} \frac{ts_i^2}{1 + ts_i} + \frac{t\|s\|^2}{1 - t\|s\|} \\
&= t\sum_{i=1}^{n} z_i^2\left(\frac{1}{1 + tz_i} + \frac{1}{1 - t\|z\|}\right) + t\sum_{i=1}^{n} s_i^2\left(\frac{1}{1 + ts_i} + \frac{1}{1 - t\|s\|}\right),
\end{aligned}$$

we have $|z_i| \leq \|z\|$ and $|s_i| \leq \|s\|$ for all i, which gives $-\|z\| \leq z_i \leq \|z\|$ and $-\|s\| \leq s_i \leq \|s\|, \forall i$. Then

$$\frac{1}{1 + t\|z\|} \leq \frac{1}{1 + tz_i} \leq \frac{1}{1 - t\|z\|} \text{ and } \frac{1}{1 + t\|s\|} \leq \frac{1}{1 + ts_i} \leq \frac{1}{1 - t\|s\|}.$$

This implies that

$$h_1'(t) \geq t\sum_{i=1}^{n} z_i^2\left(\frac{1}{1 - t^2\|z\|^2}\right) + t\sum_{i=1}^{n} s_i^2\left(\frac{1}{1 - t^2\|s\|^2}\right) \geq 0,$$

and consequently, the clip();function h_1 is increasing and as $h_1(0) = 0$, then $h_1(t) \geq 0$, $\forall t$ in its domain of definition. By the same argument we prove that $h_2(t)$ is decreasing and $h_2(0) = 0$ so $h_2(t) \leq 0$, $\forall t$. This completes the proof. □

By the first inequality in Lemma 4.2, we have

$$t(\|z\|+\|s\|)+\ln(1-t\|z\|)+\ln(1-t\|s\|)\leq -\sum_{i=1}^{n}\ln(1+tz_i)-\sum_{i=1}^{n}\ln(1+ts_i)+t\left(\sum_{i=1}^{n}z_i+\sum_{i=1}^{n}s_i\right).$$

Now, adding the term $\left[\left(\frac{t^2-2t}{2}\right)a-t\big(\|z\|^2+\|s\|^2\big)\right]$ for both sides to the previous inequality, we obtain $\gamma_3(t)\leq\gamma(t)$ for all $t\in[0,\tilde{t}_3)$.

The proof of the second majorant function is similar to the one used for $\gamma_3(t)$, where we use the second inequality in Lemma 4.2. Then it is easy to prove that $\gamma(t)\leq\gamma_4(t)$ for all $t\in[0,\tilde{t}_3)$. We mention that these two functions have the same domain of definition.

In addition, we have

$$\gamma_3'(t)=(t-1)a+\big(\|z\|+\|s\|-\|z\|^2-\|s\|^2\big)-\frac{\|z\|}{1-t\|z\|}-\frac{\|s\|}{1-t\|s\|}$$
$$\gamma_3''(t)=a+\frac{\|z\|^2}{(1-t\|z\|)^2}+\frac{\|s\|^2}{(1-t\|s\|)^2}>0$$
$$\gamma_4'(t)=(t-1)a-\big(\|z\|+\|s\|+\|z\|^2+\|s\|^2\big)+\frac{\|z\|}{1-t\|z\|}+\frac{\|s\|}{1-t\|s\|}$$

and

$$\gamma_4''(t)=a+\frac{\|z\|^2}{(1-t\|z\|)^2}+\frac{\|s\|^2}{(1-t\|s\|)^2}>0.$$

Further, the properties in (9) are satisfied by γ_3 and γ_4. Also it is clear that $\min_t\gamma_3(t)\leq\min_t\gamma(t)\leq\min_t\gamma_4(t)$ for $t\in[0,\tilde{t}_3)$.

Finally, we conclude that the computation of a displacement step $t^{(k)}$ at each iteration k, is based on the optimal minimum of the functions $\gamma_i(t), i=1,\ldots,4$. It is clear that these functions are definite and strictly convex on their domains $[0,\tilde{t}_i)$, $i=1,\ldots,4$ and therefore they reach their minimum when

$$\gamma_i'(t)=0,\ i=1,\ldots,4. \tag{12}$$

Hence, in the numerical implementation of our algorithm, we take only the root which belongs to the domain of each function.

5. Numerical results

In this section, we present some numerical results to demonstrate the efficiency of our algorithm applied to problems taken from the literature. Additionally, we compare the obtained numerical results with the classical Wolfe and Powell line search methods. In our implementation, the accuracy is set to $\epsilon=10^{-5}$. Considering the influence of the barrier parameter μ on the algorithm's performance, we experiment with different values of μ, and a threshold parameter $\rho=0.5$. The iterations were conducted using **MATLAB 7.9** software on a PC with a CPU running at **2.13 GHz** and **2 GB** of **RAM**, using double precision format. The solution and initial points are denoted by $x^\star$ and $x^{(0)}$, respectively. The terminology used is also presented below.

