




## GRAPHS WHOSE VERTICES OF DEGREE AT LEAST 2 LIE IN A TRIANGLE

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**Abstract.** A pendant vertex is one of degree one and an isolated vertex has degree zero. A neighborhood star-free (NSF for short) graph is one in which every vertex is contained in a triangle except pendant vertices and isolated vertices. This class has been considered before for several contexts. In the present paper, we study the complexity of the dominating induced matching (DIM) problem and the perfect edge domination (PED) problem for NSF graphs. We prove the corresponding decision problems are NP-Complete for several of its subclasses. As an added value of this study, we show three connected variants of planar positive 1in3SAT are also NP-Complete. Since these variants are more basic in complexity theory context than many graph problems, these results can be useful to prove that other problems are NP-Complete.

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### 1. INTRODUCTION

A graph with vertex set  $V$  and edge set  $E$  is denoted by  $G = (V, E)$ , and the number of vertices and edges is  $|V| = n$  and  $|E| = m$ , respectively. The open and closed neighborhoods of  $v \in V$  are denoted by  $N(v)$  and  $N[v] = N(v) \cup \{v\}$ , respectively. Write  $d(v) = |N(v)|$  for the degree of  $v$ . Let  $e = (u, w) \in E$  an edge, denote by  $d(e) = d(u, w) = |N(u) \cap N(w)|$  the degree of  $e$ . For  $V' \subseteq V$ ,  $G[V']$  represents the subgraph of  $G$  induced by  $V'$ . A *star* is the bipartite graph  $K_{1,t}$ , for some  $t \geq 1$ .

We say that  $G = (V, E)$  is a *neighborhood star-free* graph, NSF graph for short, if for every vertex  $v \in V$  with degree at least 2,  $G[N[v]]$ , is not a star. In other words, every vertex  $v \in V$  is contained in some triangle, for  $d(v) \geq 2$ . Such a class of graphs seems natural, in some different ways. For instance, to model human relationships, as the common case where every individual is acquainted with at least a pair of individuals, which

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are themselves acquainted. In this paper, we explore the use of this class in the problems of perfect and efficient edge domination of graphs. We remark that NSF graphs are not necessarily closed under induced subgraphs (adding 2 new vertices of degree  $n + 1$  to any  $n$ -vertex graph makes it an NSF graph). In this paper, we focus on NSF graphs and this class has been considered before on various contexts.

In [14], the authors investigate the  $F$ -domination number for all 2-stratified graphs  $F$  of order at most 3. Particularly, for two 2-stratified graphs  $K_3$ , they consider (i) graphs in which every vertex is in a triangle and (ii) graphs in which every edge is in a triangle; both are NSF graph subclasses. A graph  $G$  is *2-stratified graph* if its vertex set is partitioned into two non-empty color classes: red and blue. Let  $F$  be a 2-stratified graph rooted at some blue vertex  $v$ . The  $F$ -domination number  $\gamma_F(G)$  of a graph  $G$  is the minimum number of red vertices of  $G$  in a red-blue coloring of the vertices of  $G$  such that every blue vertex  $v$  of  $G$  belongs to a copy of  $F$  rooted at  $v$ .

In a very recent paper [1], it is shown that every vertex of any optimal IC-planar graph is in a triangle. An  $n$ -vertex graph  $G$  is *IC-Planar* if it has a drawing in the plane such that each edge is crossed by at most one edge and every vertex is incident to at most one crossed edge. Such IC-planar graph  $G$  is *optimal* if every  $n$ -vertex graph  $G'$  with more edges than  $G$  is non-IC-Planar.

Also, any  $k$ -tree for fixed  $k \geq 2$  is an NSF graph. A complete graph on  $k$  vertices is a  $k$ -tree, fixed  $k$ . If  $G$  is a  $k$ -tree, fixed  $k$  and  $C$  is a  $k$ -clique of  $G$ , then the graph formed by adding a new vertex to  $G$ , and making it adjacent to all vertices of  $C$  is a  $k$ -tree, fixed  $k$ . Several characterizations of  $k$ -trees are given in [34]. As maximal outerplanar graphs are 2-trees [29], they are NSF graphs.

Most of our proposed results on this paper are on edge domination problems. For a graph  $G = (V, E)$ , every edge  $e \in E$  *dominates* itself and all edges adjacent to  $e$ . A subset  $E' \subseteq E$  is an *edge dominator* of  $G$ , if  $E'$  dominates every edge  $e \in E$  of  $G$ . In particular, if each edge  $e \in E$  is dominated exactly by one edge then  $E'$  is called an *efficient edge dominating set (EED)*. On the other hand, if we relax the definition, and let each of  $e \in E \setminus E'$  be dominated exactly by one edge then  $E'$  is called a *perfect edge dominating set (PED)*. It follows from the definitions that every EED is also a PED. Observe that a graph does not necessarily contain an EED and in case of having several EEDs, they all have the same number of edges. However, every graph always contains a PED. In particular, the entire edge set  $E$  is always a PED, called *trivial*. Every PED which is not trivial and not an EED is called *proper*. As many domination problems, the interest is to find a PED (EED) of smaller size or weight if the graph has weights assigned to its edges. All these problem variants are NP-hard for graphs in general. In particular, determining whether or not a graph has an EED is NP-Complete [17]. Below is a summary table of the complexity status of this problem by different graph classes and their references. Also in [13], the authors define a limit class  $\mathcal{S} = \bigcap \mathcal{S}_k$  (see Tab. 1) for EED problem and they claim that unless  $P = NP$ , the problem is solvable in polynomial time in an hereditary graph class  $X = \text{Free}(M)$  only if  $X$  excludes all classes  $\mathcal{S}_k$ , *i.e.*, only if  $M \cap \mathcal{S}_k \neq \emptyset$ , for each  $k$ .

In contrast, there is less research done on the PED problem (Tab. 2).

Also, a dichotomy theorem is given in [26] which establishes the complexity of the PED problem for every graph class  $\mathcal{G}(H, d)$  which is  $H$ -free graphs with degree  $\leq d$  for some fixed  $d \geq 3$  and  $H$  a graph. The problem is polynomial time solvable for graphs in  $\mathcal{G}(H, d)$  if  $H$  is a linear forest (union of induced paths) and NP-Complete otherwise.

We can see that most of the studies done on these problems so far focus their attention on the hereditary graph classes. It would be of great interest to study them for a non-hereditary graph class like NSF graphs.

