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OPTIMAL REINSURANCE STRATEGY WITH MEAN-VARIANCE PREMIUM PRINCIPLE AND RELATIVE PERFORMANCE CONCERN

Shumin Chen[1,*] and Haixiang Yao[2]

Abstract. This paper investigates the optimal reinsurance strategies for n insurers who compete with each other within the non-zero-sum game framework, as well as the optimal reinsurance premium loadings under the Stackelberg framework. The reinsurance premium is determined in accordance with the mean-variance principle. The insurers' objectives are to maximize their utility of relative wealth over a finite decision horizon. Firstly, utilizing the dynamic programming technique, we derive a system of coupled Hamilton–Jacobi–Bellman (HJB) equations and characterize the equilibrium reinsurance strategies. We also obtain explicit solutions in the special case where the insurers possess exponential utility functions and present numerical examples to illustrate our theoretical findings. Secondly, leveraging the outcomes from the first section, we derive the optimal premium loadings for the reinsurer. We formulate the HJB equation and, for the special case of exponential utility, we numerically and explicitly obtain optimal decisions. Furthermore, we provide numerical examples to illustrate the impact of model parameters on the optimal reinsurance premium loadings.

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1. Introduction

Recently, the optimal reinsurance problem of insurers has received a great deal of attention, for examples, see Zeng *et al.* [44], Li and Young [27], Han *et al.* [24], Guan and Li [20], Meng *et al.* [35], Azcue *et al.* [1], Dong *et al.* [15], and the references therein. These studies accurately depict how insurers optimally operate, using reinsurance to transfer their risk of claims from policyholders.

In the industry, there are plenty of insurers in the market who compete with each other. Thus, interactions among competing insurers have aroused great interest and have been studied extensively in recent years. For example, Taksar and Zeng [40] investigate the optimal reinsurance strategies for a pair of competing insurers sharing a single reward function that is hinged on the surpluses of both insurers. They describe the Nash equilibrium of the zero-sum game, prove a verification theorem, and present explicit solutions for a probability-maximizing game. Bensoussan *et al.* [3] investigate the optimal reinsurance-investment problem for two insurers under the stochastic differential non-zero-sum game framework . They assume that each insurer has a different utility function and that both insurers' surpluses are modulated by a continuous-time Markov chain and a

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[1] School of Management, Guangdong University of Technology, Guangzhou 510006, P.R. China.
[2] School of Finance, Guangdong University of Foreign Studies, Guangzhou 510006, P.R. China.
[*]Corresponding author: chenshumin@gdut.edu.cn

market index. Along this line, Meng *et al.* [33] consider an optimal reinsurance problem when the two insurers' surpluses are subject to quadratic risk controls. Pun and Wong [37] consider a reinsurance game problem for two ambiguity-averse insurers and obtain a Nash equilibrium under a worst-case scenario framework for exponential utility functions. Deng *et al.* [14] study a non-zero-sum game between two insurers who can invest in a risk-free asset, a risky asset with Heston's stochastic volatility, and a defaultable corporate bond. More recently, Yang *et al.* [43] and Wang *et al.* [41] study the competition among n insurers.

On the other hand, how insurers make reinsurance decisions under different premium principles is also a topic of academic concern. Commonly, it is assumed that reinsurance premiums are calculated according to the expected value principle, variance principle, or exponential principle (see [14, 26, 38, 46]). In these studies, by using the dynamic programming principle approach, optimal reinsurance strategies are derived under different premium principles. Based on the explicit results, comparative static analyses are performed to show the impact of model parameters on the insurers' optimal decisions. Recently, the mean-variance reinsurance premium principle, which is a combination of the expected-value and variance premium principles, has become the focus of attention in research, see Han *et al.* [23, 24], Li and Young [29], Cao *et al.* [7, 8], Cao and Young [6], Dong *et al.* [15], Liang *et al.* [31]. The above research considers reinsurance strategies of insurance companies from different perspectives. For example, Cao *et al.* [7] study a dynamic Stackelberg differential game between an insurer and a reinsurer where both parties are ambiguous about the intensity of the loss. Cao and Young [6] consider a similar problem by using a diffusion process serving as a valid approximation of the classical Cramér-Lundberg (CL) risk process. They show that the equilibrium for the diffusion approximation equals the limit of the equilibrium for the scaled CL process. Han *et al.* [23, 24] determine the optimal reinsurance strategy to minimize the probability of drawdown, which is defined as the probability that the insurer's surplus process reaches some fixed fraction of its maximum value to date. It is worth noting that the above research has mainly considered reinsurance problems for a single insurer with the mean-variance premium principle. However, corresponding strategies for reinsurance among competitive insurers are lacking in the literature.

The main objective of this paper is to investigate the reinsurance decisions of competitive insurers (including property insurers and life insurers) and the optimal risk loadings for the reinsurer when the reinsurance premiums are determined by the mean-variance principle. Our model is similar to Elie *et al.* [16], Han *et al.* [25], Bensoussan *et al.* [3], Deng *et al.* [14], and Dong *et al.* [15]. In line with Yang *et al.* [43] and Wang *et al.* [41], we assume that there are $n(\geq 2)$ insurers who evaluate his performance relative to the average performance of the insurance market[1]. Since all the insurers operate similar insurance businesses, they are subject to the same common shock to the claims. For example, the weather and traffic conditions in a particular region have a significant influence on the number of claims filed against car insurance companies in that area. Inspired by Cao *et al.* [7, 8] and Cao and Young [6], we assume that the reinsurance premium rate is calculated according to the mean-variance principle. In addition to the insurance risk, the insurers may be worried that the insurance model is misspecified; that is, the model uncertainty (ambiguity)[2]. We formulate the insurers' optimization problems as a robust non-zero sum stochastic differential game. By using the dynamic programming approach, we write down a system of Hamilton–Jacobi–Bellman (HJB) equations and the verification theorem. When the value functions of the insurers exist and are twice continuously differentiable, we present some delicate analyses to show the existence and the structure of the equilibrium reinsurance strategies.

We consider the special case where insurers have exponential utilities. We derive robust equilibrium reinsurance strategies and equilibrium value functions in semi-explicit forms. When there are two insurers in the insurance market, we rigorously demonstrate that: (1) With a common shock, competition in the insurance market leads to a decrease in the demand for reinsurance protection. Stiffer competition leads to less reinsur-

[1] This model can also be applied to insurance companies, including property insurance companies and life insurance companies, with different lines of insurance businesses. For example, a car insurance company includes sections such as family car insurance, commercial bus insurance, and truck insurance, and each section uses the company's average performance as the benchmark for performance evaluation.

[2] Research on the robust reinsurance strategy is extensive, see a.s. Zhang and Siu [45], Yi *et al.* [42], Zeng *et al.* [44], Guan and Liang [21], Bäuerle and Leimcke [2], and Guan *et al.* [22].

ance being purchased by both insurers. (2) Insurers purchase less reinsurance as it becomes more expensive or as they become less ambiguous about the risk model. When $n > 2$, we also perform some comparative static analyses numerically on the equilibrium strategies to demonstrate our theoretical results and to provide more economic insights. Second, based on the insurers' optimal decisions, we consider the reinsurer's optimal reinsurance premium loadings under the Stackelberg framework. We write down the HJB equation for the reinsurer and derive the optimal reinsurance premium loadings implicitly. We also provide numerical examples to illustrate the impact of model parameters on the reinsurer's optimal reinsurance premium loadings.

The contributions of this paper are threefold. (i) We examine the optimal decision problem for n competing insurers and a reinsurer, a topic seldom explored in the literature. Additionally, we consider additional features pertinent to the optimal reinsurance problem, including the mean-variance principle in characterizing the reinsurance premium, relative performance concern, and model uncertainty. (ii) We first determine the optimal reinsurance strategies for the insurers and then the optimal reinsurance premium loadings for the reinsurer. We characterize the optimal reinsurance strategies semi-explicitly through a system of non-linear equations. Furthermore, we rigorously demonstrate the properties of the optimal reinsurance strategies and provide a algorithm for characterizing the optimal reinsurance premium loadings. (iii) We conduct numerical experiments to further illustrate the impact of model parameters on the insurers and the reinsurer's optimal decisions.

Recently, several papers have also addressed the optimal reinsurance problem within the non-zero-sum framework and the inclusion of model uncertainty; for instance, Deng *et al.* [14], Meng *et al.* [34], and Dong *et al.* [15]. However, these papers focus solely on the game between insurers and do not consider the optimal reinsurance premium loadings. Additionally, some papers investigate the Stackelberg differential game problem between an insurer and a reinsurer (*e.g.*, [7], Zhang *et al.* [47]), but they overlook the competition between insurers. Our paper is closely related to Gu *et al.* [19], who consider a similar problem. Our paper differs from Gu *et al.* [19] in three key aspects. First, Gu *et al.* [19] model the claims of policyholders using a Cramér-Lundberg model, whereas we employ a diffusion model, which makes our model easier to solve and facilitates detailed analysis of the solution. Second, contrary to the mean-value principle utilized by Gu *et al.* [19], we assume that premiums are determined under the mean-variance premium principle, which has garnered considerable attention recently. Third, we consider many competing insurers in the insurance market, making our model more realistic. Our paper is also related to Yang *et al.* [43] and Wang *et al.* [41], who also consider optimal reinsurance decisions for n insurers. However, Yang *et al.* [43] assume that the insurers employ proportional reinsurance to mitigate their claim risk, and the reinsurance premiums are determined using the variance value principle. Wang *et al.* [41], on the other hand, consider n insurers utilizing proportional reinsurance and derive optimal time-consistent reinsurance-investment strategies for the insurers, as well as the optimal reinsurance prices and investment strategy for the reinsurer.

We organize the remainder of this paper as follows: Section 2 introduces optimization problems for n insurers. First, we formulate the surplus processes for the insurers and define the non-zero-sum stochastic differential game with model ambiguity. Then, we derive a system of HJB equations for the stochastic differential game and analyze the equilibrium strategies. Finally, we provide semi-explicit solutions when the insurers have exponential utility functions. In Section 3, we study the optimization problem for the reinsurer to derive the optimal reinsurance premium rate based on the insurers' strategies determined in Section 2. Under the Stackelberg game framework, we determine the optimal premium loadings. Sections 2 and 3 present numerical analyses that illustrate our theoretical results. Section 4 concludes the paper. All the proofs are in the appendix.

2. Optimization problems for the insurers

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$ be a filtered complete probability space, where the information set $\mathcal{F}_t$ stands for the information available up to time t, T is the terminal of the insurers' decision horizon, and $\mathbb{P}$ is a reference probability measure. We assume that all stochastic processes and stochastic variables given in the following are well-defined and adapted to this probability space. For readability purposes, we summarize in Table 1 the key notations in the article hereafter.

TABLE 1. Summary table of important notations.

Symbol	Parameter description	Symbol	Parameter description
Insurers			
n	The number of insurers	p_i	Insurer i's premium rate
$N_i(t)+N(t)$	The number claims against insurer i up to time t	$\lambda+\lambda_i$	The arrival rate of claims to insurer i
$N(t)$	The common shock against all insurers	λ	The intensity of $N(t)$
$Z_{i,k}$	The size of k-th claim against insurer i	R_i	Insurer i's self-retention level
$\pi_i(t)$	Insurer i's reinsurance premium rate	$X_i(t)$	Insurer i's surplus at time t
m_i	The intensity of competition of insurer i	$W_i(t)$	Insurer i's relative surplus
U_i	Insurer i's utility function	r	Risk-free investment return
$\boldsymbol{\phi}_i(t)=(\phi_{ij}(t))$	Insurer i's probability distortion function	ψ_{ij}	Degree of insurer i's model ambiguity
Reinsurer			
$(\boldsymbol{\theta},\boldsymbol{\eta})=(\theta_i,\eta_i)$	Reinsurance risk loadings	$Y(t)$	Reinsurer's surplus at time t
U	The reinsurer's utility function	$(\tilde{\psi_i})$	Degree of the reinsurer's model ambiguity

2.1. The risk model

The insurance market consists of n competing insurers, denoted as insurer $i \in \{1,\cdots,n\}$, whose surpluses are depicted by the classic risk model[3]:

$$X_i(t) = x_i + p_i t - \sum_{k=1}^{N_i(t)+N(t)} Z_{i,k}, \tag{1}$$

where $x_i \geq 0$ is the initial surplus, $p_i > 0$ is the premium rate, $N_i(t)+N(t)$ represents the number of claims against insurer i up to time t, $\{Z_{i,k}\}$ are the positive claims against insurer i. The surplus process (1) indicates that all insurers are subject to a common impact that is represented by $N(t)$ [4, 5, 13, 30]. $N(t)$ represents a systematic insurance risk, while $N_i(t)$ represents insurer i's idiosyncratic insurance risk. $N_1(t),\cdots,N_n(t)$, and $N(t)$ are $n+1$ mutually independent homogeneous Poisson processes with positive intensity $\lambda_1,\cdots,\lambda_n$ and λ. The claim sizes $\{Z_{i,k}\}$ are independent identically distributed (i.i.d.) random variables that are independent of $N_i(t), N(t)$, and $\{Z_{j,k}\}, j \neq i$. In addition, we assume that $\{Z_{i,k}\}$ has common distribution function $F_i(\cdot)$ and

$$\mu_i := \mathbb{E}[Z_{i,k}] < \infty, \quad \sigma_i^2 := \mathbb{E}[Z_{i,k}^2] < \infty.$$

Assume that the reinsurer offers per-loss reinsurance to both insurers in exchange for a continuously payable premiums that are calculated using the *mean-variance premium principle* (see [23], Li and Young [28], [7]). Specifically, let $R_i(t,z)$ denote the retained claim of insurer i for a loss z at time t, then $z - R_i(t,z)$ is the amount of claim that is transferred to the reinsurer. Let $\bar{\lambda}_i := \lambda + \lambda_i$. Then, for $t \in [0,T]$, the reinsurance premium rate is given by

$$\pi_i(t) = \bar{\lambda}_i \mathbb{E}\left[(1+\theta_i(t))(Z_i - R_i(t,Z_i)) + \frac{\eta_i(t)}{2}(Z_i - R_i(t,Z_i))^2\right], \tag{2}$$

in which $\theta_i(t)$ and $\eta_i(t)$ are non-negative reinsurance premium loadings (or risk loadings).