MIN1: The strategy based on the first minorant function.
MAJ1: The strategy based on the first majorant function.
MIN1: The strategy based on the second minorant function.
MAJ2: The strategy based on the second majorant function.
WOLFE: The strategy based on the classical Wolfe et Powell line search.
ITER: The number of iterations produced by the algorithm.

Table 1. Number of iterations and CPU time for problems 1 and 2 with $\mu = 0.5$.

	MIN1		MAJ1		MIN2		MAJ2		WOLFE	
Problem	ITER	CPU	ITER	CPU	ITER	CPU	ITER	CPU	ITER	CPU
Problem 1	2	0.0015	4	0.0049	5	0.0063	4	0.0046	7	0.0372
Problem 2	8	0.0087	8	0.0083	9	0.0118	7	0.0076	12	0.0924

CPU: The required time (in seconds).

Our testing problem set contains three examples of monotone LCPs. The first two are of small size, while the third one is with a large size.

Problem 1. The matrix M and the vector q for the monotone LCP are given by

$$M = \begin{pmatrix} 3 & -2 & -1 \\ -2 & 2 & 1 \\ -1 & 1 & 1 \end{pmatrix}, \quad q = (14, -11, -7)^T.$$

Problem 2. In this example M and q are given as follows

$$M = \begin{pmatrix} 2 & 1 & 0 & 3 & -3 \\ 1 & 4 & 0 & 4 & 2 \\ 0 & 0 & 6 & -2 & 1 \\ -3 & -4 & 2 & 0 & 0 \\ 3 & -2 & -1 & 0 & 0 \end{pmatrix}, \quad q = (1,\, -2,\, 4,\, 10,\, 2)^T.$$

The initial point for Problem 1 is $x^{(0)} = (0.1646, 4.2045, 3.1691)^T$ with $x^\star = (0,\, 4,\, 3)^T$.

However, for Problem 2, $x^{(0)} = e \in \mathbb{R}^5$ with $x^\star = (0,\, 0.5,\, 0,\, 0,\, 0)^T$ (Tab. 1).

Problem 3. Consider the monotone LCP where M and q are given by:

$$M = (m_{ij}) = \begin{cases} 4 & \text{if} \quad i = j, \\ -1 & \text{if} \quad |i - j| = 1, \\ 0 & \text{otherwise}, \end{cases} \qquad q = (-1,\, 1,\, \ldots,\, 1,\, -1)^T \in \mathbb{R}^n.$$

The initial point taken in Problem 3 is $x^{(0)} = e \in \mathbb{R}^n$. The details of our obtained numerical results for different size of n are given in Table 2.

5.1. Comments

Based on the numerical results presented in the tables above, we observe that determining the displacement step using the new technique of minorant and majorant approximation functions is efficient for solving monotone LCPs. It is noteworthy that the most favorable numerical results (in terms of the number of iterations and CPU time) are achieved with the first majorant function (MAJ1). Furthermore, this new technique of majorant and minorant functions presents a significant challenge to classical line search methods such as Wolfe's rule.

6. Conclusion and future work

In this paper, we present a logarithmic barrier method for solving monotone LCP problems. We have transformed the LCP into an equivalent optimization problem. We demonstrate the existence and uniqueness of optimal solutions for this transformed problem and show that the limit of its optimal solutions tends to a

Table 2. Number of iterations and CPU time for Problem 3 where $\mu = 0.4$.

	MIN1		MAJ1		MIN2		MAJ2		WOLFE	
n	ITER	CPU	ITER	CPU	ITER	CPU	ITER	CPU	ITER	CPU
100	36	0.3934	32	0.1273	98	0.6274	92	0.5104	105	0.6408
500	45	9.6555	42	5.2072	229	34.9834	203	21.2705	242	37.1834
1000	72	26.2486	67	23.4821	253	64.3724	244	56.2613	256	59.0025
2000	113	64.2154	105	51.7283	381	115.4927	367	98.2279	380	113.6798
3000	187	103.4915	174	98.9736	472	197.6292	459	172.6638	475	212.2719
4000	318	256.0387	294	228.3717	609	293.4173	582	286.8815	623	309.5198
5000	462	493.7629	438	478.1089	842	613.7291	804	592.0026	810	625.6813
6000	603	635.5107	588	697.9995	959	846.3992	951	839.9431	974	881.5193
7000	861	877.0591	806	932.5519	1228	1014.0937	1207	978.3921	1231	993.7192
8000	1036	1106.6692	1004	1065.4921	1443	1387.6639	1427	1255.7002	1431	1312.0048

solution of the LCPs. We apply Newton's method to determine the descent direction. The novel strategy of using minorant and majorant approximating functions is employed for computing the displacement step in this algorithm. The numerical results obtained are very promising and demonstrate the effectiveness of this new strategy. This encourages us to apply this new approach to other classes of nonlinear optimization problems and helps us identify promising future research subjects. Additionally, constructing other simple minorant and majorant approximating functions remains a good topic for further research. Finally, the convergence of the algorithm remains a good topic of research in the future.

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