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EFFECTS OF MEMORY ON INVENTORY CONTROL AND PRICING POLICY FOR IMPERFECT PRODUCTION WITH REWORK PROCESS

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Abstract. Fractional calculus is a pertinent way to study the memory effect in an inventory model for investigating its dynamical behavior. Since inventory management is a memory-dependent process, fractional calculus approach may be employed to discover some fruitful insights and can help to gain more profit. In realistic scenarios, the manufacturing process cannot be perfect, and it delivers some faulty units due to many inevitable reasons. In literature, an imperfect production inventory problem under the memory effect has not been studied. Our study aims to investigate the memory effect on a production inventory system. In this article, a fractional order inventory control problem is formulated by considering an imperfect manufacturing process and price-sensitive demand. The faulty units are repaired through a rework process. Caputo fractional derivatives and integrals are used to consider the memory effect. Due to the nonlinear cost elements in the formulated problem, optimal pricing and production policies are investigated by using a quasi-Newton optimization algorithm and particle swarm optimization approach. The managerial implications of the proposed study are discussed with the help of numerical illustrations. The numerical outcomes suggest that consideration of memory in the inventory system boosts the profitability of the firm.

Mathematics Subject Classification. 26A33, 90B05.

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1. Introduction

Fractional calculus generalizes the classical calculus of integer order. Lacroix [1] introduced the idea of fractional calculus around 1819 by defining fractional ordered derivative of $y(x) = x^k$ as:

$$y^{(l)} = \frac{\Gamma(k+1)}{\Gamma(k-l+1)} x^{k-l}$$

where $k \geq l$ and l is any real number.

Fractional calculus (FC) adds more dimensions into classical calculus. FC is being used in several areas of science and technology. The economic growth of the G7 countries was analyzed by Tejado *et al.* [2] for the period of 1973–2016 with the help of fractional calculus. The outcomes studied by making comparisons with the classical calculus approach inferred that the fractional models have better predictions than the classical models.

Keywords. Inventory, memory effect, fractional calculus, imperfect production, price sensitive demand.

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Inventory is an important asset for any business and can be viewed as an assembly of goods (used in production) or the finished goods held by a firm. The process of inventory management involves monitoring and controlling the inventory level of the firm to ensure adequate replenishment at the desired time. Several constraints are incorporated by the inventory managers of a company to minimize inventory costs and improve the customer service. The practitioners are interested to develop an appropriate inventory model so as to suggest an optimal inventory policy. A good model helps a company in the judicious use of its resources, thereby improving the profit percentage. Basically, two types of inventory models are considered in the literature, *i.e.*, (i) deterministic in which demand for a product can be predicted accurately and (ii) stochastic in which demand for a product cannot be determined precisely.

The prime goal of developing an inventory model is to reduce inventory costs, including ordering and holding costs. Most of the lot-sizing models available in the literature deal with perfect production system, relaxing the assumption of faulty units in the production lot. But in a practical scenario, a production process cannot be 100% perfect, due to the fact that the presence of defectives is unavoidable. The imperfect units are generated in most common scenarios due to many reasons, such as faulty raw materials, poor maintenance policy, unskilled labor, machine disruption, etc. Manufacturers are reluctant to convert defective items to scrap which increases the cost of the manufactured products. They prefer a rework on the defective items with additional costs to make them usable or of the same quality as those of the perfect quality products.

Markovian property states that the past information of a system does not affect its current state. Fractional order derivatives and integrals possess non-Markovian property. Fractional calculus is a helpful approach to study the process in which decisions are highly influenced by past information (memory). Inventory is a memory-affected system, and it can be studied more efficiently by using fractional calculus methods. Memory parameters in inventory models facilitate insights to understand the present state of a system without neglecting its past states.

Both external and internal factors play a vital role in an inventory system. For example, a firm's profit depends on many factors, such as physical environment of the firm, reputation in the business world, hospitality, and many other factors. The profit of any firm depends on the customer's willingness to engage in business with the firm. The customers usually make the decision to do business after considering their past experiences with the company's products and the quality of the goods. If a customer does not have a good experience with the firm, he/she may not be willing to do business with the company, which directly affects the firm's profit. Therefore, past experience or memory has a great impact on an inventory system. The classical inventory models suffer from amnesia as the stock levels are studied by ordinary differential equations of first order. The temporal rate of change of integer orders is specified by the property of differentiable functions of time only in an indefinitely small neighborhood of the time epoch. Therefore, the classical calculus approach does not include past effects of the system [3]. Motivated by the importance of memory based models, the authors employed fractional calculus approach to evaluate the memory effect in the concerned inventory system [3]. Here fractional-order means the index of memory or the strength of memory. The current study attempts to investigate memory impact in an imperfect production inventory model with price-dependent demand.

The rest of the study is arranged in different sections. A review of relevant studies of production-inventory problems is presented in Section 2. The motivations, research gaps, and contributions are outlined in this section. Section 3 presents some basic functions and results used in fractional calculus. The model description, assumptions, and notations used are stated in Section 4. The inventory model formulation and cost function, including various cost elements, are discussed in Section 5. Numerical simulations and sensitivity of parameters are provided in Section 6. The key managerial insights are also discussed in this section. Section 7 highlights the noble features, limitations, and further extension possibilities of the proposed study.

2. Literature review

The development of fractional calculus started at the end of the 17th century. As an early development, Leibnitz came up with an idea to calculate the derivative of order $\alpha = \frac{1}{2}$ in 1965 [1]. Since then, FC has influenced many researchers from various areas of mathematics, physics, economics, and biology. Samko *et al.* [4] provided a detailed investigation of various aspects of FC. In the literature, different fractional differentiation approaches have been developed. Riemann–Liouville (RL), Caputo, Grunwald–Letnikov, Hadamard, and Riesz are just a few such approaches [5].

Firms use inventory control techniques to optimize their profit and reduce operating costs. Researchers have progressively done a good deal of investigation to improve inventory management policies. Singh *et al.* [6] discussed the significance of the issues related to inventory management. The authors presented a methodology for inventory control and further encouraged empirical validation of the method and research to reduce the overall cost without compromising the quality. Inegbedion *et al.* [7] investigated inventory management techniques to improve a firm's inventory decisions. Yamini and Gajanand [8] provided an overview of inventory control tactics to encourage new perspectives in inventory and supply chain management.

In reality, no production system is 100% perfect as there are always some faulty units in a production lot. Sivashankari and Panayappan [9] developed an Economic Production Quantity (EPQ) model with defective products being reworked during the same production cycle and gave insights into the impact of such scenarios. Ghosh *et al.* [10] proposed a reverse logistics-based imperfect EPQ model, where the imperfect units were used as raw material in the remanufacturing process. The recycling process used reduces the production cost and makes the process environment-friendly. Khedlekar *et al.* [11] presented an EPQ model having imperfect production along with a perfect rework process after the usual production to find the optimal pricing and ordering policy. Yadav *et al.* [12] developed a multi-item production inventory (PI) model under a supply chain consisting of two players, *viz.*, manufacturer and retailer. It showcased that profit is sensitive concerning the learning phenomenon in the proportion of defective units in the production system. Pal *et al.* [13] considered an imperfect EPQ model under the assumption of price-sensitive demand and time-dependent partially backlogged shortages. A review of the inventory problems in procurement and production activities based on the complexity and optimization approaches was investigated by Utama *et al.* [14]. Sarkar *et al.* [15] studied the reliability of a production system by considering the cross-price elasticity of demand.

Rework is one of the key factors in the manufacturing process as it is usually more profitable than disposing of defective items. Al-Salamah [16] developed a mathematical model with a flawed manufacturing process producing faulty items. The authors highlighted the importance of a rework strategy on the profit percentage by considering synchronous and asynchronous flexible rework process. Sanjai and Periyasamy [17] developed two EPQ models depicting two operational policies based on rework and shortages. Khanna *et al.* [18] considered carbon emissions an imperfect manufacturing that undergo inspection and rework. Further, they proposed two strategies to deal with defective items: selling them at a lower price or employing a rework strategy and selling the units at their pre-determined price. Sarkar *et al.* [19] investigated the influence of random defective rate in an imperfect production system with multiple products, giving scope for studying the effect of learning phenomenon on the model. Dey *et al.* [20] examined a flawed production inventory model for the assembled products, identifying machine failure or labour issues related to defective parts. To replicate realistic situations, Cardenas-Barron *et al.* [21] investigated an EPQ model by considering price-sensitive demand and different holding costs for perfect and defective units in the lot. Gautam *et al.* [22] studied inventory-related issues of imperfect production processes and provided a discussion on EPQ models with imperfect production. Alfares *et al.* [23] developed an EPQ model with non-constant input parameters to reflect a true production scenario. Salmasnia *et al.* [24] presented an integrated model with the rework process for deteriorating products. Sana [25] studied an imperfect PI policy under preventive maintenance. The author suggested an optimal buffer quantity to reduce the total expenditure of the firm. Sarkar *et al.* [26] proposed an automation policy to detect defective units in an imperfect manufacturing system. Two different trading channels, *viz.* offline and online, are considered to boost the firm's revenue. Singh *et al.* [27] provided a brief review of the literature on imperfect production inventory problems using metaheuristic optimization techniques. Jain *et al.* [28] investigated an imperfect PI problem under carbon emission constraints. The authors studied the impact of selling price sensitive demand under shortages by employing bat algorithm (BA). Khan *et al.* [29] proposed a framework to achieve optimal production levels using variable circularity and reduction in carbon emissions. Sebatjane and Adetunji

[30] developed a multi-echelon supply chain model under an imperfect manufacturing system. Dey *et al.* [31] explored how retail industries can maximize profits by optimizing production rates, costs, and green attributes of products while reducing carbon emissions. Singh and Mishra [32] developed an EPQ model for the deteriorating products using machine learning and fuzzy variables to handle uncertainty. It aims to optimize ordering and replenishment decisions while considering carbon emission costs.