[3] This model can be applied to property insurance companies; it can also be applied to life insurance companies, including most non-life insurance branches, as well as general types of life insurance; see Grandell [18].

Remark 1. When R_i is larger, insurer i bears a greater portion of the claim risk on his own, and as a result, according to equation (2) he pays the reinsurer a lower premium. When R_i is lower, however, he must pay a higher premium for reinsurance to manage the risk. We call R_i as the reinsurance strategy of insurer i. To simplify our notations, in the sequel, when there is no confusion we will write $R_i(t,z), \theta_i(t), \eta_i(t)$ as R_i, θ_i and η_i respectively.

Definition 1. A reinsurance strategy R_i is admissible if it is a Borel-measurable function of $(t,z) \in [0,T] \times \mathbb{R}_+$ such that $R_i \in [0,z]$. Let $\mathcal{R}$ denote the set of all admissible reinsurance strategies.

Assume that the insurer invests his wealth in the financial market with fixed return $r > 0$. Given risk loadings (θ_i, η_i), if insurer i adopts strategy $R_i \in \mathcal{R}$, the surplus evolves according to

$$X_i^{R_i}(t) = x_i + \int_0^t \Big(rX_i^{R_i}(s) + p_i - \pi_i(s) \Big)\, \mathrm{d}s - \sum_{k=1}^{N_i(t)+N(t)} R_i(Z_{i,k}), \qquad X_i^{R_i}(0) = x_i, \tag{3}$$

where π_i is given by equation (2). According to Grandell [18], Li and Young [27], and Han *et al.* [23],

$$R_i \mathrm{d}(N_i(t) + N(t)) \approx \bar{\lambda}_i \mathbb{E}[R_i]\, \mathrm{d}t - \sqrt{\bar{\lambda}_i \mathbb{E}[R_i^2]}\, \mathrm{d}B_i(t), \tag{4}$$

in which $\approx$ indicates that the risk process on the left hand side is approximated in law by the drifted Brownian motion on the right hand side, and converges as $\bar{\lambda}_i \to \infty$. $B_i(t)$ is a standard Brownian motion with $\mathbb{E}[\mathrm{d}B_i(t)\mathrm{d}B_j(t)] = \rho_{ij}\mathrm{d}t, j \neq i$, where ρ_{ij} is the correlation coefficient given by

$$\rho_{ij} := \frac{\lambda \mathbb{E}[R_i]\mathbb{E}[R_j]}{\sqrt{(\lambda_i + \lambda)(\lambda_j + \lambda)\mathbb{E}[R_i^2]\mathbb{E}[R_j^2]}}. \tag{5}$$

Thus, the stochastic process in equation (3) can be approximated by the following diffusion process:

$$\mathrm{d}X_i^{R_i}(t) = \Big(rX_i^{R_i}(t) + p_i - \pi_i(t) - \bar{\lambda}_i \mathbb{E}[R_i] \Big)\, \mathrm{d}t + \sqrt{\bar{\lambda}_i \mathbb{E}[R_i^2]}\, \mathrm{d}B_i(t), \qquad X_0(0) = x_i. \tag{6}$$

Inspired by Espinosa and Touzi [17], Chen *et al.* [12], and Dong *et al.* [15], we assume that both insurers' objectives are to maximize their expected utilities of relative performances at the terminal time T. Thus, given the strategy $(R_j)_{j\neq i}$, insurer i will choose an admissible reinsurance strategy such that the utility of relative suplus at the terminal time T is maximized:

$$\sup_{R_i \in \mathcal{R}} \mathbb{E}\Big[U_i\Big((1-m_i)X_i^{R_i}(T) + m_i(X_i^{R_i}(T) - \bar{X}(T)) \Big) \Big] = \sup_{R_i \in \mathcal{R}} \mathbb{E}\Big[U_i\Big(X_i^{R_i}(T) - m_i \bar{X}(T) \Big) \Big],$$

where U_i is the utility function of insurer i, $\bar{X}(T) = \frac{1}{n}\sum_{i=1}^n X_i^{R_i}(T)$ is the average surplus of the insurance industry at time T, $m_i \in [0,1]$ measures the sensitivity of insurer i to the performance of the insurance industry. With a larger m_i, insurer i pays more attention to his relative performance to the industry, and the insurance market becomes more competitive.

Given $R_j \in \mathcal{R}$, let $W_i(t) := X_i^{R_i}(t) - m_i \bar{X}(t)$ be the relative surplus process of insurer i, for $i = 1, \cdots, n$. It follows from equation (6) that

$$\begin{aligned} \mathrm{d}W_i(t) &= \mathrm{d}X_i^{R_i}(t) - m_i \mathrm{d}\bar{X}(t) \\ &= \left[rW_i(t) + p_i - m_i \frac{1}{n}\sum_{i=1}^n p_i - \left(\pi_i(t) - \frac{m_i}{n}\sum_{j=1}^n \pi_j(t) \right) - \left(\bar{\lambda}_i \mathbb{E}[R_i] - \frac{m_i}{n}\sum_{j=1}^n \bar{\lambda}_j \mathbb{E}R_j \right) \right] \mathrm{d}t \end{aligned}$$

$$
\begin{aligned}
&+\sqrt{\bar\lambda_i\mathbb{E}[R_i^2]}\,\mathrm{d}B_i(t)-\frac{m_i}{n}\sum_{j=1}^{n}\sqrt{\bar\lambda_j\mathbb{E}[R_j^2]}\,\mathrm{d}B_j(t)\\
&=\left[rW_i(t)+\chi_i-\left(\bar M_i\pi_i(t)-M_i\sum_{j\neq i}\pi_j\right)-\left(\bar M_i\bar\lambda_i\mathbb{E}[R_i]-M_i\sum_{j\neq i}\bar\lambda_j\mathbb{E}R_j\right)\right]\mathrm{d}t\\
&+\bar M_i\sqrt{\bar\lambda_i\mathbb{E}[R_i^2]}\,\mathrm{d}B_i(t)-M_i\sum_{j\neq i}\sqrt{\bar\lambda_j\mathbb{E}[R_j^2]}\,\mathrm{d}B_j(t),\qquad W_i(0)=w_i:=x_i-m_i\frac{1}{n}\sum_{j=1}^{n}x_j,
\end{aligned}
\tag{7}
$$

where $\chi_i := p_i - m_i\frac{1}{n}\sum_{i=1}^{n} p_i$, $M_i := \frac{m_i}{n}$, $\bar M_i := 1-\frac{m_i}{n}$.

2.2. Model ambiguity

In this section, we further assume that all insurers are uncertain about the drift terms of model (7) and are ambiguity-averse. To describe this model ambiguity, referring to Chen and Epstein [9] and Maenhout [32], we introduce a suitable probability distortion function $\Lambda_i(t)$ to define an alternative probability measure that is locally equivalent to $\mathbb{P}$, and consider insurer i's optimization problem *via* the new probability measure. $\Lambda_i(t)$ is defined by

$$
\frac{1}{\Lambda_i(t)}\,\mathrm{d}\Lambda_i(t) = -\sum_{j=1}^{n}\phi_{ij}(t)\,\mathrm{d}B_j(t),\qquad \Lambda_i(0)=1,
\tag{8}
$$

where the process $\boldsymbol{\phi}_i = (\phi_{ij}(t))_{t\in[0,T]}$ is called the *probability distortion function.*

Definition 2. A probability distortion function $\boldsymbol{\phi}_i = (\phi_{ij}(t))_{t\in[0,T]}$ is admissible if it is adapted to the filtration $\mathcal{F}_t$ and satisfies the Novikov condition

$$
\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^t\sum_{j=1}^{n}\phi_{ij}^2(s)\,\mathrm{d}s\right)\right] < +\infty
$$

for all $t \geq 0$. The set of all admissible probability distortion functions is denoted as Φ.

According to Radon–Nykodym theorem (see [36]), for any admissible probability distortion function $\boldsymbol{\phi}_i$, an alternative probability measure $\mathbb{Q}_{\boldsymbol{\phi}_i}$ is defined by

$$
\frac{\mathrm{d}\mathbb{Q}_{\boldsymbol{\phi}_i}}{\mathrm{d}\mathbb{P}}\bigg|_{\mathcal{F}_t} = \Lambda_i(t).
\tag{9}
$$

Here, we allow the insurers to have multi possibly different distortion functions that define n alternative probability measures according to equation (9). By Girsanov's theorem, given an admissible probability distortion function $\boldsymbol{\phi}_i(t) = (\phi_{ij}(t))$,

$$
B_j^{\boldsymbol{\phi}_i}(t) := B_j(t) + \int_0^t \phi_{ij}(s)\,\mathrm{d}s
$$

is a standard $\mathbb{Q}_{\boldsymbol{\phi}_i}$-Brownian motion. Thus, under the alternative probability measure $\mathbb{Q}_{\boldsymbol{\phi}_i}$, the surplus process equation (7) can be written as

$$
\begin{aligned}
\mathrm{d}W_i(t) = \Bigg[&rW_i(t)+\chi_i-\left(\bar M_i\pi_i(t)-M_i\sum_{j\neq i}\pi_j\right)-\left(\bar M_i\bar\lambda_i\mathbb{E}[R_i]-M_i\sum_{j\neq i}\bar\lambda_j\mathbb{E}R_j\right)\\
&-\bar M_i\sqrt{\bar\lambda_i\mathbb{E}R_i^2}\phi_{ii}(t)+M_i\sum_{j\neq i}\sqrt{\bar\lambda_j\mathbb{E}R_j^2}\phi_{ij}(t)\Bigg]\mathrm{d}t
\end{aligned}
$$

$$+ \bar{M}_i \sqrt{\bar{\lambda}_i \mathbb{E}[R_i^2]}\, \mathrm{d}B_i^{\boldsymbol{\phi}_i}(t) - M_i \sum_{j\neq i} \sqrt{\bar{\lambda}_j \mathbb{E}[R_j^2]}\, \mathrm{d}B_j^{\boldsymbol{\phi}_i}(t).$$

With model ambiguity, we determine the optimal reinsurance strategy R_i^* for insurer i under the worst-case prior, which is the so-called robust strategy, given that the other insurers adopt the optimal strategy $R_j, j \neq i$.

Problem 1. With ambigtuity-aversion, insurer i's problem is

$$J^i(t, w_i) = \sup_{R_i\in\mathcal{R}} \inf_{\boldsymbol{\phi}_i\in\Phi} \mathbb{E}^{\boldsymbol{\phi}_i} \left\{ U_i(W_i(T)) + \frac{1}{2}\int_t^T \sum_{j=1}^n \frac{\phi_{ij}^2(s)}{\psi_{ij}(s)}\, \mathrm{d}s \right\}, \qquad i = 1, \cdots, n, \tag{10}$$

where $\mathbb{E}^{\boldsymbol{\phi}_i}$ is the expectation under probability $\mathbb{Q}_{\boldsymbol{\phi}_i}$.

Problem 1 is a typical example of the non-zero-sum game among the n insurers. Consequently, the solution to Problem 1 is the Nash equilibrium of the game. In this problem, each insurer takes into account the impact of model ambiguity on his decisions by taking the worst-case scenario approach, as characterized by taking infimum over the set Φ. $\frac{1}{2}\int_t^T \sum_{j=1}^n \frac{\phi_{ij}^2(s)}{\psi_{ij}(s)}\, \mathrm{d}s$ in equation (10) is the penalty term, representing the disparity between the probability measures $\mathbb{Q}_{\boldsymbol{\phi}_i}$ and $\mathbb{P}$. It emphasizes that an alternative probability measure $\mathbb{Q}_{\boldsymbol{\phi}_i}$ should not deviate too far away from the reference probability measure $\mathbb{P}$, since $\mathbb{P}$ is the priori probability that should not be easily discarded. (ψ_{ij}) are non-negative weights to characterize the degree of insurer i's model ambiguity. Larger (ψ_{ij}) indicate that a given deviation from the reference probability measure $\mathbb{P}$ is less penalized and that insurer i has less faith in $\mathbb{P}$. Thus, insurer i's ambiguity aversion is increasing in ψ_{ij}.

2.3. General results

In this section, given admissible reinsurance premium loadings (θ_i, η_i), we determine the optimal reinsurance strategies and probability distortion functions for the insurers.

According to the principle of stochastic dynamic programming (SDP, see *e.g.*, [36]), if other insurers adopt optimal reinsurance strategies $(R_j)_{j\neq i}$ and the value function J^i is twice continuously differentiable with respect to w_i, it satisfies the following HJB equation:

$$J_t^i(t, w_i) + \sup_{R_i} \inf_{\boldsymbol{\phi}_i} \left\{ \mathcal{L}^{R_i, \boldsymbol{\phi}_i} J^i(t, w_i) + \sum_{j=1}^n \frac{\phi_{ij}^2}{2\psi_{ij}} \right\} = 0, \qquad i = 1, 2, j \neq i, \tag{11}$$

with terminal condition $J^i(T, w_i) = U_i(w_i)$, where $\mathcal{L}^{R_i,\boldsymbol{\phi}_i}$ is the infinitesimal generator of W_i with the admissible strategy R_i of insurer i and the distortion function $\boldsymbol{\phi}_i$:

$$\begin{aligned} \mathcal{L}^{R_i,\boldsymbol{\phi}_i} J^i(t, w_i) := J_{w_i}^i(t, w_i) \Bigg[& rw_i + \chi_i - \bar{M}_i\pi_i - \bar{M}_i\bar{\lambda}_i\mathbb{E}R_i \\ & + M_i \sum_{j\neq i} \pi_j + M_i \sum_{j\neq i} \bar{\lambda}_j \mathbb{E}R_j^* - \bar{M}_i \sqrt{\bar{\lambda}_i \mathbb{E}R_i^2}\phi_{ii} + M_i \sum_{j\neq i} \sqrt{\bar{\lambda}_j \mathbb{E}(R_j^*)^2}\phi_{ij} \Bigg] \\ & + \frac{1}{2}\Bigg[\bar{M}_i^2\bar{\lambda}_i\mathbb{E}R_i^2 + M_i^2 \sum_{j\neq i} \bar{\lambda}_j \mathbb{E}R_j^2 - 2M_i\bar{M}_i\rho_{ij}\sqrt{\bar{\lambda}_i\bar{\lambda}_j\mathbb{E}R_i^2\mathbb{E}R_j^2} \Bigg] J_{w_iw_i}^i(t, w_i), \end{aligned}$$

where the subscripts denote partial derivatives with respect to w_i.