In [26], the authors proved that connected NSF graphs do not have any proper edge perfect dominating set. The only possible PEDs of these graphs are (i) the trivial PED, and (ii) EEDs. We give a brief idea of this proof. Given a perfect edge dominating set  $E' \subseteq E$  of a connected graph  $G = (V, E)$ , we can classify each vertex of  $V$  in three types according to the number of its incident edges in  $E'$ : white, 0 edges; yellow, 1 edge; black, 2 or more edges. The following statements are true.

- Black vertices have no white neighbors.
- If  $(u, v) \in E$ , then  $(u, v) \in E'$  if and only if  $u$  and  $v$  are both non-white vertices.
- Any neighbor of a white vertex is yellow.

TABLE 1. Complexity status for existence of EED problem by graph classes.

Graph class	Complexity	Ref.
Cubic graphs	NP-Complete	[20]
$r$ -regular graphs for fixed $r \geq 3$	NP-Complete	[12]
Bipartite graphs	NP-Complete	[27]
Planar bipartite graphs	NP-Complete	[28]
Planar bipartite graphs with degree $\leq 3$	NP-Complete	[10]
$\mathcal{S}_k = (C_3, \dots, C_k, H_1, \dots, H_k)$ -free bipartite graphs with degree $\leq 3$	NP-Complete	[13]
Circular-arc graphs	Linear	[24]
Claw-free graphs	$O(n)$	[13, 23]
Convex graphs	Linear	[10, 13]
Biconvex graphs	$O(n)$	[10, 23]
Bounded clique-width graphs	Polynomial	[13]
Bipartite permutation graphs	$O(n)$	[23, 27]
Generalized series-parallel graphs	Linear	[28]
Chordal graphs	$O(n)$	[23, 28]
Dually chordal graphs	$O(n)$	[11, 23]
Hole-free graphs	Polynomial	[10]
Chordal bipartite graphs	Linear	[10]
$P_7$ -free graphs	Linear	[3]
$P_8$ -free graphs	Polynomial	[4]
$P_9$ -free graphs	Polynomial	[8]
$S_{1,2,2}$ -free graphs	Polynomial	[19]
$S_{1,2,3}$ -free graphs	Polynomial	[19]
$S_{1,2,4}$ -free graphs	Polynomial	[5]
$S_{2,2,2}$ -free graphs	Polynomial	[18]
$S_{2,2,3}$ -free graphs	Polynomial	[6]
$S_{1,1,5}$ -free graphs	Polynomial	[7]

TABLE 2. Complexity status for the PED problem by graph classes.

Graph class	Complexity	Ref.
Bipartite graphs	NP-Complete	[28]
Claw-free graphs with degree $\leq 3$	NP-Complete	[26]
$r$ -regular graphs for fixed $r \geq 3$	NP-Complete	[26]
Graphs with degree $\leq r$ and girth $\geq k$ for fixed $r, k \geq 3$	NP-Complete	[26]
Generalized series-parallel graphs	Linear	[28]
Chordal graphs	Linear	[28]
Circular-arc graphs	Linear	[24]
Cubic claw-free graphs	$O(n)$	[26]
$P_5$ -free graphs	Linear	[26]

- Any yellow vertex has exactly one non-white neighbor.
- For any triangle  $T$  of  $G$ ,  $T$  has either all 3 black vertices or 2 of them are yellow and the third one is white.
- Given a black vertex  $u$  and some of its neighbors  $v$ , if  $v$  is in some triangle then  $v$  is also black.
- If  $|V| = 1$ ,  $E'$  is a trivial PED.
- If  $|V| \geq 2$ ,  $E'$  is a trivial PED if and only if there is no white vertex.
- $E'$  is an EED if and only if there is no black vertex.

- If  $E'$  is a proper PED ( $E'$  is not trivial PED nor EED) then there are some black vertex and some white vertex. Clearly, every degree one vertex can not be black. There is some black vertex  $u \in V$  with degree at least degree 2. If  $G$  is an NSF graph, then  $v$  is in some triangle  $T$  of  $G$  and all vertices of degree 2 or more are black since for each vertex  $v$  of degree at least 2, there is a path connecting  $u$  with it. Every vertex in this path is in some triangle. Consequently, all degree one vertices are yellow and there is no white vertex which is a contradiction. Then  $E'$  is not a proper PED or  $G$  is not an NSF graph.

In this article, we show that deciding if a connected NSF graph contains an EED is an NP-Complete problem. In fact, we describe NP-Completeness proofs for several subclasses of connected NSF graphs.

We briefly describe some known results related to the present paper.

There is a concept of bi-coloring associated to an EED set [13] as follows. Given a graph  $G = (V, E)$  and an EED  $E' \subseteq E$  of  $G$ , vertices of  $G$  can be classified into two color classes.

**Black vertices:** if they have exactly one incident edge of  $E'$ . We denote this subset of vertices by  $B$ .

**White vertices:** if they do not have any incident edge of  $E'$ . We denote this subset of vertices by  $W$ .

We call  $(B, W)$  the *bi-coloring associated to  $E'$* .

**Observation 1.** The subset  $B$  of  $G$  induces in  $G$  the EED  $E'$ . Moreover,  $E'$  is an induced matching of  $G$ .

For the above reason, an EED is also called *dominating induced matching (DIM)* of  $G$ . From now on, we will use either the terms EED or DIM, with the same meaning.

The following rules are useful to verify if a bi-coloring of a graph  $G$  is associated to some DIM of  $G$ . In the positive case, we say that it is a *valid* bi-coloring of  $G$ .

**Theorem 1 ([13]).** *Given a graph  $G$  and a bi-coloring  $(B, W)$  of  $G$ . Then  $(B, W)$  is valid if and only if it verifies the following conditions.*

- $G[B]$  is 1-regular which means every black vertex has exactly one black neighbor.
- $W$  is an independent set. That is, all neighbors of white vertices have black color.

As in [25], assigning one of the two possible colors to vertices of  $G$  is called a *coloring* of  $G$ . A coloring is *partial* if only part of the vertices of  $G$  has been assigned colors, otherwise is *total*.

**Definition 1 ([25]).** A partial coloring of  $G$  is *valid* if verifies the following conditions.

- White vertices form an independent set.
- Every black vertex has at most one black neighbor.
- Every black vertex without black neighbors has an uncolored neighbor.

It is clear that any invalid partial coloring cannot be extended to a valid total coloring.

For the next definitions, we need some more basic terminology.