The effect of memory or past information on inventory models has broadened the scope of research in developing inventory control policies. The importance of memory-based inventory models to portray realistic dynamic scenarios has motivated us to study different inventory models with memory effect. Pakhira *et al.* [3, 33–38] have developed and studied various inventory control models using fractional calculus to investigate how memory affects the inventory system. These studies are concerned with the influence of long and short memory on the inventory systems with variable memory indices. Pakhira *et al.* [33] studied three economic order quantity models taking into account the memory effect. They considered the optimization issues and solved them by using geometric programming (GP) approach. Further, the authors generalized the memory-based model by considering shortage, memory effect, and linear demand rate [35]. Pakhira *et al.* [37] studied an inventory control aspect by including the features of memory and price-dependent demand. Pakhira *et al.* [38] also studied a fractional order inventory model for deteriorating items. To study the implication of memory effect in production quantity models, Rahaman *et al.* [39] considered memory effects in EPQ models with or without deterioration, showing that classical EPQ is a particular case of the generalized (fractional) EPQ model. Due to its complex nature, Rahaman *et al.* [40] optimized the EPQ model with deterioration using the artificial bee colony (ABC) algorithm. Jana and Das [41] showed that the long-memory effect may be helpful for a firm towards policy making rather than short or memoryless systems. Rahaman *et al.* [42] studied the joint impact of memory and learning on the decision-making for inventory systems operating under an Economic Order Quantity (EOQ) policy using fuzzy fractional differential equations. Further, Rahaman *et al.* [43] developed a fractional order inventory control problem by considering a price and stock sensitive demand. To develop a fractional order EOQ model, Kumar *et al.* [44] considered a time-sensitive demand, reliability, and promotional activities. They discussed an optimal framework for inventory modeling to generate the best possible decisions. A memory based demand function along with price-sensitivity factor was studied to establish optimal inventory decisions for non-instantaneously deteriorating items [45].

The real-life inventory problems have become tedious with the advancement in production and logistic technology. To resolve the memory based inventory related issues, the classical integer derivative approaches are not capable. Due to the non-linearity, the classical methods fail to provide optimal solutions. Metaheuristics are widely used by the researchers to solve complex optimization issues. Particle swarm optimization (PSO) is a metaheuristic optimization approach that is popular among practitioners to handle real-world optimization issues. PSO was proposed by Kennedy and Eberhart [46] based on the study of the behavior of swarm of birds. The PSO algorithm is devised on the social behviour of bird flocking. In PSO, the particles are treated as solutions, and during each iteration, they try to find the optimum position in the search space. Several researchers have been using PSO as an optimization tool to solve real-life inventory control problems. Kumar *et al.* [47] used different variants of PSO to study an imperfect production-inventory model. A collaboration policy for a two-echelon supply chain was proposed by Sharma *et al.* [48] *via* PSO. A pricing and inventory policy for deteriorating items was investigated by Jain and Singh [49] using PSO, genetic algorithm (GA) and differential evolution (DE) metaheuristics. An inventory control model with random deterioration start under preservation investment and price-sensitive demand was proposed by employing PSO and DE metaheuristics [50].

2.1. Motivation and research gaps

Inventory control issues incorporating the memory effect have received less attention from the researchers. Only a few inventory scenarios have been investigated using fractional calculus in the inventory literature [33,36,38,41]. To the best of our knowledge, Rahaman *et al.* [39,40] are the only researchers who have attempted to investigate production-inventory model using a fractional calculus approach. The authors assumed that the production process is perfect, which is inappropriate for many real-world production problems. There is a

TABLE 1. A brief review and comparison of relevant studies.

Authors	Price sensitive demand	EPQ	Imperfect production	Rework	Memory effect	Optimization technique
Pakhira *et al.* [3]					✓	Classical
Sivashankari and Panayappan [9]		✓	✓	✓		Classical
Ghosh *et al.* [10]		✓	✓	✓		Classical
Khedlekar *et al.* [11]	✓	✓	✓	✓		Classical
Yadav *et al.* [12]	✓	✓	✓			Classical
Pal *et al.* [13]	✓	✓	✓	✓		Classical
Al-Salamah [16]		✓	✓	✓		Classical
Sanjai and Periyasamy [17]		✓	✓	✓		Classical
Khanna *et al.* [18]	✓	✓	✓	✓		Classical
Sarkar *et al.* [19]		✓	✓	✓		Classical
Dey *et al.* [20]		✓	✓	✓		Classical
Cardenas *et al.* [21]	✓		✓	✓		Classical
Alfares *et al.* [23]	✓	✓				Search
Salmasnia *et al.* [24]		✓	✓	✓		PSO
Sana [25]		✓	✓	✓		Classical
Sarkar *et al.* [26]		✓	✓	✓		Classical
Jain *et al.* [28]	✓	✓	✓	✓		BA
Dey *et al.* [31]	✓	✓	✓	✓		Classical
Pakhira *et al.* [33]					✓	GP
Pakhira *et al.* [34]					✓	GP
Pakhira *et al.* [35]					✓	GP
Pakhira *et al.* [36]					✓	GP
Pakhira *et al.* [37]	✓				✓	GP
Pakhira *et al.* [38]					✓	GP
Rahaman *et al.* [39]		✓			✓	Search
Rahaman *et al.* [40]	✓	✓			✓	ABC
Jana and Das [41]					✓	GP
Rahaman *et al.* [42]	✓				✓	Search
Rahaman *et al.* [43]	✓				✓	Search
Singh and Jain [45]	✓				✓	PSO
Present article	✓	✓	✓	✓	✓	PSO, QNA

considerable research gap on the memory based studies for the production-inventory problems. The defectives in the manufacturing process must be addressed to formulate fractional order production-inventory problems. A comparison of the relevant studies is summarized in Table 1.

2.2. Contributions

Motivated by the research in the area of memory based inventory related problems, the following key features have been studied in the present article:

- An imperfect production process is considered, where the produced units may be faulty. A constant rate of defective production is assumed.
- The revenue loss due to defective units is managed by the rework process of these defectives.

– A fractional order initial value problem is formulated to investigate the imperfect EPQ model. The past inventory information is utilized by implementing Caputo fractional order derivative approach.
– A price-based demand function is considered to obtain optimal pricing decisions.
– An optimization problem is formulated by using an integral memory index (β). The optimal decision parameters and respective optimum profit are obtained by PSO and quasi-Newton algorithm (QNA).

3. Preliminaries of FC

Some important functions and definitions that will help us to solve fractional differential equations involved in the concerned inventory model are described in this section. A detailed introduction to FC can be seen in Miller [1] and Kilbas [5].

3.1. Definitions and theorems

Definition 3.1 (RL fractional integrals)**.** Consider a finite interval $[a,b]$ on the real line $\mathbb{R}$. The RL left-sided $I^{\alpha}_{a+}g$, and right-sided $I^{\alpha}_{b-}g$ fractional integrals of the function $g(x)$ of order $\alpha \in \mathbb{C}(\Re(\alpha) > 0)$ are respectively, defined by

$$\left(I^{\alpha}_{a+}f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)\,\mathrm{d}t}{(x-t)^{1-\alpha}}, \qquad (x > a; \Re(\alpha) > 0) \tag{1}$$

and

$$\left(I^{\alpha}_{b-}f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{f(t)\,\mathrm{d}t}{(x-t)^{1-\alpha}}, \qquad (x < b; \Re(\alpha) > 0). \tag{2}$$

where $[\Re(\alpha)]$ means the integral part of $\Re(\alpha)$.

Definition 3.2 (RL fractional derivatives)**.** Let $[a,b]$ be a finite interval on the real axis $\mathbb{R}$. The RL fractional derivatives $D^{\alpha}_{a+}\nu(\mu)$ and $D^{\alpha}_{b-}\nu(\mu)$ of order $\alpha \in \mathbb{C}(\Re(\alpha) > 0)$ are defined by

$$\begin{aligned}\left(D^{\alpha}_{a+}\nu\right)(\mu) &= \left(\frac{\mathrm{d}}{\mathrm{d}\mu}\right)^n \left(I^{n-\alpha}_{a+}\nu\right)(\mu)\\ &= \frac{1}{\Gamma(n-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}\mu}\right)^n \int_a^{\mu} \frac{\nu(\tau)\,\mathrm{d}\tau}{(\mu-\tau)^{\alpha-n+1}}, \qquad (n = [\Re(\alpha)] + 1; \mu > a)\end{aligned} \tag{3}$$

and

$$\begin{aligned}\left(D^{\alpha}_{b-}\nu\right)(\mu) &= \left(-\frac{\mathrm{d}}{\mathrm{d}\mu}\right)^n \left(I^{n-\alpha}_{b-}\nu\right)(\mu)\\ &= \frac{1}{\Gamma(n-\alpha)}\left(-\frac{\mathrm{d}}{\mathrm{d}\mu}\right)^n \int_{\mu}^{b} \frac{\nu(\tau)\,\mathrm{d}\tau}{(\mu-\tau)^{\alpha-n+1}}, \qquad (n = [\Re(\alpha)] + 1; \mu < b),\end{aligned} \tag{4}$$

Definition 3.3 (Caputo fractional derivatives)**.** Let $[a,b]$ be a finite interval on real axis $\mathbb{R}$. The Caputo fractional derivatives $({}^{C}D^{\alpha}_{a+}y)(x)$ and $({}^{C}D^{\alpha}_{b-}y)(x)$ of order $\alpha \in \mathbb{C}(\Re(\alpha) > 0)$ are defined by using the RL derivatives as:

$$\left({}^{C}D^{\alpha}_{a+}y\right)(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{y^{(n)}(t)\,\mathrm{d}t}{(x-t)^{\alpha-n+1}} = (I^{n-\alpha}_{a+}D^n y)(x) \tag{5}$$

and

$$\left({}^{C}D^{\alpha}_{b-}y\right)(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b \frac{y^{(n)}(t)\,\mathrm{d}t}{(t-x)^{\alpha-n+1}} = (-1)^n(I^{n-\alpha}_{b-}D^n y)(x), \tag{6}$$

where $n = [\Re(\alpha)] + 1$ for $\alpha \notin \mathbb{N}_0$; $n = \alpha$ for $\alpha \in \mathbb{N}_0$.