Conversely, the following verification lemma shows that under certain conditions, solution to equation (11) coincides with the value function J^i.

Lemma 1. *For $i = 1, 2, \cdots, n$, suppose $\bar{J}^i \in \mathbb{C}^{1,2}([0,T] \times \mathbb{R})$ solves HJB equation (11), then $J^i = \bar{J}^i$. Moreover, let $\boldsymbol{\phi}_i^* \in \Phi$ denote the probability distortion that achieves the infimum in equation (11), and let $R_i^* \in \mathcal{R}$ denote the self retention level that achieves the supremum in equation (11). Then, R_i^* is the optimal reinsurance strategy, $\boldsymbol{\phi}_i^* = (\phi_{ij}^*)$ is the optimal probability distortion function for insurer i.*

According to this lemma, we need to find solutions $\bar{J}^1, \cdots, \bar{J}^n$ to a system of coupled HJB equation (11) and characterize the corresponding equilibrium strategies $R_1^*, \cdots, R_n^*$ and the probability distortion functions $\boldsymbol{\phi}_1^*, \cdots, \boldsymbol{\phi}_n^*$. Before this, we explore some properties of the equilibrium strategies R_i^*. The following theorem reports our main results.

Theorem 1. *Suppose that $J^i \in \mathbb{C}^{1,2}([0,T] \times \mathbb{R}), i = 1, \cdots, n$ are strictly increasing and concave. Let*

$$\kappa_i := -\frac{J^i_{w_i w_i}}{J^i_{w_i}} + \bar{M}_i \psi_{ii} J^i_{w_i}, \tilde{M}_i := -M_i \frac{\lambda}{\bar{\lambda}_i} \frac{J^i_{w_i w_i}}{J^i_{w_i}} \tag{12}$$

and $\boldsymbol{\phi}_i^* = (\phi_{ij}^*)$, where

$$\begin{cases} \phi_{ii}^* = \bar{M}_i \sqrt{\bar{\lambda}_i \mathbb{E}(R_i^*)^2} \psi_{ii} J^i_{w_i}, \\ \phi_{ij}^* = -M_i \sqrt{\bar{\lambda}_j \mathbb{E}(R_j^*)^2} \psi_{ij} J^i_{w_i}, j \neq i \end{cases} \tag{13}$$

and

$$R_i^* = \frac{\theta_i + \tilde{M}_i \sum_{j \neq i} \mathbb{E} R_j^* + \eta_i z}{\eta_i + \kappa_i} \wedge z. \tag{14}$$

If $\boldsymbol{\phi}_i^* = (\phi_{ij}^*)$ satisfies the Novikov condition in Definition 2, $(R_1^*, \cdots, R_n^*)$ are the optimal reinsurance strategies and $\boldsymbol{\phi}_i^*$ is the optimal probability distortions of insurer i.

Proof. See Appendix A.1. □

Remark 2. Without competition or common shock ($m_i = 0$ or $\lambda = 0$), we have

$$R_i^* = \frac{\theta_i + \eta_i z}{\eta_i + \kappa_i} \wedge z,$$

i.e., the reinsurance strategy of insurer i is independent of insurer j. By comparing this finding to equation (14), it is evident that insurer i tends to retain greater risk on his own when he is concerned about the performance of the industry and when there exists a common shock. This outcome is in line with our general intuition and is consistent with Chen *et al.* [12]. Thus, in a competitive insurance market, with a common shock, the demand for reinsurance is weakened. In this case, all insurers face a larger risk exposure and should be subject to stricter risk monitoring imposed by the regulatory authority.

Remark 3. When the reinsurance premium is determined according to the expected value principle (*i.e.*, θ_i is constant and $\eta_i \equiv 0$), then the optimal reinsurance strategy R_i^* in equation (14) reduces to an excess-of-loss reinsurance strategy, specifically,

$$R_i^* = \frac{\theta_i + \tilde{M}_i \sum_{j \neq i} \mathbb{E} R_j^*}{\kappa_i} \wedge z.$$

This result is well reported in a series literature (see [7, 29]). However, these papers only consider the optimal reinsurance problem for one insurer.

From equation (14), it follows that

$$\mathbb{E}(R_i^*) = \mu_i - \frac{\kappa_i}{\kappa_i + \eta_i} \int_0^\infty \left(z - \frac{\theta_i + \tilde{M}_i \sum_{j \neq i} \mathbb{E} R_j^*}{\kappa_i} \right) \mathbf{1}_{z > \frac{\theta_i + \tilde{M}_i \sum_{j \neq i} \mathbb{E} R_j^*}{\kappa_i}} \, \mathrm{d}F_i(z), \qquad i = 1, \cdots, n. \tag{15}$$

Once $J^1, \cdots, J^n$ are given, according to the following lemma, $\mathbb{E}R_1^*, \cdots, \mathbb{E}R_n^*$ are determined uniquely and then $R_1^*, \cdots, R_n^*$ are also determined by equation (14).

Lemma 2. *Assume* $J^i \in \mathbb{C}^{1,2}([0,T]\times\mathbb{R})$. *Then,* $\mathbb{E}R_1^*,\cdots,\mathbb{E}R_n^*$ *are uniquely determined.*

Proof. See Appendix A.2. □

The proof of Lemma 2 also provides a iterative numerical scheme for the characterization of $\mathbb{E}R_1^*,\cdots,\mathbb{E}R_n^*$, see Algorithm 1 below.

Algorithm 1: The algorithm for characterizing $\mathbb{E}R_1^*,\cdots,\mathbb{E}R_n^*$.

Result: $\mathbb{E}R_1^*,\cdots,\mathbb{E}R_n^*$
Data: model parameters
initialization: let $(x_1^0,\cdots,x_n^0)=(\mu_1,\cdots,\mu_n),\delta=1$;
while $\delta>\varepsilon$ **do**
  for $i=1,\cdots,n$, calculate $x_i^1=\mu_i+\int_0^\infty \frac{\theta_i+\tilde{M}_i\sum_{j\neq i}x_j^0-\kappa_i z}{\kappa_i+\eta_i}\mathbf{1}_{z>\frac{\theta_i+\tilde{M}_i\sum_{j\neq i}x_j^0}{\kappa_i}}\mathrm{d}F_i(z)$;
  $\delta=\max(|x_1^1-x_1^0|\cdots,|x_n^1-x_n^0|)$;
  if $\delta<\varepsilon$ **then**
    $(\mathbb{E}R_1^*,\cdots,\mathbb{E}R_n^*)=(x_1^1,\cdots,x_n^1)$;
    break;
  else
    let $(x_1^0,\cdots,x_n^0)=(x_1^1,\cdots,x_n^1)$;
  end
end

2.4. The case of exponential utility

Next, we derive semi-closed form expressions of the optimal reinsurance strategies and probability distortions for the insurers in the special case of exponential utility. That is, we assume that all insurers have utility functions:

$$U_i(W_i)=-\frac{1}{\gamma_i}e^{-\gamma_i W_i},\qquad i=1,2,\cdots,n, \tag{16}$$

where $\gamma_i>0$ is the risk-aversion parameter of insurer i. This utility function is well adopted in actuarial research, see Chen *et al.* [12], and Deng *et al.* [14]. Besides, for analytical tractability, following Yi *et al.* [42] we assume that (ψ_{ij}) are non-negative, state-dependent, and inversely proportional to the value function such that

$$\psi_{ij}(s)=-\frac{\alpha_i}{\gamma_i J^i(s,W_i)}, \tag{17}$$

where α_i is a non-negative parameter that captures insurer i's attitude of ambiguity aversion. When α_i is larger, insurer i is more averse to model ambiguity. When $\alpha_i=0$, insurer i is ambiguity neutral.

Theorem 2. *Assume equations* (16) *and* (17) *hold. For insurer* i, *the value function is given by*

$$\bar{J}^i(t,w_i)=-\frac{1}{\gamma_i}\exp\{-\gamma_i w_i e^{r\tau}+A_i(\tau)\},$$

where $\tau=T-t$ *and* A_i *is given by equation* (A.8). *The optimal reinsurance strategy is given by*

$$R_i^*=\frac{\theta_i+\tilde{M}_i\sum_{j\neq i}\mathbb{E}[R_j^*]+\eta_i z}{\eta_i+\kappa_i}\wedge z, \tag{18}$$

where $\tilde{M}_i=M_i\gamma_i e^{r\tau}\frac{\lambda}{\bar{\lambda}_i},\kappa_i=(\gamma_i+\bar{M}_i\alpha_i)e^{r\tau}$, *and the optimal probability distortion* $\boldsymbol{\phi}_i^*$ *is given by*

$$\begin{cases}\phi_{ii}^*(\tau)=\bar{M}_i\alpha_i e^{r\tau}\sqrt{\bar{\lambda}_i\mathbb{E}(R_i^*)^2},\\ \phi_{ij}^*(\tau)=-M_i\alpha_i e^{r\tau}\sqrt{\bar{\lambda}_j\mathbb{E}(R_j^*)^2}, j=1,\cdots,n\end{cases} \tag{19}$$

Proof. See Appendix A.3. □

Specially, we consider the case where the market has two insurers. In this special case, the two insurers are competing with each other. According to (15), their optimal reinsurance strategies are determined by

$$\mathbb{E}R_1^* = \mu_1 - \frac{\kappa_1}{\kappa_1+\eta_1}\int_0^\infty \left(z - \frac{\theta_1 + \tilde{M}_1\mathbb{E}R_2^*}{\kappa_1}\right)\mathbf{1}_{z>\frac{\theta_1+\tilde{M}_1\mathbb{E}R_2^*}{\kappa_1}}\,\mathrm{d}F_1(z), \tag{20}$$

$$\mathbb{E}R_2^* = \mu_2 - \frac{\kappa_2}{\kappa_2+\eta_2}\int_0^\infty \left(z - \frac{\theta_2 + \tilde{M}_2\mathbb{E}R_1^*}{\kappa_2}\right)\mathbf{1}_{z>\frac{\theta_2+\tilde{M}_2\mathbb{E}R_1^*}{\kappa_2}}\,\mathrm{d}F_2(z). \tag{21}$$

Based on the system of equations, we are able to rigorously demonstrate the impact of model parameters on the insurers' reinsurance decisions.

Lemma 3. *Assume $\lambda > 0$ and $m_1 m_2 > 0$. For $i \in \{1,2\}$,*

(i) *R_i^* strictly increases with the common shock λ.*
(ii) *R_i^* strictly increases as the extent of competition m_i or m_j increases.*

Proof. See Appendix A.4. □

Lemma 3 indicates that, as insurer i is more concerned about the relative performance (*i.e.*, m_i is larger), he tends to keep more claims by himself and buy less reinsurance. This outcome is logical; as the insurer gets more competitive, he decides to take on more risk in order to earn more money and outperform the industry in terms of relative performance. Whereas, it is interesting to see that insurer i also buys less reinsurance when insurer j becomes more concerned about his relative performance (*i.e.*, m_j is larger). The reason is that as insurer j becomes more competitive, his surplus is expected to be larger at terminal time T. Thus insurer i is encouraged to act similarly to get more profit[4]. It appears that insurers' level of competition has a significant influence on reinsurance decisions, and this should not be ignored. The argument further indicates how risk homogeneity raises the market's overall risk.

On the other hand, both insurers are more connected to one another when they are both subject to a greater degree of common shock represented by λ. Both insurers decide to raise their retention levels and pay more claims on their own because a higher level of reinsurance protection in this situation will not boost their relative terminal surplus.

Lemma 4. *For $i \in \{1,2\}$, R_i^* strictly decreases with both α_i and $\alpha_j (j \neq i)$.*

Proof. See Appendix A.5. □

Lemma 4 shows that insurer i pays less claims and acquires more reinsurance for risk management when he is more ambiguity averse. On the other hand, insurer j would behave similarly when he is more ambiguous and will have less surplus at time T; then, insurer i will concentrate on his own risk management and increase the reinsurance purchases as a result of the decreased relative performance pressure.

Lemma 5 demonstrates the impact that reinsurance premium loadings has on the insurers' reinsurance strategies.

Lemma 5. *For $i \in \{1,2\}$, R_i^* increases as $\theta_i, \eta_i, \theta_j$ or $\eta_j, j = 1, 2$, increases.*

Proof. See Appendix A.6. □

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[4]Bensoussan *et al.* [3], Siu *et al.* [39], Chen *et al.* [12], and others have reported similar results. However, in these articles, the result is not rigorously proved, but is illustrated numerically.

TABLE 2. Model parameters.