A vertex is *universal* if it is neighbor of all other vertices of the graph. A vertex is *simplicial* if the subgraph induced by its neighborhood is complete. Next we define some of the graphs depicted in Figure 1. A *butterfly* is an NSF graph that consists of exactly two triangles sharing a single vertex. We call this vertex its *central* vertex. If we remove any vertex except the central vertex, the resulting graph is a *paw*. A *pendant* or *leaf* vertex is a vertex of degree one and its unique neighbor is a *preleaf* vertex.  $P_k$  is a connected graph with exactly  $k - 2$  vertices of degree 2 and two leaves.  $C_k$  is a connected graph of  $k$  vertices of degree 2. A *wheel*  $W_k$  is a  $C_k$  plus a universal vertex (see Fig. 2). A *diamond* is the resulting graph after deleting an edge from a  $K_4$ . A *gem* is a  $P_4$  plus a universal vertex. An  $H$  graph is two copies of  $P_3$  plus an edge connecting both middle vertices of  $P_3$ 's. The latter edge is the *middle* edge of  $H$ .  $H_k$  is obtained by replacing the middle edge  $(v, u)$  of  $H$  by a path  $P_{k+1}$  that connects  $v$  and  $u$ . Exactly  $k - 1$  new vertices of degree 2 are added. A *snail* is obtained by adding 3 pendant vertices to a triangle, in such a way that two of them become adjacent to the same vertex of the triangle. A *press* graph consists of two copies of paw plus an edge connecting a degree 2 vertex of each paw.

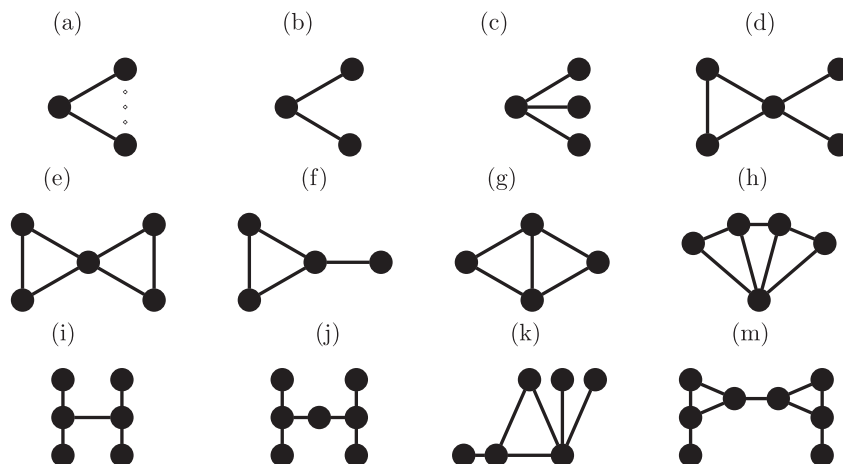


FIGURE 1. Some graphs of interests. (a)  $k$ -star =  $K_{1,k}$ . (b) 2-star =  $K_{1,2}$ . (c) 3-star =  $K_{1,3}$  = claw. (d) Cricket. (e) Butterfly. (f) Paw. (g) Diamond. (h) Gem. (i)  $H = H_1$ . (j)  $H_2$ . (k) Snail. (l) Press.

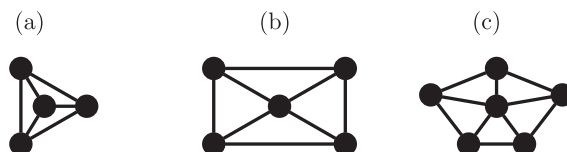


FIGURE 2. Some wheels. (a)  $K_4 = W_3$ . (b)  $W_4$ . (c)  $W_5$ .

There are some structures and properties related to valid bi-colorings.

**Theorem 2** ([13]). *Given a graph  $G$  and any valid bi-coloring  $(B, W)$  of  $G$*

- Any triangle of  $G$  has exactly one white vertex.
- The central vertex of an induced butterfly  $F$  of  $G$  is the unique white vertex of  $F$ .
- The odd degree vertices in an induced paw of  $G$  have different colors.
- In an induced diamond, the vertices of degree 2 are white and the other two vertices are black.

There is some more bibliography to add to the already vast literature [2, 9, 25, 31, 33, 35] on dominating induced matchings. As we have seen before, the papers on perfect edge domination are less frequent. There is a paper [16] where the authors describe ILP formulations for the PED problem, together with some experimental results.

## 2. NP-COMPLETENESS RESULTS

We consider in this work five variants of the lin3SAT problem, two of them are well-known NP-Complete problems, and we prove that the other three are NP-Complete. Then, we reduce them to the existence of DIMs on some subclasses of connected NSF graphs.

Given a formula  $F$  in conjunctive normal form (CNF), its associated graph  $G(F)$  is a bipartite graph where one of the classes of the bipartition corresponds to the clauses and the other one to the variables. We follow the convention of previous articles in this field, using rectangles to represent vertices corresponding to clauses and circles for those that are associated to variables. In  $G(F)$ , two vertices are neighbors if one of them corresponds

to a variable  $x_j$  (circle), the other one corresponds to a clause  $C_i$  (rectangle), and  $x_j$  or  $\neg x_j$  is a literal of  $C_i$ . As we work with 1in3SAT, every clause of  $F$  has exactly three different literals which implies that every clause (rectangle) vertex has degree 3 in  $G(F)$ . We say that an assignment is *valid* if in every clause there is exactly one true literal. Two formulas  $F$  and  $F'$  in CNF are equivalent if there is a valid assignment for  $F$  if and only if there is a valid assignment for  $F'$ . A formula  $F$  is *positive* if all literals in  $F$  are positive.

On the other hand, it is highly probable that the associated graphs of hard instances (to determine if there is a valid assignment) contain some induced  $C_4$ . In many NP-completeness proofs variants of 1in3SAT (for instance [30]), the reduction adds two clauses that repeat two literals in order to assure that the third literal has the same value in any valid assignment. This action makes the associated graph to contain induced  $C_4$ 's.

Below are the six decision problems that we deal with in this article. The first two of them are known NP-Complete variants of 1in3SAT.

#### PLANAR POSITIVE 1in3SAT

INPUT: A positive formula  $F$  in CNF, each clause is a disjunction of exactly three different literals and the associated graph  $G(F)$  is planar.

QUESTION: Is there a valid assignment for  $F$ ?

The NP-Completeness proof of this variant can be found in [21, 32].

#### CUBIC PLANAR POSITIVE 1in3SAT

INPUT: A positive formula  $F$  in CNF, each clause is a disjunction of exactly three different literals and the associated graph  $G(F)$  is cubic planar.

QUESTION: Is there a valid assignment for  $F$ ?