Definition 3.4 (Mittag–Leffler function)**.** Mittag–Leffler (ML) function is defined as

$$E_{\alpha,\beta}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + \beta)} \tag{7}$$

where $z \in \mathbb{C}(\Re(\alpha) > 0, \Re(\beta) > 0)$. In case when $\beta = 1$, it is abbreviated as $E_\alpha(z)$.

Theorem 3.1 (Cauchy problem)**.** *Consider the fractional initial value problem (FIVP) of order $\alpha(0 < \alpha \leq 1)$*

$$\left({}^C D^\alpha_{a+} y\right)(x) - \lambda y(x) = f(x), y(a) = b, (b \in \mathbb{R}) \tag{8}$$

where $\lambda \in \mathbb{R}$.

Proof. The solution of the above FIVP (8) can be obtained by using ML functions as [1]:

$$y(x) = bE_\alpha[\lambda(x-a)^\alpha] + \int_a^x (x-t)^{\alpha-1} E_{\alpha,\alpha}[\lambda(x-t)^\alpha] f(t)\,\mathrm{d}t. \tag{9}$$

□

3.2. Conversion in differ-integral form

An ordinary differential equation in interval $[0, x_1]$ of the form

$$\frac{\mathrm{d}(y(x))}{\mathrm{d}x} = -f(x) \tag{10}$$

can be transformed with memory integral equation as:

$$\frac{\mathrm{d}(y(x))}{\mathrm{d}x} = -\int_0^x k(x-t) f(t)\,\mathrm{d}t \tag{11}$$

where $k(x-t)$ is a kernel function. The kernel is equal to the dirac delta function $\delta(x-t)$ in case of a Markov process. To incorporate memory in a dynamical system, we consider $k(x-t) = \frac{1}{\Gamma(1-\alpha)}(x-t)^{\alpha-2}$, where $\Gamma(.)$ denotes the gamma function [1] and $0 < \alpha \leq 1$. Then, equation (11) can be rewritten as

$$\frac{\mathrm{d}y(x)}{\mathrm{d}x} = -{}_0D_x^{-(\alpha-1)}(f(x)). \tag{12}$$

Now, if left RL derivative of order $(\alpha - 1)$ is applied on equation (12), we have:

$$\left(D^\alpha_{a+} y\right)(x) = -f(x), \qquad 0 < \alpha \leq 1, \quad 0 \leq x \leq x_1. \tag{13}$$

Here, left RL derivative and RL fractional integral are considered as inverse operators.

4. Model description

The present study is concerned with memory-based inventory model for a manufacturing system to solve production and lot-sizing issues of the firm. The firm observes a price-driven market demand for such products. A price-sensitive demand function is considered to depict the real-time market scenario. A fraction of the produced units cannot be handed over to the customers due to the quality concern. An estimation of the defect rate is made using the past data for the defectives generated during the production process. The firm also aims to minimize the loss due to defectives, which are to be handled using rework. The firm's past information about stock depletion can be used to enhance the accuracy of the production and inventory decisions. The optimal production-inventory decisions are to be made by formulating a suitable profit function that includes sales revenue, cost associated with production, rework, and holding the produced units in the warehouse. The pictorial overview of the proposed problem is shown in Figure 1.

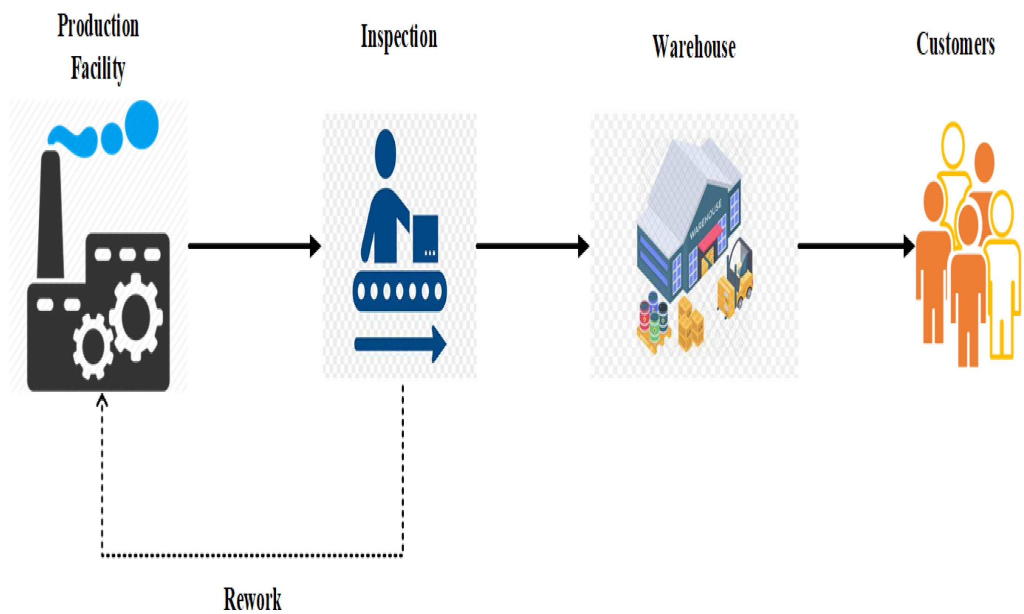


FIGURE 1. Graphical representation of the problem.

4.1. Notations

The notations that are used for the formulation of the mathematical model are as follows:

P	Items manufactured per unit time (units/week)
P_d	Rate at which defective units are produced (units/week)
r	Rate of reworking of defective items (units/week)
S	Selling price of an item ($/unit)
$D(S)$	Demand of an item per unit time (units/week)
$I(t)$	Inventory level at time t (units)
S_c	Setup cost per order ($/setup)
C_p	Production cost per unit ($/unit)
h_p	Holding cost for each perfect item ($/unit/week)
h_d	Holding cost for each imperfect unit ($/unit/week)
C_r	Repair cost per unit ($/unit)
t_1	Production time (week)
α	Differential memory index
β	Integral memory index
$Pbest_k^i$	Personal best position of the i^{th} particle in the k^{th} generation [46]
$Gbest_k$	Global best position in the k^{th} generation
ϕ_1, ϕ_2	Acceleration factors for velocity update equation
r_1, r_2	Random numbers in the interval $(0, 1)$

4.2. Assumptions

The following assumptions are made to formulate the inventory control model [13]:

(i) The planning horizon is infinite.
(ii) Demand ($D(S)$) is a positive, non-increasing function of the selling price.
(iii) Production rate (P) and production rate of defective items (P_d) are constant. Also, ($P - P_d > D$) (see Eq. (28b)).
(iv) The past information on the stock level is modeled by making use of fractional order derivatives. The Caputo derivative and Riemann–Liouville fractional integrals are incorporated to develop the mathematical model with differential memory α and integral memory β [45].
(v) The production process is not completely reliable. During the production process, a proportion of defective units generates along with the perfect units.
(vi) The produced units are screened along with the production process. The screening cost per unit item is included in the production cost (C_P).
(vii) The defective units are repaired at the rate r ($r > D$) (see Eq. (28a)) after the production time. No faulty unit is generated during the rework process.
(viii) No shortage is considered.

5. The analysis

5.1. Stock level for perfect items

Figure 2 depicts the nature of stock level with time. The production process initiates at time $t = 0$ and stops at time t_1. Some defective units are also generated during the production with rate P_d. The imperfect units are repaired at the rate r during the time interval $t_1 \leq t \leq t_2$. After time $t = t_2$, stock depletes due to the demand only. The stock level ceases to zero at time epoch $t = t_3$.

By using (13), the stock level for perfect items with memory α ($0 < \alpha \leq 1$) is governed by the following fractional initial value problems (FIVPs):

$$ {}_0^C D_t^\alpha (I_1(t)) = P - P_d - D, \qquad 0 \leq t \leq t_1 \tag{14} $$

subject to the initial condition $I_1(0) = 0$.

$$ {}_{t_1}^C D_t^\alpha (I_2(t)) = r - D, \qquad t_1 \leq t \leq t_2 \tag{15} $$

subject to the condition $I_2(t_1) = I_1(t_1)$.

$$ {}_{t_2}^C D_t^\alpha (I_3(t)) = -D, \qquad t_2 \leq t \leq t_3 \tag{16} $$

subject to the condition $I_3(t_3) = 0$.

Solving equation (14) and using Theorem 3.1, we get

$$ I_1(t) = \frac{(P - P_d - D)}{\Gamma(\alpha + 1)} t^\alpha. \tag{17} $$

From equation (17), we have

$$ I_1(t_1) = \frac{(P - P_d - D)}{\Gamma(\alpha + 1)} t_1^\alpha. \tag{18} $$

Using Theorem 3.1, equation (15) gives

$$ I_2(t) = \frac{(P - P_d - D)}{\Gamma(\alpha + 1)} t_1^\alpha + \frac{(r - D)}{\Gamma(\alpha + 1)} (t - t_1)^\alpha. \tag{19} $$

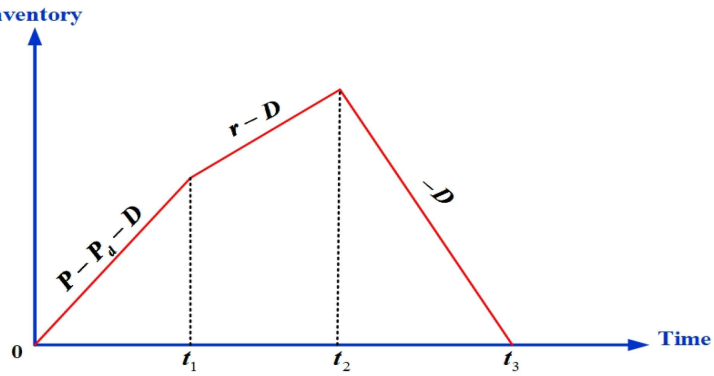


Figure 2. Inventory behavior for perfect items with rework.