	ξ_i	λ_i	γ_i	m_i	θ_i	η_i	α_i
$i=1$	1	1	0.3	0.5	0.2	0.5	1
$i=2$	1	0.5	0.5	0.7	0.3	0.5	1
$i=3$	1	1	0.5	0.5	0.2	0.5	1

Lemma 5 shows that insurer i tends to accept more claims on his own as his reinsurance protection gets more expensive. This result is intuitive. However, it is interesting to see that insurer i also tends to retain more risk when insurer j's reinsurance contract becomes more expensive. In fact, with a more expensive reinsurance contract insurer j tends to buy less reinsurance and is likely to have more surplus at terminal time T. Thus, insurer i is forced to retain more risk by himself.

Thus, the demand for reinsurance protection declines as it becomes more expensive. This is likely to have a negative effect on the profit of the reinsurer, which would have an adverse effect on the reinsurer's ability to provide reinsurance protection. Therefore, it is important that reinsurance prices are kept to reasonable levels in order to ensure the stability of reinsurance market. We will explore this problem in Section 3.

Example 1. We assume that claims for both insurers are exponentially distributed[5], *i.e.*,

$$\mathrm{d}F_i(z) = \xi_i e^{-\xi_i z}\,\mathrm{d}z, \qquad i = 1, 2, 3, \tag{22}$$

where $\xi_i > 0$. Then $\mathbb{E}[Z_i] = \frac{1}{\xi_i}, \mathbb{E}[Z_i^2] = \frac{2}{\xi_i^2}$, and

$$\begin{aligned}\mathbb{E}(R_i^*) &= \frac{1}{\xi_i} - \frac{(\gamma_i + \bar{M}_i\alpha_i)e^{r\tau}}{(\gamma_i + \bar{M}_i\alpha_i)e^{r\tau} + \eta_i}\int_0^\infty \left(z - \frac{\theta_i + \tilde{M}_i \sum_{j\neq i}\mathbb{E}R_j^*}{(\gamma_i + \bar{M}_i\alpha_i)e^{r\tau}}\right)\mathbf{1}_{z > \frac{\theta_i + \tilde{M}_i \sum_{j\neq i}\mathbb{E}R_j^*}{(\gamma_i + \bar{M}_i\alpha_i)e^{r\tau}}}\,\xi_i e^{-\xi_i z}\,\mathrm{d}z \\ &= \frac{1}{\xi_i}\left(1 - \frac{(\gamma_i + \bar{M}_i\alpha_i)e^{r\tau}}{(\gamma_i + \bar{M}_i\alpha_i)e^{r\tau} + \eta_i} e^{-\xi_i \frac{\theta_i + \tilde{M}_i \sum_{j\neq i}\mathbb{E}R_j^*}{(\gamma_i + \bar{M}_i\alpha_i)e^{r\tau}}}\right).\end{aligned}$$

Moreover, we set $\lambda = 0.5$ and $T = 5$. Other parameters are close to those given in the existing literature, see *e.g.*, Chen and Shen [10,11], Yang *et al.* [43], Gu *et al.* [19]. They are presented in Table 2. Using Algorithm 1, we obtain $\mathbb{E}R_1^*, \mathbb{E}R_2^*$ and $\mathbb{E}R_3^*$ and then the insurers' optimal reinsurance strategies R_1^*, R_2^* and R_3^*, see Figure 1.

Figure 2 shows the optimal reinsurance strategies of the insurers as functions of model parameters. We fix the claim size $z = 1$. Panels (a) and (c) of Figure 2 verify our arguments in Lemmas 3 and 4. As the common shock (λ) and competition levels (m_1) increase, all insurers tend to keep more risk to themselves; as the insurers are more uncertain about the model, they tend to transfer more risk to the reinsurers. These results are consistent with our theoretical results. It is worth noting that insurer 1's demand for reinsurance, $Z^1 - R_1^*$, increases dramatically as α_1 increases, indicating that the insurer's model ambiguity has a significant impact on his decision making. However, the effect of α_1 on insurers 2 and 3 is negligible. Finally, panel (d) indicates that Insurer 1's risk aversion has a significant and decreasing effect on his risk retention. By comparing the first and second columns of the figure, we find that λ and m_1 depict the relationship between insurer i and the overall insurance market, and their changes also have a significant impact on the overall insurance market. However, model uncertainty α_1 and risk preference γ_1 mainly characterize insurer i's heterogeneity preferences, which have a significant impact on the insurer's decision-making but have little effect on the overall insurance market.

Figure 3 shows the impact of reinsurance loadings on the reinsurance strategies. As θ_1 and η_1 increase, the reinsurance protection becomes more expensive and the insurers tend to keep more risk to themselves. The

[5] Exponential distributions are also consider by Grandell [18], Schmidli [38], Gu *et al.* [19], among others. Although we can also take other distributions into consideration, they are popular in actuarial mathematics study due to their ease of calculation.

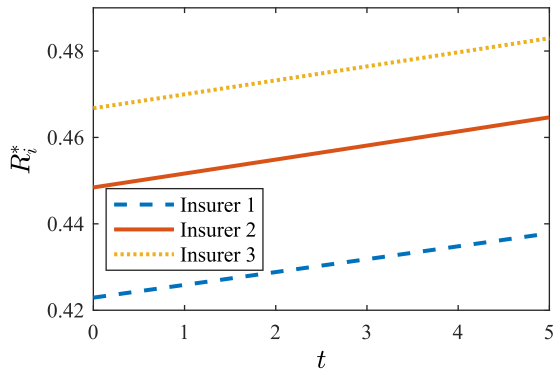

FIGURE 1. The optimal reinsurance strategies R_1^*, R_2^*, R_3^* when $U_i(W) = -\frac{1}{\gamma_i} e^{-\gamma_i W}$.

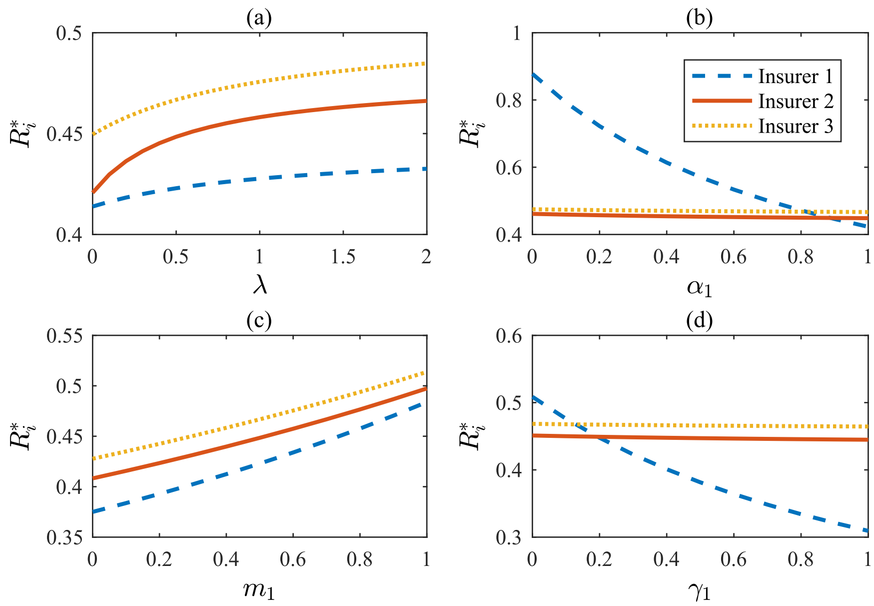

FIGURE 2. The optimal reinsurance strategies R_1^*, R_2^*, R_3^* as the functions of $\lambda, \alpha_1, m_1, \gamma_1$.

results are intuitive and are consistent with our argument in Lemma 5. Furthermore, it is noted that θ_1 and η_1 have a greater impact on R_1^* while having a smaller impact on R_2^* and R_3^* because their influences on insurers 2 and 3 are indirect.

3. THE CHARACTERIZATION OF REINSURANCE PRICES

In this section, we characterize the optimal reinsurance premium parameters for the reinsurer in a Stackelberg differential game framework based on the results of Section 2.

By taking reinsurance business, the reinsurer receives $\sum_{i=1}^n \pi_i(t)$ for each unit of time and pays $Z_{i,k} - R_i^*$ for a claim $Z_{i,k}$. Given a set of risk loadings $(\boldsymbol{\theta}, \boldsymbol{\eta}) = (\theta_i, \eta_i)_i$, let $Y(t)$ denote the reinsurer's surplus at time t and $\bar{Z}_{i,k} := Z_{i,k} - R_i^*$. Besides, the reinsurer invests his wealth in the financial market with the same return rate r.

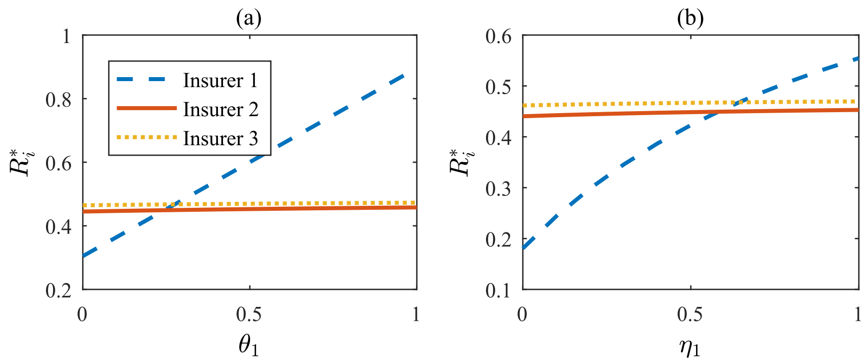

FIGURE 3. The optimal reinsurance strategies (R_1^*, R_2^*) under different premium loadings.

Then $Y(t)$ evolves according to

$$\mathrm{d}Y(t) = \left[rY(t) + \sum_{i=1}^{n} \pi_i(t)\right]\mathrm{d}t - \mathrm{d}\sum_{i=1}^{n}\sum_{k=1}^{N_i(t)+N(t)} \bar{Z}_{i,k}, \qquad Y(0) = y, \tag{23}$$

where y is the reinsurer's surplus at time $t = 0$,

$$\pi_i(t) = \bar{\lambda}_i \mathbb{E}\Big[(1+\theta_i)\bar{Z}_i + \frac{\eta_i}{2}\bar{Z}_i^2\Big]$$

is the reinsurance premium from insurer i. Again, we approximate the compound Poisson processe in equation (23) by a diffusion process for tractability:

$$\bar{Z}_i \mathrm{d}(N_i(t) + N(t)) \approx \bar{\lambda}_i \mathbb{E}\big[\bar{Z}_i\big]\,\mathrm{d}t - \sqrt{\bar{\lambda}_i \mathbb{E}\big[\bar{Z}_i\big]^2}\,\mathrm{d}\tilde{B}_i(t),$$

where $(\tilde{B}_i(t))$ are Brownian motions with correlation coefficients

$$\tilde{\rho}_{ij} = \frac{\lambda \mathbb{E}[\bar{Z}_i]\mathbb{E}[\bar{Z}_j]}{\sqrt{\bar{\lambda}_i \bar{\lambda}_j \mathbb{E}[\bar{Z}_i^2]\mathbb{E}[\bar{Z}_j^2]}}, \qquad i \neq j.$$

Thus, the reinsurer's surplus process in equation (23) is approximated by

$$\mathrm{d}Y(t) = \left(rY(t) + \sum_{i=1}^{n} \bar{\lambda}_i \mathbb{E}\Big(\theta_i \bar{Z}_i + \frac{\eta_i}{2}(\bar{Z}_i)^2\Big)\right)\mathrm{d}t + \sum_{i=1}^{n}\sqrt{\bar{\lambda}_i \mathbb{E}\big[\bar{Z}_i^2\big]}\,\mathrm{d}\tilde{B}_i(t), \qquad Y(0) = y. \tag{24}$$

Assume that the reinsurer is ambiguity averse. Given probability distortion function $\tilde{\boldsymbol{\phi}} = (\tilde{\phi}_i) \in \Phi$, where Φ is defined in Definition 2, we may define an equivalent probability measure $\mathbb{Q}_{\tilde{\boldsymbol{\phi}}}$ by equations (8) and (9). Under the alternative probability measure $\mathbb{Q}_{\tilde{\boldsymbol{\phi}}}$,

$$\tilde{B}_i^{\tilde{\boldsymbol{\phi}}}(t) := \tilde{B}_i(t) + \int_0^t \tilde{\phi}_i(s)\,\mathrm{d}s, \qquad i = 1, \cdots, n$$

are standard Brownian motions. Thus, under $\mathbb{Q}_{\tilde{\boldsymbol{\phi}}}$, the reinsurer's surplus in equation (24) can be written as

$$\mathrm{d}Y(t) = \left(rY(t) + \sum_{i=1}^{n} \bar{\lambda}_i \mathbb{E}\Big(\theta_i \bar{Z}_i + \frac{\eta_i}{2}\bar{Z}_i^2\Big) - \sum_{i=1}^{n}\sqrt{\bar{\lambda}_i \mathbb{E}\big[\bar{Z}_i^2\big]}\tilde{\phi}_i\right)\mathrm{d}t$$

$$+\sum_{i=1}^{n}\sqrt{\bar{\lambda}_i\mathbb{E}\big[\bar{Z}_i^2\big]}\,\mathrm{d}\tilde{B}_i^{\tilde{\phi}}(t), \qquad Y(0)=y.$$

The reinsurer's objective is to seek the robust optimal premium loadings $(\boldsymbol{\theta}^\star, \boldsymbol{\eta}^\star)$ to maximize the expected value of terminal surplus. Thus, the reinsurer's objective function is given by

$$\tilde{J}(t,y;\boldsymbol{\theta},\boldsymbol{\eta}) = \mathbb{E}^{\tilde{\phi}}\left[U(Y(T)) + \frac{1}{2}\sum_{j=1}^{n}\int_t^T \frac{\tilde{\phi}_j^2(s)}{\tilde{\psi}_j(s)}\,\mathrm{d}s\right] \tag{25}$$

where $\mathbb{E}^{\tilde{\phi}}$ is the expectation under probability $\mathbb{Q}_{\tilde{\boldsymbol{\phi}}}$, $(\tilde{\psi}_j)_j$ are non-negative weights to characterize the degree of the reinsurer's model ambiguity.