The NP-Completeness proof of this more restricted variant can be found in [30] where a polynomial reduction is presented to transform an input instance  $F$  of PLANAR POSITIVE 1in3SAT to an instance  $F'$  of CUBIC PLANAR POSITIVE 1in3SAT. It is important to see that if  $G(F)$  is connected then  $G(F')$  is also connected.

Now, we consider the new connected versions of these two previous 1in3SAT variants.

#### CONNECTED PLANAR POSITIVE 1in3SAT

INPUT: A positive formula  $F$  in CNF, each clause is a disjunction of exactly three different literals and the associated graph  $G(F)$  is connected and planar.

QUESTION: Is there a valid assignment for  $F$ ?

We prove this restricted variant 1in3SAT is also NP-Complete. Let  $F$  be an input instance of PLANAR POSITIVE 1in3SAT, we will construct a positive formula  $F^*$  in CNF, input instance of CONNECTED PLANAR POSITIVE 1in3SAT, and show that  $F$  and  $F^*$  are equivalent. This construction is iterative and will stop when the obtained formula is a valid input instance for CONNECTED PLANAR POSITIVE 1in3SAT. We will show in every step, the new formula  $F^*$  of the step is a valid input instance of PLANAR POSITIVE 1in3SAT, the number of connected components of  $G(F^*)$  has decreased by 1 compared to the previous formula and both formulas are equivalent. Initially, let  $F^* = F$ . Repeat the following PROCEDURE while  $G(F^*)$  is not connected.

#### PROCEDURE

- (1) Let  $R$  be a planar representation of  $G(F^*)$ .
- (2) Select  $G_1$  and  $G_2$ , two different connected components of  $G(F^*)$ .

- (3) As  $G(F^*)$  is a bipartite graph and every rectangle (clause) vertex has exactly 3 circle (variable) vertices as neighbors. Therefore, every connected component of  $G(F^*)$  has some circle vertex which lies in the external region of  $R$ . Let  $v_1$  (corresponds to some variable  $x$ ) and  $v_2$  (corresponds to some variable  $y$ ), circle vertex with this property for  $G_1$  and for  $G_2$ , respectively.
- (4) Add 3 variables ( $x', y'$  and  $z$ ) and 2 clauses ( $x \vee z \vee x'$  and  $y \vee z \vee y'$ ) to  $F^*$ .

It is clear that after the execution of the above procedure, the number of connected components of  $G(F^*)$  has decreased by 1 since there is a path connecting  $v_1$  and  $v_2$  passing through vertices corresponding to  $x \vee z \vee x'$ ,  $z$  and  $y \vee z \vee y'$ . It is not hard to see that the new  $G(F^*)$  is still planar (the five new vertices and the six new edges can be drawn in the external region of  $R$  without edge crossing) and  $F^*$  is still a valid input instance of PLANAR POSITIVE 1in3SAT. Now, if the new  $F^*$  is satisfiable then the old  $F^*$  is satisfiable because all clauses of the old one are clauses of the new one. If the old  $F^*$  is satisfiable, we can extend any valid assignment for it to a valid assignment for the new  $F^*$  as below: setting *FALSE* to  $z$ , the value of  $\neg x$  to  $x'$  and  $\neg y$  to  $y'$ . With this the proof is completed.

#### CONNECTED CUBIC PLANAR POSITIVE 1in3SAT

INPUT: A positive formula  $F$  in CNF, each clause is a disjunction of exactly three different literals and the associated graph  $G(F)$  is connected and cubic planar.

QUESTION: Is there a valid assignment for  $F$ ?

The polynomial reduction given in [30] transforms connected instances of PLANAR POSITIVE 1in3SAT to connected instances of CUBIC PLANAR POSITIVE 1in3SAT. The NP-Completeness of this variant is guaranteed by the NP-Completeness of CONNECTED CUBIC PLANAR POSITIVE 1in3SAT and the correctness of this reduction.

The following is our last new proposed variant of 1in3SAT.

#### CONNECTED SUBCUBIC PLANAR $C_4$ -FREE POSITIVE 1in3SAT

INPUT: A positive formula  $F$  in CNF, each clause is a disjunction of exactly three different literals and the associated graph  $G(F)$  is connected, planar,  $C_4$ -free, with degree at most 3.

QUESTION: Is there a valid assignment for  $F$ ?

Finally, our target problem.

#### EXISTENCE OF DIM FOR CONNECTED NSF GRAPHS

INPUT: Connected NSF graph  $G$ .

QUESTION: Does  $G$  contain a dominating induced matching?

Next we prove that CONNECTED SUBCUBIC PLANAR  $C_4$ -FREE POSITIVE 1in3SAT is NP-complete.

**Theorem 3.** *Connected subcubic planar  $C_4$ -free positive 1in3SAT is NP-Complete.*

*Proof.* Firstly we describe a polynomial-time reduction from an input instance of Connected cubic planar positive 1in3SAT, a formula  $F$ , to an input instance of Connected subcubic planar  $C_4$ -free positive 1in3SAT, a formula  $F'$ . Secondly, we show that  $F$  and  $F'$  are equivalent. This will prove that Connected subcubic planar  $C_4$ -free positive 1in3SAT is NP-Complete.

The reduction will consider a positive formula  $F$  as input (where all literals are positive). The associated graph  $G(F)$  is connected cubic planar bipartite. Every variable  $y_i$  of  $F$ , appears exactly in 3 clauses  $C_i^1, C_i^2, C_i^3$  as positive literal. We replace  $y_i$  by three variables  $x_i^1, x_i^2, x_i^3$  and  $x_i^j$  replaces  $y_i$  as positive literal in  $C_i^j$  for  $j = 1, 2, 3$ . In order to assure that  $x_i^1, x_i^2, x_i^3$  have the same value in every valid assignment, we add 8 new clauses and 10 additional variables as follows (see Fig. 3).

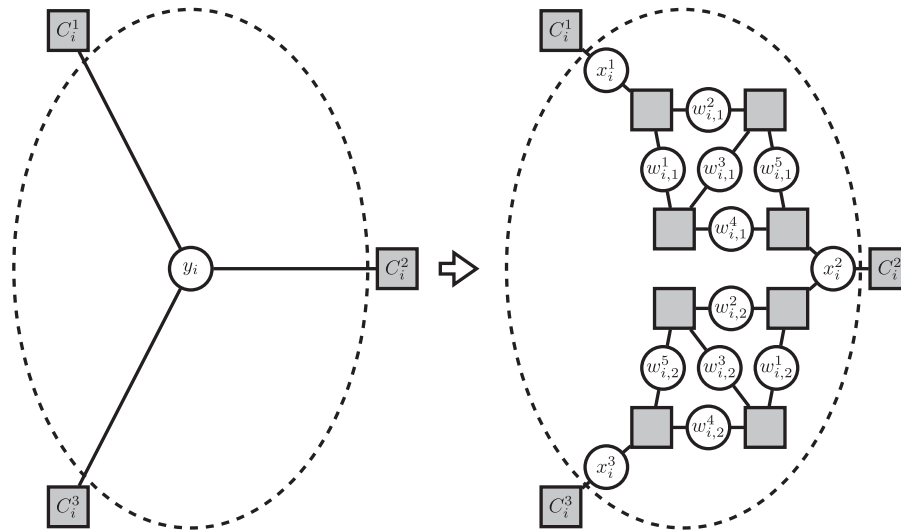


FIGURE 3. Reduction employed in the NP completeness proof.