From equation (19), we have

$$I_2(t_2) = \frac{(P - P_d - D)}{\Gamma(\alpha+1)} t_1^{\alpha} + \frac{(r - D)}{\Gamma(\alpha+1)} (t_2 - t_1)^{\alpha}. \tag{20}$$

Using Theorem 3.1, equation (16) yields

$$I_3(t) = \frac{(P - P_d - D)}{\Gamma(\alpha+1)} t_1^{\alpha} + \frac{(r - D)}{\Gamma(\alpha+1)} (t_2 - t_1)^{\alpha} - \frac{D}{\Gamma(\alpha+1)} (t - t_2)^{\alpha}. \tag{21}$$

5.2. Stock level for defective items

The stock level for defective units is shown in Figure 3. The defective units increase until the production time t_1, and then the defective units convert into perfect items by rework. By using (13), the stock level for the defective items with memory α $(0 < \alpha \leq 1)$ is framed as

$$ {}_{0}^{C}D_t^{\alpha}(I_{D1}(t)) = P_d, \qquad 0 \leq t \leq t_1 \tag{22}$$

subject to the initial condition $I_{D1}(0) = 0$.

Also,

$$ {}_{t_1}^{C}D_t^{\alpha}(I_{D2}(t)) = -r, \qquad t_1 \leq t \leq t_2 \tag{23}$$

subject to the condition $I_{D2}(t_1) = I_{D1}(t_1), \quad I_{D2}(t_2) = 0$.

Using Theorem 3.1, equation (22) yields

$$I_{D1}(t) = \frac{P_d}{\Gamma(\alpha+1)} t^{\alpha}. \tag{24}$$

Using Theorem 3.1, equation (23) gives

$$I_{D2}(t) = \frac{P_d}{\Gamma(\alpha+1)} t_1^{\alpha} - \frac{r}{\Gamma(\alpha+1)} (t - t_1)^{\alpha}. \tag{25}$$

Using $I_{D2}(t_2) = 0$ and $I_3(t_3) = 0$, we get

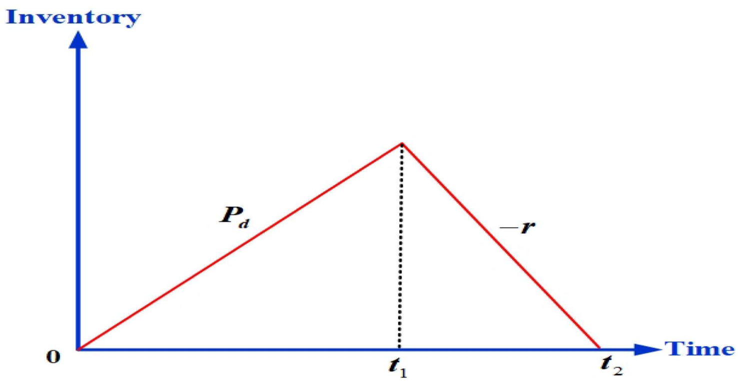


FIGURE 3. Inventory behavior for imperfect items.

$$t_2 = t_1\left\{1 + \left(\frac{P_d}{r}\right)^{\frac{1}{\alpha}}\right\} \tag{26}$$

$$t_3 = t_1\left\{\left(\frac{P}{D} - 1 - \frac{P_d}{r}\right)^{\frac{1}{\alpha}} + \left(1 + \left(\frac{P_d}{r}\right)^{\frac{1}{\alpha}}\right)\right\}. \tag{27}$$

It is noted that

$$r > D \implies \frac{P}{D} - 1 - \frac{P_d}{r} > \frac{P}{D} - 1 - \frac{P_d}{D}. \tag{28a}$$

Also,

$$P - P_d > D \implies \frac{P - P_d}{D} - 1 > 0 \implies \frac{P}{D} - 1 - \frac{P_d}{D} > 0. \tag{28b}$$

Using (28a) and (28b), we have

$$\frac{P}{D} - 1 - \frac{P_d}{r} > 0$$

which proves the validity of equation (27) and hence $t_3 > t_2$.

5.3. Inventory cost and average profit

To analyze the profitability and cost distribution of the production-inventory process, some elementary cost elements are to be computed. Such essential costs are the expenses associated with the production of a unit, inventory costs related to holding the produced units, and rework of the defective items. The collective revenue is calculated by selling the perfect units following the market demand. In this article, we have used Caputo fractional derivative to obtain the stock level by utilizing memory coefficient (α). Since the stock levels provided in equations (17), (19), (21), (24) and (25) have time and memory dependency, the average holding cost is derived by using memory-based Riemann–Liouville integral with memory coefficient (β). The RL integrals are used to enhance the cost-effectiveness of the proposed optimization problem. The sale revenue and inventory costs are computed as follows:

– **Sales Revenue**
  The total sales revenue by selling the perfect items is given as

$$\mathrm{SR} = S\int_0^{t_3} D\,\mathrm{d}t = SDt_3. \tag{29}$$

– **Holding Cost**

The holding costs for perfect and defective units in the production lot size are determined as follows:

- The β^{th} order holding cost for perfect items ($\mathrm{HC}_{\alpha,\beta}$) is defined as

$$\mathrm{HC}_{\alpha,\beta} = {}_0^C D_{t_3}^{-\beta} h_p I(t)\,\mathrm{d}t.$$

Thus,

$$\begin{aligned}\mathrm{HC}_{\alpha,\beta} = h_p\Bigg(&\frac{(P-P_d-D)}{\Gamma(\beta)\Gamma(\alpha+1)} t_3^{\alpha+\beta} B\left(\frac{t_1}{t_3},\alpha+1,\beta\right) + \frac{(P-P_d-D)}{\Gamma(\alpha+1)\Gamma(\beta+1)} t_1^{\alpha}(t_3-t_1)^{\beta}\\ &+ \frac{(r-D)}{\Gamma(\beta)\Gamma(\alpha+1)}(t_3-t_1)^{\alpha+\beta} B\left(\frac{t_2-t_1}{t_3-t_1},\alpha+1,\beta\right)\\ &+ \frac{(r-D)}{\Gamma(\alpha+1)\Gamma(\beta+1)}(t_2-t_1)^{\alpha}(t_3-t_2)^{\beta} - \frac{D}{\Gamma(\alpha+\beta+1)}(t_3-t_2)^{\alpha+\beta}\Bigg)\end{aligned} \tag{30}$$

where, $B(x,a,b)$ is the incomplete beta function [1].

- The β^{th} order holding cost for defective items ($\mathrm{HCD}_{\alpha,\beta}$) is defined as

$$\mathrm{HCD}_{\alpha,\beta} = {}_0^C D_{t_2}^{-\beta} h_d I_D(t)\,\mathrm{d}t.$$

Thus,

$$\begin{aligned}\mathrm{HCD}_{\alpha,\beta} = h_d\Bigg(&\frac{P_d}{\Gamma(\alpha+1)\Gamma(\beta)} t_2^{\alpha+\beta} B\left(\frac{t_1}{t_2},\alpha+1,\beta\right) + \frac{P_d}{\Gamma(\alpha+1)\Gamma(\beta+1)} t_1^{\alpha}(t_2-t_1)^{\beta}\\ &- \frac{r}{\Gamma(\alpha+\beta+1)}(t_2-t_1)^{\alpha+\beta}\Bigg).\end{aligned} \tag{31}$$

– **Production Cost**

The cost of producing the units in the production lot is

$$\mathrm{PC} = C_p \int_0^{t_1} P\,\mathrm{d}t = C_p P t_1. \tag{32}$$

– **Rework Cost**

The cost associated with reworking the defective units produced during the manufacturing process is

$$\mathrm{RC} = C_r \int_{t_1}^{t_2} r\,\mathrm{d}t = C_r r(t_2-t_1). \tag{33}$$

– **Average Profit**

The average profit for the complete planning cycle is obtained as

$$\Pi(t_1,S) = \frac{1}{t_3}\{\mathrm{SR} - (S_c + \mathrm{HC}_{\alpha,\beta} + \mathrm{HCD}_{\alpha,\beta} + \mathrm{PC} + \mathrm{RC})\}. \tag{34}$$

The proposed optimization problem can be stated as follows:

$$\text{Max } \Pi(t_1,S)$$
$$\text{subject to: } t_1 > 0 \text{ and } S > C_p.$$

6. Numerical analysis

The production-inventory control model developed in the previous section may be appropriate for accessories such as smartphone cases and covers. These products have a huge market demand and get outdated with time. Also, the customers intend to buy such products at considerably low prices. The past production and inventory information may be utilized to determine optimal inventory and pricing decisions. Some test data for costs and parameter values have been adopted by studying the relevant inventory literature [13].

The default parameters for numerical simulation of the proposed memory-based inventory model are set as follows:
$P = 8000$ units per week, $P_d = 150$ units per week, $r = 10000$ units per week, $S_c = \$100$ per setup, $h_d = \$2$ per unit, $h_p = \$1.2$ per unit, $C_p = \$12$ per unit, $C_r = \$3$ per unit.

For numerical analysis, the following two illustrations are considered based on linear and power demand functions:

Illustration 1 (NI_1)

$D(S) = a - bS$, where $a = 5000$ and $b = 100$.

Illustration 2 (NI_2)

$D(S) = aS^{-b}$, where $a = 20000$ and $b = 1.5$.