Problem 2. With model ambiguity, the reinsurer's problem is

$$V(t,y) = \sup_{(\boldsymbol{\theta},\boldsymbol{\eta})} \tilde{J}(t,y;\boldsymbol{\theta},\boldsymbol{\eta}). \tag{26}$$

By integrating Problems 1 and 2, we arrive at a Stackelberg differential game framework, featuring a single leader (the reinsurer) and multiple followers (the insurers), similar to the setups presented in Chen and Shen [10, 11] as well as Cao *et al.* [7, 8]. In the game, the reinsurer provides each insurer a reinsurance contract, which is characterized by a pair of risk loadings $(\theta_i,\eta_i), i=1,\cdots,n$ with the premium rate π_i. Then, with the given reinsurance contract, insurer i maximizes the objective function J^i by choosing the optimal reinsurance strategy R_i^* and optimal probability distortion function $\boldsymbol{\phi}_i^*$ that is determined in Theorem 2. Finally, the reinsurer specifies the optimal premium loadings $(\boldsymbol{\theta}^\star, \boldsymbol{\eta}^\star)$, and the optimal probability distortion functions $\tilde{\boldsymbol{\phi}}^\star$ under the worst case scenery.

If $V \in \mathbb{C}^{1,2}([0,T]\times\mathbb{R})$, according to the dynamic programming principle, it satisfies the following HJB equation:

$$\begin{aligned} V_t + \sup_{(\boldsymbol{\theta},\boldsymbol{\eta})}\inf_{\tilde{\boldsymbol{\phi}}}\Bigg\{ & ryV_y + \sum_{i=1}^{n}\left(\bar{\lambda}_i\mathbb{E}\Big[\theta_i\bar{Z}_i + \frac{\eta_i}{2}\big(\bar{Z}_i^2\big)\Big] - \sqrt{\bar{\lambda}_i\mathbb{E}\big[\bar{Z}_i^2\big]}\tilde{\phi}_i\right)V_y \\ & + \frac{1}{2}\left(\sum_{i=1}^{n}\bar{\lambda}_i\mathbb{E}\big[\bar{Z}_i^2\big] + \sum_{i\neq j}\tilde{\rho}_{ij}\sqrt{\bar{\lambda}_i\mathbb{E}\big[\bar{Z}_i^2\big]}\sqrt{\bar{\lambda}_j\mathbb{E}\big[\bar{Z}_j^2\big]}\right)V_{yy} + \sum_{i=1}^{n}\frac{\tilde{\phi}_i^2}{2\tilde{\psi}_i}\Bigg\} = 0, \end{aligned} \tag{27}$$

with terminal condition $V(T,y) = U(y)$.

Lemma 6. *Suppose $\bar{V} \in \mathbb{C}^{1,2}([0,T],\mathbb{R})$ solves HJB equation (27). Then $V = \bar{V}$. Moreover, let $\boldsymbol{\phi}^\star$ denote the probability distortion that achieves the infimum in equation (27), and let $(\boldsymbol{\theta}^\star, \boldsymbol{\eta}^\star)$ denote the premium loadings that achieve the supremum in equation (27). Then, $(\boldsymbol{\theta}^\star, \boldsymbol{\eta}^\star)$ and $\boldsymbol{\phi}^\star$ are optimal.*

3.1. The case of exponential utility

In this case, we consider the insurers have utility functions equation (16) and the reinsurer has exponential utility function defined as $U(Y) = -\frac{1}{\gamma}e^{-\gamma Y}$, where $\gamma > 0$ is a constant relative risk aversion coefficient. Assume that ψ_i is non-negative, state-dependent, and inversely proportional to the value function such that

$$\tilde{\psi}_i(s) = -\frac{\tilde{\alpha}_i}{\gamma V}, \tag{28}$$

where $\tilde{\alpha}_i$ is a non-negative parameter that captures the reinsurer's attitude of ambiguity aversion.

Based on Theorem 2, we have the following results.

Theorem 3. *Given insurers' self-retention levels $R_i^*, i = 1, \cdots, n$, in Section 2.4, the value function for the reinsurer is given by*

$$V(t,y) = -\frac{1}{\gamma} e^{-\gamma y e^{r\tau} + B(\tau)},$$

where $\tau = T - t$, B depends only on τ. Then, $\tilde{\phi}_i^\star = \tilde{\alpha}_i e^{r\tau} \sqrt{\bar{\lambda}_i \mathbb{E}[\bar{Z}_i^2]}$. The optimal risk loadings are given by

$$(\theta_i^\star, \eta_i^\star) = \arg\max_{(\theta_i,\eta_i)} \left\{ \sum_{i=1}^n \bar{\lambda}_i \left(\frac{1}{2}(\hat{\gamma}_i - \eta_i)\mathbb{E}\big[\bar{Z}_i^2\big] - \theta_i \mathbb{E}\big[\bar{Z}_i\big] \right) + \lambda\gamma e^{r\tau} \sum_{j \neq i} \mathbb{E}\big[\bar{Z}_i\big]\mathbb{E}\big[\bar{Z}_j\big] \right\}, \qquad i = 1, \cdots, n,$$

where $\hat{\gamma}_i := (\tilde{\alpha}_i + \gamma)e^{r\tau}$.

Proof. See Appendix A.7. □

According to (15), we have

$$\begin{aligned}
\mathbb{E}[\bar{Z}_i] &= \mathbb{E}\left[\left(Z_i - \frac{\theta_i + \tilde{M}_i \sum_{j\neq i} \mathbb{E}[R_j^*] + \eta_i Z_i}{\eta_i + \kappa_i} \right) \mathbf{1}_{\{Z_i > \frac{\theta_i + \tilde{M}_i \sum_{j\neq i} \mathbb{E}[R_j^*] + \eta_i Z_i}{\eta_i + \kappa_i}\}} \right] \\
&= \mathbb{E}\left[\frac{\kappa_i}{\eta_i + \kappa_i} \left(Z_i - \frac{\theta_i - \tilde{M}_i \sum_{j\neq i} \mathbb{E}[\bar{Z}_j] + \tilde{M}_i \sum_{j\neq i} \mu_j}{\kappa_i} \right) \mathbf{1}_{\{Z_i > \frac{\theta_i - \tilde{M}_i \sum_{j\neq i} \mathbb{E}[\bar{Z}_j] + \tilde{M}_i \sum_{j\neq i} \mu_j}{\kappa_i}\}} \right] \\
&= \frac{\kappa_i}{\eta_i + \kappa_i} \int_{\frac{\theta_i - \tilde{M}_i \sum_{j\neq i} \mathbb{E}[\bar{Z}_j] + \tilde{M}_i \sum_{j\neq i} \mu_j}{\kappa_i}}^{\infty} \left(z - \frac{\theta_i - \tilde{M}_i \sum_{j\neq i} \mathbb{E}\big[\bar{Z}_j\big] + \tilde{M}_i \sum_{j\neq i} \mu_j}{\kappa_i} \right) f_i(z)\,\mathrm{d}z \\
&= \frac{\kappa_i}{\eta_i + \kappa_i} \int_0^\infty z f_i \left(z + \frac{\theta_i - \tilde{M}_i \sum_{j\neq i} \mathbb{E}[\bar{Z}_j] + \tilde{M}_i \sum_{j\neq i} \mu_j}{\kappa_i} \right) \mathrm{d}z,
\end{aligned} \tag{29}$$

and

$$\mathbb{E}\big[\bar{Z}_i^2\big] = \frac{\kappa_i^2}{(\eta_i + \kappa_i)^2} \int_0^\infty z^2 f_i \left(z + \frac{\theta_i - \tilde{M}_i \sum_{j\neq i} \mathbb{E}\big[\bar{Z}_j\big] + \tilde{M}_i \sum_{j\neq i} \mu_j}{\kappa_i} \right) \mathrm{d}z, \tag{30}$$

where $f_i(z) = \frac{\mathrm{d}F_i(z)}{\mathrm{d}z}$ is the distribution density function (DDF). Note that $\mathbb{E}[\bar{Z}_i]$ and $\mathbb{E}[\bar{Z}_i^2]$ are dependent on $\theta_1, \cdots, \theta_n$ and $\eta_1, \cdots, \eta_n$. As a result, explicit solutions for the optimal safety loadings are unattainable. We have to resort to numerical solutions.

Remark 4. When $\lambda = 0$,

$$R_i^* = \frac{\theta_i + \eta_i z}{\eta_i + \kappa_i} \wedge z,$$

which is the function of (θ_i, η_i) and is independent of $(\theta_j, \eta_j), j \neq i$. Thus, we have

$$\begin{cases} (\theta_i^\star, \eta_i^\star) = \arg\max_{(\theta_i,\eta_i)} \left\{ \frac{1}{2}(\hat{\gamma}_i - \eta_i)\mathbb{E}[\bar{Z}_i^2] - \theta_i \mathbb{E}[\bar{Z}_i] \right\}, \\ \text{s.t.} \quad \mathbb{E}[\bar{Z}_i] = \frac{\kappa_i}{\eta_i + \kappa_i} \int_0^\infty z f_i(z + \frac{\theta_i}{\kappa_i})\,\mathrm{d}z, \mathbb{E}[\bar{Z}_i^2] = \frac{\kappa_i^2}{(\eta_i + \kappa_i)^2} \int_0^\infty z^2 f_i\left(z + \frac{\theta_i}{\kappa_i}\right) \mathrm{d}z \end{cases}$$

I.e., the reinsurer's optimization problem is greatly simplified.

Example 2. Assume that the claims $\{Z_{i,k}\}$ has the distribution function (22). Thus, from (29) and (30) $\mathbb{E}[\bar{Z}_i]$ and $\mathbb{E}[\bar{Z}_i^2]$ are determined by

$$\mathbb{E}[\bar{Z}_i] = \frac{\kappa_i}{\eta_i + \kappa_i} \frac{1}{\xi_i} e^{-\xi_i \left(\frac{\theta_i - \tilde{M}_i \sum_{j\neq i} \mathbb{E}[\bar{Z}_j] + \tilde{M}_i \sum_{j\neq i} \mu_j}{\kappa_i} \right)},$$

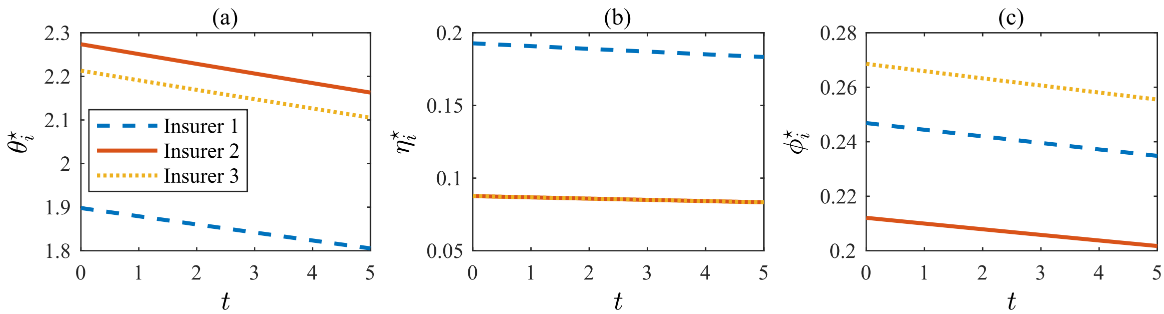

FIGURE 4. The optimal reinsurance strategies $(\theta_i^\star, \eta_i^\star)$ and $(\phi_i^\star)$.

$$\mathbb{E}[\bar{Z}_i^2] = \left(\frac{\kappa_i}{\eta_i + \kappa_i}\right)^2 \frac{1}{\xi_i^2} e^{-\xi_i\left(\frac{\theta_i - \tilde{M}_i \sum_{j\neq i} \mathbb{E}[\bar{Z}_j] + \tilde{M}_i \sum_{j\neq i} \mu_j}{\kappa_i}\right)}.$$

With the model parameters in Table 2, we use a built-in Matlab function fmincon to obtain the optimal loadings as presented in Figure 4. It indicates that, as $t \to T$, the reinsurer's premium loadings slightly decrease but overall remain at very stable levels. $\tilde{\phi}_i^\star$ also shows a similar trend. In fact, as t approaches T, the insurers tend to purchase less reinsurance and take more risk by themselves, see Figure 1. As a result, the reinsurer must lower risk loadings in order to attract insurers to purchase more reinsurance contracts.

Figure 5 presents some more numerical results that provide more insights into the nature of the solution.

From Panel (a), we see that, as Insurer 1 becomes more competitive, since he increases his risk exposure by keeping more risk for himself, the reinsurer slightly decreases the reinsurance price $\theta_1^\star$ to drag Insurer 1 back to business. Furthermore, he lowers $\theta_2^\star$ and $\theta_3^\star$ to encourage insurers 2 and 3 to buy more reinsurance. At the same time, the reinsurer increases η_1, η_2 and η_3 with m_1 (see panel (b)). The reduction of $\eta_i^\star$ compensates for the reinsurance premium losses resulting from the fall of $\theta_i^\star$ and may also be utilized to manage reinsurance risks. Panel (c) shows that $\phi_1^\star$ slightly decreases as m_1 increases. This is because, when m_1 increases, the reinsurer will receive less reinsurance business from the insurers, and thus the concerns about the model uncertainty of insurer 1 will decrease.