For  $j = 1, 2$

$$\begin{aligned}
 &x_i^j \vee w_{i,j}^1 \vee w_{i,j}^2 \\
 &w_{i,j}^1 \vee w_{i,j}^3 \vee w_{i,j}^4 \\
 &x_i^{j+1} \vee w_{i,j}^4 \vee w_{i,j}^5 \\
 &w_{i,j}^2 \vee w_{i,j}^3 \vee w_{i,j}^5.
 \end{aligned}$$

We show that  $w_{i,j}^1, w_{i,j}^5$  have the same value in any valid assignment. If  $w_{i,j}^1$  is true, then  $x_i^j, w_{i,j}^2, w_{i,j}^3, w_{i,j}^4$  should be false. Therefore,  $w_{i,j}^5$  should be the true literal in  $w_{i,j}^2 \vee w_{i,j}^3 \vee w_{i,j}^5$ . Symmetrically, if  $w_{i,j}^5$  is true, then  $x_i^{j+1}, w_{i,j}^2, w_{i,j}^3, w_{i,j}^4$  should be false. Therefore,  $w_{i,j}^1$  should be the true literal in  $w_{i,j}^3 \vee w_{i,j}^4 \vee w_{i,j}^5$ . Now, we prove that  $x_i^j, x_i^{j+1}$  have the same value in any valid assignment. In case that  $w_{i,j}^1, w_{i,j}^5$  are true, then  $x_i^j, x_i^{j+1}$  are both false. Then, we can assume that  $w_{i,j}^1, w_{i,j}^5$  are false. If  $x_i^j$  is true, then  $w_{i,j}^2$  is false and  $w_{i,j}^3$  should be the true literal in  $w_{i,j}^2 \vee w_{i,j}^3 \vee w_{i,j}^5$ . Consequently,  $w_{i,j}^4$  is false and  $x_i^{j+1}$  should be the true literal in  $x_i^{j+1} \vee w_{i,j}^4 \vee w_{i,j}^5$ . Symmetrically, if  $x_i^{j+1}$  is true,  $x_i^j$  would be true using a similar deduction. As a consequence,  $x_i^1, x_i^2, x_i^3$  have the same value in every valid assignment as we need. Note that all these variables except  $x_i^1, x_i^3$  are literals of exactly 2 of these 8 clauses.  $x_i^1, x_i^3$  appear only once. But solely,  $x_i^1, x_i^2, x_i^3$  appear in exactly one other clause ( $C_i^1, C_i^2, C_i^3$ , respectively). In conclusion, every variable appears at most in 3 clauses. Clearly, the resulting formula  $F'$  is equivalent to  $F$ . If  $F$  uses  $n$  variables and  $m$  clauses, then  $F'$  has exactly  $13n$  variables and  $m + 8n$  clauses. Therefore,  $F'$  can be obtained in polynomial time. As all clauses have exactly 3 literals, the associated graph  $G(F')$  has maximum degree 3. In case that  $G(F')$  has an induced  $C_4$  (see Fig. 4). Two of the vertices of this  $C_4$  correspond to clauses and the other two variables meaning that those two clauses have both variables as literals. This situation does not occur in the formula  $F'$ . The original  $m$  clauses have their local copies of variables. They do not have any variable in common. Any pair of the new  $8n$  clauses have at most one variable in common. An original clause and a new one can have at most one variable  $x_i^j$  in common. Therefore,  $G(F')$  is  $C_4$ -free. Given a planar representation of  $G(F)$ , this representation can be easily transformed to a planar representation of  $G(F')$  (see Fig. 3). It is clear that if  $G(F)$  is connected then  $G(F')$  also does. Finally, we can conclude that  $G(F')$  is connected subcubic planar  $C_4$ -free. Consequently, we can state that Connected subcubic planar  $C_4$ -free positive 1in3SAT is NP-Complete.  $\square$



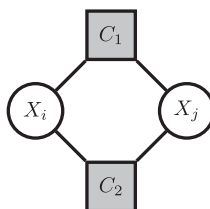


FIGURE 4. An induced  $C_4$  cannot exist in the associated graph  $G(F')$ .

Now, we give a polynomial-time reduction from an instance of input of Connected subcubic planar  $C_4$ -free positive lin3SAT, a positive formula  $F$  in CNF, to a graph  $S(G(F))$  an instance of our final target problem.

Given such positive formula  $F$  where each clause is a disjunction of exactly three different literals and its associate graph  $G(F)$  is connected planar  $C_4$ -free with degree at most 3, we construct a graph  $S(G(F))$  from  $G(F)$  as follows.

- (1) Replace every rectangle vertex  $v$  (corresponding to a clause  $C_i$ ) of  $G(F)$  by a triangle (clause triangle). Each vertex of this triangle takes a different neighbor (a circle vertex corresponding to a variable) of  $v$  as its owner (see Fig. 5). Call this resulting graph as *temporary* graph  $T(G(F))$ .
- (2) For each circle vertex  $w$ , add two mutually adjacent new vertices as its neighbors forming a triangle (vertex triangle) (see Fig. 6). This final graph is the graph  $S(G(F))$ .

For our convenience, we call this construction as *Main Transformation*.

Clearly,  $S(G(F))$  can be obtained in polynomial-time from  $F$ . If  $F$  has  $n$  variables and  $m$  clauses, then  $S(G(F))$  has exactly  $3n + 3m$  vertices and  $3n + 6m$  edges.

**Theorem 4.**  $S(G(F))$  is connected planar NSF ( $K_4$ , diamond, butterfly,  $K_{1,5}$ ,  $H$ , snail, press,  $C_4$ ,  $C_5$ ,  $C_6$ ,  $C_7$ ,  $C_8$ ,  $C_{3t+1}$ ,  $C_{3t+2}$ )-free for  $t \geq 3$  without pendant vertices.