6.1. Optimization

Since the proposed average profit function is a highly non-linear function due to the presence of fractional terms and incomplete beta function, it is difficult to solve the optimization issue by using classical approaches. In the memory-based inventory studies, the researchers have used GP approach [37] and metaheuristic optimization techniques [45]. In the proposed article, the particle swarm optimization approach is employed for the numerical solution of the non-linear optimization problem. An algorithm to implement the PSO is provided in the Appendix A. The following input parameters based on various experiments are used to employ the PSO metaheuristic algorithm:

$$\phi_1 = 2, \quad \phi_2 = 2, \quad PopSize = 50 \quad \text{and } MaxIt = 100.$$

A quasi-Newton algorithm [51] based on Broyden–Fletcher–Goldfarb–Shanno (BFGS) approach is also used to optimize the proposed average profit function. A pseudo-algorithm for the BFGS method is provided in the Appendix B. The optimal decisions are obtained by using MATLAB (R2023b). For the efficient execution of the BFGS algorithm, the maximum number of iterations ($MaxIt$) and tolerance error (Tol) to stop the optimization process are taken as $MaxIt = 1000$ and $Tol = 10^{-5}$, respectively.

The optimal solutions for the proposed production-inventory problem are obtained by both PSO and BFGS algorithms. It is observed that the optimal decisions are same by both algorithms. The optimal profit obtained by both the algorithms are depicted in Figure 4. Optimal policies for both the illustrations for a set of memory indices are provided in Table 2. The convergence dynamics of the PSO for the illustrations are also depicted in Figure 5. It can be observed that the PSO achieves convergence within 50 iterations for all the cases implemented. For the convergence purpose, normalized profit value is used to scale it in all the cases. Table 3 presents the optimal production time (t_1^*), optimal selling price (S^*), and respective optimal average profit (Π^*) for illustration 1 (NI_1) *via* different values of memory indices α and β. We have the following observations from Table 3:

- The proposed model reduces to the classic production-inventory model with rework when we fix $\alpha = 1.0$ and $\beta = 1.0$. The optimal decision parameters for the classic model (NI_1) are obtained as:
  $t_1^* = 0.082$ week (13.77 h), $S^* = \$31.08$, $\Pi^* = \$35402.97$.

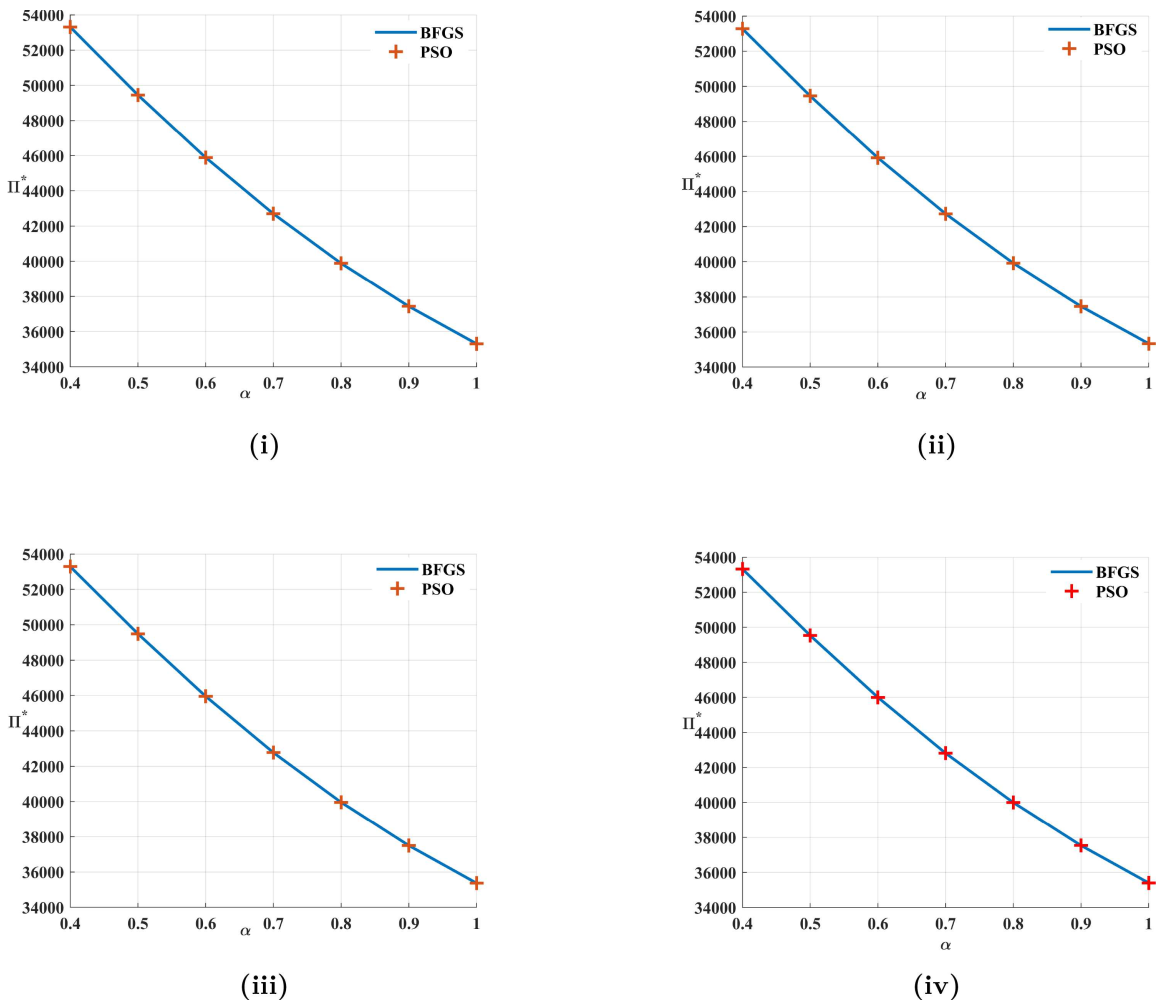


Figure 4. Comparison of the PSO and BFGS for illustration 1: (i) $\beta = 0.7$, (ii) $\beta = 0.8$, (iii) $\beta = 0.9$, (iv) $\beta = 1$.

- For $\beta = 1, 0.9$, and 0.8, the optimal production time (t_1^*) increases with the enhancement in α but in case of $\beta = 0.7$, the value of (t_1^*) first decreases and then increases with respect to the value of α. It can be concluded that the behavior of t_1^* changes significantly with an increment in the value of β after a certain threshold level.
- The value of optimal selling price (S^*) remains almost invariable with the change in β keeping α as constant. The value of S^* increases with the increment in the value of α for a wide range. The changes in S^* are insignificant, with a slight change in the value α.
- For a fixed value of β, there is a rise in the value of optimal average profit (Π^*) with the decrement in the value of α. The influence of β on Π^* is less significant than the impact of α on the optimal average profit.

Based on numerical results summarized in Table 2, we can conclude that the classical memoryless case $(\alpha = 1, \beta = 1)$ is less profitable than the case when memory is accounted. The profit value rises with the decrement in the value of α. In conclusion, the memory effect influences the optimal average profit value and is useful for decision-makers to obtain the optimal decisions.

TABLE 2. Optimal policies for the illustrations.

(α, β)	NI$_1$			NI$_2$		
	t_1^* (week)	S^* (\$)	Π^* (\$)	t_1^* (week)	S^* (\$)	Π^* (\$)
$(1, 1)$	0.0802	31.08	35 402.97	0.0148	38.68	2072.92
$(1, 0.8)$	0.0811	31.10	35 337.08	0.0178	38.77	2082.47
$(0.9, 1)$	0.0695	31.30	37 537.05	0.0129	27.92	2562.36
$(0.8, 0.9)$	0.0604	31.44	39 956.20	0.0122	20.72	3123.54
$(0.7, 1)$	0.0502	31.41	42 817.76	0.0095	15.61	3752.13
$(0.8, 0.7)$	0.0680	31.46	39 884.50	0.0164	20.77	3141.23

TABLE 3. Optimal decisions for different memory indices.

α	$\beta = 1$			$\beta = 0.9$		
	t_1^* (week)	S^* (\$)	Π^* (\$)	t_1^* (week)	S^* (\$)	Π^* (\$)
1	0.0802	31.08	35 403.00	0.0800	31.09	35 369.09
0.9	0.0696	31.30	37 537.06	0.0699	31.31	37 498.48
0.8	0.0596	31.43	39 999.37	0.0605	31.44	39 956.20
0.7	0.0502	31.41	42 817.76	0.0519	31.42	42 770.78
0.6	0.0416	31.19	46 003.69	0.0441	31.20	45 954.89
0.5	0.0337	30.69	49 535.81	0.0372	30.71	49 489.40
0.4	0.0264	29.86	53 326.21	0.0313	29.87	53 290.04
	$\beta = 0.8$			$\beta = 0.7$		
1	0.0811	31.10	35 337.08	0.0840	31.11	35 309.87
0.9	0.0716	31.32	37 462.26	0.0754	31.33	37 432.11
0.8	0.0630	31.45	39 916.30	0.0680	31.47	39 884.51
0.7	0.0554	31.44	42 728.73	0.0623	31.45	42 698.19
0.6	0.0490	31.22	45 914.09	0.0590	31.23	45 890.64
0.5	0.0443	30.73	49 456.49	0.0610	30.74	49 451.24
0.4	0.0426	29.89	53 277.08	0.0812	29.89	53 310.74

The trend of the average profit function with the decision variables is depicted in Figure 6. The changes in Π are analyzed for different combinations of (α, β). The surface plots suggest the concavity of the average profit function, which ensures that the decisions obtained by the BFGS and PSO algorithm are global optimal solutions.

6.2. Special cases

In our study, an EPQ model is investigated by considering imperfect production, rework and memory effect. This study can be deduced to some existing inventory control studies as special cases by setting appropriate parameter values as follows:

(i) Sivashankari and Panayappan [9] studied a production-inventory model for imperfect items. In particular case when parameters $D(s) = D$, $\alpha = 1$, $\beta = 1$, our study deduces to the model studied by Sivashankari and Panayappan [9].

(ii) Imperfect production inventory models under price-sensitive demand rates were investigated by Pal and Adhikari [13] & Sanjai and Periyasamy [17]. These studies did not account for the memory effects, while our study used fractional order derivatives to incorporate the memory effects. By setting $\alpha = 1$, $\beta = 1$, our model provides results of the inventory model investigated by Pal and Adhikari [13] & Sanjai and Periyasamy [17].