Panel (d) shows how the reinsurer's concern about insurer 1's model uncertainty affects reinsurance prices. It indicates that, as α_1 increases, the reinsurer reduces his reinsurance business from insurer 1 and mitigates the impact of model uncertainty by increasing the premium $\theta_1^\star$. An increased $\theta_1^\star$ results in reduced reinsurance purchase, thus diminishing the reinsurer's apprehension regarding the volatility of claims filed against him, which in turn leads to a decline in $\eta_1^\star$; see to panel (e). Panel (f) indicates that $\phi_1^\star$ increases as the reinsurer is more concerned about the model uncertainty of Insurer 1. This result is intuitive.

Panel (g) demonstrates that when the common shock escalates, the reinsurer encounters increased claim risks and consequently prefers to increase the risk loading $\theta_i^\star$. Due to inflated reinsurance premiums, insurers purchase fewer reinsurance contracts, resulting in the reinsurer assuming diminished claim risks. The rise in common shock is counterbalanced by the decline in reinsurance contracts. Consequently, it is noteworthy to note from panel (h) that the alternative risk loading $\eta_i^\star$ remains unchanged. Furthermore, as λ increases, the reinsurer has larger risk exposures, leading to less larger $\tilde{\phi}_i^\star$ and lower income; see panel (i).

4. CONCLUSION

In this paper, we study a non-zero sum stochastic differential game among n insurers who use reinsurance for risk management. We assume that the insurers are subject to a common shock of claims, exhibit ambiguity-aversion regarding the risk model, and determine the reinsurance premium rates based on the mean-variance principle. Their objectives are to maximize the expected utility of relative performance at the end of the

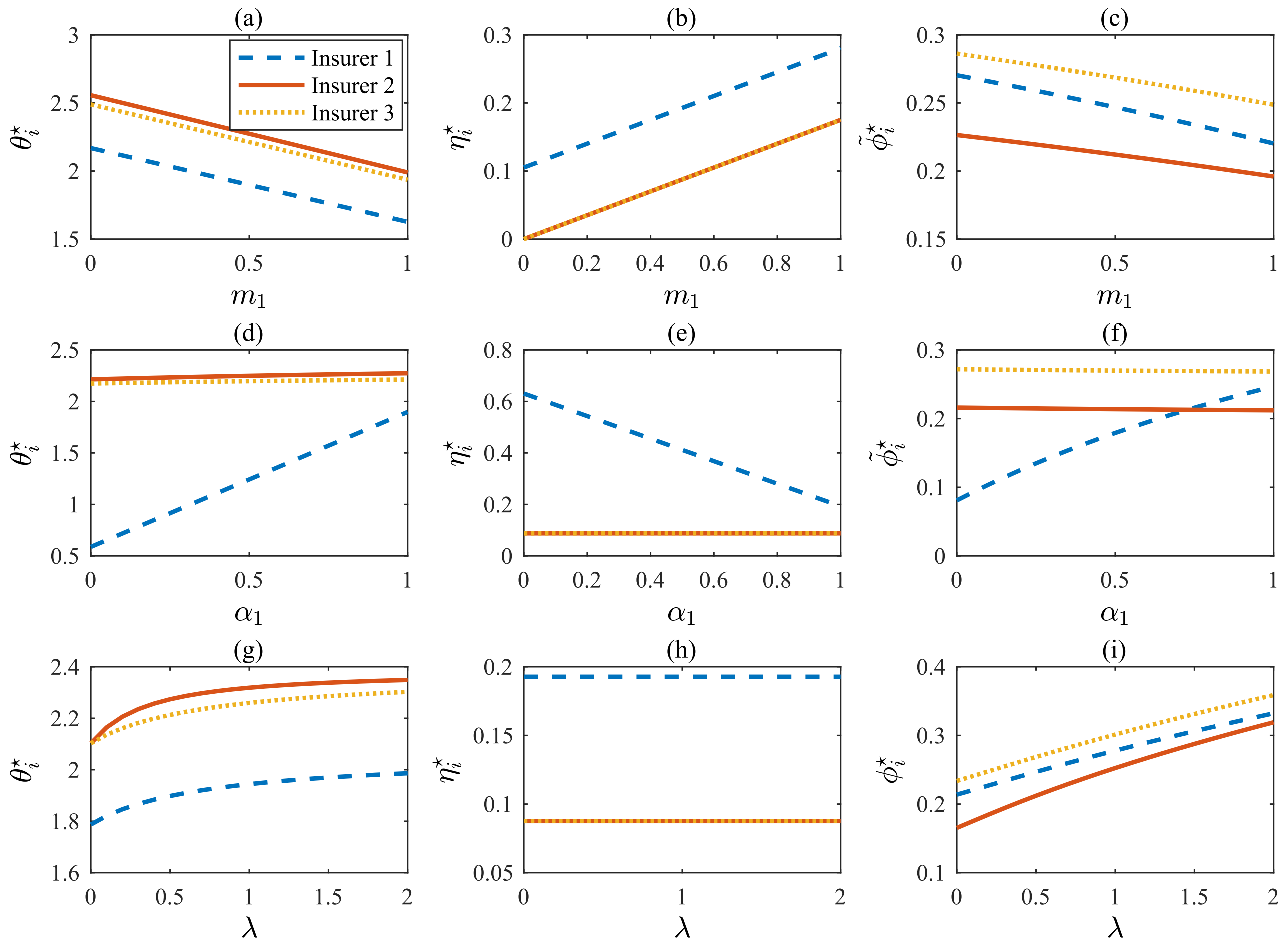

FIGURE 5. The optimal reinsurance strategies $(\boldsymbol{\theta}^\star, \boldsymbol{\eta}^\star)$ and $\tilde{\boldsymbol{\phi}}^\star$ as the functions of m_1, α_1 and λ.

decision horizon. When the insurers have general utility functions, we derive a system of coupled HJB equations and discuss the existence of equilibrium reinsurance strategies. We obtain semi-explicit solutions for cases where insurers aim to maximize the exponential utility of relative terminal surpluses. These include the value functions, the equilibrium strategies, and the probability distortion functions for the insurers, along with a number of numerical examples. Our results show that competition, common shock, and model uncertainty all have a vital impact on the insurers' decisions. Based on the results, we consider the pricing of reinsurance contracts for the insurers in a Stackelberg game. When the reinsurer has exponential utility and the claims are exponentially distributed, we conduct several numerical examples to explore the impact of competition, model uncertainty, and common shock.

Our results can be extended in two directions. First, we could consider the presence of multiple reinsurers in the insurance market. Second, one may consider risk control for the insurers by adding risk constraints to them.

Appendix A. Proofs

A.1. Proof of Theorem 1

Proof. Using (5), we may rewrite the HJB equation (11) as following:

$$0 = J_t^i + \sup_{R_i} \inf_{\boldsymbol{\phi}_i} \Bigg[J_{w_i}^i \Bigg[rw_i + \chi_i - \Big(\bar{M}_i \pi_i - M_i \sum_{j \neq i} \pi_j \Big) - \Big(\bar{M}_i \bar{\lambda}_i \mathbb{E} R_i - M_i \sum_{j\neq i} \bar{\lambda}_j \mathbb{E} R_j \Big) \\ - \bar{M}_i \sqrt{\bar{\lambda}_i \mathbb{E} R_i^2} \phi_{ii} + M_i \sum_{j \neq i} \sqrt{\bar{\lambda}_j \mathbb{E} R_j^2} \phi_{ij} \Bigg] \\ + \frac{1}{2} \Bigg[\bar{M}_i^2 \bar{\lambda}_i \mathbb{E} R_i^2 + M_i^2 \sum_{j \neq i} \bar{\lambda}_j \mathbb{E} R_j^2 - 2 M_i \bar{M}_i \sum_{j \neq i} \lambda \mathbb{E} R_i \mathbb{E} R_j J_{w_i w_i}^i \Bigg] + \sum_j \frac{\phi_{ij}^2}{2 \psi_{ij}} \Bigg], \qquad i = 1, \cdots, n. \tag{A.1}$$

Since $J_{w_i}^i > 0$ and $J_{w_i w_i}^i < 0$, using the first order condition in equation (A.1), the optimal probability distortion $\boldsymbol{\phi}_i^* = (\phi_{ij}^*)$ is given by

$$\phi_{ii}^* = \bar{M}_i \sqrt{\bar{\lambda}_i \mathbb{E} R_i^2} \psi_{ii} J_{w_i}^i, \tag{A.2}$$

$$\phi_{ij}^* = -M_i \sqrt{\bar{\lambda}_j \mathbb{E} R_j^2} \psi_{ij} J_{w_i}^i. \tag{A.3}$$

When substituting $\boldsymbol{\phi}_i^*$ into (A.1), then, the system of equations become

$$0 = J_t^i + \sup_{R_i} \Bigg[J_{w_i}^i \Bigg[rw_i + \chi_i - \Big(\bar{M}_i \pi_i - M_i \sum_{j \neq i} \pi_j \Big) - \Big(\bar{M}_i \bar{\lambda}_i \mathbb{E} R_i - M_i \sum_{j\neq i} \bar{\lambda}_j \mathbb{E} R_j \Big) \Bigg] \\ + \frac{1}{2} \Bigg[\bar{M}_i^2 \bar{\lambda}_i \mathbb{E} R_i^2 + M_i^2 \sum_{j \neq i} \bar{\lambda}_j \mathbb{E} R_j^2 - 2 M_i \bar{M}_i \sum_{j \neq i} \lambda \mathbb{E} R_i \mathbb{E} R_j \Bigg] J_{w_i w_i}^i \\ - \bar{M}_i^2 \bar{\lambda}_i \mathbb{E} R_i^2 \frac{\psi_{ii}}{2} \left(J_{w_i}^i \right)^2 - \sum_{j \neq i} \bar{\lambda}_j \mathbb{E} R_j^2 \frac{M_i^2 \psi_{ij}}{2} \left(J_{w_i}^i \right)^2 \Bigg]. \tag{A.4}$$

Since $J_{w_i}^i > 0$, we may isolate the R_i terms in the square brackets as:

$$\mathbb{E}\Big[\theta_i R_i + \eta_i Z^i R_i - \frac{\eta_i}{2} R_i^2 \Big] + \frac{1}{2} \Bigg[\mathbb{E} R_i^2 - 2 M_i \frac{\lambda}{\bar{\lambda}_i} \mathbb{E} R_i \sum_{j \neq i} \mathbb{E} R_j^* \Bigg] \frac{J_{w_i w_i}^i}{J_{w_i}^i} - \bar{M}_i \frac{\psi_{ii}}{2} \mathbb{E} R_i^2 J_{w_i}^i \\ = \mathbb{E} \Bigg[\theta_i R_i + \eta_i Z^i R_i - \frac{1}{2} (\eta_i - \frac{J_{w_i w_i}^i}{J_{w_i}^i} + \bar{M}_i \psi_{ii} J_{w_i}^i) R_i^2 - M_i \frac{\lambda}{\bar{\lambda}_i} \frac{J_{w_i w_i}^i}{J_{w_i}^i} R_i \sum_{j \neq i} \mathbb{E} R_j \Bigg] \\ = \int_0^\infty \Bigg(\theta_i R_i + \eta_i z R_i - \frac{1}{2} (\eta_i + \kappa_i) R_i^2 + \tilde{M}_i R_i \sum_{j \neq i} \mathbb{E} R_j \Bigg) \mathrm{d} F_i(z),$$

where

$$\kappa_i = -\frac{J_{w_i w_i}^i}{J_{w_i}^i} + \bar{M}_i \psi_{ii} J_{w_i}^i, \tilde{M}_i = -M_i \frac{\lambda}{\bar{\lambda}_i} \frac{J_{w_i w_i}^i}{J_{w_i}^i} \cdot$$

The above integral is maximized by maximizing the integrand z-by-z, subject to the condition $0 \le R_i \le z$. By assumption we have $\kappa_i + \eta_i > 0$, thus the integrand is strictly concave in R_i. Using the first order condition, the optimal reinsurance strategy $R_1^*, \cdots, R_n^*$ are given by

$$R_i^* = \frac{\theta_i + \tilde{M}_i \sum_{j\neq i} \mathbb{E}R_j^* + \eta_i z}{\kappa_i + \eta_i} \wedge z, \qquad i = 1, \cdots, n, \tag{A.5}$$

for all $z \ge 0$. □

A.2. Proof of Lemma 2

Proof. We solve the above system of non-linear equations (15) iteratively. To this end, let $(x_1^0, \cdots, x_n^0) = (\mu_1, \cdots, \mu_n)^\top$ be the initial values and let

$$x_i^n = \mu_i - \frac{\kappa_i}{\kappa_i + \eta_i} \int_0^\infty \left(z - \frac{\theta_i + \tilde{M}_i \sum_{j\neq i} x_j^{n-1}}{\kappa_i} \right) \mathbf{1}_{z > \frac{\theta_i + \tilde{M}_i \sum_{j\neq i} x_j^{n-1}}{\kappa_i}} \, \mathrm{d}F_i(z),$$
$$x_i^{n+1} = \mu_i - \frac{\kappa_i}{\kappa_i + \eta_i} \int_0^\infty \left(z - \frac{\theta_i + \tilde{M}_i \sum_{j\neq i} x_j^{n}}{\kappa_i} \right) \mathbf{1}_{z > \frac{\theta_i + \tilde{M}_i \sum_{j\neq i} x_j^{n}}{\kappa_i}} \, \mathrm{d}F_i(z).$$