*Proof.* Clearly, as  $G(F)$  is connected then  $S(G(F))$  is connected. Every vertex of  $S(G(F))$  is in exactly one triangle which implies that  $S(G(F))$  is NSF graph without pendant vertices and it is ( $K_4$ , diamond, butterfly)-free.

Every vertex  $u$  of a clause triangle  $T$  has exactly one neighbor  $z \notin T$  and  $z$  is in a vertex triangle ( $u$  has degree 3). Every vertex triangle has exactly two simplicial vertices and both of degree two. It is clear that the only vertices that can have degree more than 3 in  $S(G(F))$  are those corresponding to circle vertices in  $G(F)$ . They have degree at most 3 in  $G(F)$  but in  $S(G(F))$ , they have two additional neighbors. Hence, they have degree at most 5 in  $S(G(F))$ . As every vertex in  $S(G(F))$  has at most degree 5, and it is in exactly one triangle,  $S(G(F))$  is  $K_{1,5}$ -free (also, ( $K_4$ , diamond, butterfly,  $K_{1,5}$ )-free implies maximum degree 5).

Next, let us show that  $S(G(F))$  is ( $H$ , snail, press)-free.

Suppose  $S(G(F))$  has a snail as an induced subgraph. The triangle of the snail should be a vertex triangle since one of its vertices has degree at least 4. But in this case, this triangle has at most one simplicial vertex which is a contradiction. Hence,  $S(G(F))$  is snail-free.

If  $S(G(F))$  has an  $H$  as an induced subgraph. We consider two cases: (i) the middle edge is part of some triangle  $T$  and clearly,  $T$  is a vertex triangle, again  $T$  has at most one simplicial vertex (ii) the middle edge connects two triangles, one of them is a clause triangle meaning that one of the middle vertices should have degree exactly 3. All its three neighbors are in  $H$  and there is not any triangle, a contradiction. Therefore,  $S(G(F))$  is  $H$ -free.

Suppose now that  $S(G(F))$  contains a press as an induced subgraph. Clearly, one of the two triangles must be a variable triangle but both of them have at most one simplicial vertex. Again, a contradiction. So,  $S(G(F))$  is press-free.

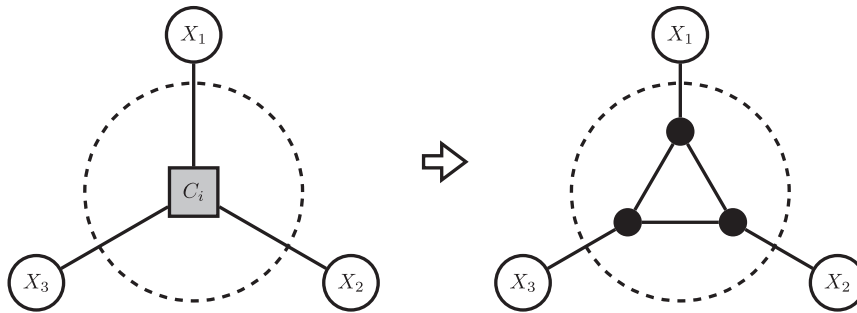


FIGURE 5. Replacing a clause by a triangle.

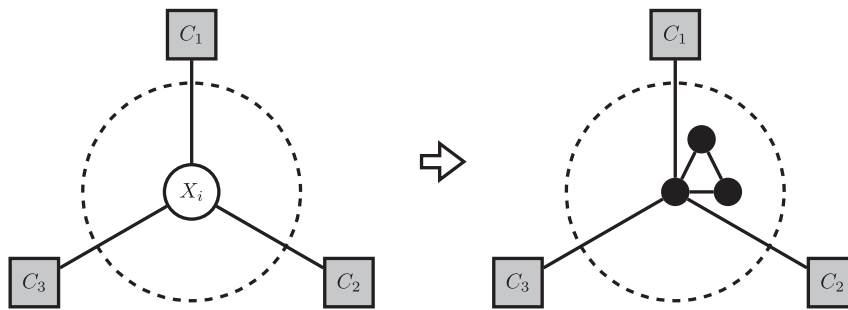


FIGURE 6. Replacing a variable by a triangle.

Next, we analyze the possible induced cycles of  $S(G(F))$  and  $G(F)$ .

Let  $C$  be an induced cycle of  $S(G(F))$  having length greater than 3. We claim that the length of  $C$  is reduced by  $2/3$ , regarding its corresponding length in  $G(F)$ . To show this claim, consider a cycle of  $G(F)$  and its corresponding cycle of  $S(G(F))$ . Let  $t$  and  $k$  represent the lengths of these cycles of  $G(F)$  and  $S(G(F))$ , respectively. Recall that  $G(F)$  is planar,  $C_4$ -free and bipartite, whose parts are formed by circle vertices and rectangle vertices, respectively.

In  $G(F)$  each rectangle vertex has exactly degree 3 because each clause is formed by 3 positive literals. Furthermore, each of these rectangle vertices is replaced by a clause triangle in  $T(G(F))$ . To extend  $T(G(F))$  into  $S(G(F))$ , we add two new adjacent vertices for each circle vertex  $u$ , both to be adjacent to  $u$ . Therefore, the vertices belonging to an induced cycle of length  $\geq 4$  in  $S(G(F))$  are circle vertices, and vertices of clause triangles. Clearly, they ought to be these both types, since the set of circle vertices forms an independent set, while the clause triangles are disjoint triangles. Clearly, we need at least two circle vertices, to form an induced cycle  $C$  of  $S(G(F))$ . For each such circle vertex, we need two vertices of clause triangles. Therefore, if  $C$  contains  $t$  circle vertices,  $C$  also contains another  $2t$  vertices from clause triangles, that is  $3t$  vertices in total. Next, every pair of vertices of clause triangles is contracted into a single vertex. Therefore,  $t = (2/3)k$ , and the length of each cycle of  $G(F)$  becomes  $2/3$  smaller, indeed.

As  $G(F)$  is bipartite and  $C_4$ -free, the only values available for  $k$  is  $3t$  with  $t \geq 3$ . Therefore,  $S(G(F))$  and  $T(G(F))$  are  $(C_4, C_5, C_6, C_7, C_8, C_{3t+1}, C_{3t+2})$ -free, for  $t \geq 3$ .