(iii) Pakhira *et al.* [37] proposed a memory-based inventory control model under price-sensitive demand. By relaxing production related parameter values, *i.e.*, $P = 0$, $P_d = 0$, $r = 0$, our model reduces to the inventory model studied by Pakhira *et al.* [37].

(iv) Rahaman *et al.* [39] investigated a generalized EPQ model of arbitrary order. The authors did not consider defectives in the production system. By setting $D(s) = D$, $P_d = 0$, $r = 0$, our study deduces to a production-inventory model with memory effect proposed by Rahaman *et al.* [39].

6.3. Sensitivity analysis

Sensitivity analysis of optimal decision parameters t_1^* and S^* along with the corresponding profit Π^* for illustration NI_1 is conducted. The various parameter values are varied by 25% for the sensitivity analysis, and the memory indices α and β are fixed at 0.5 and 0.7, respectively. The resultant effects on the values of t_1^*, S^*, and Π^* are recorded in Table 4. The following observations from Table 4 are worth noting:

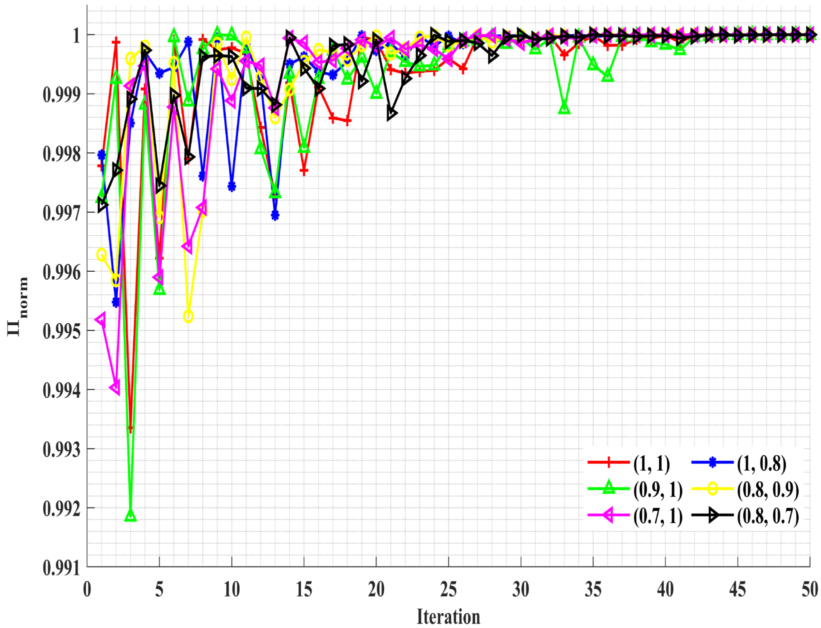
- t_1^* and S^* are highly sensitive to the production rate (P). An increment in P decreases t_1^* and S^* while the optimal average profit increases with P. Due to the high production rate, more units are produced in less time, and the higher production yield leads to higher profits.
- The changes in defective generation rate (P_d) and rework rate (r) do not significantly influence the optimal decisions. The impact of (P_d) and (r) are counterbalanced by the production time and selling price so that the resultant effect is insignificant.
- The demand rate parameters, *viz.*, a and b have a great impact on the optimal decisions. The demand plays an important part while making inventory decisions due to the fact that more demand results in more sales and, hence, more profit. The decision-makers should pay great attention to estimate the future demand more accurately.
- The holding cost for perfect items (h_p) significantly affects the optimal descriptors. The optimal production time (t_1^*) decreases whereas S^* increases with an increment in h_p. The optimal average profit (Π^*) decreases by increasing the holding cost (h_p). The holding cost for defectives (h_d) has comparatively less effect on the optimal decisions. The optimal inventory decision parameters are obtained in such a way that the effect of holding cost can be reduced.
- The optimal policies seem to be unaffected by the rework cost (C_r). The production cost (C_p) affects the optimal decision variables t_1^* and Π^* which decrease as C_p increases. It is seen that S^* increases with an enhancement in C_p.

From the above discussion, some remarkable observations are noted. The changes in rework cost (C_r), holding cost for defectives (h_d), defective generation rate (P_d) and rework rate (r) have an insignificant impact on the optimal decisions. The firm can easily estimate optimal decisions for a slight change in these parameters. The profit function is highly influenced by the demand function, which is a practical scenario. A high demand results in higher profit and *vice versa*.

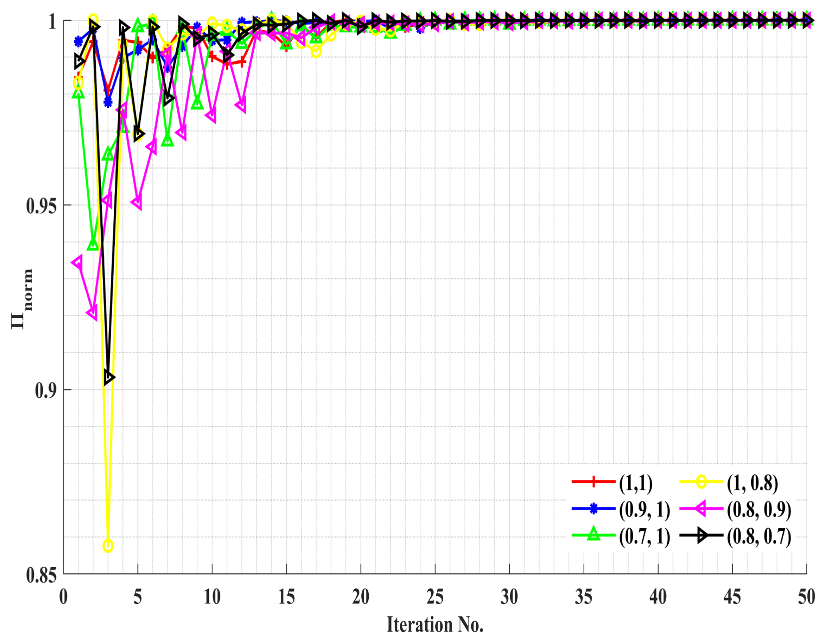
6.4. Managerial insights

The proposed memory-based production and inventory policies may be advantageous for the concerned firm's business decisions. The numerical and sensitivity results obtained offer useful information for the decision-makers of the business firm. The following significant observations are noteworthy:

- Due to the memory consideration, the optimum profit value is affected. Thus, memory is one of the crucial factors that should be involved in inventory planning.
- The production rate (P) and production cost (C_p) have significant influence on the decisions of the firm. The decision-makers should carefully consider these parameters while designing the concerned inventory system.
- The demand rate parameters also significantly affect the optimal policies. A suitable statistical analysis can be performed to estimate the demand rate and its relationship with the selling price.
- The optimal pricing policy may be helpful in reducing the production cost and increasing the firm's profitability.



(i)



(ii)

FIGURE 5. Convergence of the PSO for (i) NI_1 and (ii) NI_2.

– Defective production is an inevitable aspect of any manufacturing system. An effective rework can convert the defectives into good quality products. Rework may also be adopted to enhance the goodwill of the customers which will enhance the revenue of the concerned firm.

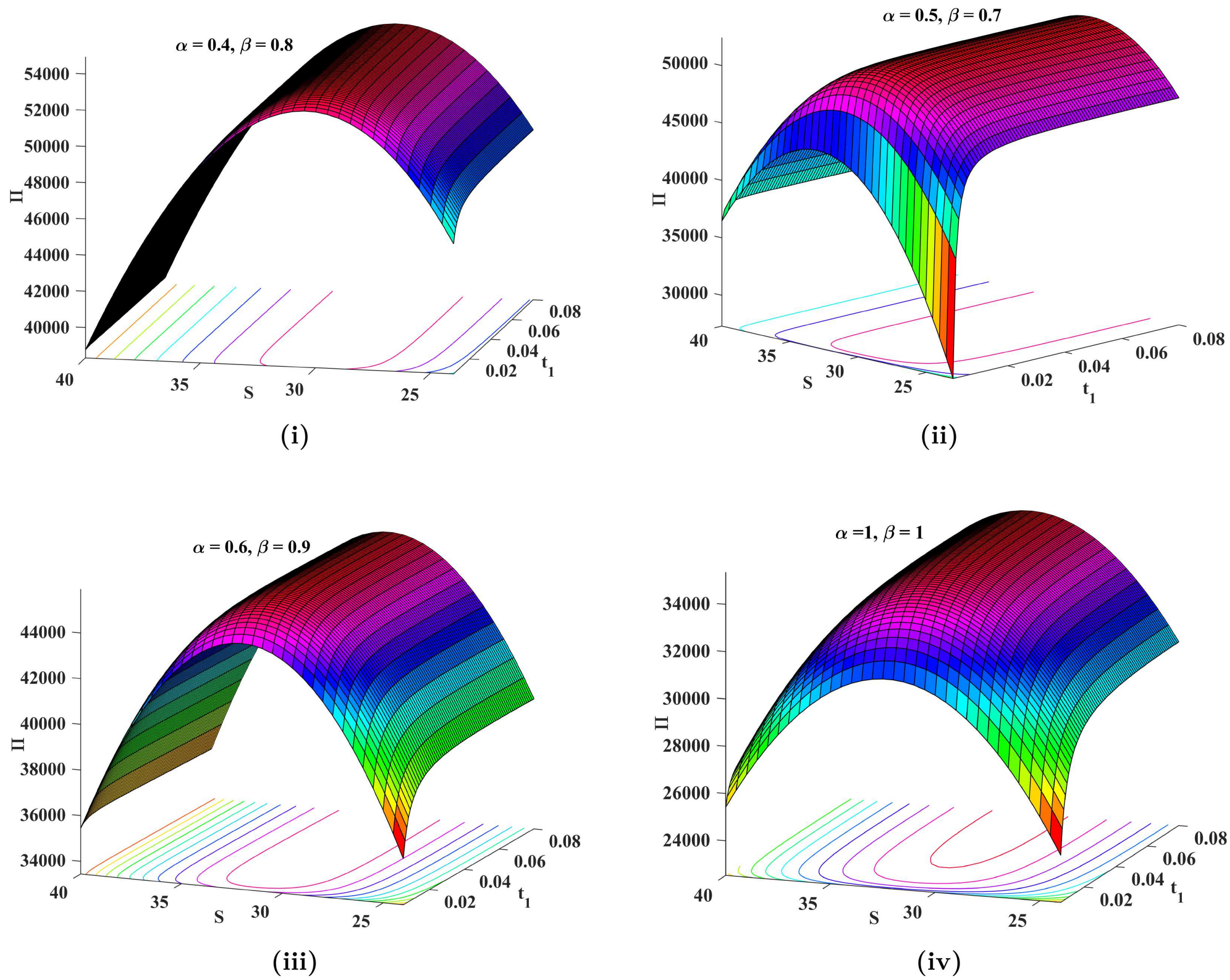


FIGURE 6. Average profit function for (i) $(\alpha, \beta) = (0.4, 0.8)$, (ii) $(\alpha, \beta) = (0.5, 0.7)$, (iii) $(\alpha, \beta) = (0.6, 0.9)$, (iv) $(\alpha, \beta) = (1, 1)$.