Let $\delta^n := \max_i |x_i^n - x_i^{n-1}|$. Then,

$$\begin{aligned}
&|x_i^{n+1} - x_i^n| \\
&= \frac{\kappa_i}{\kappa_i + \eta_i} \left| \int_{\frac{\theta_i + \tilde{M}_i \sum_{j\neq i} x_j^n}{\kappa_i}}^\infty \left(z - \frac{\theta_i + \tilde{M}_i \sum_{j\neq i} x_j^n}{\kappa_i} \right) \mathrm{d}F_i(z) - \int_{\frac{\theta_i + \tilde{M}_i \sum_{j\neq i} x_j^{n-1}}{\kappa_i}}^\infty \left(z - \frac{\theta_i + \tilde{M}_i \sum_{j\neq i} x_j^{n-1}}{\kappa_i} \right) \mathrm{d}F_i(z) \right| \\
&\le \frac{\kappa_i}{\kappa_i + \eta_i} \left| \int_{\frac{\theta_i + \tilde{M}_i \sum_{j\neq i} x_j^n}{\kappa_i}}^\infty \left(z - \frac{\theta_i + \tilde{M}_i \sum_{j\neq i} x_j^n}{\kappa_i} \right) \mathrm{d}F_i(z) - \int_{\frac{\theta_i + \tilde{M}_i \sum_{j\neq i} x_j^{n}}{\kappa_i}}^\infty \left(z - \frac{\theta_i + \tilde{M}_i \sum_{j\neq i} x_j^{n-1}}{\kappa_i} \right) \mathrm{d}F_i(z) \right| \\
&\quad + \frac{\kappa_i}{\kappa_i + \eta_i} \left| \int_{\frac{\theta_i + \tilde{M}_i \sum_{j\neq i} x_j^n}{\kappa_i}}^\infty \left(z - \frac{\theta_i + \tilde{M}_i \sum_{j\neq i} x_j^{n-1}}{\kappa_i} \right) \mathrm{d}F_i(z) - \int_{\frac{\theta_i + \tilde{M}_i \sum_{j\neq i} x_j^{n-1}}{\kappa_i}}^\infty \left(z - \frac{\theta_i + \tilde{M}_i \sum_{j\neq i} x_j^{n-1}}{\kappa_i} \right) \mathrm{d}F_i(z) \right| \\
&\le \left| \int_{\frac{\theta_i + \tilde{M}_i \sum_{j\neq i} x_j^n}{\kappa_i}}^\infty \frac{\tilde{M}_i \sum_{j\neq i} |x_j^n - x_j^{n-1}|}{\kappa_i + \eta_i} \mathrm{d}F_i(z) \right| + \frac{\kappa_i}{\kappa_i + \eta_i} \left| \int_{\frac{\theta_i + \tilde{M}_i \sum_{j\neq i} x_j^{n-1}}{\kappa_i}}^{\frac{\theta_i + \tilde{M}_i \sum_{j\neq i} x_j^n}{\kappa_i}} \left(z - \frac{\theta_i + \tilde{M}_i \sum_{j\neq i} x_j^{n-1}}{\kappa_i} \right) \mathrm{d}F_i(z) \right|.
\end{aligned}$$

Let

$$y_1 = \frac{\theta_i + \tilde{M}_i \sum_j x_j^{n-1} - \kappa_i z}{\kappa_i}, y_2 = \frac{\theta_i + \tilde{M}_i \sum_j x_j^n - \kappa_i z}{\kappa_i}.$$

Without loss generality, assume that $y_1 < y_2$ (the case $y_1 \ge y_2$ can be proved similarly). It follows that

$$\begin{aligned}
\left| x_i^{n+1} - x_i^n \right| &\le \int_{y_2}^\infty \frac{\tilde{M}_i (n-1)\delta_n}{\kappa_i + \eta_i} \mathrm{d}F_i(z) + \frac{\kappa_i}{\kappa_i + \eta_i} \int_{y_1}^{y_2} (z - y_1) \, \mathrm{d}F_i(z) \\
&\le (1 - F_i(y_2)) \frac{\tilde{M}_i (n-1)\delta_n}{\kappa_i + \eta_i} + \frac{\kappa_i}{\kappa_i + \eta_i} (y_2 - y_1)(F_i(y_2) - F_i(y_1)) \\
&= (1 - F_i(y_2)) \frac{\tilde{M}_i (n-1)\delta_n}{\kappa_i + \eta_i} + \frac{\tilde{M}_i (n-1)\delta_n}{\kappa_i + \eta_i} (F_i(y_2) - F_i(y_1)) \\
&\le \frac{\tilde{M}_i (n-1)\delta_n}{\kappa_i + \eta_i} (1 - F_i(y_1)) < \frac{\tilde{M}_i (n-1)}{\kappa_i + \eta_i} \delta_n.
\end{aligned}$$

Note that

$$\frac{\tilde{M}_i(n-1)}{\kappa_i+\eta_i} = \frac{-m_i\frac{n-1}{n}\frac{\lambda_i}{\bar{\lambda}_i}\frac{J_{w_iw_i}}{J_{w_i}}}{-\frac{J_{w_iw_i}}{J_{w_i}}+\bar{M}_i\psi_{ii}J_{w_i}+\eta_i} < \frac{-\frac{J_{w_iw_i}}{J_{w_i}}}{-\frac{J_{w_iw_i}}{J_{w_i}}+\bar{M}_i\psi_{ii}J_{w_i}+\eta_i} < 1,$$

we have $\delta_{n+1} = \max_i |x_i^{n+1} - x_i^n| < \delta_n$ and the sequence (x_i^n) converge. Let $\mathbb{E}R_i^* = \lim_{n\to\infty} x_i^n$. Then, $\mathbb{E}R_i^*, i = 1, \cdots, n$, are uniquely determined. □

A.3. Proof of Theorem 2

Proof. Inspired by Yi *et al.* [42], we conjecture that the solution to the HJB equations (11) have the following structure:

$$\bar{J}^i(t, w_i) = -\frac{1}{\gamma_i}\exp\{-\gamma_i w_i e^{r\tau} + A_i(\tau)\}, \qquad i = 1, \cdots, n, \tag{A.6}$$

where $A_i(\tau)$ is the function of τ to be determined, $\tau = T - t$. It follows that

$$\bar{J}^i_{w_i} = -\gamma_i e^{r\tau}\bar{J}^i, \bar{J}^i_{w_iw_i} = \gamma_i^2 e^{2r\tau}\bar{J}^i, \bar{J}^i_t = \gamma_i w_i r e^{r\tau}\bar{J}^i - A_i'(\tau)\bar{J}^i,$$

and

$$\kappa_i = (\gamma_i+\alpha_i)e^{r\tau}, \tilde{M}_i = M_i\gamma_i e^{r\tau}\frac{\lambda}{\bar{\lambda}_i}\cdot$$

According to Theorem 1, the optimal reinsurance strategy for insurer i is given by

$$R_i^*(\tau, z) = \frac{\theta_i + M_i\gamma_i e^{r\tau}\frac{\lambda}{\bar{\lambda}_i}\sum_{j\neq i}\mathbb{E}R_j^* + \eta_i z}{\eta_i + (\gamma_i + \bar{M}_i\alpha_i)e^{r\tau}} \wedge z,$$

and the optimal distortion function $\boldsymbol{\phi}_i^* = (\phi_{ij}^*)$ is given by

$$\begin{cases} \phi_{ii}^*(\tau) = \alpha_i e^{r\tau}\sqrt{\bar{\lambda}_i\mathbb{E}(R_i^*)^2}, \\ \phi_{ij}^*(\tau) = -m_i\alpha_i e^{r\tau}\sqrt{\bar{\lambda}_j\mathbb{E}(R_j^*)^2}, j\neq i. \end{cases} \tag{A.7}$$

Let $\pi_i^*(\tau) = \pi_i(\tau, R_i^*)$. Substituting equation (A.6) into equation (A.4) leads to

$$\begin{aligned}
&-A_i'(\tau) + (-\gamma_i)e^{r\tau}\left[\chi_i - \bar{M}_i\pi_i^*(\tau) - \bar{M}_i\bar{\lambda}_i\mathbb{E}R_i^* + M_i\sum_{j\neq i}\pi_j^*(\tau) + M_i\sum_{j\neq i}\bar{\lambda}_j\mathbb{E}R_j^*\right] \\
&+\frac{1}{2}\left[\bar{M}_i^2\bar{\lambda}_i\mathbb{E}(R_i^*)^2 + M_i^2\sum_{j\neq i}\bar{\lambda}_j\mathbb{E}(R_j^*)^2 - 2M_i\bar{M}_i\lambda\mathbb{E}R_i^*\sum_{j\neq i}\mathbb{E}R_j^*\right]\gamma_i^2e^{2r\tau} \\
&+\left(\bar{M}_i^2\bar{\lambda}_i\mathbb{E}(R_i^*)^2 + M_i^2\sum_{j\neq i}\bar{\lambda}_j\mathbb{E}\big(R_j^*\big)^2\right)\frac{\gamma_i\alpha_i}{2}e^{2r\tau} = 0 \\
\Rightarrow A_i(\tau) = &\int_0^\tau(-\gamma_i)e^{rs}\left[\chi_i - \bar{M}_i\pi_i^*(s) - \bar{M}_i\bar{\lambda}_i\mathbb{E}R_i^* + M_i\sum_{j\neq i}\pi_j^*(s) + M_i\sum_{j}\bar{\lambda}_j\mathbb{E}R_j^*\right] \\
&+\gamma_i e^{2rs}\left[\frac{\gamma_i+\alpha_i}{2}\bar{\lambda}_i\bar{M}_i\mathbb{E}(R_i^*)^2 + M_i^2\sum_{j\neq i}\bar{\lambda}_j\frac{\gamma_i+\alpha_i}{2}\mathbb{E}(R_j^*)^2 - M_i\bar{M}_i\gamma_i\lambda\mathbb{E}R_i^*\sum_{j\neq i}\mathbb{E}R_j^*\right]\mathrm{d}s.
\end{aligned} \tag{A.8}$$

It is obvious that $\bar{J}^i$ is strictly increasing and concave, thus R_i^* and $\boldsymbol{\phi}_i^*$ obtain their minimum and maximum values respectively, and $\bar{J}^i$ satisfies HJB equation (A.1). Thus, according to verification lemma, $\bar{J}^i, i = 1, 2$, is the optimal value function, and $(R_1^*, \cdots, R_2^*)$ is the Nash equilibrium for this non-zero-sum game. □

A.4. Proof of Lemma 3

Proof. According to (18), we just need to show that

$$\frac{\theta_i + \frac{m_i}{2}\gamma_i e^{r\tau}\frac{\lambda}{\lambda_i}\mathbb{E}R_j^* + \eta_i z}{\eta_i + \kappa_i}$$

strictly increases with respect to λ, m_i and m_j, where $\kappa_i = (\alpha_i \bar{M}_i + \gamma_i)e^{r\tau}$. To this end, define

$$\Gamma_i(x) := \int_{\frac{\theta_i + \tilde{M}_i x}{\kappa_i}}^{\infty} \frac{\theta_i + \tilde{M}_i x - \kappa_i z}{\kappa_i + \eta_i} \,\mathrm{d}F_i(z).$$

Then,

$$\mathbb{E}R_i^* = \mu_i + \Gamma_i(\mathbb{E}R_j^*) = \mu_i + \Gamma_i(\mu_j + \Gamma_j(\mathbb{E}R_i^*)), \qquad i = 1, 2, j \neq i. \tag{A.9}$$

Since $\frac{\partial \Gamma_i}{\partial x} > 0$, it is clear that $\mathbb{E}R_i^*$ strictly increases with $\mathbb{E}R_j^*$.

(1) $\frac{\theta_i + M_i\gamma_i e^{r\tau}\frac{\lambda}{\lambda_i}\mathbb{E}R_j^* + \eta_i z}{\eta_i + \kappa_i}$ strictly increases with λ. Since $\frac{\lambda}{\lambda_i}$ strictly increases with λ, we just need to show that $\mathbb{E}R_j^*$, or equivalently, $\mathbb{E}R_i^*$ strictly increases with λ. Direct calculation shows the following results:

$$\frac{\partial \Gamma_i}{\partial x} = \int_{\frac{\theta_i + \tilde{M}_i x}{\kappa_i}}^{+\infty} \frac{\tilde{M}_i}{\kappa_i + \eta_i} \,\mathrm{d}F_i(z) < 1,$$

$$\frac{\partial \Gamma_i}{\partial \lambda} = \int_{\frac{\theta_i + \tilde{M}_i x}{\kappa_i}}^{+\infty} \frac{\partial \tilde{M}_i}{\partial \lambda}\frac{x}{\kappa_i + \eta_i} \,\mathrm{d}F_i(z) = \frac{\partial \tilde{M}_i}{\partial \lambda} x \int_{\frac{\theta_i + \tilde{M}_i x}{\kappa_i}}^{+\infty} \frac{1}{\kappa_i + \eta_i} \,\mathrm{d}F_i(z).$$

Since $\tilde{M}_i > 0$, $0 < \frac{\partial \Gamma_i}{\partial x} < 1$ and $\frac{\partial \tilde{M}_i}{\partial \lambda} = -M_i \frac{1}{\lambda_i}\frac{J^i_{w_i w_i}}{J^i_{w_i}} > 0$. Moreover,

$$\begin{aligned}\frac{\partial \mathbb{E}R_i^*}{\partial \lambda} &= \frac{\partial \Gamma_i}{\partial x}\frac{\partial}{\partial \lambda}(\mu_j + \Gamma_j(\mathbb{E}R_i^*)) + \frac{\partial \Gamma_i}{\partial \lambda} \\ &= \frac{\partial \Gamma_i}{\partial x}\left(\frac{\partial \Gamma_j}{\partial x}\frac{\partial \mathbb{E}R_i^*}{\partial \lambda} + \frac{\partial \Gamma_j}{\partial \lambda}\right) + \frac{\partial \Gamma_i}{\partial \lambda} \\ \Rightarrow \frac{\partial \mathbb{E}R_i^*}{\partial \lambda}\left(1 - \frac{\partial \Gamma_i}{\partial x}\frac{\partial \Gamma_j}{\partial x}\right) &= \frac{\partial \Gamma_i}{\partial x}\frac{\partial \Gamma_j}{\partial \lambda} + \frac{\partial \Gamma_i}{\partial \lambda} > 0.\end{aligned}$$

Since $0 < \frac{\partial \Gamma_i}{\partial x_i}\frac{\partial \Gamma_j}{\partial x_j} < 1$ we have $\frac{\partial \mathbb{E}R_i^*}{\partial \lambda} > 0$.