As  $G(F)$  is planar, take a planar representation of  $G(F)$ , any vertex is replaced by a triangle. In case of a variable triangle, it is clear that the two adjacent new neighbors of the circle vertex maintain the planar representation (see Fig. 6). In case of a clause triangle, a similar fact holds (see Fig. 5). Therefore,  $S(G(F))$  is planar.  $\square$

**Theorem 5.** *There is a valid assignment for an input formula  $F$  of Connected subcubic planar  $C_4$ -free positive 1in3SAT if and only if the resulting graph  $S(G(F))$  of the main transformation contains some dominating induced matching.*

*Proof.* Given a valid assignment, we define a bi-coloring of  $S(G(F))$  using this assignment below.

- (1) If a variable  $x_i$  is true then color black its corresponding circle vertex  $v$ . Choose arbitrarily any neighbor  $w$  of its variable triangle and color it also black. Color white all other neighbors of  $v$ .
- (2) If a variable  $x_i$  is false then color white its corresponding circle vertex  $v$  and color black all its neighbors.

Next we prove that this bi-coloring verifies the conditions of Theorem 1.

- (1) In a variable triangle, every vertex satisfies trivially the conditions.
- (2) In a clause triangle, a vertex is colored black (white) if its unique circle vertex neighbor  $x_i$  is colored white (black) because the variable  $x_i$  is false (true). As every clause has one true literal (all literals are positive). In any clause triangle  $T$ , there is exactly one white vertex  $v$  and all its 3 neighbors have been colored black. Particularly, the other two vertices of  $T$ .

Therefore, this bi-coloring corresponds to a DIM of  $S(G(F))$ .

Conversely, given a DIM  $D$  and its associated bi-coloring, we define an assignment as follows. If a circle vertex  $x_i$  is black (white) then the variable  $x_i$  is true (false). We will prove this assignment is valid. Using the properties of Theorem 2 and considering  $T$  the variable triangle that contains  $x_i$ , every clause triangle neighbor  $w$  of  $v$  has a different color of  $x_i$  since the vertices of  $T$  and  $w$  induce a paw and  $x_i, w$  are the odd degree vertices in the induced paw. The color of  $w$  is white only if  $x_i$  is true. As every clause triangle has exactly one white vertex. Consequently, every clause has exactly one true literal.  $\square$

**Corollary 1.** *EXISTENCE DIM is NP-Complete for connected planar NSF ( $K_4$ , diamond, butterfly,  $K_{1,5}$ ,  $H$ , snail, press,  $C_4$ ,  $C_5$ ,  $C_6$ ,  $C_7$ ,  $C_8$ ,  $C_{3t+1}$ ,  $C_{3t+2}$ )-free graphs for  $t \geq 3$  without pendant vertices.*

Since connected NSF graphs do not have proper perfect dominating sets, the existence of DIM is equivalent to asking if there exists a perfect edge dominating set with at most  $m - 1$  edges where  $|E| = m$  (the trivial perfect dominating set is the set of all edges  $E$ ).

**Corollary 2.** *EXISTENCE of a perfect dominating set with at most  $m - 1$  edges is NP-Complete for Connected planar NSF ( $K_4$ , diamond, butterfly,  $K_{1,5}$ ,  $H$ , snail, press,  $C_4$ ,  $C_5$ ,  $C_6$ ,  $C_7$ ,  $C_8$ ,  $C_{3t+1}$ ,  $C_{3t+2}$ )-free graphs for  $t \geq 3$  without pendant vertices where  $m$  is the number of edges in the graph.*

A *subdivision* of an edge  $(v, w)$  in a graph consists of replacing such edge by two edges  $(v, u), (u, w)$  and a new vertex  $u$ .

**Lemma 1** ([13]). *Let  $G$  be a graph and  $e$  an edge in  $G$ . If  $G'$  is the graph obtained from  $G$  by subdividing the edge  $e$  exactly three times, then  $G$  has a dominating induced matching if and only if so has  $G'$ .*

Cardoso, *et al.* [13] use the ideas of the triple subdivision of the edges (see Lem. 1) to extend the girth of the graph without affecting the existence or not of DIM. This technique does not work for our class under consideration since this operation adds 3 new vertices of degree 2 and they are not part of triangles. Instead of this operation we propose other alternatives to avoid induced cycles of size at most  $k$  for any  $k$ . But all of them come with some cost. The main idea is replacing edges of  $S(G(F))$ , the resulting graph of Main Transformation that connects vertices of different triangles by chains of structures that assure the vertices at both ends of chains should have different colors in any valid bi-coloring. We call those edges as variable-clause edges since they connect a variable triangle to a clause triangle. It is clear that every vertex of the chains should be in a triangle. We define two types of triangles: (i) vertex link triangles, these triangles have only one vertex in the chain, and (ii) edge link triangles, which are those that have exactly two vertices in the chain. There are many alternatives to create chains.

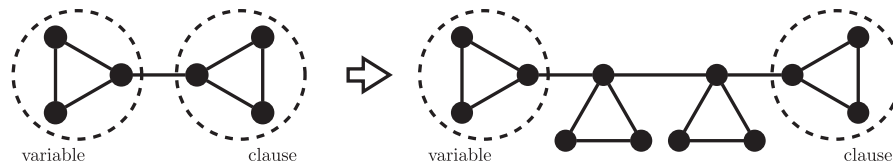


FIGURE 7. Replacing a variable-clause edge by a chain of 2 vertex link triangles.

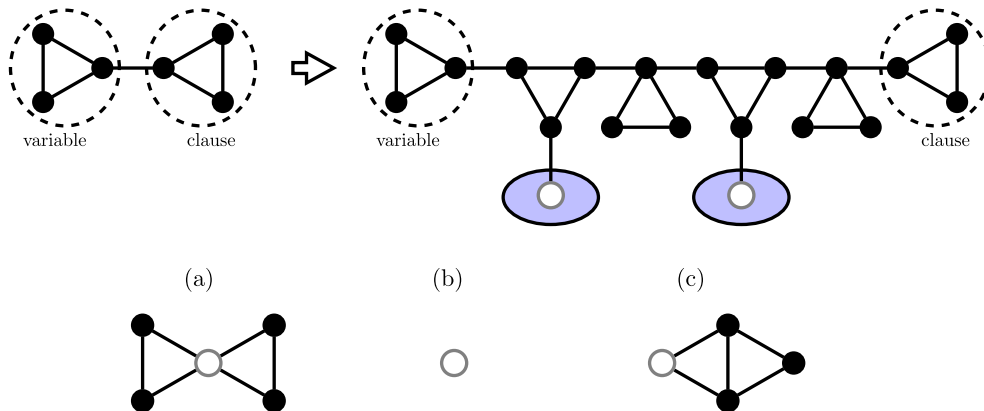


FIGURE 8. Replacing a variable-clause edge by a mixed chain of 2 vertex link triangles and 2 edge link triangles and 3 options for gadget. (a) Butterfly. (b) Pendant. (c) Diamond.