– An increment in P can generate high revenue, but enhancing the production rate beyond a certain threshold is not always feasible.

7. Conclusions

This article studies the effects of memory on an economic production quantity model under an imperfect manufacturing process. A fractional differential equation approach employed analyzes the memory effect by incorporating price-sensitive demand in the EPQ model. Caputo derivatives and Riemann–Liouville integrals are used to develop the production inventory problem. An imperfect production-inventory model with memory effect has been investigated for the first time. The feature of the rework process considered is more realistic and will be helpful to cope up with the imperfect manufacturing process. The rework of faulty units will enhance the profit of the concerned company. A selling price-sensitive demand included in our study makes the proposed model to deal with a real-life production-inventory problem.

TABLE 4. Sensitivity analysis of inventory parameters $(\alpha, \beta) = (0.5, 0.7)$.

Parameter	% Change	Optimal solution			% Change in Π^*
		t_1^* (week)	S^* ($)	Π^*($)	
P	−50	0.1992	35.23	38 841.65	−21.4629
	−25	0.1027	32.53	45 471.43	−8.0576
	0	**0.0610**	**30.74**	**49 451.24**	**0.0000**
	25	0.0401	29.54	52 002.52	5.1482
	50	0.0280	28.72	53 724.99	8.6310
P_d	−50	0.0616	30.71	49 501.08	0.0903
	−25	0.0615	30.72	49 478.78	0.0452
	0	**0.0610**	**30.74**	**49 451.24**	**0.0000**
	25	0.0612	30.75	49 433.97	−0.0454
	50	0.0611	30.76	49 411.46	−0.0909
r	−50	0.0616	30.79	49 378.54	−0.1575
	−25	0.0614	30.75	49 430.45	−0.0525
	0	**0.0610**	**30.74**	**49 451.24**	**0.0000**
	25	0.0613	30.73	49 471.99	0.0315
	50	0.0613	30.72	49 482.38	0.0525
a	−50	0.0242	14.91	12 567.60	−74.5885
	−25	0.0416	22.71	28 150.36	−43.0804
	0	**0.0610**	**30.74**	**49 451.24**	**0.0000**
	25	0.0828	38.90	76 253.95	54.1841
	50	0.1058	47.05	108 461.97	119.3082
b	−50	0.0728	56.76	110 018.41	122.4553
	−25	0.0665	39.54	69 400.32	40.3262
	0	**0.0610**	**30.74**	**49 451.24**	**0.0000**
	25	0.0570	25.34	37 722.17	−23.7264
	50	0.0532	21.67	30 056.52	−39.2262
C_p	−50	0.0723	28.47	54 507.72	10.2137
	−25	0.0663	29.69	51 813.12	4.7652
	0	**0.0610**	**30.74**	**49 451.24**	**0.0000**
	25	0.0408	31.69	47 027.62	−4.9110
	50	0.0382	32.49	45 170.96	−8.6651
h_p	−50	0.1082	30.67	49 844.93	0.7856
	−25	0.0777	30.71	49 645.16	0.3817
	0	**0.0610**	**30.74**	**49 451.24**	**0.0000**
	25	0.0510	30.77	49 275.56	−0.3657
	50	0.0438	30.79	49 100.82	−0.7190
h_d	−50	0.0618	30.73	49 463.28	0.0139
	−25	0.0616	30.73	49 459.84	0.0069
	0	**0.0610**	**30.74**	**49 451.24**	**0.0000**
	25	0.0611	30.74	49 452.98	−0.0069
	50	0.0609	30.74	49 449.56	−0.0139
C_r	−50	0.0613	30.74	49 456.72	0.0006
	−25	0.0613	30.74	49 456.57	0.0003
	0	**0.0610**	**30.74**	**49 451.24**	**0.0000**
	25	0.0613	30.74	49 456.25	−0.0003
	50	0.0613	30.74	49 456.10	−0.0006

The complexity and non-linearity of the proposed optimization problem make it difficult to solve by using the classical optimization approach. We have used a numerical optimization technique, *viz.*, quasi-Newton method and particle swarm optimization so as to determine the optimal decisions. The optimization algorithms suggested may handle non-linear optimization in a cost-effective manner and are easy to implement. The effects of memory indices on the proposed problem suggest that the consideration of memory indices will be beneficial to set the design parameters optimally, which can also increase the profitability of the firm. There are only a very few research contributions on the memory-based inventory control models in the literature. Further, most of the studies available in the literature deal with basic EOQ models with memory effect. The proposed model is the sole article that analyzes an imperfect production inventory problem by including the memory effect. To broaden insights into the memory effect on inventory control, fractional calculus can be used to investigate many complex inventory problems related to the supply chain or warehouse inventory control problems. Our study can be further enriched by including real-life scenarios like deterioration, memory-affected demand, and shortages. A statistical estimation approach to determine memory indices can also be used. In future work, a real-world case study may be attempted so as to examine the continuous time behavior of the concerned inventory model with imperfect production and price-sensitive demand.

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References

[1] K.S. Miller and B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations. John Wiley & Sons Inc., New York (1993).

[2] I. Tejado, E. Pérez and D. Valério, Fractional calculus in economic growth modelling of the group of seven. *Fract. Calculus Appl. Anal.* **22** (2019) 139–157.

[3] R. Pakhira, U. Ghosh and S. Sarkar, Study of memory effect between two memory dependent inventory models. *J. Fract. Calculus App.* **11** (2020) 203–218.

[4] S.G. Samko, A.A. Kilbas and O.I. Marichev, Fractional Integrals and Derivatives. Vol. 1. Gordon and Breach Science Publishers, Yverdon Yverdon-les-Bains, Switzerland (1993).

[5] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, Theory and Applications of Fractional Differential Equations. Vol. 204. Elsevier (2006).

[6] D. Singh and A. Verma, Inventory management in supply chain. *Mater. Today: Proc.* **5** (2018) 3867–3872.

[7] H. Inegbedion, S.C. Eze, A.J. Asaleye and A.I. Lawal, Inventory management and organisational efficiency. *J. Soc. Sci. Res.* **5** (2019) 756–763.

[8] S. Yamini and M.S. Gajanand, Inventory decision-making biases: A review and suggestions for future research. *Benchmarking Int. J.* **29** (2021) 1889–1912.

[9] C.K. Sivashankari and S. Panayappan, Production inventory model with reworking of imperfect production, scrap and shortages. *Int. J. Manage. Sci. Eng. Manage.* **9** (2014) 9–20.

[10] P.K. Ghosh and J.K. Dey, An imperfect production inventory model with remanufacturing of return and defective items. *Int. J. Comput. App.* **150** (2016) 06–13.

[11] U.K. Khedlekar and R.K. Tiwari, Imperfect production model for sensitive demand with shortage. *Reliab. Theory App.* **13** (51) (2018) 43–54.

[12] D. Yadav, S.R. Singh and Vandana, Integrated model with imperfect production process under the effect of learning. *Malaya J. Mat. (MJM)* **1** (2018) 90–101.

[13] B. Pal and S. Adhikari, Price-sensitive imperfect production inventory model with exponential partial backlogging. *Int. J. Syst. Sci. Oper. Logistics* **6** (2019) 27–41.

[14] D.M. Utama, I. Santoso, Y. Hendrawan and W.A.P. Dania, Integrated procurement-production inventory model in supply chain: A systematic review. *Oper. Res. Perspect.* **9** (2022) 100221.

[15] B. Sarkar, H. Seok, T.K. Jana and B.K. Dey, Is the system reliability profitable for retailing and consumer service of a dynamical system under cross-price elasticity of demand? *J. Retail. Consum. Serv.* **75** (2023) 103439.

[16] M. Al-Salamah, Economic production quantity in an imperfect manufacturing process with synchronous and asynchronous flexible rework rates. *Oper. Res. Perspect.* **6** (2019) 100103.

[17] M. Sanjai and S. Periyasamy, An inventory model for imperfect production system with rework and shortages. *Int. J. Oper. Res.* **34** (2019) 66–84.

[18] A. Khanna, P. Gautam, A. Hasan and C.K. Jaggi, Inventory and pricing decisions for an imperfect production system with quality inspection, rework and carbon-emissions. *Yugoslav J. Oper. Res.* **30** (2020) 339–360.

[19] B. Sarkar, B.K. Dey, S. Pareek and M. Sarkar, A single-stage cleaner production system with random defective rate and remanufacturing. *Comput. Ind. Eng.* **150** (2020) 106861.