(2) $\frac{\theta_i + M_i\gamma_i e^{r\tau}\frac{\lambda}{\lambda_i}\mathbb{E}R_j^* + \eta_i z}{\eta_i + \kappa_i}$ strictly increases with m_i and m_j. Since $\lambda > 0$,

$$\frac{\partial \Gamma_i}{\partial m_i} = \int_{\frac{\theta_i + \tilde{M}_i x}{\kappa_i}}^{\infty} \frac{\partial}{\partial m_i}\frac{\tilde{M}_i x}{\kappa_i + \eta_i} \,\mathrm{d}F_i(z) = \int_{\frac{\theta_i + \tilde{M}_i x}{\kappa_i}}^{\infty} \frac{\lambda}{2\bar{\lambda}_i}\frac{-J^i_{w_i w_i}}{J^i_{w_i}}\frac{x}{\kappa_i + \eta_i} \,\mathrm{d}F_i(z) > 0.$$

Thus,

$$\frac{\partial \mathbb{E}R_i^*}{\partial m_i} = \frac{\partial \Gamma_i}{\partial x}\frac{\partial}{\partial m_i}\Big(\mu_j + \Phi_j(\mathbb{E}R_i^*)\Big) + \frac{\partial \Gamma_i}{\partial m_i}$$

$$= \frac{\partial \Gamma_i}{\partial x}\frac{\partial \Gamma_j}{\partial x}\frac{\partial \mathbb{E}R_i^*}{\partial m_i} + \frac{\partial \Gamma_i}{\partial m_i}$$
$$\Rightarrow \frac{\partial \mathbb{E}R_i^*}{\partial m_i}\Big(1 - \frac{\partial \Gamma_i}{\partial x}\frac{\partial \Gamma_j}{\partial x}\Big) = \frac{\partial \Gamma_i}{\partial m_i} > 0 \Rightarrow \frac{\partial \mathbb{E}R_i^*}{\partial m_i} > 0.$$

Similarly, we have

$$\frac{\partial \mathbb{E}R_i^*}{\partial m_j} = \frac{\partial \Gamma_i}{\partial x}\frac{\partial}{\partial m_j}\Big(\mu_j + \Gamma_j(\mathbb{E}R_i^*)\Big)$$
$$= \frac{\partial \Gamma_i}{\partial x}\Big(\frac{\partial \Gamma_j}{\partial x}\frac{\partial \mathbb{E}R_i^*}{\partial m_j} + \frac{\partial \Gamma_j}{m_j}\Big) \Rightarrow \frac{\partial \mathbb{E}R_i^*}{\partial m_j} > 0.$$

The proof is completed. □

A.5. Proof of Lemma 4

Proof. We just need to show that $\frac{\theta_i + \tilde{M}_i \mathbb{E}R_j^* + \eta_i z}{\eta_i + (\gamma_i + \bar{M}_i\alpha_i)e^{r\tau}}$ strictly decreases with respect to α_i and α_j.

Since $\frac{\partial \Gamma_i}{\partial x}\frac{\partial \Gamma_j}{\partial x} < 1$, from equation (A.9) it follows that

$$\frac{\partial \mathbb{E}R_j^*}{\partial \alpha_i} = \frac{\partial \Gamma_j}{\partial \alpha_i} + \frac{\partial \Gamma_j}{\partial x_j}\frac{\partial \Gamma_i}{\partial x_i}\frac{\partial \mathbb{E}R_j^*}{\partial \alpha_i} = \frac{\partial \Gamma_j}{\partial x_j}\frac{\partial \Gamma_i}{\partial x_i}\frac{\partial \mathbb{E}R_j^*}{\partial \alpha_i} \Rightarrow \frac{\partial \mathbb{E}R_j^*}{\partial \alpha_i} = 0.$$

Thus, $\frac{\theta_i + \tilde{M}_i \mathbb{E}R_j^* + \eta_i z}{\eta_i + (\gamma_i + \bar{M}_i\alpha_i)e^{r\tau}}$ strictly decreases with α_i.

On the other hand, it is easy to see that $\frac{\theta_j + \tilde{M}_j x - e^{r\tau}(\bar{M}_j\alpha_j + \gamma_j)z}{\eta_j + e^{r\tau}(\bar{M}_j\alpha_j + \gamma_j)}$ strictly decreases with α_j. Thus,

$$\frac{\partial \Gamma_j}{\partial \alpha_j} = \int_{\frac{\theta_j + \tilde{M}_j x}{\gamma_j + \bar{M}_j\alpha_j}}^{\infty} \frac{\partial}{\partial \alpha_j}\left(\frac{\theta_j + \tilde{M}_j x - e^{r\tau}(\bar{M}_j\alpha_j + \gamma_j)z}{\eta_j + e^{r\tau}(\bar{M}_j\alpha_j + \gamma_j)}\right)\mathrm{d}F_j(z) < 0.$$

It follows that

$$\frac{\partial \mathbb{E}R_j^*}{\partial \alpha_j} = \frac{\partial \Gamma_j}{\partial \alpha_j} + \frac{\partial \Gamma_j}{\partial x_j}\frac{\partial \Gamma_i}{\partial x_i}\frac{\partial \mathbb{E}R_j^*}{\partial \alpha_j} \Rightarrow \frac{\partial \mathbb{E}R_j^*}{\partial \alpha_j} < 0,$$

indicating that $\frac{\theta_i + \tilde{M}_i \mathbb{E}R_j^* + \eta_i z}{\eta_i + (\gamma_i + \bar{M}_i\alpha_i)e^{r\tau}}$ strictly decreases with α_j. □

A.6. Proof of Lemma 5

Proof. First, we show that R_i^* increases with θ_i and η_i. Direct calculation shows

$$\frac{\partial \Gamma_i}{\partial \theta_i} = \int_{\frac{\theta_i + \tilde{M}_i x}{\kappa_i}}^{\infty} \frac{1}{\kappa_i + \eta_i}\,\mathrm{d}F_i(z) > 0, \frac{\partial \Gamma_i}{\partial \theta_j} = 0,$$
$$\frac{\partial \Gamma_i}{\partial \eta_i} = -\int_{\frac{\theta_i + \tilde{M}_i x}{\kappa_i}}^{\infty} \frac{\theta_i + \tilde{M}_i x - \kappa_i z}{(\kappa_i + \eta_i)^2}\,\mathrm{d}F_i(z) > 0, \frac{\partial \Gamma_i}{\partial \eta_j} = 0.$$

Thus,

$$\frac{\partial \mathbb{E}R_j^*}{\partial \theta_i} = \frac{\partial \Gamma_j}{\partial x_j}\frac{\partial \Gamma_i}{\partial x_i}\frac{\partial \mathbb{E}R_j^*}{\partial \theta_i} + \frac{\partial \Gamma_j}{\partial \theta_i} = \frac{\partial \Gamma_j}{\partial x_j}\frac{\partial \Gamma_i}{\partial x_i}\frac{\partial \mathbb{E}R_j^*}{\partial \theta_i} \Rightarrow \frac{\partial \mathbb{E}R_j^*}{\partial \theta_i} = 0, \tag{A.10}$$

and

$$\frac{\partial}{\partial \theta_i}\left(\frac{\theta_i + \tilde{M}_i \mathbb{E}R_j^* + \eta_i z}{\eta_i + \kappa_i}\right) = \frac{1}{\eta_i + \kappa_i} > 0,$$

which means that R_i^* strictly increases with θ_i. Similarly, we have $\frac{\partial \mathbb{E}R_j^*}{\partial \eta_i} = 0$. When $\frac{\theta_i + \tilde{M}_i \mathbb{E}R_j^* + \eta_i z}{\eta_i + \kappa_i} < z$, we have $\theta_i + \tilde{M}_i \mathbb{E}R_j^* - \kappa_i z < 0$ and hence

$$\frac{\partial}{\partial \eta_i}\left(\frac{\theta_i + \tilde{M}_i \mathbb{E}R_j^* + \eta_i z}{\eta_i + \kappa_i}\right) = \frac{\kappa_i z_i - (\theta_i + \tilde{M}_i \mathbb{E}R_j^*)}{(\eta_i + \kappa_i)^2} > 0,$$

indicating that R_i^* strictly increases with η_i.

Next, we proceed to show that R_i^* increases with θ_j and η_j. Similar to equation (A.10), we have

$$\frac{\partial \mathbb{E}R_j^*}{\partial \theta_j} = \frac{\partial \Gamma_j}{\partial x_j}\frac{\partial \Gamma_i}{\partial x_i}\frac{\partial \mathbb{E}R_j^*}{\partial \theta_j} + \frac{\partial \Gamma_j}{\partial \theta_j} \Rightarrow \frac{\partial \mathbb{E}R_j^*}{\partial \theta_j} = \frac{\frac{\partial \Gamma_j}{\partial \theta_j}}{1 - \frac{\partial \Gamma_j}{\partial x_j}\frac{\partial \Gamma_i}{\partial x_i}} > 0,$$

and $\frac{\partial \mathbb{E}R_j^*}{\partial \eta_j} > 0$. From equation (18) it follows that R_i^* strictly increases with (θ_j, η_j). □

A.7. Proof of Theorem 3

Proof. By first order condition, we have

$$\tilde{\phi}_i^\star = \sqrt{\bar{\lambda}_i \mathbb{E}[\bar{Z}_i]^2} V_y \tilde{\psi}_i$$

and

$$\begin{aligned} V_t + \sup_{(\boldsymbol{\theta},\boldsymbol{\eta})}\Bigg\{ & ryV_y + \sum_{i=1}^n \Big(\bar{\lambda}_i \mathbb{E}\Big[\theta_i \bar{Z}_i + \frac{\eta_i}{2}(\bar{Z}_i)^2\Big]\Big) V_y + \sum_{i=1}^n \frac{\tilde{\alpha}_i}{2\gamma V} V_y^2 \bar{\lambda}_i \mathbb{E}[\bar{Z}_i]^2 \\ & + \frac{1}{2}\left(\sum_{i=1}^n \bar{\lambda}_i \mathbb{E}[\bar{Z}_i]^2 + \lambda \sum_{i \neq j} \mathbb{E}[\bar{Z}_i]\mathbb{E}[\bar{Z}_j]\right) V_{yy} \Bigg\} = 0, \qquad V(T,y) = U(y). \end{aligned} \tag{A.11}$$

We postulate that

$$V(t,y) = -\frac{1}{\gamma}\exp\{-\gamma y e^{r\tau} + B(\tau)\}.$$

It follows that $V_y = -\gamma e^{r\tau} V, V_{yy} = \gamma^2 e^{2r\tau} V, V_t = \gamma y r e^{r\tau} V - B'(\tau) V$, and

$$\begin{aligned} -B'(\tau) + \sup_{(\boldsymbol{\theta},\boldsymbol{\eta})}\Bigg\{ & \sum_{i=1}^n \Big(\bar{\lambda}_i \mathbb{E}\Big[\theta_i \bar{Z}_i + \frac{\eta_i}{2}(\bar{Z}_i^2)\Big]\Big)(-\gamma e^{r\tau}) + \sum_i \frac{\gamma \tilde{\alpha}_i}{2} \bar{\lambda}_i e^{2r\tau} \mathbb{E}[\bar{Z}_i^2] \\ & + \frac{1}{2}\left(\sum_i \bar{\lambda}_i \mathbb{E}[\bar{Z}_i^2] + \lambda \sum_{i \neq j} \mathbb{E}[\bar{Z}_i]\mathbb{E}[\bar{Z}_j]\right)\gamma^2 e^{2r\tau} \Bigg\} = 0, \qquad B(0) = 0. \end{aligned} \tag{A.12}$$

Let

$$\begin{aligned} F(\boldsymbol{\theta},\boldsymbol{\eta}) = & -\sum_{i=1}^n \bar{\lambda}_i \Big(\theta_i \mathbb{E}[\bar{Z}_i] + \frac{\eta_i}{2}\mathbb{E}[\bar{Z}_i]^2\Big) \\ & + \frac{1}{2}\left(\sum_{i=1}^n \bar{\lambda}_i \mathbb{E}[\bar{Z}_i]^2 + \lambda \sum_{i \neq j} \mathbb{E}[\bar{Z}_i]\mathbb{E}[\bar{Z}_j]\right)\gamma e^{r\tau} + \sum_{i=1}^n \frac{\tilde{\alpha}_i}{2}\bar{\lambda}_i e^{r\tau}\mathbb{E}[\bar{Z}_i]^2 \\ = & -\sum_{i=1}^n \bar{\lambda}_i \theta_i \mathbb{E}[\bar{Z}_i] + \frac{\lambda \gamma e^{r\tau}}{2}\sum_{i \neq j}\mathbb{E}[\bar{Z}_i]\mathbb{E}[\bar{Z}_j] + \sum_{i=1}^n \frac{\bar{\lambda}_i}{2}((\tilde{\alpha}_i + \gamma)e^{r\tau} - \eta_i)\mathbb{E}[\bar{Z}_i^2]. \end{aligned}$$

Then, the optimal reinsurance premium loadings are determined by

$$(\boldsymbol{\theta}^\star, \boldsymbol{\eta}^\star) = \arg\max_{(\boldsymbol{\theta},\boldsymbol{\eta})} F(\boldsymbol{\theta}, \boldsymbol{\eta}).$$

It follows that

$$-B'(\tau) + F(\boldsymbol{\theta}^\star, \boldsymbol{\eta}^\star)\gamma e^{r\tau} = 0 \Rightarrow B(\tau) = \int_0^\tau F(\boldsymbol{\theta}^\star, \boldsymbol{\eta}^\star)\gamma e^{rs}\,\mathrm{d}s. \tag{A.13}$$

The proof is complete. □

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