- (1) Every chain has exactly  $2k_1$  vertex link triangles (see Fig. 7 for  $k_1 = 1$ ). It is clear that every pair of consecutive vertices in the chain are odd-degree vertices in some induced paw. By Theorem 2, they have different colors in any valid bi-coloring. Since the chain has  $2k_1 + 1$  edges, then both ends of the chain have different colors in any valid bi-coloring.
- (2) Every chain has alternately edge link triangles and vertex link triangles if we start from the variable triangle side, the first triangle is an edge link triangle. The number of each type of triangle is  $2k_2$  (see Fig. 8 for  $k_2 = 1$ ). Some additional structure is needed for edge link triangles. Each edge link triangle has exactly one vertex that is not in the chain. We should connect this vertex to some contact vertex of a gadget. There are three options for the gadget structure (see Fig. 8).
  - (a) A butterfly, where the contact vertex is its central vertex.
  - (b) A  $K_1$  and the contact vertex is the unique one. This vertex will become a pendant vertex.
  - (c) A diamond, where the contact vertex is any vertex of degree 2 in the diamond.

In any case, the contact vertex is colored white in any valid bi-coloring by Theorem 2 and its neighbor in the edge link triangle should be black, again by Theorem 2. In consequence, the vertices of the edge link triangle in the chain should have different colors. All other edges of the chain form part of some induced paw and their incident vertices have degree odd in such paw. Therefore, every pair of consecutive vertices in the chain have different colors in any valid bi-coloring. The number of edges in the chain is exactly  $6k_2 + 1$ , an odd number, which implies that both ends of the chain have a different color in any valid bi-coloring.

It is clear that  $S(G(F))$  admits a DIM if and only if the resulting graph  $Q$ , after the replacement of variable-clause edges by the chains described above, has a DIM. Furthermore, every variable-clause edge of  $S(G(F))$  is part of some induced paw and its incident vertices have odd degree in such induced paw. Consequently, they should have different colors in any valid bi-coloring, by Theorem 2. Then from any valid bi-coloring of  $Q$ , we can obtain trivially a valid bi-coloring of  $S(G(F))$ . Conversely, any valid bi-coloring of  $S(G(F))$ , can be extended

easily to a bi-coloring of  $Q$ . Consequently, we can define new reductions from Subcubic planar  $C_4$ -free positive 1in3SAT to the DIM problem, concatenating the Main Transformation and one of these types of replacement of variable-clause edges.

Next, let us examine the resulting graph class of each type of replacement. The number of variable-clause edges in an induced cycle in  $S(G(F))$  is at least 6 and the number of clause triangle edges is at least 3 since the induced cycles in the associated graph  $G(F)$  have at least 3 clause vertices.

- (1) In this case, the only forbidden subgraph of Corollaries 1 and 2 that can appear is the  $H$  graph. All induced cycles of  $Q$  should have at least  $6 \cdot (2k_1 + 1) + 3 = 12k_1 + 9$  vertices. The number of vertices of  $Q$  is  $n + (m - n) \cdot 3 \cdot 2k_1 = 6k_1m - (6k_1 - 1) \cdot n$  and  $Q$  has  $m + (m - n) \cdot 4 \cdot 2k_1 = (8k_1 + 1) \cdot m - 8k_1n$  edges.
- (2) Independently of the gadget employed, all induced cycles of  $Q$  should have at least  $6 \cdot (3 \cdot 2k_2 + 1) + 3 = 36k_2 + 9$  vertices.
  - (a) In this case, the only forbidden subgraph of Corollaries 1 and 2 that can appear is the butterfly. The number of vertices of  $Q$  is  $n + (m - n) \cdot (8 + 3) \cdot 2k_2 = 22k_2m - (22k_2 - 1) \cdot n$  and  $Q$  has  $m + (m - n) \cdot (11 + 4) \cdot 2k_2 = (30k_2 + 1) \cdot m - 30k_2n$  edges.
  - (b) In the present situation,  $Q$  has pendant vertices. The number of vertices of  $Q$  is  $n + (m - n) \cdot (4 + 3) \cdot 2k_2 = 14k_2m - (14k_2 - 1) \cdot n$  and  $Q$  has  $m + (m - n) \cdot (5 + 4) \cdot 2k_2 = (18k_2 + 1) \cdot m - 18k_2n$  edges.
  - (c) The only forbidden subgraph of Corollaries 1 and 2 that can appear is the diamond. We remark that there are no gems, nor  $W_4$ 's. The number of vertices of  $Q$  is  $n + (m - n) \cdot (7 + 3) \cdot 2k_2 = 20k_2m - (20k_2 - 1) \cdot n$  and  $Q$  has  $m + (m - n) \cdot (10 + 4) \cdot 2k_2 = (28k_2 + 1) \cdot m - 28k_2n$  edges.

Also, in any case,  $Q$  will still be a planar graph of degree at most five. We remark that cycles of size between 4 and  $k$ , in  $Q$ , for fixed  $k \geq 4$ , are forbidden. For each type of replacement, we can determine the smallest required value of  $k_1$  or  $k_2$  to assure this.

**Theorem 6.** *For any fixed value  $k \geq 4$ , EXISTENCE DIM and EXISTENCE of PED with at most  $m - 1$  edges are NP-Complete in the following graph classes.*

- Connected planar NSF ( $K_4$ , diamond, butterfly,  $K_{1,5}$ , snail, press,  $C_4, \dots, C_k$ )-free without pendant vertices.
- Connected planar NSF ( $K_4$ , diamond,  $K_{1,5}$ ,  $H$ , snail, press,  $C_4, \dots, C_k$ )-free without pendant vertices.
- Connected planar NSF ( $K_4$ , diamond, butterfly,  $K_{1,5}$ ,  $H$ , snail, press,  $C_4, \dots, C_k$ )-free.
- Connected planar NSF ( $K_4$ , gem,  $W_4$ , butterfly,  $K_{1,5}$ ,  $H$ , snail, press,  $C_4, \dots, C_k$ )-free without pendant vertices.

### 3. CONCLUSIONS

We conclude with an open question.

We wonder if there is some algorithmic relation between efficient and perfect edge domination. More specifically, we remark that there are graph classes which admit polynomial time solutions for solving the efficient edge domination problem while being hard for solving the perfect edge domination problem. However, we ask whether there is some graph class for which there exists a polynomial time algorithm for solving the perfect edge domination problem while being hard for the efficient edge domination problem.

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