[20] B.K. Dey, B. Sarkar and H. Seok, Cost-effective smart autonomation policy for a hybrid manufacturing–remanufacturing. *Comput. Ind. Eng.* **162** (2021) 107758.

[21] L.E. Cárdenas-Barrón, M.J.L. Plaza-Makowsky, M.A. Sevilla-Roca, J.M. Núñez-Baumert and B. Mandal, An inventory model for imperfect quality products with rework, distinct holding costs, and nonlinear demand dependent on price. *Mathematics* **9** (2021) 1362.

[22] P. Gautam, S. Maheshwari, A. Kausar and C.K. Jaggi, Inventory models for imperfect quality items: A two-decade review, in Advances in Interdisciplinary Research in Engineering and Business Management. Asset Analytics. Springer, Singapore (2021) 185–215.

[23] H.K. Alfares and A.M. Ghaithan, A generalized production-inventory model with variable production, demand, and cost rates. *Arab. J. Sci. Eng.* **47** (2022) 3963–3978.

[24] A. Salmasnia, Z. Hajihosseini and M.R. Maleki, An economic manufacturing quantity model with rework process for deteriorating products under maintenance-quality policy. *Int. J. Modell. Simul.* **42** (2022) 946–965.

[25] S.S. Sana, Optimum buffer stock during preventive maintenance in an imperfect production system. *Math. Methods Appl. Sci.* **45** (2022) 8928–8939.

[26] B. Sarkar, B.K. Dey, M. Sarkar and S.J. Kim, A smart production system with an autonomation technology and dual channel retailing. *Comput. Ind. Eng.* **173** (2022) 108607.

[27] N. Singh, M. Jain and P. Singh, Nature-inspired optimization for inventory model with imperfect production, in Data Analytics and Artificial Intelligence for Inventory and Supply Chain Management. Springer Nature Singapore (2022) 23–44.

[28] M. Jain, N. Sharma and P. Singh, Sustainable inventory prediction with random defect and rework using Bat algorithm. *RAIRO: Oper. Res.* **57** (2023) 481–501.

[29] M.A.A Khan, L.E. Cárdenas-Barrón, G. Treviño-Garza and A. Céspedes-Mota, Optimal circular economy index policy in a production system with carbon emissions. *Expert Syst. App.* **212** (2023) 118684.

[30] M. Sebatjane and O. Adetunji, A four-echelon supply chain inventory model for growing items with imperfect quality and errors in quality inspection. *Ann. Oper. Res.* **335** (2024) 327–359.

[31] B.K. Dey, H. Seok and K. Chung, Optimal decisions on greenness, carbon emission reductions, and flexibility for imperfect production with partial outsourcing. *Mathematics* **12** (2024) 654.

[32] R. Singh and V.K. Mishra, Inventory model using Machine Learning for demand forecast with imperfect deteriorating products and partial backlogging under carbon emissions. *Ann. Oper. Res.* **335** (2024) 361–383.

[33] R. Pakhira, U. Ghosh and S. Sarkar, Study of memory effects in an inventory model using fractional calculus. *Appl. Math. Sci.* **12** (2018) 797–824.

[34] R. Pakhira, U. Ghosh and S. Sarkar, Application of memory effects in an inventory model with linear demand and no shortage. *Int. J. Res. Advent. Technol.* **6** (2018) 1853–1871.

[35] R. Pakhira, U. Ghosh and S. Sarkar, Study of memory effect in an inventory model with linear demand and shortage. *Int. J. Math. Sci. Comput.* **5** (2019) 54–70.

[36] R. Pakhira, U. Ghosh and S. Sarkar, Study of memory effect in an inventory model with quadratic type demand rate and salvage value. *Appl. Math. Sci.* **13** (2019) 209–223.

[37] R. Pakhira, U. Ghosh and S. Sarkar, Study of memory effect in an inventory model with price dependent demand. *J. Appl. Econ. Sci.* **14** (2019) 360–367.

[38] R. Pakhira, S. Sarkar and U. Ghosh, Study of memory effect in an inventory model for deteriorating items with partial backlogging. *Comput. Ind. Eng.* **148** (2020) 106705.

[39] M. Rahaman, S.P. Mondal, A.A. Shaikh, P. Pramanik, S. Roy, M.K. Maiti, R. Mondal and D. De, Arbitrary-order economic production quantity model with and without deterioration: generalized point of view. *Adv. Differ. Equ.* **2020** (2020) 1–30.

[40] M. Rahaman, S.P. Mondal, A.A. Shaikh, A. Ahmadian, N. Senu and S. Salashour, Artificial bee colony optimization-inspired synergetic study of fractional-order economic production quantity model. *Soft Comput.* **24** (2020) 15341–15359.

[41] D.K. Jana and A.K. Das, A memory dependent partial back-logging inventory model for non instantaneous deteriorating item with stock dependent demand. *Int. J. Appl. Comput. Math.* **7** (2021) 199.

[42] M. Rahaman, S.P. Mondal and S. Alam, An estimation of effects of memory and learning experience on the EOQ model with price dependent demand. *RAIRO: Oper. Res.* **55** (2021) 2991–3020.

[43] M. Rahaman, R.M.S. Abdulaal, O.A. Bafail, M. Das, S. Alam and S.P. Mondal, An insight into the impacts of memory, selling price and displayed stock on a retailer's decision in an inventory management problem. *Fractal Fract.* **6** (2022) 531.

[44] A. Kumar, P.K. Santra and G.S. Mahapatra, Fractional order inventory system for time-dependent demand influenced by reliability and memory effect of promotional efforts. *Comput. Ind. Eng.* **179** (2023) 109191.

[45] P. Singh and M. Jain, Inventory policy for degrading items under advanced payment with price and memory sensitive demand using metaheuristic techniques. *Oper. Res. Int. J.* **24** (2024) 43.

[46] J. Kennedy and R. Eberhart, Particle swarm optimization, in Proceedings of ICNN'95 – 680 International Conference on Neural Networks. IEEE (1995) 1942–1948.

[47] N. Kumar, A.K. Manna, A.A. Shaikh and A.K. Bhunia, Application of hybrid binary tournament-based quantum-behaved particle swarm optimization on an imperfect production inventory problem. *Soft Comput.* **25** (2021) 11245–11267.

[48] N. Sharma, M. Jain and D. Sharma, Collaboration policy for two-echelon green industrial supply chain with carbon emissions using particle swarm optimization. *Nat. Acad. Sci. Lett.* (2023) 1–5. DOI: 10.1007/s40009-023-01368-y.

[49] M. Jain and P. Singh, Pricing, prepayment and preservation strategy for inventory model with deterioration using metaheuristic algorithms. *Soft Comput.* **28** (2024) 3415–3430.

[50] P. Singh and M. Jain, Metaheuristics for inventory control policies with random deterioration start, preservation investment and pre-payment. *Arab. J. Sci. Eng.* (2024). DOI: 10.1007/s13369-024-09444-1.

[51] S.S. Rao, Engineering Optimization: Theory and Practice, 4th edition. John Wiley and Sons, New-Jersey (2009).

Appendix A. Particle Swarm Optimization Algorithm

The notations used for particle swarm optimization (PSO) algorithm are provided in Section 4.1. The search procedure of the particle swarm optimization is as follows:

Step I: Input Parameters
Intialize the input parameters ϕ_1, ϕ_2, Population size ($PopSize$) and maximum number of iterations ($MaxIt$).
Step II: Population Initialization
Randomly initiate particle positions and corresponding velocities in the search space with **Pbest** and **Gbest** values. Set the generation number $k = 1$.
Step III: Update Position and Velocity
Update the velocities and positions of the ith particle for $(k+1)^{\text{th}}$ generation using the following equations: $v_{k+1}^i = v_k^i + \phi_1 r_1 (Pbest_k^i - x_k^i) + \phi_2 r_2 (Gbest_k - x_k^i)$ and $x_{k+1}^i = x_k^i + v_{k+1}^i$.
Step IV: Fitness Calculation
Calculate fitness function, **Pbest** and **Gbest** values for the updated population.

Step V: Convergence
If the convergence criterion is satisfied.
Gbest is the optimal solution.
STOP
Else go to **Step III**

Appendix B. Broyden–Fletcher–Goldfarb–Shanno (BFGS) Approach

To find the value of X such that the multi-variable function $f(X)$ attains minima at X. The algorithmic steps for Quasi-Newton algorithm, namely BFGS are provided as follows:

Step I: Initialization
Take arbitrary initial point X_1 and matrix $[A_1]$, where $[A_1]$ is the approximation for Hessian Matrix $[J_1]$ which is composed of the second partial derivatives of f.

Fix tolerance for stopping criterion $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$, maximum number of iterations $(MaxIt)$.

Set $i = 1$.
Step II: Calculate the Search direction for i^{th} iteration S_i using

$$S_i = -[A_i]^{-1} \nabla f(X_i).$$

Step III: Take optimal step length α_i^* such that

$$\alpha_i^* = \arg\min_{\alpha_i}(f(X_i + \alpha_i S_i)).$$

Step IV: Compute next approximation of X using

$$X_{i+1} = X_i + \alpha_i^* S_i.$$

Step V: Test the new approximation X_{i+1} for optimality.

If $\nabla f(X_{i+1})^T \nabla f(X_{i+1}) \leq \varepsilon_1$, **STOP**.

If $|f(X_{i+1}) - f(X_i)| \leq \varepsilon_2$, **STOP**.

If $\Delta X_i^T \Delta X_i \leq \varepsilon_3$, **STOP**.

Else go to **Step VI**

Step VI: Update approximation for Hessian matrix,

$$[A_{i+1}] = [A_i] + \frac{Y_i Y_i^T}{Y_i^T \Delta X_i} + \frac{\nabla f(X_{i+1})^T \nabla f(X_{i+1})}{\nabla f(X_i)^T S_i}$$

where

$$Y_i = \nabla f(X_{i+1}) - \nabla f(X_i)$$
$$\Delta X_i = X_{i+1} - X_i.$$

$i \to i + 1$, go to **Step II**